


Turning a Newtonian analogy for FLRW cosmology into a relativistic problem

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A Newtonian uniform ball expanding in empty space constitutes a common heuristic analogy for FLRW cosmology. We discuss possible implementations of the corresponding general-relativistic problem and a variety of new cosmological analogies arising from them. We highlight essential ingredients of the Newtonian analogy, including the fact that the Hawking-Hayward quasilocal mass that is always “Newtonian” in the sense that the magnetic part of the Weyl tensor does not contribute to it. A symmetry of the Einstein-Friedmann equations produces another one in the original Newtonian system.

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I. INTRODUCTION

A Newtonian analogy is often used to introduce Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology to beginners and to provide physical intuition (e.g., [1–4]—see [5] for a different version). The analogy is based on a ball of uniform density expanding in empty space and on the conservation of energy for a test particle on the surface of this ball. The Newtonian energy conservation equation is formally analogous to the Friedmann equation of relativistic cosmology for a universe filled with a perfect fluid with zero pressure (a dust). Of course, this analogy is not realistic, and the proper description of the universe requires general relativity (GR). Moreover, the analogy has limitations because the Newtonian ball can only produce the analogue of a dust-filled universe, while FLRW cosmology includes a much richer variety of matter content (radiation, dark energy, scalar fields, etc.). Most presentations in the literature and unpublished lecture notes available on the internet do caution that this is only an analogy. *A posteriori*, it is interesting to revisit this analogy from the GR point of view. Does the corresponding GR problem still lead to an analogy with FLRW cosmology? Does a relativistic isolated ball still expand like a FLRW universe, like a dust-dominated universe, or are there other possibilities? A similar situation involves the black hole concept, first introduced by Michell and Laplace in a naive Newtonian context [6,7], and then rediscovered in the Schwarzschild solution of the Einstein equations.

Here, we revisit the Newtonian analogy by considering an expanding (or contracting) isolated ball of uniform density in empty space in GR, and two possible ways to

extend the Newtonian analogy. The first, and easiest, way consists of cheating the difficulties and looking at the radial motion of a test particle just outside the surface of the ball, i.e., using the radial timelike geodesic equation in the vacuum, spherical geometry (which is necessarily Schwarzschild). The radial timelike geodesic equation is still formally analogous to the Friedmann equation for a dust-filled universe. This “easy” way to derive an analogy has significant limitations, which are discussed below. However, it can be generalized to many static and spherically symmetric geometries, producing a host of new cosmological analogies.

The second, and proper, way to address the problem consists of looking for exact solutions of the Einstein equation that are spherically symmetric, time-dependent, and asymptotically flat and are sourced by a ball of perfect fluid with uniform density and pressure, expanding or contracting *in vacuo*. Due to the Birkhoff theorem, the solution outside the ball is the Schwarzschild geometry, while the interior solution, to be determined, must be matched to it on the surface of the ball by imposing the Darmois-Israel junction conditions [8–10]. The interior of the ball will necessarily be described by a FLRW solution (hence, this system would be useless to introduce FLRW cosmology, but this is no longer the motivation).

The solution of this GR problem can be obtained as a special case of exact solutions of the Einstein equations describing spherical objects embedded in FLRW spaces or external fluids. A well-known one corresponds to the Oppenheimer-Snyder solution for dust collapse [11], in which the ball interior is positively curved FLRW. This is, however, only one possibility, and one would like to consider expanding spheres, possibly with a range of equations of state. The situation resembles that of an expanding fireball or spherical explosion, and there are

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already in the literature analytical solutions of the Einstein equations that can be used to solve our problem.

Assuming the FLRW line element,

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_{(2)}^2 \right), \quad (1.1)$$

in comoving coordinates $(t, r, \vartheta, \varphi)$, where the constant K is the curvature index and $d\Omega_{(2)}^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the line element on the unit two-sphere, the Einstein-Friedmann equations for a universe sourced by a perfect fluid with a stress-energy tensor $T_{ab} = (P + \rho)u_a u_b + P g_{ab}$ (where ρ is the energy density, P is the pressure, and u^c is the 4-velocity of comoving observers) are¹

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \rho - \frac{K}{a^2}, \quad (1.2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} (\rho + 3P), \quad (1.3)$$

$$\dot{\rho} + 3H(P + \rho) = 0, \quad (1.4)$$

where an overdot denotes differentiation with respect to the comoving time t and $H \equiv \dot{a}/a$ is the Hubble function. Only two of these three equations are independent.

We proceed as follows: the next section recalls the Newtonian analogy; Sec. III points out the analogy between radial timelike geodesics of Schwarzschild and the Friedmann equation and generalizes it to many spherical static geometries. Section IV discusses the proper GR problem, highlighting essential features of the Newtonian analog, while Sec. V discusses the Newtonian character of the quasilocal mass used. Section VI uses a little known symmetry of the Einstein-Friedmann equations to generate (only for zero energy/spatially flat universes) a new symmetry of the Newtonian ball, while Sec. VII contains the conclusions.

II. NEWTONIAN ANALOGY FOR FLRW UNIVERSES

Consider, in Newtonian physics,² a uniform expanding sphere with a radius $R(t)$, homogeneous density $\rho(t)$, and total mass $M = 4\pi R^3 \rho/3$, and a test particle of mass m on its surface. The total mechanical energy of this particle is

$$E = \frac{1}{2} m \dot{R}^2 - \frac{GMm}{R}, \quad (2.1)$$

and is constant. Rearranging this energy integral, we write

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} \rho + \frac{2E}{mR^2}. \quad (2.2)$$

By introducing the quantities,

$$H \equiv \frac{\dot{R}}{R}, \quad K \equiv -\frac{2E}{mc^2}, \quad (2.3)$$

where c is the speed of light, Eq. (2.2) becomes

$$H^2 = \frac{8\pi G}{3} \rho - \frac{Kc^2}{R^2}. \quad (2.4)$$

This equation is analogous to the Friedmann equation of relativistic cosmology for a universe filled by nonrelativistic matter.

If $E > 0$, which corresponds to $K < 0$ and to $v > v_{\text{escape}}$, where

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} \quad (2.5)$$

is the escape velocity from the surface of the ball, then the particle will escape to $R = +\infty$ with a residual velocity \dot{R}_∞ given by the limit,

$$0 < E = \frac{1}{2} m \dot{R}^2 - \frac{GMm}{R} \rightarrow \frac{1}{2} m \dot{R}_\infty^2. \quad (2.6)$$

If $E = 0$, corresponding to $K = 0$ and to $v = v_{\text{escape}}$, the particle barely escapes to infinity with a zero velocity \dot{R} , according to

$$0 = \frac{1}{2} m \dot{R}^2 - \frac{GMm}{R} \rightarrow \frac{1}{2} m \dot{R}_\infty^2 = 0. \quad (2.7)$$

If instead $E < 0$, corresponding to $K > 0$ and to $v < v_{\text{escape}}$, the particle reaches a maximum radius and then falls back reversing its velocity. In this case, one cannot take the limit $R \rightarrow +\infty$. The maximum radius is attained when $\dot{R} = 0$ just before the particle reverses its velocity,

$$E = -\frac{GMm}{R_{\text{max}}}, \quad (2.8)$$

which yields

¹We follow the notation of Ref. [12] and use units in which the speed of light and Newton's constant G are unity.

²We restore Newton's constant G and the speed of light c in this section.

$$R_{\max} = \frac{GMm}{|E|}. \quad (2.9)$$

Analytical solutions of the energy integral (2.3) in parametric form are well known. Define the new time variable η by

$$d\eta = \sqrt{\frac{2|E|}{m}} \frac{dt}{R(t)}; \quad (2.10)$$

then the solutions are

$$R = \frac{GMm}{2|E|} (1 - \cos \eta), \quad (2.11)$$

$$\pm(t - t_0) \sqrt{\frac{2|E|}{m}} = \frac{GMm}{2|E|} (\eta - \sin \eta), \quad (2.12)$$

if $E < 0$, or

$$R(t) \propto (t - t_0)^{2/3}, \quad (2.13)$$

if $E = 0$ (in this case, one can eliminate the parameter), or

$$R = \frac{GMm}{2|E|} (\cosh \eta - 1), \quad (2.14)$$

$$\pm(t - t_0) \sqrt{\frac{2E}{m}} = \frac{GMm}{2E} (\sinh \eta - \eta), \quad (2.15)$$

if $E > 0$.

Because the energy E is conserved, it is not possible for a closed universe to become an open (*i.e.*, expanding forever) one and *vice versa*, or for a closed or open universe to become flat and *vice versa*.

In this Newtonian analogy for the universe, the solution for $E < 0$ corresponds to an open universe that expands forever, the case $E > 0$ to a universe that reaches a maximum size and recollapses, and the $E = 0$ case to a critically open universe. This is only an analogy: in relativistic cosmology, the meaning of the variables is different, and there is no center of expansion for the universe. Nevertheless, this analogy recurs frequently in the literature and is even used as a toy model for quantum cosmology [13,14].

III. TEST PARTICLE STARTING ABOVE THE BALL SURFACE

Now let us promote the Newtonian problem to a general-relativistic one. We must derive a new, relativistic equation for the motion of the surface of a uniform ball of fluid,

which is a nontrivial task best postponed to the next section. Here, we consider a simpler alternative: in the Newtonian analogy, things proceed unchanged if the test particle is just above the surface of the ball instead of being located on it, and it shoots radially away from this surface. Things are unchanged as long as the expanding ball does not overtake the particle or the falling particle returning from a failed escape hits the ball (whether this happens depends on how the uniform ball expands, which in turn depends on the nature of the fluid in it, the equation of motion it satisfies, and the initial velocities of the test particle and ball surface). That this does not happen is certainly not warranted *a priori* and will be discussed in the following sections. For now, let us proceed by assuming that the test particle and ball do not meet, at least for a certain period of time.

In GR, nobody is forbidden to consider a particle *outside* the ball, that starts out radially close to it, as long as the surface of the ball does not overtake the particle. For a contracting ball, this does not happen at least until the particle reaches its maximum height and comes back (if it does). If it comes back, it would have to fall radially faster than the ball contracts, which is possible if the matter in the ball has pressure and the fluid does not follow geodesics. Or, a falling particle could hit an expanding ball. It is also possible that the surface of the expanding ball moves outward faster than the massive test particle, overtaking it. This will, again, happen if the fluid has pressure, or if its initial velocity is higher than that of the particle, *etc.* Separating the test particle from the surface of the ball opens up these new possibilities.

Assuming, for the moment, that the test particle and the ball surface do not collide, our massive test particle moves along a radial timelike geodesic of the Schwarzschild geometry,

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right) dt^2 + \frac{d\bar{r}^2}{1 - \frac{2m}{\bar{r}}} + \bar{r}^2 d\Omega_{(2)}^2. \quad (3.1)$$

The equation of radial geodesics is [12]

$$\dot{\bar{r}}^2 + \left(1 - \frac{2m}{\bar{r}}\right) \left(\frac{L^2}{\bar{r}^2} + \kappa\right) = E^2, \quad (3.2)$$

for $\bar{r} > 2m$, where an overdot now denotes differentiation with respect to the proper time τ or the affine parameter along the geodesic, and E and L are the conserved energy and angular momentum per unit mass of the particle, which are (apart from the sign) contractions of the particle 4-velocity u^a with the timelike and rotational Killing vector fields, respectively. In Eq. (3.2), $\kappa = 1$ for timelike geodesics, and $\kappa = 0$ for null geodesics.

A. Massive test particle

In the timelike case, the radial geodesic equation can be written as

$$\left(\frac{\dot{\bar{r}}}{\bar{r}}\right)^2 = \frac{E^2 - 1}{\bar{r}^2} + \frac{2m}{\bar{r}^3}. \quad (3.3)$$

This equation is analogous to the Friedmann equation (1.2) for a FLRW universe, where $K = 1 - E^2$ plays the role of the curvature index, $\rho = \rho_0/a^3$ corresponds to a radiation fluid with equation of state $P = \rho/3$, and $\rho_0 = \frac{3m}{4\pi}$. Therefore, there is a straightforward analogy between the motion of a test particle in the field of the ball and the Friedmann equation, as in the Newtonian situation. All three possible signs of the curvature K of the analogous FLRW universe are possible, but only an analogous universe containing a dust fluid can be obtained.

B. Generalization to any static spherical geometry

As a digression, we note that the cosmological analogy for timelike geodesics can often be generalized to static and spherically symmetric geometries. For such spacetimes, the line element can always be written in the Abreu-Nielsen-Visser gauge [15,16] as

$$ds^2 = -e^{-2\Phi(R)} \left(1 - \frac{2M(R)}{R}\right) dt^2 + \frac{dR^2}{1 - \frac{2M(R)}{R}} + R^2 d\Omega_{(2)}^2, \quad (3.4)$$

where $T^a \equiv (\partial/\partial t)^a$ is the timelike Killing vector and $M(R)$ is the Misner-Sharp-Hernandez mass [17,18] (to which the Hawking-Hayward quasilocal mass [19,20] reduces in spherical symmetry [21]). Consider radial timelike geodesics: the energy E of a particle of mass m and four-momentum $p^c = mu^c$ along such a geodesic is conserved, $p_c T^c = -E$, which yields

$$u^0 = \frac{dt}{d\tau} = \frac{\bar{E}e^{2\Phi}}{1 - \frac{2M}{R}}, \quad (3.5)$$

where $\bar{E} = E/m$ is the energy per unit mass. The normalization $u_c u^c = -1$ gives

$$g_{00} \left(\frac{dt}{d\tau}\right)^2 + g_{11} \left(\frac{dR}{d\tau}\right)^2 = -1; \quad (3.6)$$

substituting Eq. (3.5) to obtain $(dR/d\tau)^2$ and dividing the result by R^2 , one obtains

$$\left(\frac{1}{R} \frac{dR}{d\tau}\right)^2 = \frac{\bar{E}^2 e^{2\Phi}}{R^2} - \frac{1}{R^2} + \frac{2M(R)}{R^3}. \quad (3.7)$$

Many spherical spacetimes of interest in GR and in alternative theories of gravity satisfy the condition

$g_{tt}g_{RR} = -1$ (or $\Phi \equiv 0$), which is associated to special algebraic properties [22,23]; in this case, Eq. (3.7) reduces to

$$\left(\frac{1}{R} \frac{dR}{d\tau}\right)^2 = \frac{(\bar{E}^2 - 1)}{R^2} + \frac{2M(R)}{R^3}, \quad (3.8)$$

which can be analogous to the Friedmann equation (1.2) for a universe with curvature index $K = 1 - \bar{E}^2$ and energy density $\rho = \frac{3}{4\pi} \frac{M(a)}{a^3}$. To complete the analogy, the cosmic fluid must satisfy the covariant conservation equation (1.4), which yields

$$P = -\rho - \frac{a}{3} \frac{d\rho}{da} = -\rho - \frac{a}{4\pi} \left(\frac{M'}{a^3} - \frac{3M}{a^4}\right) = -\frac{M'}{4\pi a^2}. \quad (3.9)$$

This effective equation of state can be written in the time-dependent form,

$$P = -\frac{M'(a)}{4\pi a^2} \equiv w(a)\rho. \quad (3.10)$$

Examples of cosmic analogies derived from radial timelike geodesics of static spherical geometries are listed in the Appendix.

IV. EXACT GR SOLUTION FOR AN EXPANDING RELATIVISTIC BALL

Now let the massive particle sit on the surface of the ball and be a particle of the fluid composing the ball, which is always larger than its Schwarzschild radius. For a general fluid, this particle does not follow a timelike geodesic. In fact, in the presence of pressure, the four-gradient $\nabla_a P$ moves fluid particles away from geodesics. Only dust ($P = 0$) is geodesic in the absence of external forces.

Since we cannot ignore the matter at radii below the initial particle radius $R(0)$, we must now find, and solve, the equation of motion for the boundary of the relativistic ball.

A static ball with uniform density is described by the well-known Schwarzschild interior solution [24], but it is of no use here. We want instead a ball of uniform density that expands/contracts while remaining uniform. The metric must be asymptotically flat (we assume a zero cosmological constant). Due to Birkhoff's theorem, an observer outside the matter distribution (or on the ball surface) sees the Schwarzschild vacuum. What is the equation of the surface of the ball in this case? Is this motion geodesic or does the normal to the ball surface deviate from a geodesic vector? Moving on to time-dependent, asymptotically, flat fluid spheres, some exact solutions were provided early on by Vaidya [25]. They contain the Schwarzschild interior solution [24] and the Oppenheimer-Snyder [11] solution as special cases. The most useful geometries here are probably

those of Smoller and Temple [26], which contain the solution of the Einstein equations describing our situation as a special case and reproduce results of previous literature, that we briefly review here.

Mashhoon and Partovi [27] found that the unique solution for the spherical and shear-free motion of an uncharged perfect fluid obeying an equation of state is the FLRW one. However, another hypothesis must be added for the theorem to hold, namely that the energy density is uniform, $\partial\rho/\partial r = 0$, which is one of our needed assumptions [28]. Therefore, the interior of the ball can only be FLRW. *A posteriori*, this fact legitimates the use of a uniform ball in the Newtonian analogy. A nonuniform ball would lead to a spherically symmetric, but inhomogeneous, universe. Contracting balls with pressure were considered by Thompson and Whitrow [29,30] and Bondi [31,32], mainly to study gravitational collapse to black holes.

The Newtonian analogy requires also that the particles composing the ball, as well as the test particle on its surface, have no pressure and are subject only to gravity: the fluid composing the ball must be a dust for the interior geometry to match the exterior Schwarzschild one.

A. A special case of the Smoller-Temple shock wave solution

Smoller and Temple [26] consider exact solutions of the Einstein equations representing a spherical shock wave expanding into a gas, with interior and exterior matching on a zero thickness shell with no material on it (no jump). Because of uniformity, $\rho = \rho(t)$, $P = P(t)$, the metric inside the shock wave must be FLRW (all values of the curvature index K are possible),

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_{(2)}^2 \right). \quad (4.1)$$

Outside the shock wave, the geometry is that of a static and spherical Oppenheimer-Tolman solution (usually employed to describe the interior of a relativistic star),

$$d\bar{s}^2 = -B(\bar{r})dt^2 + \frac{d\bar{r}^2}{1 - 2M/\bar{r}} + \bar{r}^2 d\Omega_{(2)}^2, \quad (4.2)$$

in coordinates³ $(t, \bar{r}, \vartheta, \varphi)$, where $M(\bar{r})$ is the mass contained inside the sphere of radius \bar{r} , or

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\rho(\bar{r}), \quad (4.3)$$

$\rho(\bar{r})$ is the energy density at radius \bar{r} , and

³The comoving coordinate r inside the ball is not the same as the curvature coordinate \bar{r} outside.

$$\frac{B'(\bar{r})}{B(\bar{r})} = -\frac{2\bar{P}'(\bar{r})}{\bar{\rho} + \bar{P}}. \quad (4.4)$$

\bar{P} is the outside pressure, and a prime denotes differentiation with respect to this radial coordinate. The mass $M(\bar{r})$ coincides with the Misner-Sharp-Hernandez mass [17,18] at a radius \bar{r} , which is defined in any spherically symmetric spacetime by

$$1 - \frac{2M_{\text{MSH}}}{R} = \nabla^c R \nabla_c R \doteq g^{RR}, \quad (4.5)$$

where R is the areal radius and the last equality holds if the areal radius is employed as the radial coordinate [which is the case for the line element (4.2)].

The interior and exterior solutions are matched on the surface of a ball (the shock wave) by imposing the Darmois-Israel junction conditions [8–10] so that there is no jump in the first and second fundamental forms, and there is no material layer on the shell. The junction conditions for an outgoing shock wave give [26]

$$r\dot{a} = \sqrt{1 - Kr^2}\sqrt{1 - \Theta}, \quad (4.6)$$

$$\dot{r}a = \sqrt{1 - Kr^2} \frac{\sqrt{1 - \Theta}}{\gamma\Theta - 1}, \quad (4.7)$$

where

$$\Theta = \frac{1 - 2M/r}{1 - Kr^2}, \quad (4.8)$$

$$\gamma = \frac{\rho + \bar{P}}{\bar{\rho} + \bar{P}}. \quad (4.9)$$

A coordinate transformation between interior and exterior coordinates is found in [26], with the simple result,

$$\bar{r} = a(t)r, \quad (4.10)$$

implying that the areas of two-spheres of symmetry change smoothly across the shock wave [*i.e.*, the areal radius $R = a(t)r$ of the FLRW interior matches the areal radius \bar{r} of the Schwarzschild exterior] and that the surface of the ball comoves with its FLRW interior. Now we impose that the exterior is vacuum, $\bar{\rho} = \bar{P} = 0$, then the spherical and asymptotically flat exterior metric necessarily reduces to Schwarzschild. This situation is reported as a special case of the outgoing shock wave in [26]. It requires that the pressure P vanishes inside the entire ball in order to match the vanishing pressure at the boundary: the interior FLRW

fluid can only be dust.⁴ In the Newtonian analogy, one wants a particle on the ball surface subject only to gravity: a pressure gradient would complicate the Newtonian description, requiring the specification of an equation of state and could make the ball overtake a test particle placed on it (these complications would ruin the simplicity of the Newtonian analogy).

The vanishing of the outside pressure $\bar{P} \rightarrow 0$ corresponds to the limit $\gamma \rightarrow \infty$ and implies [26]

$$r = r_0 = \text{const} \quad (4.11)$$

at the surface of the ball. Then the density must scale as

$$\rho(t) = \frac{3M}{4\pi r_0^3 a^3}, \quad (4.12)$$

consistent with a dust, and the Friedmann equation at this surface reduces to

$$\dot{a}^2 = \frac{2M}{r_0^3 a} - K. \quad (4.13)$$

Apparently unknown to Smoller and Temple, the conclusion that only a uniform ball of dust can be matched to Schwarzschild was reached already by McVittie [34], Bondi [31], Mansouri [35], Mashhoon and Partovi [27], and Glass [36]. The surface of the ball expands into the surrounding Schwarzschild vacuum while comoving with its interior. Further, since $P = 0$, the fluid particles follow radial geodesics because of spherical symmetry, and the unit normal to the comoving ball surface is a timelike geodesic vector (radial geodesic congruences are normal to the surface of the ball because, due again to spherical symmetry, there is no vorticity). The radial geodesics of the interior geometry join smoothly the radial geodesics of the Schwarzschild exterior, provided that the ball surface comoves with its interior [26].

All possibilities for the spatial curvature of the FLRW space inside the ball are studied in [26]: for $K > 0$, the well-known Oppenheimer-Snyder solution describing the collapse of a ball of dust [11] is recovered by considering an ingoing shock wave [and changing the signs of the right-hand sides of Eqs. (4.6), (4.7)] with the ball boundary describing the cycloid,

$$a(\eta) = \frac{1}{2}(1 + \cos \eta), \quad (4.14)$$

$$t(\eta) = \frac{1}{2\sqrt{K}}(\eta + \sin \eta), \quad (4.15)$$

⁴A similar situation is encountered in Swiss-cheese models [33].

where the initial conditions $a(0) = 1, \dot{a}(0) = 0$ have been imposed and the curvature index has been normalized to $K = 2M/r_0^3$ [26]. For $K < 0$, one finds the solution [26],

$$\sqrt{a + a^2} - \frac{1}{2} \ln [1 + 2(a + \sqrt{a + a^2})] = \sqrt{|K|}t, \quad (4.16)$$

with the big bang initial condition $a(0) = 0$ while, for $K = 0$, the scale factor and the comoving surface follow the Einstein-de Sitter scaling $a(t) = (9\pi M/2)^{1/3} t^{2/3}$.

The surface of the ball, as well as the scale factor of its FLRW dust, satisfies the Newtonian energy equation (2.1). Indeed Bondi [31] remarks the similarity between the relativistic equation for the ball expansion $R(t)$ and Eq. (2.1), while Mashhoon and Partovi [27] refer explicitly to the Newtonian analogy for FLRW universes (certain errors in [27,35] were corrected in [28,36]).

V. RICCI AND WEYL TENSORS AND QUASILOCAL ENERGY

In GR, gravity is curvature and is described by the Riemann tensor R_{abcd} , which splits into a Ricci part constructed with R_{ab} and a Weyl part C_{abcd} [12],

$$R_{abcd} = C_{abcd} + g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{\mathcal{R}}{3}g_{a[c}g_{d]b}, \quad (5.1)$$

where $\mathcal{R} \equiv R^c_c$ is the Ricci scalar. Further, the Weyl tensor is decomposed into an electric and a magnetic part with respect to a chosen timelike observer. The electric part E_{ab} has a Newtonian analogue, while the magnetic part H_{ab} does not [37], and it contains the true (propagating) degrees of freedom of the gravitational field.

Let the timelike vector u^a be the four-velocity of an observer: following the definition of [38] (which differs from that of [37] in the magnetic part of the Weyl tensor, correcting a sign [38], see also p. 87 of [39]) the electric and magnetic parts of the Weyl tensor are [38,39]

$$E_{ac}(u) = C_{abcd}u^b u^d, \quad (5.2)$$

$$H_{ac}(u) = \frac{1}{2}\eta_{abpq}C^{pq}_{ce}u^b u^e, \quad (5.3)$$

respectively, where $\eta_{abcd} = \sqrt{-g}\epsilon_{abcd}$, ϵ_{abcd} is the Levi-Civita symbol and g is the determinant of the metric tensor g_{ab} , so that $\eta^{abcd} = \eta^{[abcd]}$ and $\eta^{0123} = 1/\sqrt{-g}$. According to the observer u^a , E_{ab} and H_{ab} are purely spatial and symmetric, trace-free tensors,

$$E_{ab}u^a = E_{ab}u^b = H_{ab}u^a = H_{ab}u^b = 0, \quad (5.4)$$

$$E_{ab} = E_{(ab)}, \quad H_{ab} = H_{(ab)}, \quad (5.5)$$

$$E^a_a = H^a_a = 0. \quad (5.6)$$

The Weyl tensor is reconstructed from its electric and magnetic parts according to [37–39]

$$C_{abcd} = (g_{abef}g_{cdpq} - \eta_{abef}\eta_{cdpq})u^e u^p E^{fq} - (\eta_{abef}g_{cdpq} + g_{abef}\eta_{cdpq})u^e u^p H^{fq} \quad (5.7)$$

with

$$g_{abef} \equiv g_{ae}g_{bf} - g_{af}g_{be}, \quad (5.8)$$

giving

$$C_{abcd} = u_a u_c E_{bd} - u_a u_d E_{bc} - u_b u_c E_{ad} + u_b u_d E_{ac} - \eta_{abef}\eta_{cdpq}u^e u^p E^{fq} - \eta_{abef}u_c u^e H_d^f + \eta_{abef}u^e u_d H_c^f - u_a u^p \eta_{cdpq}H_b^q + u_b u^p \eta_{cdpq}H_a^q. \quad (5.9)$$

By construction, in the GR extension of the uniform Newtonian ball, the Riemann tensor is purely Ricci in the interior and purely Weyl outside. In fact, all FLRW metrics are conformally flat, and the Weyl tensor C_{abcd} is conformally invariant [12]; therefore, it vanishes in FLRW, leaving only the Ricci part of R_{abcd} inside the ball. In the exterior vacuum region, the Ricci tensor $R_{ab} = 8\pi(T_{ab} - g_{ab}T/2)$ is identically zero, leaving only the Weyl part of R_{abcd} . Therefore, Ricci and Weyl tensors switch roles when crossing the ball boundary.

At the ball surface, both Ricci and Weyl are discontinuous. Focusing on the respective scalars $\mathcal{R} \equiv R^a_a$ and $C_{abcd}C^{abcd}$, the Ricci scalar $\mathcal{R} = -\rho + 3P = -\rho$ jumps discontinuously to zero at the ball surface, while the Weyl scalar is identically zero inside and jumps to the Schwarzschild value $C_{abcd}C^{abcd} = 48m^2/r^6$.

Since our GR problem originates in Newtonian physics, it is fit to discuss the Newtonian character of the (quasi-local) mass used in the discussion.

The interior mass (4.3) matches the exterior Schwarzschild mass m at the surface of the ball. For any K , the mass M at areal radius $R \leq R_0$ in the FLRW interior is

$$M^{(-)}(R) = \frac{4\pi\rho}{3}R^3, \quad (5.10)$$

and, as noted, it coincides with the Misner-Sharp-Hernandez quasilocal energy. In the limit $R \rightarrow a(t)r_0$ to the surface of the ball, which is comoving, $M^{(-)}$ becomes constant, and this is possible only because the energy density has the dust scaling $\rho \sim 1/a^3$, which is necessary to match the constant mass m of the Schwarzschild exterior. More generally, the Misner-Sharp-Hernandez mass of a comoving sphere of radius R in any FLRW space satisfies [40,41]

$$\dot{M}_{\text{MSH}} + 3H\frac{P}{\rho}M_{\text{MSH}} = 0, \quad (5.11)$$

and the constancy of M_{MSH} goes hand in hand with the vanishing of the pressure (it is a peculiarity of the Misner-Sharp-Hernandez mass to depend on ρ but not on P , while \dot{M}_{MSH} depends on P but not on ρ).

In principle, a different dependence of the energy density on the scale factor $\rho(a) = \rho_0/a^{3(w+1)}$, which occurs for a fluid equation of state $P = w\rho$, $w = \text{const}$, could be compensated if the ball boundary expands in the non-comoving way $R_0(t) \simeq t^{w+1}$ to keep $M^{(-)}$ constant. The Misner-Sharp-Hernandez mass of a sphere of radius R_0 (not comoving in general) satisfies [40,41]

$$\dot{M}_{\text{MSH}} + 4\pi\rho R_0^3 \left[H \left(1 + \frac{P}{\rho} \right) - \frac{\dot{R}_0}{R_0} \right] = 0, \quad (5.12)$$

and M_{MSH} remains constant if $\dot{R}_0/R_0 = (w+1)H$ for $P = w\rho$ with $w = \text{const}$. However, this choice would make the pressure P discontinuous at the ball boundary, this surface would no longer follow a timelike geodesic, and a test particle initially placed on it would immediately detach from it. This situation is unacceptable because then radial timelike geodesics inside and outside, together with the areal radii R and \bar{r} , would not match, signaling a discontinuity in the geometry. A discontinuity in P is associated with a material layer at the ball surface, which becomes a membrane with its own pressure foreign to the original Newtonian situation.

As noted, the mass used in the Oppenheimer-Tolman exterior is the Misner-Sharp-Hernandez mass common in spherical fluid dynamics [17,18]. The more general Hawking-Hayward quasilocal mass [19,20] reduces to it in spherical symmetry [21] and to the Arnowitt-Deser-Misner mass, if further spacetime is asymptotically flat. Several other inequivalent quasilocal energy constructions populate the literature (see [42] for a review). In our case, when the exterior Oppenheimer-Tolman metric is reduced to Schwarzschild, the Hawking-Hayward/Misner-Sharp-Hernandez mass $M_{\text{MSH}}^{(+)}$ reduces to the Schwarzschild mass, *i.e.*, the constant m appearing in the Schwarzschild metric (3.1).

In general, the Hawking-Hayward quasilocal mass splits into a contribution from the matter stress-energy tensor T_{ab} and a “pure gravity” contribution from the Weyl tensor. The latter arises solely from the electric part of the Weyl tensor, while the magnetic part does not contribute [43].

Specifically, the Hawking-Hayward quasilocal mass enclosed by a two-surface \mathcal{S} is defined as follows [19,20]. In a spacetime with metric g_{ab} , let \mathcal{S} be a spacelike, embedded, compact, and orientable two-surface. h_{ab} and $\mathcal{R}^{(h)}$ denote the two-metric and Ricci scalar induced on \mathcal{S} by g_{ab} . Let μ be the volume two-form on \mathcal{S} and A the area of \mathcal{S} . The congruences of ingoing (–) and outgoing (+) null

geodesic emanate from \mathcal{S} and have expansion scalars $\theta_{(\pm)}$ and shear tensors $\sigma_{ab}^{(\pm)}$. Let ω^a denote the projection onto \mathcal{S} of the commutator of the null normal vectors to \mathcal{S} , *i.e.*, the anoholonomicity [20]. The Hawking-Hayward quasilocal mass is [19,20]

$$M_{\text{HH}} \equiv \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\mathcal{S}} \mu \left(\mathcal{R}^{(h)} + \theta_{(+)}\theta_{(-)} - \frac{1}{2} \sigma_{ab}^{(+)} \sigma_{ab}^{(-)} - 2\omega_a \omega^a \right). \quad (5.13)$$

By splitting the Riemann tensor into Ricci and Weyl and, further, the latter into its electric and magnetic parts with respect to an observer with four-velocity parallel to the unit normal to the two-sphere \mathcal{S} , the Hawking-Hayward quasilocal mass splits as [43]

$$M_{\text{HH}} = \sqrt{\frac{A}{16\pi}} \int_{\mathcal{S}} \mu \left(h^{ab} T_{ab} - \frac{2T}{3} \right) - \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\mathcal{S}} \mu \eta_{abcdef} \eta_{cdpq} h^{ac} h^{bd} u^e u^p E^{fq}, \quad (5.14)$$

where the ‘‘pure gravity’’ contribution to M_{HH} [the second integral in the right-hand side of Eq. (5.14)] comes only from the electric part E_{ab} of the Weyl tensor, while the magnetic part H_{ab} does not contribute. In this sense, the Hawking-Hayward quasilocal mass is ‘‘Newtonian.’’

In the spherical spacetime corresponding to the Newtonian ball, the two-surface \mathcal{S} is a sphere of radius R or \bar{r} and Eq. (5.14) reduces to

$$M_{\text{MSH}}(R) = \frac{4\pi R^3}{3} \rho \theta_H(R_0 - R) + m \theta_H(\bar{r} - \bar{r}_0), \quad (5.15)$$

where R stands for the areal radius [$R = a(t)r$ inside and \bar{r} outside] and $\theta_H(x)$ is the Heaviside step function. In the interior of the ball, M_{MSH} is given entirely by the first term on the right-hand side, while the second term vanishes. In the exterior region, these two terms switch roles. The matching between interior and exterior makes the two terms equal on the ball boundary and guarantees the continuity of M_{MSH} . At all times, this quasilocal energy remains ‘‘Newtonian.’’⁵ Ultimately, the fact that a Newtonian analogy exists for FLRW cosmology is made possible by the fact that the corresponding GR manifold has vanishing magnetic part of the Weyl tensor everywhere according to static observers.

⁵A Newtonian character for the Misner-Sharp-Hernandez quasilocal mass is claimed also in Ref. [44] based on the study of the timelike radial geodesics of Schwarzschild.

VI. BACK TO NEWTONIAN GRAVITY FROM FLRW: SYMMETRIES

Symmetries of the Einstein-Friedmann equations have been explored in detail, mostly with the goal of generating new exact solutions (*e.g.*, [45–52]). For FLRW universes fueled by a perfect fluid, in general, one can rescale appropriately the time and scale factor or Hubble parameter, while changing to a different barotropic fluid, leaving the Einstein-Friedmann equations unchanged. Here we are concerned, in particular, with the symmetry found in Ref. [48] for spatially flat ($K = 0$) FLRW universes,⁶

$$\rho \rightarrow \bar{\rho} = \bar{\rho}(\rho), \quad (6.1)$$

$$H \rightarrow \bar{H} = \sqrt{\frac{\bar{\rho}}{\rho}} H, \quad (6.2)$$

$$P \rightarrow \bar{P} = -\bar{\rho} + \sqrt{\frac{\bar{\rho}}{\rho}} (P + \rho) \frac{d\bar{\rho}}{d\rho}, \quad (6.3)$$

[where $\bar{\rho}(\rho)$ is a free but positive and differentiable function], which leaves the Einstein-Friedmann equations invariant. Since the relativistic analogue of the Newtonian ball requires the interior FLRW metric to match the exterior Schwarzschild and, therefore, the pressure P to vanish everywhere, in order for this transformation to be a symmetry of the Newtonian analogue, it must be $P = \bar{P} = 0$. In this case, Eq. (6.3) becomes, using the new variable $z \equiv 1/\sqrt{\rho}$,

$$\frac{d\bar{z}}{dz} = \frac{\bar{\rho}^{-3/2} d\bar{\rho}}{\rho^{-3/2} d\rho} = 1, \quad (6.4)$$

and is trivially integrated to $\bar{z}(z) = z - z_0$, or

$$\frac{\bar{\rho}}{\rho} = \frac{1}{1 + \frac{\rho}{\rho_0} - 2\sqrt{\frac{\rho}{\rho_0}}}, \quad (6.5)$$

where ρ_0 (or z_0) is an integration constant. Therefore, the particular transformation,

$$\rho \rightarrow \bar{\rho} = \frac{\rho}{1 + \frac{\rho}{\rho_0} - 2\sqrt{\frac{\rho}{\rho_0}}}, \quad (6.6)$$

$$H \rightarrow \bar{H} = \frac{H}{1 + \frac{\rho}{\rho_0} - 2\sqrt{\frac{\rho}{\rho_0}}}, \quad (6.7)$$

preserves the zero-pressure condition and does not change the physics of the Newtonian ball [Eq. (2.1) with $E = 0$]

⁶The fact that this symmetry also relates de Sitter and anti-de Sitter spaces apparently went unnoticed in the literature.

used in the cosmological analogy. Since we have $K = 0$ and a dust, $R(t) = R_0 t^{2/3}$, and one obtains

$$\bar{\rho}(t) = \frac{3M}{4\pi R_0^3} \frac{1}{t^2 - 2\sqrt{\frac{3M}{4\pi R_0^2 \rho_0} t} + \frac{3M}{4\pi R_0^2 \rho_0}}, \quad (6.8)$$

$$\overline{\left(\frac{\dot{R}}{R}\right)} = \left(\frac{\dot{R}}{R}\right) \frac{t^2}{t^2 - 2\sqrt{\frac{3M}{4\pi R_0^2 \rho_0} t} + \frac{3M}{4\pi R_0^2 \rho_0}}. \quad (6.9)$$

In spite of the fact that uniform balls are often used in teaching Newtonian physics and in well-known problems such as gravity tunnels [53–58] and the terrestrial brachistochrone [54,56,59] (also discussed in view of futuristic technological applications [60]), this symmetry seems to have escaped attention.

Finally, another aspect of the Einstein-Friedmann equations that has been used extensively in FLRW cosmology is the fact that, in the presence of a single barotropic fluid with a constant equation of state $P = w\rho$, the Friedmann equation can be reduced to a Riccati equation by changing comoving into conformal time ([61–63]). This means that applying the change of variable for a dust-dominated FLRW universe to the Newtonian equation of motion of a test particle, the same reduction occurs. Indeed, the change,

$$r = s^2, \quad (6.10)$$

$$dt = rd\eta = s^2 d\eta, \quad (6.11)$$

(where the last equation is analogous to the change from comoving to conformal time $dt = ad\eta$ of FLRW cosmology) reduces the Newtonian equation of motion,

$$\frac{d^2 \vec{r}}{dt^2} = -\frac{GM}{r^3} \vec{r}, \quad (6.12)$$

to

$$\frac{d^2 \vec{s}}{d\eta^2} + \frac{|E|}{2} \vec{s} = 0, \quad (6.13)$$

where $E = \frac{1}{2} (d\vec{r}/dt)^2 - \frac{GM}{r} < 0$ is the particle energy. The change of variables (6.10), (6.11) reduces the Coulomb force problem to that of two decoupled harmonic oscillators. This change of variables for the Newtonian problem has been known since Euler and has been discussed or rediscovered many times through the years [64].

VII. DISCUSSION AND CONCLUSIONS

Abandoning the original pedagogical motivation for the Newtonian analogy to FLRW universes, one turns it into a rather interesting fully relativistic problem. It is natural, with a slight deviation from the original theme, to consider as the first model, the radial geodesic trajectories of

massive particles starting out just above the ball surface. This is very easy to do, but it has the drawback that the ball surface could meet and engulf the test particle. A formal analogy between the radial timelike geodesic equation and the Friedmann equation ensues, and the analogous FLRW universe can only be dust dominated. More general static and spherically symmetric spacetimes generate new cosmological analogies through their timelike radial geodesics: we have listed several of them in the Appendix.

Finally, the full general relativistic problem of a uniform fluid ball expanding (or contracting) in a surrounding Schwarzschild vacuum can be considered. This situation is a special case of more general setups in the literature on fluid spheres expanding in a (possibly cosmological) fluid. One quickly learns that two assumptions of the Newtonian analogy, usually not explained in the introductory literature, are crucial: (i) *uniform ball*. A nonuniform ball leads to a spherical inhomogeneous universe, which is not an analogy for FLRW. (ii) *test fluid*. As a consequence of the Darmois-Israel matching conditions, only a dust interior can match the Schwarzschild exterior. Moreover, a test particle initially placed on the surface of a ball of fluid other than dust would immediately detach from it, compromising the analogy. For a massive test particle to remain on the surface of the ball, the fluid in it must be dust.

The pressure must vanish at the surface of the ball to avoid a matter layer on it, while it is accepted as a necessary evil that the density is discontinuous there, as in the Schwarzschild interior solution.

To summarize, it is only natural to wonder about the fully relativistic counterpart of the Newtonian ball analogy. This problem has been mentioned sporadically, but repeatedly, in the literature on perfect fluid solutions of GR and/or fluid collapse since the 1960s but, to the best of our knowledge, this is the first time that it is approached systematically. The cosmological analogy highlights certain Newtonian features of the relativistic spaces used as examples, which do not seem to be appreciated in the literature. A novel feature is the role that the Hawking-Hayward quasilocal mass, which has Newtonian character in the sense discussed, plays in the exact solution analogous to the Newtonian ball. Furthermore, thanks to the analogy with FLRW cosmology and to relatively recent results on the symmetries of the Einstein-Friedmann equations, we have pointed out a new symmetry of the Newtonian problem.

To broaden the scope, one can, in principle, consider other physical situations. For example, one can study a discontinuous pressure associated with a spherical matter layer enclosing a fluid with $P \neq 0$. At this point, however, there is no longer a need for the ball, and one can retain only the spherical shell since, due to the Birkhoff theorem, the exterior geometry is still Schwarzschild. Now the problem bears little resemblance to an expanding universe mimicked by a uniform ball.

Inhomogeneous universes analogous to nonuniform balls with $\rho = \rho(t, r)$, $P = P(t, r)$ [33] also do not

resemble FLRW ones. In principle, one could also consider theories of gravity alternative to GR to evade the Birkhoff theorem and match an interior ball solution to an exterior geometry that is not Schwarzschild. All these options are found wanting because of one crucial point: the Schwarzschild solution is the unique solution of the vacuum Einstein equations that are spherically symmetric and asymptotically flat. In all the alternatives mentioned above, instead, there is no unique solution to the field equations; hence, these models are not as compelling. Looking for such alternatives, one goes further and further away from the simple Newtonian analogy, creating progressively more complicated and physically unjustified situations. To conclude, extensions to GR of the heuristic Newtonian ball problem have reasonably straightforward solutions and do create new formal analogies with FLRW universes and other GR solutions. These analogies highlight Newtonian features of the relativistic spacetimes, which provides a better understanding of the latter, as well as a new symmetry of the purely Newtonian system.

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APPENDIX: EXAMPLES OF ANALOGIES FROM RADIAL TIMELIKE GEODESICS OF STATIC SPHERICAL GEOMETRIES

Here, we provide examples of cosmic analogies arising from the radial timelike geodesics of well-known static and spherically symmetric metrics, beginning with situations in which $\Phi \equiv 0$ and $g_{tt}g_{RR} = -1$.

1. Reissner-Nordström metric

The first obvious candidate is the Reissner-Nordström metric,

$$ds^2 = -\left(1 - \frac{2m}{R} + \frac{Q^2}{R^2}\right)dt^2 + \frac{dR^2}{1 - \frac{2m}{R} + \frac{Q^2}{R^2}} + R^2 d\Omega_{(2)}^2, \quad (\text{A1})$$

where the constants m and Q are the mass and charge parameters. The Misner-Sharp-Hernandez mass is $M(R) = m - Q^2/(2R)$, and the analogous energy density in the Friedmann equation is

$$\rho(a) = \frac{3}{4\pi} \frac{m}{a^3} - \frac{3}{8\pi} \frac{Q^2}{a^4}. \quad (\text{A2})$$

The second term on the right-hand side corresponds to a radiation fluid with an unphysical negative energy density, which spoils the analogy. However, when $Q = 0$, the Reissner-Nordström metric reduces to Schwarzschild and the energy density coincides with the first term, a positive

dust density $\rho = \rho_0/a^3$ for a universe with any possible sign of the curvature index. The fact that the analogy holds with a completely nonrelativistic fluid (a dust) highlights the closeness of the radial geodesic motion in Schwarzschild spacetime to its Newtonian analogue, at least when the areal radius (also called ‘‘curvature coordinate’’) is used.

2. (Anti-)de Sitter space

The (anti-)de Sitter line element in locally static coordinates is

$$ds^2 = -(1 \mp H^2 R^2)dt^2 + \frac{dR^2}{(1 \mp H^2 R^2)} + R^2 d\Omega_{(2)}^2, \quad (\text{A3})$$

where the upper (respectively, lower) sign refers to de Sitter (respectively, anti-de Sitter) space. The Misner-Sharp-Hernandez mass is $M(R) = \pm H^2 R^3/2$, and the energy density and pressure of the analogous FLRW universe are

$$\rho(a) = \pm \frac{3H^2}{8\pi}, \quad (\text{A4})$$

$$P = -\frac{M'}{4\pi a^2} = \mp \frac{3H^2 a^2}{8\pi a^2} = -\rho, \quad (\text{A5})$$

that is, the energy density and pressure of a cosmological constant $\Lambda = \pm 3H^2$. Thus, the timelike geodesics of (anti-)de Sitter produce analogous universes with the same cosmological constant and any value of the curvature index. For $K = 0, -1$, respectively, this procedure reproduces the same (anti-)de Sitter space used as an analogue generator. In retrospect, this is due to the fact that a cosmological constant can be introduced in Newtonian theory through the Newtonian potential $\phi_N = H^2 R^2 = \Lambda R^2/3$, provided that the curvature (areal) radius is used. Again, this analogy highlights certain similarities between the relativistic space and a Newtonian situation.

3. Schwarzschild-de Sitter/Kottler geometry

The Schwarzschild-de Sitter/Kottler line element,

$$ds^2 = -\left(1 - \frac{2m}{R} - H^2 R^2\right)dt^2 + \frac{dR^2}{1 - \frac{2m}{R} - H^2 R^2} + R^2 d\Omega_{(2)}^2, \quad (\text{A6})$$

has Misner-Sharp-Hernandez mass $M = m + H^2 R^3/2$, generating the energy density of the FLRW analogue cosmic fluid,

$$\rho(a) = \frac{3m}{4\pi a^3} + \frac{3H^2}{8\pi}, \quad (\text{A7})$$

corresponding to a cosmological constant plus a dust. This was to be expected since the Schwarzschild black hole generates a dust-dominated analogous FLRW universe, while de Sitter space has itself as a Newtonian analogue.

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