Global asymptotic dynamics of cosmological Einsteinian cubic gravity

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We investigate the cosmological dynamics of an up to cubic curvature correction to general relativity (GR) known as cosmological Einsteinian cubic gravity, whose vacuum spectrum consists of the graviton exclusively. Its cosmology is well posed as an initial value problem. We are able to uncover the global asymptotic structure of the phase space of this theory. It is revealed that an inflationary, matter-dominated big bang is the global past attractor, which means that inflation is the starting point of any physically meaningful cosmic history. Given that higher-order curvature corrections to GR are assumed to influence the cosmological dynamics at early times—the high energy/large curvature limit—late-time inflation is possible only if one considers a nonvanishing cosmological constant term.

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I. INTRODUCTION

Higher curvature corrections to general relativity have now become popular, at a time when we are looking for a "compass" to point us in the right direction to find answers to the many unsolved puzzles of contemporary physics. Higher-order corrections to general relativity (GR) are required for a renormalization procedure to work [1–6]. Generalizations of general relativity are considered gravitational alternatives for unified description of the early time inflation with late-time cosmic acceleration in [7]. Among the modified theories considered are $F(R)$ and Horava-Lifshitz $F(R)$ gravity, scalar-tensor theory, string-inspired and Gauss-Bonnet theory, nonlocal gravity, nonminimally coupled models, and power-counting renormalizable covariant gravity. It was shown in that Ref. [7] that some versions of the theories mentioned may be consistent with local tests and may provide a qualitatively reasonable unified description of inflation with a dark energy epoch.

The higher curvature modifications of GR are characterized by the high complexity of their mathematical structure. In this case only through given approximations may one retrieve some useful analytic information on the cosmological dynamics. Otherwise one either has to perform a numeric investigation or apply the tools of dynamical systems theory. By means of the dynamical systems tools one obtains very useful information on the asymptotic dynamics of the above-mentioned cosmological models. The asymptotic dynamics is characterized by (i) attractor solutions into which the system evolves for a wide range of initial conditions, (ii) saddle equilibrium configurations that attract the phase space orbits in one direction but repel them in another direction, (iii) source critical points which may be pictured as past attractors, or (iv) limit circles, among others.

Although the use of dynamical systems is especially useful when one deals with scalar-field cosmological models—see Refs. [8–14] for a very small but representative sample of related research—their usefulness in other contexts has been explored as well [15–18]. In [15] by means of a combined use of the type Ia supernovae and $H(z)$ data tests, together with the study of the asymptotic properties in the equivalent phase space, Avelino *et al.* demonstrated that the bulk viscous matter-dominated scenario is not a good model for explaining the accepted cosmological paradigm. Meanwhile, in [16] García-Salcedo *et al.* explored the entire phase space of the so-called Veneziano/QCD ghost dark energy models,

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where the dynamics of the inner trapping horizon is ignored, and also the more realistic models where the time dependence of the horizon was taken into consideration. In a similar way, in [17], through exploration of the asymptotic properties of the corresponding cosmological model in the phase space, it was investigated to which extent noncommutativity—a property of quantum nature—may influence the cosmological dynamics at late times/large scales. The dynamical systems tools were also applied in [18] to study the asymptotic properties of a cosmological model based on a nonlinear modification of general relativity in which the standard Einstein-Hilbert action is replaced by one of Dirac-Born-Infeld type containing higher-order curvature terms.

In a recent paper [19] an up to cubic curvature correction to GR was proposed, with the following features: (i) its vacuum spectrum consists of a transverse massless graviton exclusively, just as in GR, (ii) it possesses well-behaved black hole solutions which coincide with those of Einstein cubic gravity (ECG) [20–22], (iii) its cosmology is well posed as an initial value problem, and (iv) it entails a geometric mechanism triggering an inflationary period in the early Universe (driven by radiation) with a graceful exit to a late-time cosmology arbitrarily close to Λ cold dark matter (ΛCDM).

In this paper we shall look for the global asymptotic dynamics of the cosmological Einsteinian cubic gravity (CECG) model proposed in [19] as a further generalization of the ECG. Our aim is to correlate the generic solutions of the model with past and future attractors as well as with saddle equilibrium configurations in some state space. This will give a solid mathematical basis to several statements made in [19]. It will be confirmed, in particular, that nonstandard matter-dominated inflationary Friedmann evolution is the global past attractor of any phase space orbits that represent viable cosmic histories. Our results will show that the graceful exit to a late-time ΛCDM cosmology in the CECG model is a consequence not of the proposed curvature modification of GR but of the presence of a nonvanishing cosmological constant term. The existence of a phase of decelerated expansion in the model, allowing for the correct amount of cosmic structure to form, depends on the initial conditions. A similar study of so-called $f(P)$ cubic gravity was presented in [23].

We have organized the paper as follows. In Sec. II we expose the basic elements of the CECG model, including the cosmological equations of motion. In Sec. III we trade the second-order cosmological field equations for a set of autonomous ordinary differential equations on some phase space variables, which we identify with the dynamical system of the model. In that section we find the critical points of the resulting dynamical system and study their existence and stability properties. To illustrate our study with numeric computations a phase portrait of the model is drawn. The particular case of the CECG model without the cosmological constant is explored in Sec. IV to elucidate the role of vacuum energy in global asymptotic dynamics. In Sec. V we discuss the most important physical aspects resulting from the dynamical system investigation, and in Sec. VI brief conclusions are drawn.

II. THE FORMALISM

The ECG formalism [20–22] is the outcome of an approach based on a D-dimensional theory involving arbitrary contractions of the Riemann tensor and the metric given by the action

$$S = \int d^D x\sqrt{|g|}\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\sigma\lambda}), \tag{1}$$

with equations of motion

$$\mathcal{E}_{\mu\nu} = P_{\mu\sigma\rho\lambda}R_{\nu}^{\sigma\rho\lambda} - \frac{1}{2}g_{\mu\nu}\mathcal{L} - 2\nabla^{\lambda}\nabla^{\sigma}P_{\mu\lambda\sigma\nu} = 0, \tag{2}$$

where $\mathcal{E}_{\mu\nu}$ is the Euler-Lagrange tensor and

$$P^{\mu\nu\sigma\lambda} \equiv \frac{\partial\mathcal{L}}{\partial R_{\mu\nu\sigma\lambda}}\bigg|_{g_{\alpha\beta}}$$

contains derivatives of the metric up to fourth order. The linearization of Eq. (2) around maximally symmetric backgrounds with Riemann tensor

$$R^{(0)}_{\mu\nu\sigma\lambda} = \Lambda[g^{(0)}_{\mu\sigma}g^{(0)}_{\lambda\nu} - g^{(0)}_{\mu\lambda}g^{(0)}_{\sigma\nu}],$$

where the metric gets small perturbations of the type $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ (here $g^{(0)}_{\mu\nu}$ is the background metric, while $h_{\mu\nu} \ll 1$ are the small perturbations). This yields a gravitational spectrum consisting of[1] (i) a massless graviton, (ii) a massive (ghost) graviton with mass squared,

$$m_g^2 = \frac{2(D-3)\Lambda a - e}{2a + c},$$

and (iii) a massive scalar mode with mass squared,

$$m_s^2 = \frac{(D-2)e - 4\Lambda[a + (D-1)(Db + c)]}{2a + Dc + 4(D-1)b},$$

where a, b, c, and e are constants.

Let us apply the above-mentioned linearization procedure to a general D-dimensional cubic Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}R - \Lambda_0 + \sum_{i=1}^{3}\alpha_i\mathcal{L}_i^{(2)} + \sum_{i=1}^{8}\beta_i\mathcal{L}_i^{(3)}, \tag{3}$$

where

[1]For full details of the linearization procedure, see [20].

$$\mathcal{L}_1^{(2)} = R^2, \quad \mathcal{L}_2^{(2)} = R_{\mu\nu}R^{\mu\nu}, \quad \mathcal{L}_3^{(2)} = R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda},$$
$$\mathcal{L}_1^{(3)} = R_{\mu\ \ \nu}^{\ \sigma\ \ \lambda}R_{\sigma\ \ \lambda}^{\ \rho\ \ \delta}R_{\rho\ \ \delta}^{\ \mu\ \ \nu}, \quad \mathcal{L}_2^{(3)} = R_{\mu\nu}^{\ \ \ \sigma\lambda}R_{\sigma\lambda}^{\ \ \ \rho\delta}R_{\rho\delta}^{\ \ \ \mu\nu},$$
$$\mathcal{L}_3^{(3)} = R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma}{}_{\rho}R^{\lambda\rho}, \quad \mathcal{L}_4^{(3)} = R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}R,$$
$$\mathcal{L}_5^{(3)} = R_{\mu\nu\sigma\lambda}R^{\mu\sigma}R^{\nu\lambda}, \quad \mathcal{L}_6^{(3)} = R_{\mu}^{\ \nu}R_{\nu}^{\ \sigma}R_{\sigma}^{\ \mu},$$
$$\mathcal{L}_7^{(3)} = R_{\mu}^{\ \nu}R_{\nu}^{\ \mu}R, \quad \mathcal{L}_8^{(3)} = R^3.$$

The constant coefficients α_i and β_i are related to the constants D, Λ, a, b, c, and e (see [20]). Then let us consider the limit $|m_g| \to \infty$, $|m_s| \to \infty$ of the cubic theory, which means that $2a + c = 4b + c = 0 \Rightarrow 4\alpha_1 = 4\alpha_3 = -\alpha_2$. In this case the massive vacuum modes become infinitely heavy and decouple from the spectrum of the theory, leaving the massless graviton as the only propagating vacuum degree of freedom, as in general relativity. Additionally, it is necessary that the coefficients β_i's be independent of the dimension D of the space. It was demonstrated in [20] that the most general theory with the same form as Eq. (3), possessing dimension-independent couplings, and sharing the same spectrum as GR reads

$$\mathcal{L} = \frac{1}{2}R - \Lambda_0 + \alpha\mathcal{X}_4 + \beta\mathcal{X}_6 + \lambda\mathcal{P}, \tag{4}$$

where, in four dimensions ($D = 4$), the quadratic Lovelock term $\mathcal{X}_4$ is topological, while the cubic Lovelock term $\mathcal{X}_6$ vanishes identically. The cubic term $\mathcal{P}$,

$$\mathcal{P} = 12R_{\mu\ \ \lambda}^{\ \nu\ \ \sigma}R_{\nu\ \ \sigma}^{\ \tau\ \ \rho}R_{\tau\ \ \rho}^{\ \mu\ \ \lambda} + R_{\mu\lambda}^{\ \ \ \nu\sigma}R_{\nu\sigma}^{\ \ \ \tau\rho}R_{\tau\rho}^{\ \ \ \mu\lambda} - 12R_{\mu\lambda\nu\sigma}R^{\mu\nu}R^{\lambda\sigma} + 8R_{\mu}^{\ \lambda}R_{\lambda}^{\ \nu}R_{\nu}^{\ \mu}, \tag{5}$$

is neither trivial nor topological in four dimensions.

A. The CECG model

In [19] a cubic modification of Einstein's GR was proposed which generalizes the ECG (4). It is known as CECG and is based on the following action[2]:

$$S = \frac{1}{2}\int d^4x\sqrt{|g|}[R - 2\Lambda + 2\beta(\mathcal{P} - 8\mathcal{C}) + 2\mathcal{L}_m], \tag{6}$$

where Λ is the (non-negative) cosmological constant, β is a non-negative free parameter,[3] $\mathcal{L}_m$ is the Lagrangian of the matter degrees of freedom, and

$$\mathcal{C} = R_{\mu\lambda\nu\sigma}R^{\mu\lambda\nu}{}_{\tau}R^{\sigma\tau} - \frac{1}{4}RR_{\mu\lambda\nu\sigma}R^{\mu\lambda\nu\sigma} - 2R_{\mu\lambda\nu\sigma}R^{\mu\nu}R^{\lambda\sigma} + \frac{1}{2}RR_{\mu\nu}R^{\mu\nu}, \tag{7}$$

is another cubic invariant. Although this invariant was previously found in [22], in that reference Hennigar *et al.* were interested in static spherically symmetric spaces where $\mathcal{C}$ vanishes. The equations of motion that follow from (6) read

$$2\mathcal{E}_{\mu\nu} = G_{\mu\nu} + g_{\mu\nu}\Lambda + 2\beta\left[\left(\frac{\partial\mathcal{R}_{(3)}}{\partial R^{\mu\alpha\beta\sigma}}\right)R_{\nu}^{\ \alpha\beta\sigma} - \frac{1}{2}g_{\mu\nu}\mathcal{R}_{(3)} - 2\nabla^{\alpha}\nabla^{\beta}\left(\frac{\partial\mathcal{R}_{(3)}}{\partial R^{\mu\alpha\beta\nu}}\right)\right] = 0, \tag{8}$$

where $\mathcal{R}_{(3)} \equiv \mathcal{P} - 8\mathcal{C}$. In general Eq. (8) is fourth order, so the Lovelock theorem [26], which requires second-order differential equations on any background [27], is not violated by the CECG theory.[4] As is clearly stated in [19], the limit $|m_g| \to \infty$, $|m_s| \to \infty$ means that the massive modes do not propagate in vacuum. However, the theory may develop instabilities beyond the linearized regime or around other backgrounds.

In a Friedmann-Robertson-Walker (FRW) space,

$$ds^2 = -dt^2 + a^2(t)\delta_{ik}dx^i dx^k,$$

the field equation (8) is second order [19],

$$3H^2(1 + 16\beta H^4) = \rho_m + \Lambda,$$
$$2\dot{H}(1 + 48\beta H^4) = -(p_m + \rho_m), \tag{9}$$

together with the continuity equation $\dot{\rho}_m = -3H(\rho_m + p_m)$. In what follows, for simplicity, we assume the following equation of state: $p_m = \omega_m\rho_m$, where the constant ω_m is the equation of state parameter. That the equations of motion are second order entails a well-posed initial value problem, i.e., that the resulting FRW cosmology is well behaved.[5]

[2]In [24] it was shown that the combination of cubic invariants defining five-dimensional quasitopological gravity, when written in four dimensions, reduces to CECG. It also introduced a quartic version of the CECG and a combination of quintic invariants with the properties of the above-mentioned theory. Meanwhile, in [25] it was shown how to construct invariants up to eighth order in the curvature.

[3]As we shall see in Sec. V the Hubble rate H is a real quantity only if $\beta \geq 0$.

[4]Take, for instance, the plane-symmetric Bianchi I spacetime with line element

$$ds^2_{\rm BI} = -dt^2 + a^2dx^2 + b^2\delta_{ab}dy^a dy^b, \qquad a, b = 2, 3,$$

where $a = a(t)$ and $b = b(t)$ are the scale factors. As shown in Appendix A of [28], the CECG equations of motion for vacuum in this case contain up to fourth-order time derivatives of the scale factors: $\ddddot{a}$, $\ddddot{b}$.

[5]See, however, Ref. [28], where the well posedness of the CECG cosmological model is challenged.

III. DYNAMICAL SYSTEM

We introduce the following bounded variables of a given phase space:

$$x \equiv \frac{16\beta H^4}{1+16\beta H^4}, \qquad y \equiv \frac{\Lambda}{3H^2+\Lambda}, \tag{10}$$

where $0 \le x \le 1$ and $0 \le y \le 1$. The modified Friedmann constraint—the first equation in Eq. (9)—can be written as follows:

$$\Omega_m \equiv \frac{\rho_m}{3H^3} = \frac{1-(2-x)y}{(1-x)(1-y)}. \tag{11}$$

Meanwhile,

$$\frac{\dot{H}}{H^2} = -\frac{3(\omega_m+1)(1-x)}{2(1+2x)}\Omega_m. \tag{12}$$

In terms of the phase space variables x, y, the second-order cosmological equations (9) may be traded for the following two-dimensional autonomous dynamical system:

$$\begin{aligned} \frac{dx}{dv} &= \frac{6(\omega_m+1)x(1-x)[(2-x)y-1]}{1+2x}, \\ \frac{dy}{dv} &= -\frac{3(\omega_m+1)y(1-y)[(2-x)y-1]}{1+2x}, \end{aligned} \tag{13}$$

where we have introduced the time variable $v = \int(1+\Lambda/3H^2)Hdt$.

The phase space in which to look for critical points of the dynamical system in Eq. (13) is given by the following region of the (x, y) plane:

$$\Psi = \{(x,y): 0 \le x \le 1, 0 \le y \le (2-x)^{-1}\}. \tag{14}$$

The boundary

$$\partial\Psi = \{(x,y): 0 \le x \le 1, y = (2-x)^{-1}\} \tag{15}$$

separates the physically meaningful region of the phase space where $\Omega_m \ge 0$ from the unphysical region where $\Omega_m < 0$ (the gray region in the left panel of Fig. 1).

Another curve of physical interest is the one related to the change of sign of the deceleration parameter:

$$q \equiv -1 - \dot{H}/H^2, \tag{16}$$

i.e., the curve that follows from the condition $q = 0$,

$$y = \frac{3(\omega_m+1) - 2(1+2x)}{3(\omega_m+1)(2-x) - 2(1+2x)}. \tag{17}$$

This curve separates regions with accelerated expansion from regions with decelerated expansion (the magenta regions in Fig. 1).

A. Critical points and their properties

The critical points $P_i:(x_i, y_i)$ of the dynamical system in Eq. (13) in the phase space Ψ, as well as their stability properties, are listed and briefly discussed below.

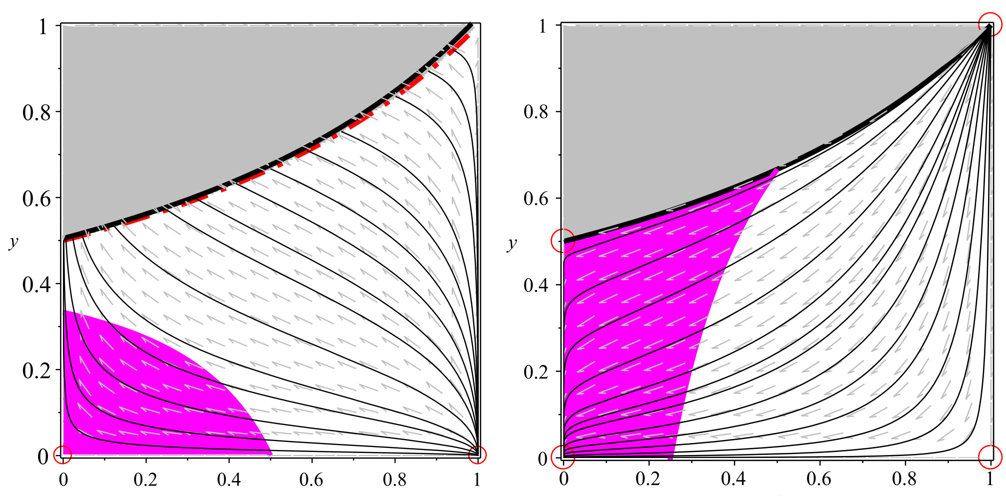

FIG. 1. Phase portrait (left panel) of the dynamical system in Eq. (13) for radiation, $\omega_m = 1/3$, and (right panel) of the dynamical system in Eq. (21), where $\Lambda = 0$. The physically meaningful region of the phase space Ψ [Eqs. (14) and (25)] lies below the boundary $\partial\Psi$ (black solid curve). The gray region in the drawings does not represent physically meaningful cosmological evolution. The magenta region which is bounded by the curve [(left panel) Eq. (17), (right panel) Eq. (24)] is the subspace where the expansion is decelerated. The critical points of the dynamical systems appear to be enclosed by the small circles. The de Sitter attractor manifold $\mathcal{M}_{dS}$ in Eq. (15), which exists only for the dynamical system in Eq. (13), is represented by the thick dash-dotted curve that coincides with the upper boundary $\partial\Psi$ in the left panel.

(1) Inflationary big bang solution, $P^{\rm infl}_{\rm bb}:(1,0)$. The eigenvalues of the linearization matrix evaluated at this point are

$$\lambda_1 = \omega_m + 1, \quad \lambda_2 = 2(\omega_m + 1).$$

Hence, this is the source point (global past attractor), and it is characterized by

$$x = 1 \Rightarrow H^4 \gg (16\beta)^{-1}, \quad y = 0 \Rightarrow 3H^2 \gg \Lambda,$$

which leads to the following modified Friedmann equation:

$$48\beta H^6 = \rho_m. \tag{18}$$

In this case Ω_m is undefined while

$$\dot{H}/H^2 = -(\omega_m + 1)/2,$$

so for the deceleration parameter (16) we get $q = (\omega_m - 1)/2$. Since, for physically meaningful matter, $0 \le \omega_m \le 1 \Rightarrow -1/2 \le q \le 0$, this means that the critical point $P^{\rm infl}_{\rm bb}$ is to be associated with accelerated expansion. This is why we call it the "inflationary big bang" to differentiate it from the standard big bang.

(2) Matter domination, $P_{\rm mat}:(0,0)$. Given that the eigenvalues of the linearization matrix at $P_{\rm mat}$,

$$\lambda_1 = -6(\omega_m + 1), \quad \lambda_2 = 3(\omega_m + 1),$$

are of different signs, this means that the matter-dominated solution is a saddle critical point. At this solution $\Omega_m = 1 \Rightarrow 3H^2 = \rho_m$ and

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(\omega_m + 1) \Rightarrow q = \frac{3\omega_m + 1}{2}.$$

(3) de Sitter attractor manifold:

$$\mathcal{M}_{\rm dS}:\left(x, \frac{1}{2-x}\right), \quad 0 \le x \le 1.$$

For points in $\mathcal{M}_{\rm dS}$ we obtain the following eigenvalues of the corresponding linearization matrix:

$$\lambda_1 = 0, \quad \lambda_2 = 3(\omega_m + 1)\left(\frac{x-1}{2-x}\right).$$

The vanishing eigenvalue is associated with an eigenvector that is tangent to the manifold at each point. The second eigenvalue is always a nonpositive quantity. This means that, as seen in Fig. 1, each one of the critical points in $\mathcal{M}_{\rm dS}$ is a local attractor, i.e., the manifold itself is a global attractor of orbits in Ψ. For each point in the de Sitter attractor manifold, $\dot{H} = 0$, $\Omega_m = 0 \Rightarrow q = -1$.

Notice that all three of the above critical points always exist.

In the left panel of Fig. 1 the phase portrait of the dynamical system in Eq. (13) is shown. The critical points $P^{\rm inf}_{\rm bb}$ and $P_{\rm mat}$ appear to be enclosed by the small circles, while the de Sitter attractor $\mathcal{M}_{\rm dS}$ is represented by the dash-dotted curve, which coincides with the upper boundary $\partial\Psi$ of the physically meaningful phase space (black solid curve). The gray region above the boundary is unphysical since $\Omega_m < 0$. The magenta region of the phase space, which is bounded by the curve [Eq. (17)], represents the subspace where the expansion of the Universe is decelerated. Hence, given that decelerated expansion is required for the formation of the amount of observed cosmic structure to happen, only those orbits that go across the latter region represent viable cosmic histories. These orbits cross the boundary represented by Eq. (17) twice, and the corresponding cosmic histories show two periods of accelerated expansion separated by a period of decelerated expansion when the cosmic structure forms. As seen, the natural exit from early time inflation to a decelerated expansion period where the cosmic structure forms depends on the initial conditions.

IV. CECG MODEL WITHOUT THE COSMOLOGICAL CONSTANT

Let us investigate the role of the cosmological constant in the global asymptotic dynamics of the CECG model. For this purpose we shall study the model with a mix of two fluids, dust and radiation, without the cosmological constant. In this case the cosmological equations of the CECG model read

$$3H^2(1 + 16\beta H^4) = \rho_d + \rho_r,$$
$$2\dot{H}(1 + 48\beta H^4) = -\rho_d - \frac{4}{3}\rho_r,$$
$$\dot{\rho}_d = -3H\rho_d, \quad \dot{\rho}_r = -4H\rho_r, \tag{19}$$

where ρ_d and ρ_r represent the energy densities of the dust and the radiation, respectively.

We shall trade the above system of second-order differential equations for a two-dimensional dynamical system. For this purpose we choose the phase space coordinate x defined in Eq. (10) and the new y coordinate:

$$y = \frac{\Omega_r}{1 + \Omega_r}, \quad \Omega_r \equiv \frac{\rho_r}{3H^2}. \tag{20}$$

Then Eq. (19) is equivalent to the dynamical system

$$\frac{dx}{dv} = -\frac{2x(1-x)[3-(2+x)y]}{1+2x},$$
$$\frac{dy}{dv} = -2y(1-y)\left[2-2y-\frac{3-(2+x)y}{2(1+2x)}\right], \quad (21)$$

where we have introduced the following time variable, $v=\int(1+\rho_r/3H^2)Hdt$ and, to eliminate the Ω_d terms, we have used the following relationship:

$$\Omega_d = \frac{1}{1-x} - \frac{y}{1-y}. \quad (22)$$

Other useful equations are

$$\frac{\dot{H}}{H^2} = \frac{(x+2)y-3}{2(1+2x)(1-y)}, \quad (23)$$

and the equation that follows from requiring that the deceleration parameter vanishes:

$$y=(4x-1)/3x. \quad (24)$$

As above, this curve separates the region where the expansion is accelerated from the region where it is decelerated.

A. Equilibrium states

The critical points $P_i:(x_i,y_i)$ of the dynamical system in Eq. (21) are found in the region of the phase space

$$\Psi = \{(x,y): 0\le x\le 1, 0\le y\le (2-x)^{-1}\}, \quad (25)$$

where $\Omega_d \ge 0$. Here we list the existing critical points and briefly comment on their properties, including their stability.

(1) Inflationary radiation-dominated big bang, $P^{\rm infl}_{\rm rad}:(1,1)$. This solution is the global past attractor to which every orbit of the phase space converges to the past (this is confirmed numerically). In this case the cosmic dynamics is governed by a modified Friedmann equation,

$$48\beta H^6 = \rho_r \Rightarrow a(t)\propto t^{3/2},$$

where we have taken into account that $\rho_r \propto a^{-4}$.

(2) Radiation-dominated Friedmann expansion solution, $P_{\rm rad}:(0,1/2)$. In this case the eigenvalues of the linearization matrix are $\lambda_1=1/2$ and $\lambda_2=-4$, so this is a saddle equilibrium point. We have that the cosmic dynamics is governed by the standard Friedmann equation $3H^2=\rho_r$.

(3) Inflationary nonstandard dust-dominated solution, $P^{\rm infl}_{\rm dust}:(1,0)$. It is a saddle critical point since the eigenvalues of the corresponding linearization matrix, $\lambda_1=2$ and $\lambda=-3$, are of opposite signs. According to this solution the cosmic dynamics is governed by the modified Friedmann equation

$$48\beta H^6 = \rho_d \Rightarrow a(t)\propto t^2,$$

where we have taken into account that $\rho_d \propto a^{-3}$.

(4) Dust-dominated solution, $P_{\rm dust}:(0,0)$. The eigenvalues of the linearization matrix, $\lambda_1=-1$, $\lambda_2=-6$, are both negative quantities, so the dust-dominated solution, $3H^2=\rho_d$, is the global attractor.

In the right panel of Fig. 1 the phase portrait of the dynamical system in Eq. (21) is drawn. The critical points are enclosed in the small circles. The black solid curve divides the phase space into a physically meaningful region Ψ [Eq. (25)] (below the curve) and a region that does not represent physically meaningful cosmic behavior (the gray region above the boundary). The magenta region represents the subspace where the expansion is decelerating.

As can be seen in the phase portraits in Fig. 1, there are two types of cosmic evolution. To the first type belong those orbits that, after emerging from the inflationary radiation-dominated past attractor, approach the (also inflationary) nonstandard dust-dominated solution to finally end up at the global attractor: the standard decelerated expansion matter-dominated solution. The second type consists of orbits that, after emerging from the global past attractor, approach the standard decelerated expansion radiation-dominated solution and then are attracted by the standard matter-dominated solution (the global future attractor). The first type of orbits lead to a not as well motivated kind of cosmic evolution as that in the second type since there is not a period of standard radiation-dominated decelerated expansion.

V. DISCUSSION

One of the most interesting consequences of the present scenario is the existence of a matter-dominated inflationary big bang—critical point $P^{\rm infl}_{\rm bb}:(1,0)$—which is the global past attractor. Hence, all of the orbits emerge from this unstable equilibrium inflationary state. Given the modified Friedmann equation (18) and that $\rho_m \propto a^{-3(\omega_m+1)}$, it follows that the scale factor evolves with the cosmic time as $a(t)\propto t^{\frac{2}{\omega_m+1}}$ so that

$$\frac{\ddot{a}}{a} = \frac{2(1-\omega_m)}{(1+\omega_m)^2}t^{-2}, \quad H(t) = \frac{2}{\omega_m+1}t^{-1}. \quad (26)$$

Our results confirm in rigorous mathematical terms the conclusion of [19]: that in the CECG scenario primordial inflation is a natural stage from which any plausible cosmic history—depicted by given orbits in the phase space—starts. This is to be expected since higher curvature corrections, such as cubic ones, are expected to modify the dynamics at early times, i.e., at very high energy/curvature. However, the late-time acceleration in the model is due to the cosmological constant term and has nothing to do with the geometric properties of the CECG setup. In this regard in Sec. IV we studied the CECG model with a mix of two fluids, radiation and dust, and with the vanishing cosmological constant $\Lambda = 0$. It is confirmed that, as stated in [19], the inflationary radiation-dominated stage driven by the nonstandard Friedmann equation,

$$48\beta H^6 = \rho_r,$$

is the global past attractor, i.e., it is the starting point of any orbit in the phase space. However, the global future attractor is the standard dust-dominated decelerated expansion driven by the Friedmann equation $3H^2 = \rho_d$. This means that the late-time de Sitter solution in the CECG model is possible only if we consider a nonvanishing cosmological constant $\Lambda \neq 0$, so it is not a curvature effect.

The latter result can be understood in an analytical way. Let us start with the following equation—Eq. (9) of [19]:

$$\frac{\ddot{a}}{a} = H^2 - \frac{(\omega_m + 1)\rho_m}{2(1 + 48\beta H^4)}, \qquad (27)$$

which is obtained by combining $\ddot{a}/a = H^2 + \dot{H}$ with Eq. (9). Then we substitute H^2 from Eq. (9) into Eq. (27) to get

$$\frac{\ddot{a}}{a} = \frac{\rho_m + \Lambda}{3(1 + 16\beta H^4)} - \frac{(\omega_m + 1)\rho_m}{2(1 + 48\beta H^4)}. \qquad (28)$$

From Eq. (28) it follows that at early times/high curvature when $H^4 \gg 1/48\beta$,

$$\frac{\ddot{a}}{a} \approx \frac{(1 - \omega_m)\rho_m + 2\Lambda}{96\beta H^4},$$

so $\ddot{a}/a \geq 0$ and the expansion is accelerated. At late times, when the density of matter has diluted enough with the course of the cosmic expansion, $\rho_m \propto a^{-3(\omega_m+1)}$, i.e., in the limit $\rho_m \to 0$, from Eq. (28) it follows that

$$\frac{\ddot{a}}{a} \to \frac{\Lambda}{3(1 + 16\beta H^4)},$$

so the expansion is accelerated only for nonvanishing $\Lambda > 0$. The same conclusion is obtained if we solve the algebraic Friedmann equation—the first equation in Eq. (9)—in terms of the Hubble rate:

$$H = \pm\sqrt{\frac{(2\sqrt{3\beta}\alpha + \sqrt{1 + 12\beta\alpha^2})^{2/3} - 1}{4\sqrt{3\beta}(2\sqrt{3\beta}\alpha + \sqrt{1 + 12\beta\alpha^2})^{1/3}}}, \qquad (29)$$

where for simplicity we have introduced the notation $\alpha \equiv \rho_m + \Lambda$. It is seen from Eq. (29) that at early times, i.e., in the formal limit when $\rho_m \to \infty \Rightarrow \alpha \gg 1/\sqrt{\beta}$, we get that

$$H = \pm\frac{\alpha^{1/6}}{(4\sqrt{3\beta})^{1/3}} \Rightarrow 48\beta H^6 = \alpha,$$

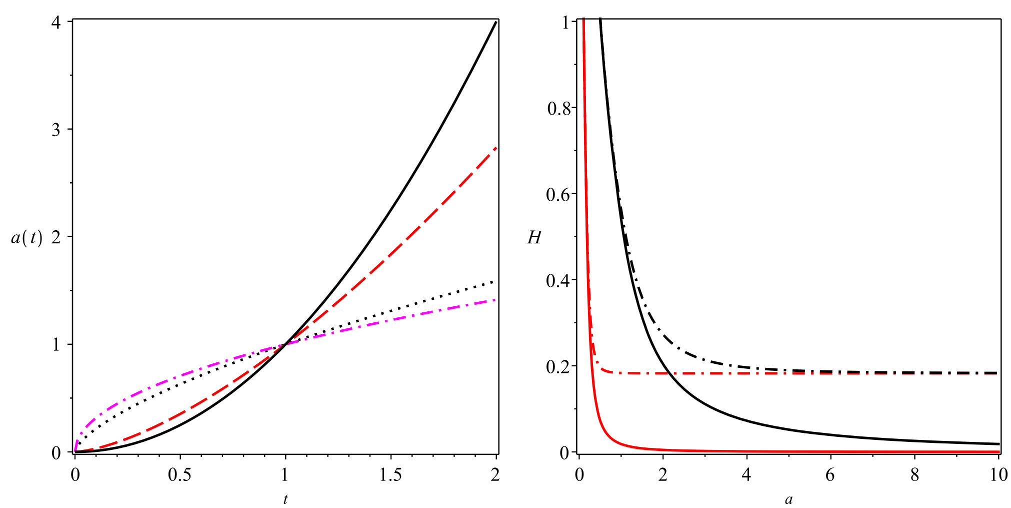

FIG. 2. Left panel: types of cosmic evolution. (1) Nonstandard (inflationary) Friedmann radiation-dominated evolution (dashed curve). (2) Nonstandard (inflationary) dust-dominated expansion (solid curve). (3) Standard (decelerated) radiation-dominated expansion (dash-dotted curve). (4) Standard dust-dominated evolution (dotted curve). Right panel: drawing of the Hubble rate H vs the scale factor a—according to Eq. (29)—shown for the cases in which the cosmological constant Λ vanishes (solid curves) and when it is a nonvanishing quantity (dash-dotted curves). We consider expanding cosmology only so that in Eq. (29) we take the positive sign. The darker curves are for dust ($\rho_d \propto a^{-3}$), while the remaining ones are for radiation ($\rho_r \propto a^{-4}$). It is seen that for vanishing $\Lambda = 0$ the Hubble rate asymptotically vanishes, i.e., the end point of the expansion in this case is the static Universe.

as expected. Meanwhile, at late times, i.e., in the formal limit $\rho_m \to 0 \Rightarrow \alpha = \Lambda$, the Hubble rate is nonvanishing only if $\Lambda \neq 0$. Actually, if we take $\Lambda = 0$, i.e., $\alpha = 0$ in Eq. (29), we obtain $H = 0$ (static Universe). This is illustrated in the right panel of Fig. 2, where the drawing of the Hubble rate H vs the scale factor a, according to Eq. (29), is shown for the cases in which the cosmological constant vanishes and in which it is a nonvanishing quantity.

VI. CONCLUSION

In this paper we have put on solid mathematical ground the result of previous works [19,25] showing that primordial inflation is the natural starting point of any plausible cosmic history within the framework of the CECG scenario. We have done this on the basis of the dynamical systems analysis of the CECG model. Dynamical systems offer unique robust information on the generic solutions of the cosmological equations of motion, i.e., those that are preferred by the differential equations according to their structural stability properties.

In the same rigorous manner we have shown that the late-time accelerated de Sitter expansion in the CECG model is a result of considering a nonvanishing cosmological constant and is not related in any way to the effects of the higher curvature contribution $\propto \mathcal{P} - 8\mathcal{C}$. Our result is natural in the sense that the higher curvature modifications of GR are supposed to have impact in the high energy, large curvature regime exclusively, i.e., at early times in the cosmic evolution.

As discussed in [28], certain instabilities may be present in this purely cubic model. In this regard it could be very interesting to explore how the results of this study are modified when all orders of curvature are included, particularly because such a configuration could avoid the kind of instabilities found in [28] in the CECG model. This was the subject of the companion paper [29].

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