Quasiequilibrium self-gravitating systems

Kamel Ourabah [*]

Theoretical Physics Laboratory, Faculty of Physics, University of Bab-Ezzouar, USTHB, Boite Postale 32, El Alia, Algiers 16111, Algeria

(Received 16 June 2020; accepted 3 August 2020; published 25 August 2020)

Because of the long-range nature of the gravitational interaction, self-gravitating systems never reach thermal equilibrium in the thermodynamic limit but remain trapped in nonequilibrium stationary states, or quasiequilibrium states. Here, we deal with quasiequilibrium self-gravitating systems by representing them as a collection of smaller subsystems that remain infinitely close to equilibrium. These subsystems represent regions from where thermalization spreads later over the whole system. Such a methodological attitude allows representing their statistical properties as a superposition of statistics, i.e., superstatistics. It has the advantage of producing Tsallis distributions, widely used in fitting observational data, as a special case of a more general family of distributions, while relying only on conventional statistical mechanics. Focusing on the three universality classes of superstatistics, namely, χ^2, inverse-χ^2, and log-normal superstatistics, we discuss the velocity distributions arising in this picture and confront them with independent numerical simulations. Then, we study the consequences on typical phenomena arising in self-gravitating systems. We discuss the Jeans instability in the classical regime, for a static and an expanding universe, and extend our results to the quantum regime by applying the Wigner-Moyal procedure. Our results reveal that quasiequilibrium systems remain stable for larger perturbations, as compared to equilibrium systems, meaning that a larger mass is needed to initiate the gravitational collapse. This is particularly relevant for Bok globules because their mass is of the same order as their Jeans mass; hence, a small deviation from equilibrium may lead to a different prediction for their stability. We also discuss the Chandrasekhar dynamical friction in a quasiequilibrium medium and analyze the consequences on the decay of globular orbits. Our results suggest that the superstatistical picture may offer a partial solution to the problem of the large timescales shown by numerical N-body simulations and semianalytical models.

DOI: 10.1103/PhysRevD.102.043017

I. INTRODUCTION

Enormous progress has been achieved in dealing with self-gravitating systems when, despite the apparent difficulties associated with the long-range nature of the gravitational interaction, statistical mechanics tools have been employed. Although such a progress is unquestionable, our current understanding of self-gravitating systems remains far from complete, and whether actual statistical mechanics theories can adequately explain their dynamical behavior is still a matter of debate. At the heart of this debate lie a number of peculiar—and even exotic—properties exhibited by self-gravitating systems [1], such as *negative specific heats*, the *inequivalence of statistical ensembles*, and *nonergodicity*. Furthermore, in the infinite particle limit, $N \to \infty$, self-gravitating systems never reach the thermodynamic equilibrium, characterized by the Maxwell-Boltzmann (MB) velocity distribution, but remain trapped in a stationary nonequilibrium state [2], i.e., a quasiequilibrium or quasi-stationary state.

For finite N, the evolution of a self-gravitating system proceeds in two steps [3,4]. Starting from a nonstationary initial condition, the system first undergoes a relaxation regime that drives it to a stationary state which, in general, is nonthermalized. The system stays in this quasiequilibrium state for a certain time, say $\tau(N)$. Then, the quasi-stationary state starts evolving in a second regime, and the system is expected to evolve toward the maximum entropy configuration, that is, to thermal equilibrium when this is well defined. In the limit $N \to \infty$, however, the lifetime of the stationary state diverges, i.e., $\tau \to \infty$, and the thermodynamic equilibrium is never reached. This requires going beyond the realm of equilibrium statistical mechanics and exploring statistical methods inherent to nonequilibrium systems.

One fruitful route that has been widely explored in the past years consists in changing the paradigm with which one approaches such systems to the so-called nonextensive statistical mechanics (NSM) [5]. The latter is an extension of statistical mechanics based on the Tsallis entropy, i.e., a one-parameter generalization of the entropy. The motivation behind this program is that self-gravitating systems

[*]kam.ourabah@gmail.com

constitute a special case of a wider class of systems characterized by long-range (and unscreened) interactions; that is, d-dimensional systems where the interparticle potential decays at large distance r as $1/r^{\alpha}$ with $\alpha \leq d$. Beyond gravitational systems, these include, for example, unscreened Coulomb systems [6], two-dimensional vortices [7], or wave-particle interactions [8]. In this case, the interactions are of a nonextensive nature; i.e., their energy grows faster than linearly with the volume, making the employment of NSM quite natural.

A number of successes have been achieved in this direction, either in reproducing observational data [9–15] or revisiting key phenomena [16–25] and clarifying fundamental statistical and thermodynamic aspects [26,27] of gravitational systems. Despite these successes, such a program faces some limitations as well. On the one hand, Tsallis distributions arising in the framework of NSM seem not universal enough to cover the rich behavior of self-gravitating systems, as indicated by a number of observations [28] and simulations [29]. On the other hand, the recourse to NSM, as a universal framework for self-gravitating systems, is not immune from a number of conceptual difficulties that call into question its universal validity. For instance, the origin of the parameter q that underpins these distributions remains to date unclear, and it is generally taken as a free parameter. Besides, velocity distributions arising in NSM are characterized by diverging moments for some parameter values [30,31], and the nonadditive nature of the entropy may potentially result in thermodynamic inconsistencies [32,33].

Here, we suggest a different perspective that allows reproducing Tsallis distributions as a special case of a more general family of distributions, while relying solely on conventional statistical mechanics. In this approach, quasiequilibrium states are taken quite literally; that is, the self-gravitating system in a quasi-stationary state is represented as a collection of smaller regions, considered large enough to be treated statistically, that remain infinitely close to equilibrium. These regions form subsystems from where thermalization spreads later over the entire system. Clearly, this picture assumes a form of slow modulation [34]; the temperature is assumed to vary on a long timescale which is much larger than the local relaxation time for each region to reach local equilibrium. This allows decomposing the dynamics of the system over two scales: at the level of these small regions, the temperature is nearly constant, and equilibrium statistical mechanics holds. At a larger scale, however, one has to account for the temperature distribution across different regions. Such a methodological attitude is known as *superstatistics* since it requires dealing with a superposition of statistics. The main idea has a long tradition in the statistical mechanics literature [35–38], but it was first crystallized in a single formalism in Ref. [39].

The paper is organized as follows. In Sec. II, we present the three universality classes of distributions arising in this picture and validate them by independent numerical simulations of self-gravitating systems. We also work out some results for future needs. In the subsequent sections, we analyze the consequences of these quasiequilibrium distributions in typical phenomena arising in self-gravitating systems. In Sec. III, we analyze the gravitational instability in the classical regime, for a static and an expanding universe. In Sec. IV, we extend our results to the quantum regime, by using the Wigner-Moyal procedure and considering quantum statistics. In Sec. V, we examine the Chandrasekhar dynamical friction and compute the dynamic friction timescale for globular clusters spiraling to the galactic center. We summarize our main results and discuss prospects for future research in Sec. VI.

II. QUASIEQUILIBRIUM DISTRIBUTIONS

The superstatistics concept can nicely be introduced by considering the adiabatic ansatz [40]: consider a system that, during its evolution, travels within its state space X, which is divided up into small cells, each characterized by a sharp value of some parameter β. Within each cell, the system is described by the conditional probability $p(A|\beta)$ to be found in a specific state $A \in X$. As β varies adiabatically across the different cells, the joint distribution of finding the system in the state A with a sharp value of β is $p(A,\beta) = p(A|\beta)p(\beta)$, viz., the *De Finetti-Kolmogorov relation*. The probability $p(A)$ for finding the system in the state A is obtained by eliminating the nuisance parameter β through marginalization and reads as a superposition of statistics. That is,

$$p(A) = \int p(A|\beta)p(\beta)d\beta. \tag{1}$$

Here, we are mainly interested in the velocity distributions (that is, $A \equiv v$) that are characteristic of a quasiequilibrium system, i.e., a system made up of small regions at different (inverse) temperatures, that is, $\beta \equiv 1/k_B T$ (hereafter, we set $k_B = 1$). Note, however, that the formalism is much more general and such a parameter may equally well represent any other fluctuating intensive quantity. One may think, for instance, of a fluctuating chemical potential or an energy dissipation rate in a turbulent fluid.

It is clear from Eq. (1) that the temperature distribution determines the form of the emergent velocity distribution $p(v)$. While, in each small region, one has local equilibrium and $p(v|\beta)$ corresponds to a Gaussian (MB distribution), the emergent distribution $p(v)$ may deviate from it significantly. Then, the question is which forms of $p(\beta)$ are relevant for quasiequilibrium systems.

In principle, there are infinitely many possible temperature distributions, but it is known that three universality classes arise as universal limit statistics in majority of known systems; the χ^2 distribution, the inverse-χ^2 distribution, and the log-normal distribution. These universality classes can be obtained from purely probabilistic arguments, relying on the Central Limit Theorem [41] or from the entropy maximization scheme [42]. The emergence of these three classes in typical situations can also be understood from the shape of the velocity distributions they produce, as discussed below. These three classes are the following:

(a) *χ^2 superstatistics.*—In this case, the inverse temperature β follows a χ^2 distribution of degree n,

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})}\left(\frac{n}{2\beta_0}\right)^{n/2}\beta^{n/2-1}e^{-\frac{n\beta}{2\beta_0}}, \tag{2}$$

where $\beta_0 \equiv \langle\beta\rangle$ is the average of β. The corresponding (d-dimensional) emergent velocity distribution, i.e., marginal distribution, follows from Eq. (1) as

$$B(v) = \int_0^\infty d\beta f(\beta)\left(\frac{\beta m}{2\pi}\right)^{d/2}\exp\left[-\frac{\beta m v^2}{2}\right] = \left(\frac{\beta_0 m}{\pi n}\right)^{d/2}\frac{\Gamma(\frac{n+d}{2})}{\Gamma(\frac{n}{2})}\left(1+\frac{\beta_0}{n}mv^2\right)^{-\frac{n+d}{2}}. \tag{3}$$

The latter can be mapped onto the Tsallis distribution (q Gaussian), emerging within the formalism of NSM, with an entropic index $q \coloneqq 1 + 2/(n+d)$ and an effective inverse temperature $\tilde{\beta} \coloneqq \beta_0(n+d)/n$. In the statistics literature, distributions in the form of Eq. (3) are known as Students t distributions. They constitute a special case of the Burr-type III distribution [43].

(b) *Inverse-χ^2 superstatistics.*—In this case, the temperature (β^{-1}) itself is χ^2 distributed. Then, β follows an inverse-χ^2 distribution,

$$f(\beta) = \frac{\beta_0}{\Gamma(\frac{n}{2})}\left(\frac{n\beta_0}{2}\right)^{n/2}\beta^{-n/2-2}e^{-\frac{n\beta_0}{2\beta}}. \tag{4}$$

The corresponding velocity distribution in this case reads as

$$B(v) = \frac{2\beta_0}{\Gamma(\frac{n}{2})}\left(\frac{m}{2\pi}\right)^{d/2}\left(\frac{\beta_0 n}{2}\right)^{n/2}\left(\frac{mv^2}{\beta_0 n}\right)^{\frac{2-d+n}{4}} \times \mathcal{K}_{\frac{2-d+n}{2}}(\sqrt{nm\beta_0}|v|), \tag{5}$$

where $\mathcal{K}_\alpha(x)$ is the modified Bessel function of the second kind.

(c) *Log-normal superstatistics.*—In this case, β follows a log-normal distribution,

$$f(\beta) = \frac{1}{\sqrt{2\pi}s\beta}\exp\left\{\frac{-(\ln\frac{\beta}{\mu})^2}{2s^2}\right\}, \tag{6}$$

with an average of β given by $\beta_0 = \mu e^{s^2/2}$. In this last case, there is no closed-form expression for the corresponding velocity distribution $B(v)$, but it can easily be computed numerically.

Note that there is a substantial and increasing empirical evidence for these three universality classes, i.e., Eqs. (2), (4), and (6), in nonequilibrium systems: χ^2 superstatistics corresponds to the statistics arising from NSM and has been observed in many situations [5]. Experimental evidence of log-normal superstatistics has been reported in the context of Lagrangian and Eulerian turbulence [41,44,45], space plasmas [31], and other systems [46], while candidate systems for inverse-χ^2 superstatistics have been discussed in Refs. [47,48].

The universality of these three classes can be understood from the velocity (or energy) distributions they generate, that cover the main families of distributions encountered in nature: for large $|\mathbf{v}|$, velocity distributions corresponding to χ^2 superstatistics exhibit power-law tails, those associated with inverse-χ^2 superstatistics decay exponentially, and log-normal superstatistics produces truncated power laws. Such rich behavior may offer new perspectives, that go beyond the usual scenario of NSM, in covering typical non-Gaussian distributions observed in self-gravitating systems.

Examples of (one-dimensional) velocity distributions emerging from the three universality classes, i.e., χ^2 [Eq. (3)], inverse-χ^2 [Eq. (5)], and log-normal (computed numerically), are shown in Fig. 1. To facilitate the comparison between the different classes, we parametrize the distributions via the parameter $q \coloneqq \langle\beta^2\rangle/\langle\beta\rangle^2$ (different from the entropic index used in NSM, as explained below). The latter is a measure of the temperature inhomogeneity in the system and reduces to 1 for a completely thermalized state, i.e., when $f(\beta)$ shrinks to a Dirac delta, recovering therefore a Gaussian distribution for $B(v)$. It can easily be expressed, for the three classes, in terms of the parameters of $f(\beta)$, as follows:

$$q \coloneqq \frac{\langle\beta^2\rangle_{\chi^2}}{\beta_0^2} = 1 + \frac{2}{n} \quad (n > 2), \qquad q \coloneqq \frac{\langle\beta^2\rangle_{\mathrm{inv}\chi^2}}{\beta_0^2} = \frac{n}{n-2}, \qquad q \coloneqq \frac{\langle\beta^2\rangle_{LN}}{\beta_0^2} = e^{s^2}. \tag{7}$$

As it can be seen from Fig. 1, the superstatistical velocity distributions, especially those associated with the χ^2 and log-normal classes, exhibit heavy tails; a typical feature of quasiequilibrium self-gravitating systems, as indicated by observations and simulations. To test the ability of these distributions to account for deviations from the MB distribution in quasiequilibrium self-gravitating systems, we confront them by the results of simulations presented in Ref. [4], that show a substantial deviation from the Gaussian distribution. The latter are produced using the

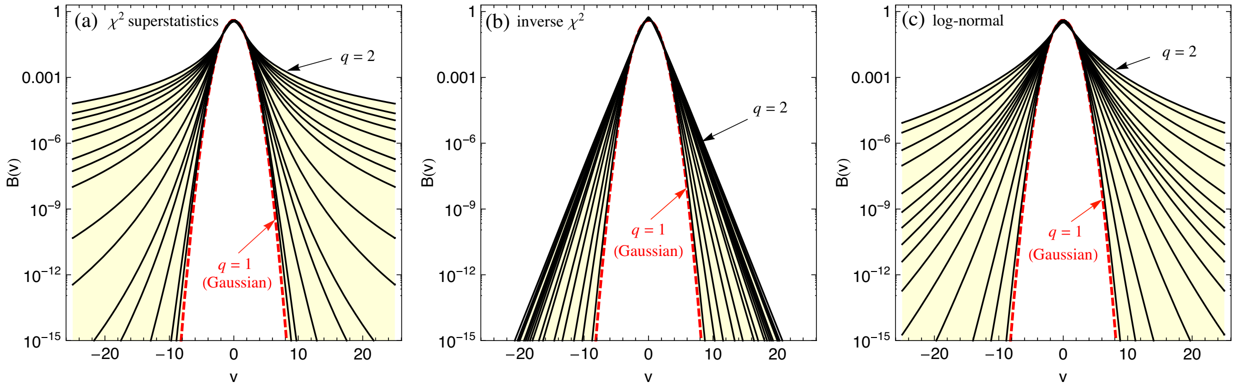

FIG. 1. Examples of one-dimensional superstatistical velocity distributions for (a) χ^2, (b) inverse-χ^2, and (c) log-normal superstatistics, with different values of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$ [cf. Eq. (7)], in a logarithmic scale to better highlight the tails. We set $\beta_0 m = 1$.

widely employed public code GADGET-2 (available here http://www.mpa-garching.mpg.de/gadget/), by fixing different initial conditions according to the so-called water-bag model. In Fig. 2, we show the best fits obtained with the three universality classes of superstatistics, i.e., χ^2 [Eq. (3)], inverse-χ^2 [Eq. (5)], and log-normal (computed numerically). One may see that the three classes exhibit similar profiles, typical of quasi-stationary states.

At this stage, one should note that, even in the case where the superstatistical distributions are not accessible in closed form, as in the case of log-normal superstatistics, the velocity moments $\langle v^l\rangle$ are accessible in an analytical form. The latter can be simply expressed as

$$\langle v^l\rangle \equiv \int v^l B(v) d^d v = \langle\langle v^l\rangle_{\mathrm{MB}}\rangle_{f(\beta)}, \tag{8}$$

where $\langle\bullet\rangle_{\mathrm{MB}}$ stands for an average over the (d-dimensional) MB velocity distribution and $\langle\bullet\rangle_{f(\beta)}$ is an average over the (inverse) temperature distribution $f(\beta)$. By combining the moments of the three distributions, i.e., Eqs. (2), (4), and (6),

$$\langle\beta^l\rangle_{\chi^2} = \frac{\Gamma(\frac{n}{2}+l)}{\Gamma(\frac{n}{2})}\left(\frac{2}{n}\right)^l \beta_0^l,$$
$$\langle\beta^l\rangle_{\mathrm{inv}\chi^2} = \frac{\Gamma(\frac{n}{2}+1-l)}{\Gamma(\frac{n}{2})}\left(\frac{n}{2}\right)^{l-1} \beta_0^l,$$
$$\langle\beta^l\rangle_{LN} = e^{l(l-1)s^2/2}\beta_0^l, \tag{9}$$

with those of the MB (Gaussian) distribution

$$\langle v^l\rangle_{\mathrm{MB}} = \frac{(l+d-2)!!}{(\beta m)^{l/2}} \tag{10}$$

(l even), one readily obtains all superstatistical velocity moments in an exact form. Especially useful for future

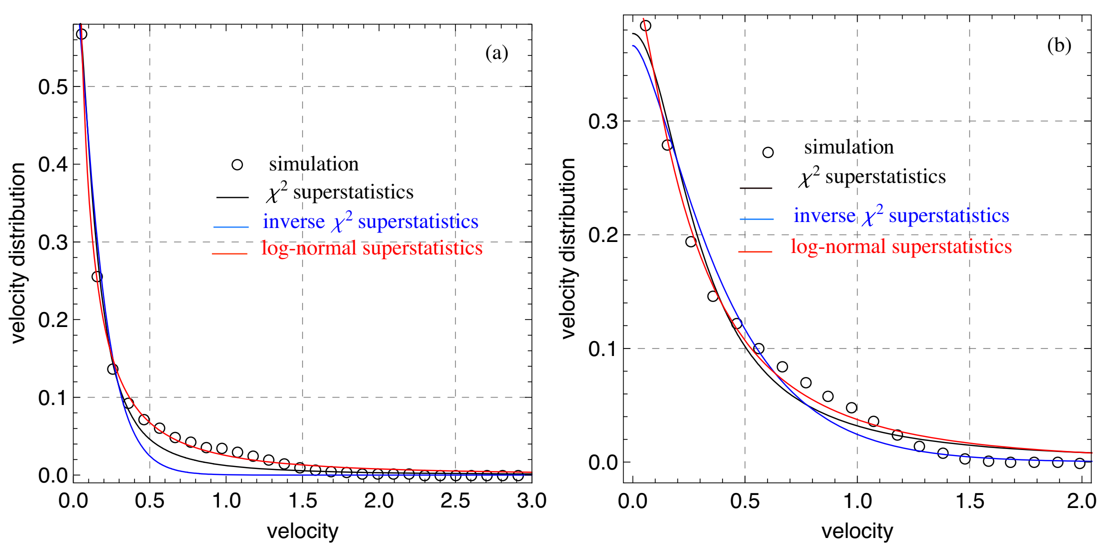

FIG. 2. Examples of non-Gaussian velocity distributions associated with quasiequilibrium self-gravitating systems. The open circles correspond to time-averaged velocity distributions, simulated using the public code GADGET-2 [4], with different water-bag initial conditions. The solid lines represent the best fits obtained with χ^2 (black), inverse-χ^2 (blue), and log-normal (red) superstatistics.

needs are the second-order moments. The latter can be expressed (assuming isotropic pressure, so that the dyadic product can be contracted: $\mathbf{v}\otimes\mathbf{v}\to v^2$), for the three universality classes, as

$$\langle v^2\rangle_{\chi^2} = \frac{n}{n-2}\langle v^2\rangle_{\mathrm{MB}} \quad (n>2),$$
$$\langle v^2\rangle_{\mathrm{inv}\chi^2} = \frac{n+2}{n}\langle v^2\rangle_{\mathrm{MB}},$$
$$\langle v^2\rangle_{LN} = e^{s^2}\langle v^2\rangle_{\mathrm{MB}}, \tag{11}$$

which can be rewritten in terms of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$ [cf. Eq. (7)] as

$$\langle v^2\rangle_i = d\cdot\phi_i(q)\frac{T_0}{m} \quad (i=1,2,3), \tag{12}$$

where $T_0 \equiv \beta_0^{-1}$ is the mean temperature and

$$\phi_1(q) \equiv \frac{1}{2-q} \quad (1<q<2),$$
$$\phi_2(q) \equiv \frac{2q-1}{q},$$
$$\phi_3(q) \equiv q, \tag{13}$$

with $i = 1$, 2, and 3 corresponding, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. Note that, for the three superstatistics, one has $\phi_i(q) \geq 1$, indicating that the presence of temperature inhomogeneities tends to increase the velocity dispersion. In the limit of a fully thermalized state, i.e., $q\to 1$, the distribution $f(\beta)$ shrinks to a Dirac delta centered at β_0 and $\langle v^2\rangle_i$ ($i = 1$, 2, 3) reduces to the MB second-order moment, i.e., $\langle v^2\rangle = d\cdot T_0/m$.

Before closing this section, a few remarks are in order. First, note that we are implicitly adopting here the so-called type-B formulation of superstatistics, i.e., we are considering *locally normalized* equilibrium distributions that are averaged over $f(\beta)$ [see, for example, Eq. (3)]. The other alternative, known as type-A superstatistics (see, for instance, Refs. [39,49]), consists in working with *unnormalized* canonical distributions and normalizing the marginal distribution at the very end, by introducing a multiplicative factor. One may, however, easily switch from one formulation to the other by properly redefining the distribution $f(\beta)$. Note also that, because we are adopting type-B formulation, the parameter q used here [Eq. (7)], in the case of χ^2 superstatistics, differs from the entropic index used in NSM. The latter is obtained via the transformation $q\to 1+[2(q-1)]/[2+d(q-1)]$.

III. JEANS MECHANISM IN THE CLASSICAL REGIME

In this section, we analyze the consequences of the superstatistics scenario on the Jeans instability. The latter is the mechanism causing the gravitational collapse of interstellar gas clouds and subsequent star formation. It occurs when the internal pressure of a region filled with matter is not strong enough to prevent gravitational collapse. The earliest understanding of the mechanism can be traced back to 1902 with the seminal work of Jeans [50], who showed the existence of a physical cutoff, today known as the Jeans wavelength λ_J, such that perturbations with wavelengths larger than λ_J may grow exponentially in time. The phenomenon, although known for a long time, has attracted recently increased attention either from the perspective of accounting for new effects that may significantly influence the collapse condition, such as the presence of dissipation [51] and viscosity [52], or revisiting the mechanism in other paradigms, such as modified gravity theories [53,54] and the framework of NSM [17,21].

We discuss here the Jeans mechanism in a quasiequilibrium self-gravitating system, as defined above. To do so, we consider an infinite self-gravitating collisionless neutral gas, subjected to a constant gravitational potential $\Phi_0 = cste$. Initially, the gas is assumed at rest in a quasiequilibrium state, with a constant mass density ρ_0 and a mean temperature T_0. In such a stationary nonequilibrium state, the distribution has the form of $B(v) = \langle f_{\mathrm{MB}}(v)\rangle_{f(\beta)}$ [cf. Eq. (1)], where $f_{\mathrm{MB}}(v)$ is the equilibrium MB distribution and $\langle\bullet\rangle_{f(\beta)}$ denotes the average over the temperature distribution $f(\beta)$. In the collisionless regime, the space-time evolution of the one-particle distribution function $f(\mathbf{r},\mathbf{v};t)$ is given by the collisionless Boltzmann (Vlasov) equation

$$\frac{\partial f(\mathbf{r},\mathbf{v};t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f(\mathbf{r},\mathbf{v};t)}{\partial\mathbf{r}} - \nabla\Phi\cdot\frac{\partial f(\mathbf{r},\mathbf{v};t)}{\partial\mathbf{v}} = 0. \tag{14}$$

We restrict ourselves to the case of linear perturbations that remains tractable analytically and write the distribution function as

$$f(\mathbf{r},\mathbf{v};t) = B(v) + \delta f(\mathbf{r},\mathbf{v};t), \tag{15}$$

where $|\delta f| \ll B$. In this case, one may linearize Eq. (14) and couple it with the Poisson equation for the gravitational potential to form a closed set of equations as

$$\frac{\partial\delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial\delta f}{\partial\mathbf{r}} - \nabla\Phi\cdot\frac{\partial B}{\partial\mathbf{v}} = 0,$$
$$\nabla^2\Phi = 4\pi G\rho_0\int\delta f d^3v, \tag{16}$$

where G is the gravitational constant and Φ denotes the perturbation of the gravitational field. Note that, in Eq. (16), we used the so-called Jeans swindle by considering that the gravitational potential is sourced only by the fluctuations around the uniform background density. Indeed, one may observe that the condition $\Phi_0 = cste.$ ($\nabla\Phi_0 = 0$) does not satisfy the Poisson equation, because it would imply a vanishing mass density. The usual way to overcome this inconsistency is to use the Jeans swindle by assuming that the homogeneous density ρ_0 does not contribute to the gravitational potential. Note, however, that such a trick can be avoided, either by the introduction of a container and the consideration of an inhomogeneous distribution of matter [55] or by taking into account the expansion of the Universe (to be discussed next).

The above set of equations (16) may be solved simultaneously by performing a decomposition in Fourier modes. That is,

$$\delta f \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \text{and} \quad \Phi \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{17}$$

We consider here, without loss of the generality, the x axis to be along the direction of the wave-vector $\mathbf{k}$ and let $v \equiv v_x$. Equation (16) yields

$$1 - \frac{\Omega^2}{k^2}\int \frac{\partial B/\partial v}{v-\omega/k} d^3v = 0, \tag{18}$$

where $\Omega = (4\pi G\rho_0)^{1/2}$ is the Jeans frequency. Note that, according to Eq. (17), for $\omega^2 > 0$ (ω real), one has an oscillatory regime, while for $\omega^2 < 0$ (ω imaginary), one has an exponential regime, responsible for the Jeans instability. The boundary separating the two regimes is obtained by setting $\omega = 0$ in Eq. (18). Bearing in mind that

$$\frac{\partial B(v)}{\partial v} = \frac{\partial \langle f_{\mathrm{MB}}(v)\rangle_{f(\beta)}}{\partial v} = \left\langle \frac{\partial f_{\mathrm{MB}}(v)}{\partial v} \right\rangle_{f(\beta)}, \tag{19}$$

and using Eq. (12), one readily obtains the Jeans wave number for the tree universality classes of superstatistics as

$$k_J^{(q)} = \phi_i(q)^{-1/2} k_J, \qquad k_J \equiv \sqrt{\frac{4\pi G m\rho_0}{T_0}}, \tag{20}$$

where k_J is the Jeans wave number for an equilibrium configuration, i.e., assuming the MB velocity distribution. Note that in the particular case of χ^2 superstatistics Eq. (20) reduces to the Jeans wave number arising in the framework of NSM, addressed by many authors from the kinetic [17] or the hydrodynamic [22] point of view.

One may observe that, since thermal fluctuations tend to increase the second-order velocity moment [see Eq. (12)], one has in general $k_J^{(q)} < k_J$, indicating that the presence of temperature inhomogeneities tend to stabilize the self-gravitating system for smaller values of k. From Eq. (20), one may define the Jeans wavelength $\lambda_J^{(q)} \equiv 2\pi/k_J^{(q)}$, such that perturbations with wavelengths $\lambda < \lambda_J^{(q)}$ yield harmonic oscillations while perturbations with wavelengths $\lambda > \lambda_J^{(q)}$ experience exponential growth. One may equivalently express the Jeans mass, defined as the mass initially contained in a sphere of diameter $\lambda_J^{(q)}$, as

$$M_J^{(q)} = \phi_i^{3/2} M_J, \qquad M_J \equiv \frac{\pi}{6}\sqrt{\frac{1}{\rho_0}\left(\frac{\pi T_0}{Gm}\right)^3}. \tag{21}$$

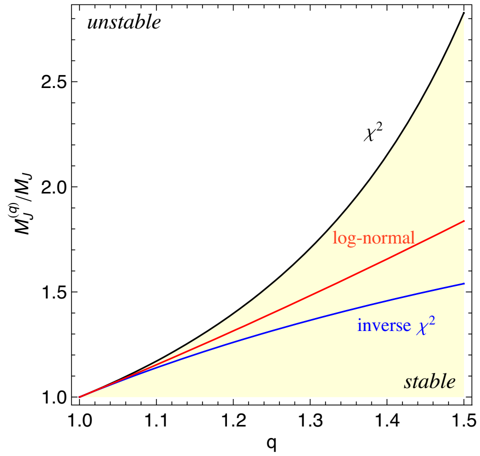

FIG. 3. Effect of temperature inhomogeneities on the Jeans mass for the three universality classes of superstatistics, as a function of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$.

The latter is of a more practical usefulness in probing the stability of interstellar clouds; if the mass of the cloud exceeds the Jeans mass $M_J^{(q)}$, the cloud will experience gravitational collapse. Figure 3 shows the Jeans mass (21), for the three universality classes, as a function of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$. One may observe that the temperature inhomogeneities tend to increase the Jeans mass, reducing therefore the process of star formation. For the same strength of fluctuations (the same q), the class of χ^2 superstatistics induces the most significant effect on the Jeans mass, while the class of inverse-χ^2 superstatistics has the least impact on it.

It is customary to write the Jeans mass/length in terms of the mean molecular mass μm_p, where μ is the mean molecular weight and m_p is the proton mass. The Jeans length and the Jeans mass can therefore be written as

$$\lambda_J^{(q)} \sim \frac{1.06pc}{\mu^2}\left(\frac{\phi_i(q)T_0}{10\ \mathrm{K}}\right)^{1/2}\left(\frac{n}{10^4\ \mathrm{cm}^{-3}}\right)^{-1/2}$$
$$M_J^{(q)} \sim \frac{155.98}{\mu^2}M_\odot\left(\frac{\phi_i(q)T_0}{10\ \mathrm{K}}\right)^{3/2}\left(\frac{n}{10^4\ \mathrm{cm}^{-3}}\right)^{-1/2}. \tag{22}$$

In Table I, we give estimations of the Jeans mass, in the absence and presence of temperature inhomogeneities, for a number of astrophysical systems. For the sake of illustration, we choose the intermediate value $q = 1.1$. One may appreciate that even relatively small deviations from the equilibrium state, i.e., $q = 1$, may lead to a significant modification of the Jeans mass. This is particularly relevant

TABLE I. Estimation of the Jeans mass of different astrophysical objects, for the three universality classes of superstatistics, with $q = 1.1$.

Object	T_0 (K)	n (10^8 m^{-3})	$M_J(M_\odot)$	$M_J^{\chi^2}(M_\odot)$	$M_J^{\mathrm{inv}\chi^2}(M_\odot)$	$M_J^{\mathrm{LN}}(M_\odot)$
Bok globules	10	100	11.24	13.16	12.81	12.97
Diffuse molecular clouds	30	50	82.63	96.77	94.15	95.33
Diffuse hydrogen clouds	50	5.0	795.13	931.27	905.98	917.33
Giant molecular clouds	15	1.0	206.58	241.95	235.38	238.33
HII regions	10^4	10^{-3}	3.79×10^9	4.44×10^9	4.32×10^9	4.37×10^9
Fermi bubbles	10^8	10^{-4}	3.79×10^{17}	4.44×10^{17}	4.32×10^{17}	4.37×10^{17}
Intracluster medium	10^7	10^{-5}	1.20×10^{18}	1.40×10^{18}	1.37×10^{18}	1.38×10^{18}

for Bok globules (clouds of interstellar gas and dust with temperatures of around 10 K) because their mass is of the same order as their corresponding Jeans mass; thus, a small deviation from the equilibrium configuration may lead to a different prediction for their stability.

As is clear from the discussion above, the Jeans criterion for gravitational collapse is not sensitive to the exact form of the velocity distribution but depends only on its first moments [cf. Eq. (20)]. Also, it is worth analyzing more generally the dispersion relation (18) for the different superstatistics and studying the instability growth rate. Since we are mainly interested in the unstable modes, we set $\omega = i\gamma$ ($\gamma \geq 0$) and substitute into Eq. (18). After simple algebraic rearrangements, the dispersion relation (18) can be written as

$$\frac{k^2}{k_J^2} = 1 - \sqrt{\frac{\pi}{2}} \int_0^\infty d\beta f(\beta) \left(\frac{\beta}{\beta_0}\right)^{3/2} \frac{\gamma k_J}{\Omega k} e^{\frac{\beta}{2\beta_0} \frac{\gamma^2 k_J^2}{\Omega^2 k^2}} \times \left[1 - \mathrm{erf}\left(\sqrt{\frac{\beta}{2\beta_0}} \frac{\gamma k_J}{\Omega k}\right)\right], \tag{23}$$

where $\mathrm{erf}(x)$ is the Gauss error function, defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt. \tag{24}$$

Note that in the limit of thermal equilibrium, i.e., for $f(\beta) \to \delta(\beta - \beta_0)$, it reduces to the standard result [56]:

$$\frac{k^2}{k_J^2} = 1 - \sqrt{\pi} \frac{1}{\sqrt{2}} \frac{\gamma k_J}{\Omega k} e^{\frac{\gamma^2 k_J^2}{2\Omega^2 k^2}} \left[1 - \mathrm{erf}\left(\frac{1}{\sqrt{2}} \frac{\gamma k_J}{\Omega k}\right)\right]. \tag{25}$$

We have solved Eq. (23) numerically for $f(\beta)$ corresponding to the three universality classes of superstatistics, i.e., Eqs. (2), (4), and (6). The results are shown in Fig. 4. One may appreciate that the three classes produce qualitatively the same effect; i.e., they tend to decrease the growth rate, a feature that is commonly attributed [17] to the special class of Tsallis statistics arising in the context of NSM.

It is worth examining two extreme situations where the dispersion relation (18) can be analyzed in closed form, for the three superstatistics. On the one hand, in the limit of high-frequency ω, as there are no singularities, it is possible to integrate Eq. (18) along the real axis. By Taylor expanding $(1 - vk/\omega)^{-1}$ and using the moments of $f(\beta)$, i.e., Eq. (12), one has

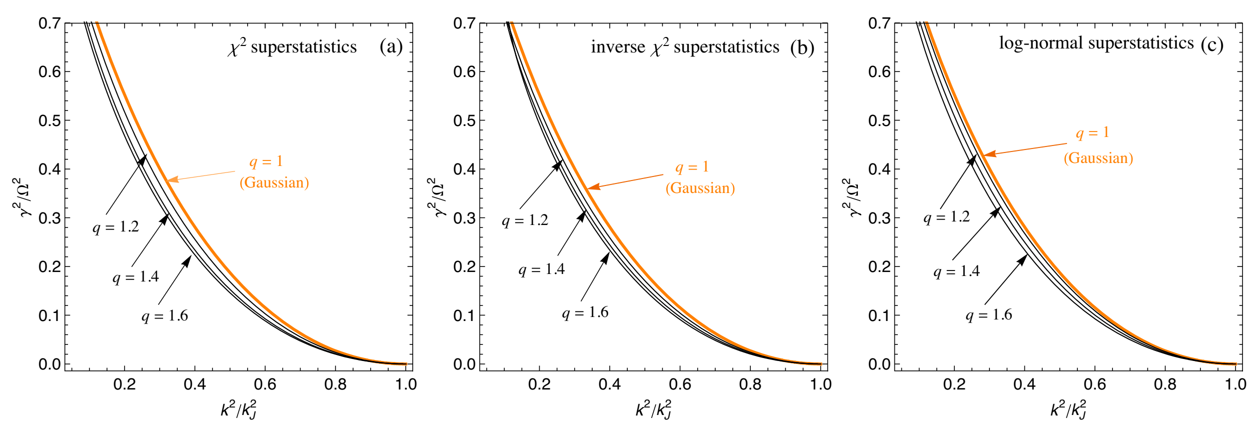

FIG. 4. Unstable branches of the dispersion relation (23) computed numerically for the three universality classes of superstatistics. One may see that the three classes have qualitatively the same effect, i.e., they tend to decrease the growth rate.

$$\frac{\omega^2}{\Omega^2} = 3\phi_i(q)\frac{k^2}{k_J^2} - 1, \tag{26}$$

which reduces to the standard dispersion relation for $q = 1$. In the other extreme, corresponding to low-frequency perturbations, by using the asymptotic relation [54]

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x\mathrm{e}^{-\frac{x^2}{2}}}{b-x}\mathrm{d}x \underset{b\ll 1}{\approx} 1 + i\sqrt{\frac{\pi}{2}}b \tag{27}$$

and considering unstable modes ($\omega = i\gamma$), Eq. (18) can be written as

$$\frac{\gamma}{\Omega} = \alpha_i(q)\sqrt{\frac{2}{\pi}}\frac{k}{k_J}\left[1 - \frac{k^2}{k_J^2}\right], \tag{28}$$

where we have defined

$$\alpha_1(q) \equiv \frac{\Gamma[\frac{1}{q-1}]}{\Gamma[\frac{3}{2}+\frac{1}{q-1}]}\frac{1}{(q-1)^{3/2}},$$
$$\alpha_2(q) \equiv \frac{\Gamma[\frac{q}{q-1}]}{\Gamma[\frac{q}{q-1}-\frac{1}{2}]}\sqrt{\frac{q-1}{q}},$$
$$\alpha_3(q) \equiv q^{-3/8}, \tag{29}$$

with $i = 1$, 2, and 3 corresponding, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. One may easily check that $\alpha_i(q) < 1$ ($i = 1, 2, 3$), indicating that temperature inhomogeneities tend to decrease the growth rate, as observed in Fig. 4.

Note that, besides the kinetic approach adopted here, the other route for studying the Jeans instability involves the hydrodynamic (fluidlike) formulation. It is instructive to briefly comment on this alternative approach in the superstatistics scenario. In the hydrodynamic formulation, the Poisson equation is coupled to the continuity equation and Euler equation, as

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0$$
$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\Phi, \tag{30}$$

where $\mathbf{v}$ is the fluid velocity and P is the pressure, related to the temperature via an equation of state (EoS). In the superstatistics formalism, one may use Eq. (12) to express the pressure as follows:

$$P = \frac{1}{3}\rho\langle v^2\rangle_i = \phi_i(q)\frac{\rho}{m}T_0. \tag{31}$$

That is, the pressure is linked to the mean temperature T_0 through an *effective* isothermal EoS. By combining Eq. (31) with the hydrodynamic model (30), one readily ends up with the Jeans wave number (20) obtained from our kinetic approach.

Finally, and for the sake of completeness, let us discuss the Jeans mechanism in an expanding universe background and show how the formalism of superstatistics can be implemented in this scenario. We will be mostly following Ref. [57] but considering a quasiequilibrium state. For an expanding universe, one has to account for the cosmic scale factor $a(t)$ through the Friedmann and acceleration equations. For a matter dominated Universe, the latter read as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho, \tag{32}$$

where the dots denote derivatives with respect to time t. Equation (32) can be solved to give the mass density ρ as a function of the cosmic scale factor as

$$\rho = \rho_0\left(\frac{a_0}{a}\right)^3, \tag{33}$$

where ρ_0 and a_0 are the values of the mass density and the scale factor at $t = 0$. Note that, accordingly, ρ is only a function of time and the unperturbed gravitational potential is now depending on the space coordinate and reads as

$$\Phi_0(\mathbf{r}, t) = \frac{2\pi}{3}G\rho r^2, \tag{34}$$

such that the recourse to the Jeans swindle is no longer necessary. From another hand, one has also to account for the universe expansion in the distribution function itself. In the superstatistical scenario, one has locally equilibrium MB distributions, that are averaged over $f(\beta)$, i.e., $B(v) \equiv \langle f_{\mathrm{MB}}\rangle_{f(\beta)}$. However, since each region is experiencing the universe expansion, by virtue of the Hubble's law $\dot{\mathbf{r}} = (\dot{a}/a)\mathbf{r}$, the local equilibrium distribution must read as [57]

$$f_{\mathrm{MB}}(\mathbf{r}, \mathbf{v}; t) = \left(\frac{m\beta(t)}{2\pi}\right)^{3/2}\exp\left\{-\frac{m\beta(t)}{2}\left(\mathbf{v} - \frac{\dot{a}}{a}\mathbf{r}\right)^2\right\}. \tag{35}$$

Notice the time dependence of the inverse temperature β, due to the universe expansion. In fact, during the universe expansion, the dispersion velocity is proportional to the inverse of the cosmic scale factor, i.e., $\sigma(t) \propto 1/a(t)$, and, accordingly, the inverse temperature scales as $\beta(t) \propto a^2(t)$. Bearing this in mind and proceeding as previously (see, for example, Ref. [57] for the detailed calculations in the case of the MB distribution), one easily obtains the evolution of the density contrast $\delta_\rho \equiv \bar{\rho}/\rho$, defined as the ratio of the perturbed and unperturbed densities, as

$$\tau^2\delta_\rho'' + \frac{4}{3}\tau\delta_\rho' - \frac{2}{3}\left(1 - \frac{3\lambda_J^2}{5\lambda_0^2\tau^{\frac{2}{3}}}\right)\delta_\rho - \frac{4\lambda_J^2}{25\lambda_0^2\tau^{\frac{2}{3}}} = 0, \tag{36}$$

where

$$\tau \equiv \sqrt{\frac{3}{2}}\Omega t \quad \lambda_0 \equiv \frac{2\pi a_0}{|\mathbf{q}|}, \quad \lambda_J^{(q)} \equiv \frac{10\pi}{3\Omega}\sqrt{\frac{\phi_i(q)T_0}{m}}. \tag{37}$$

Above, the primes represent derivatives with respect to the dimensionless time τ, and $\mathbf{q}$ is the comoving wave number, i.e., related to the physical wave number $\mathbf{k}$ through $\mathbf{k} \equiv \mathbf{q}/a(t)$. The solution of Eq. (36) involves Bessel functions of the first kind $J_\alpha(x)$ and is given by [57]

$$\delta_\rho = \tau^{-\frac{1}{6}}\left[C_1 J_{\frac{5}{2}}\left(\frac{\Lambda}{\tau^{\frac{1}{3}}}\right) + C_2 J_{-\frac{5}{2}}\left(\frac{\Lambda}{\tau^{\frac{1}{3}}}\right)\right] + \frac{2}{5}\left(1 + \frac{5\tau^{\frac{2}{3}}}{3\Lambda^2}\right), \tag{38}$$

where C_1 and C_2 are integration constants and

$$\Lambda \equiv \sqrt{\frac{18}{5}}\frac{\lambda_J^{(q)}}{\lambda_0}. \tag{39}$$

For large values of Λ, the Bessel functions produce oscillations, while for small values of Λ, the first term in Eq. (38) gives two solutions, one decaying as $1/\tau$ and the other growing as $\tau^{2/3}$. Beside, because of the last term, the density contrast δ_ρ grows with time for small values of Λ, giving rise to Jeans instability. When $q \coloneqq \langle\beta^2\rangle/\beta_0^2$ departs from the equilibrium value $q = 1$, Λ increases, preventing therefore the Jeans instability from occurring, as discussed previously for a static background.

Note that we are considering here, for simplicity, the case of neutral matter, making therefore abstraction of the potential role played by electromagnetic fields. In many astrophysical situations, however, electromagnetic fields play an important role and may significantly modify the Jeans criterion (see, for instance, Refs. [58–62]). This has important implications in the dynamics of comets, planetary rings, and the formation of stars and planets. Depending on the electrical conductivity of the matter, the pressure produced by magnetic fields tends to counterbalance the gravitational attraction, yielding larger threshold wavelengths for the onset of instability [59]. The effect of electromagnetic fields is, however, independent on temperature inhomogeneities. It can be accounted for, in the kinetic picture adopted here, by introducing the corresponding potential in the Vlasov equation (14). Although this would alter the Jeans criterion (20), the conclusion drawn here, i.e., that thermal fluctuations tend to stabilize the system, remains nonetheless valid.

IV. JEANS MECHANISM IN THE QUANTUM REGIME: WAVE-KINETIC APPROACH

In the preceding section, we discussed the Jeans mechanism in a classical context. It is worth extending the previous analysis to the quantum regime. In modeling quantum self-gravitating systems, one usually adopts the Schrödinger-Newton (SN) model (sometimes referred to as the Schrödinger-Poisson model), by coupling the Schrödinger equation with the Poisson equation, describing the self-gravitating potential, as follows:

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\nabla^2 + m\Phi\right)\psi, \\ \nabla^2\Phi &= 4\pi m G|\psi|^2. \end{aligned} \tag{40}$$

Such an approach was first considered in studying self-gravitating boson stars [63]. It also enters in dark matter models [64]; it describes fuzzy dark matter and approximates classical cold dark matter for masses sufficiently large.

In the commonly adopted treatment [65–67], Eq. (40) is written in a hydrodynamic form upon introducing the so-called Madelung transformation [68]. Such an approach, however, does not account for kinetic effects as those associated with a finite temperature and temperature fluctuations. Kinetic effects can nonetheless be introduced by adopting a wave-kinetic approach (see, for instance, Ref. [69] for a pedagogical review in the context of cold atomic gases). To elaborate on this, let us rewrite the above system (40) as a single integrodifferential equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 - m^2 G\int\frac{|\psi(\mathbf{r}', t)|^2}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}r'\right]\psi. \tag{41}$$

A kinetic treatment, similar to the one presented in Sec. III for the Vlasov dynamics, can be realized by relying on the use of the Wigner function $W(\mathbf{r}, \mathbf{q}, t)$,

$$W(\mathbf{r}, \mathbf{q}, t) = \int \psi^*(\mathbf{r}-\mathbf{s}/2, t)\psi(\mathbf{r}+\mathbf{s}/2, t)\exp(i\mathbf{q}\cdot\mathbf{s})\mathrm{d}\mathbf{s}, \tag{42}$$

where $\mathbf{q}$ is the particle momentum. The Wigner function ([70]) is simply the Fourier transform of the autocorrelation function corresponding to the wave function $\psi(\mathbf{r}, t)$. Note in passing that the Wigner function is not a bona fide distribution and should be regarded as a quasi-distribution, since it can take negative values. It is nonetheless a very useful mathematical tool, especially well suited for understanding the quantum/classical transition. By applying the so-called Wigner-Moyal procedure [71], Eq. (41) provides the evolution of the Wigner function as (detailed calculations can be found, for instance, in Refs. [70,72])

$$i\hbar\left(\frac{\partial}{\partial t} + \mathbf{v}_\mathbf{q}\cdot\nabla\right)W = -4\pi m^2 G\int\frac{n(\mathbf{k}, t)}{\mathbf{k}^2}\Delta W e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}, \tag{43}$$

where $\mathbf{v}_\mathbf{q} \equiv \hbar\mathbf{q}/m$ is the particle velocity and

$$\Delta W = W^- - W^+, \qquad W^\pm \equiv W(\mathbf{q}\pm\mathbf{k}/2, \mathbf{r}, t). \tag{44}$$

In Eq. (43), $n(\mathbf{k}, t)$ is the Fourier transform of the (number) density, defined as

$$n(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2 = \int W(\mathbf{r}, \mathbf{q}, t) \frac{\mathrm{d}\mathbf{q}}{(2\pi)^3}. \quad (45)$$

Equation (43) provides the full phase-space description of the self-gravitating system and can be treated similarly to the Vlasov equation (14). As in the classical regime, we restrict ourselves to linear perturbations and write the Wigner distribution as $W = W_0 + \delta W$ ($\delta W \ll W_0$), where W_0 is the stationary distribution describing the quasiequilibrium self-gravitating system and δW is a small perturbation assumed to evolve in space and time as approximately $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Accordingly, the density perturbation follows as $\delta n \sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Then, Eq. (43) gives

$$\delta W = -\frac{4\pi G m^2}{k^2} \frac{\Delta W_0}{\hbar(\omega - \mathbf{k}\cdot\mathbf{v}_q)} \delta n(\mathbf{k}). \quad (46)$$

Integrating over the momentum and using Eq. (45), one has

$$1 + \frac{4\pi m^2 G}{\mathbf{k}^2} \int \frac{\Delta W_0}{h(\omega - \mathbf{k}\cdot\mathbf{v}_q)} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^3} = 0. \quad (47)$$

Let us introduce the parallel and perpendicular components of the particle velocity and momentum as

$$\mathbf{v}_q = v\frac{\mathbf{k}}{|\mathbf{k}|} + \mathbf{v}_\perp, \qquad \mathbf{q} = q_\parallel \frac{\mathbf{k}}{|\mathbf{k}|} + \mathbf{q}_\perp \quad (48)$$

and write Eq. (47) in terms of the parallel velocity distribution

$$G_0(q_\parallel) \equiv \int W_0(q_\parallel, \mathbf{q}_\perp) \frac{\mathrm{d}\mathbf{q}_\perp}{(2\pi)^2} \quad (49)$$

as follows:

$$1 + \frac{\Omega^2}{\omega^2} \int \frac{G_0(v)\mathrm{d}v}{(1 - kv/\omega)^2 - \hbar^2 k^4/4m^2\omega^2} = 0. \quad (50)$$

The latter is the quantum analog of the integral dispersion relation (18). By Taylor expanding the integrand in Eq. (50) for $v \ll \omega/k$, one has

$$\omega^2 = -\Omega^2 + 3\langle v^2\rangle k^2 + \frac{\hbar^2 k^4}{4m^2}, \quad (51)$$

where $\langle v^2\rangle$ is the second-order moment of the stationary distribution G_0. Equation (51) is the quantum counterpart of the dispersion relation (26). It indicates that, in addition to thermal effects, the Jeans instability is also saturated by quantum effects that scale as approximately k^4.

Assuming a quasiequilibrium state, described by a superstatistical distribution, i.e., $G(v)$ reads as an equilibrium distribution averaged over the temperature distribution [cf. Eq. (1)], one may, using Eq. (51), analyze the effect of the temperature inhomogeneity on the Jeans criterion in the quantum regime. For classical (distinguishable) particles, the equilibrium distribution corresponds to the MB distribution, and $\langle v^2\rangle$ is given by Eq. (12). It is, however, worth studying the case of quantum particles by identifying the equilibrium distribution, associated with each small region $p(\mathbf{v}|\beta)$, with the quantum-mechanical expressions for the equilibrium distribution, namely [73], $p(\mathbf{v}|\beta) \equiv 2m^3\hat{f}(\mathbf{q})/2\pi\hbar^3$ with $\hat{f}(\mathbf{q})$ corresponding to the Fermi-Dirac (FD) or the Bose-Einstein (BE) distribution.

The case of fermions is particularly relevant. It is of interest, for instance, in dealing with fermionic clouds of dark matter [74] supposed to be associated with the known dwarf spheroidal galaxies in the vicinity of the Milky Way [75,76]. In this case, using the FD distribution and working out $\langle v^2\rangle$ as previously (see, for instance, Ref. [73]), one has

$$\langle v^2\rangle = \left\langle \sqrt{\frac{9v_F^4}{25} + \left(\frac{3}{\beta m}\right)^2} \right\rangle_{f(\beta)}, \quad (52)$$

where v_F is the Fermi velocity, i.e., related to the Fermi energy via $E_F = mv_F^2/2$. Equation (52) remains valid from the case of completely degenerate (zero temperature) fermions, where the velocity dispersion arises solely from the Pauli exclusion principle, that is, $\langle v^2\rangle = 3v_F^2/5$, all the way to the classical limit, that is, $\langle v^2\rangle_i = 3\phi_i(q)T_0/m$ [cf. Eq. (12)]. For intermediate cases (quasi-degenerate fermions), where both thermal and quantum statistical effects come into play, one may obtain closed-form expressions by retaining the appropriate leading terms: when thermal effects are dominant, one may expand Eq. (52) to find

$$\langle v^2\rangle_i \approx 3\left[\phi_i(q)\frac{T_0}{m} + \frac{v_F^4}{50}\frac{m}{T_0}\right], \quad (53)$$

which shows a quantum correction to the classical result [cf. Eq. (12)]. In the opposite limit (for highly degenerate fermions), one may also expand Eq. (52) to obtain

$$\langle v^2\rangle_i \approx 3\left[\frac{v_F^2}{5} + \frac{5}{2v_F^2}\xi_i(q)\left(\frac{T_0}{m}\right)^2\right], \quad (54)$$

where $\xi_i(q) \geq 1$ read as

$$\xi_1(q) \equiv \frac{1}{6 - 7q + 2q^2} \quad (1 < q < 3/2),$$
$$\xi_2(q) \equiv 6 + \frac{2}{q^2} - \frac{7}{q},$$
$$\xi_3(q) \equiv q^3, \quad (55)$$

with $i = 1$, 2, and 3 corresponding, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. Here, again, one may check that the temperature inhomogeneities tend to increase the velocity dispersion, i.e., Eqs. (53) and (54), that remain larger than their equilibrium counterparts ($q = 1$), hence stabilizing the self-gravitating system in the quantum regime as well.

One may follow similar lines of reasoning for bosons, by identifying $\hat{f}(\mathbf{q})$ with the BE distribution. At low temperature, however, bosons are expected to form Bose-Einstein condensates, in which case temperature effects and, *a fortiori*, temperature fluctuations are irrelevant. When temperature effects are dominant, one has, at this order of approximation, essentially Eq. (53), upon identifying the Fermi energy with an equivalent energy scale [77] $\epsilon_q \equiv \frac{\hbar^2}{2m}(6\pi^2 n/g)^{2/3}$, where g is the factor accounting for the degeneracy of the particles spin states.

Before closing this section, it is instructive to note the generality of the present approach. In fact, the applicability domain of the SN equation (41) goes far beyond the scenario of a self-gravitating system discussed here and covers a wide spectrum of laboratory systems that can be described by a formally identical equation [78]. These systems include, for instance, quantum plasmas [79] and atomic molasses in magneto-optical traps [80]. The superstatistical wave-kinetic approach we went through here remains equally well applicable for those systems and allows for the introduction of temperature fluctuations or inhomogeneities in such different contexts.

V. CONSEQUENCES ON DYNAMICAL FRICTION

When a massive body of mass M moves through a cloud of lighter bodies of typical mass m, the light bodies tend to accelerate (due to the gravitational interaction), gaining therefore momentum and kinetic energy. By virtue of the conservation of energy and momentum, the heavier body will loose kinetic energy and momentum by an amount to compensate, resulting in slowing its motion. This loss of energy and momentum of a moving body, through gravitational interactions with surrounding matter in space, is known as dynamical friction (DF). It plays a fundamental role in the evolution of many-body astrophysical systems, such as globular clusters (GCs) [81], radio galaxies in galaxy clusters [82], and nonlinear gaseous media [83].

The first understanding of this process is due to Chandrasekhar [84] who showed that the DF deceleration on a body of mass M moving with velocity v_M in a homogeneous and isotropic distribution of identical particles of mass m reads as

$$\frac{d\mathbf{v_M}}{dt} = -16\pi^2(\ln\Lambda)G^2 M m n_0 \frac{\int_0^{v_M} f(v)v^2 dv}{v_M^3}\mathbf{v_M}, \quad (56)$$

where $\ln\Lambda$ is the Coulomb logarithm, i.e., the factor by which small-angle collisions are more effective than large-angle collisions, n_0 is the number density of the surrounding particles, and $f(v)$ is their velocity distribution. In the usual treatment of DF, the medium is considered to be completely thermalized, and the distribution $f(v)$ is identified with the equilibrium MB distribution, although the effect of the more general class of Tsallis distributions arising in the framework of NSM has been recently analyzed in the literature [25]. In this section, we will study the DF mechanism in a quasiequilibrium medium, where the velocity distribution corresponds to one of the three universality classes of superstatistics, and examine the consequences of these distributions on the decay of globular orbits.

To proceed, we identify the velocity distribution $f(v)$ with one of the families of superstatistical distributions, i.e., $f(v) \equiv B(v)$. Integrating Eq. (56), and using the definition of $B(v)$, we obtain the generic equation

$$\frac{d\mathbf{v}_M}{dt} = -\frac{4\pi\ln\Lambda G^2 M\rho(r)}{v_M^3} H_i(X_M)\mathbf{v}_M, \quad (57)$$

where $i = 1$, 2, and 3 refer, respectively, to χ^2, inverse-χ^2, and log-normal superstatistics. For the first two classes of superstatistics, we obtain closed-form expressions for the functions $H_i(x)$, accounting explicitly for the effect of temperature inhomogeneities: For χ^2 superstatistics, we have

$$H_1(x) = \frac{(q-1)^{3/2}}{\sqrt{\pi}} \frac{4x^3\Gamma(\frac{3}{2}+\frac{1}{q-1})}{3\Gamma(\frac{1}{q-1})} \times {}_2F_1\left(\frac{3}{2}, \frac{3}{2}+\frac{1}{q-1}; \frac{5}{2}; (1-q)x^2\right), \quad (58)$$

while for inverse-χ^2 superstatistics, we obtain

$$H_2(x) = \frac{B_q(x)x^3 + C_q(x)x^{\frac{4q-2}{q-1}}}{A_q}, \quad (59)$$

where

$$A_q \equiv \frac{3\sqrt{\pi}}{2}(2q-1)\Gamma\left(\frac{q}{q-1}\right),$$

$$B_q(x) \equiv (4q-2)\sqrt{\frac{q}{q-1}}\Gamma\left(\frac{q+1}{2q-2}\right) \times {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; \frac{3}{2} - \frac{q}{q-1}; \frac{qx^2}{q-1}\right),$$

$$C_q(x) \equiv 3(q-1)\left(\frac{q}{q-1}\right)^{q/(q-1)}\Gamma\left(\frac{q+1}{2-2q}\right) \times {}_2F_1\left(1+\frac{q}{q-1}, \frac{1}{2}+\frac{q}{q-1}; 2+\frac{q}{q-1}; \frac{qx^2}{q-1}\right). \quad (60)$$

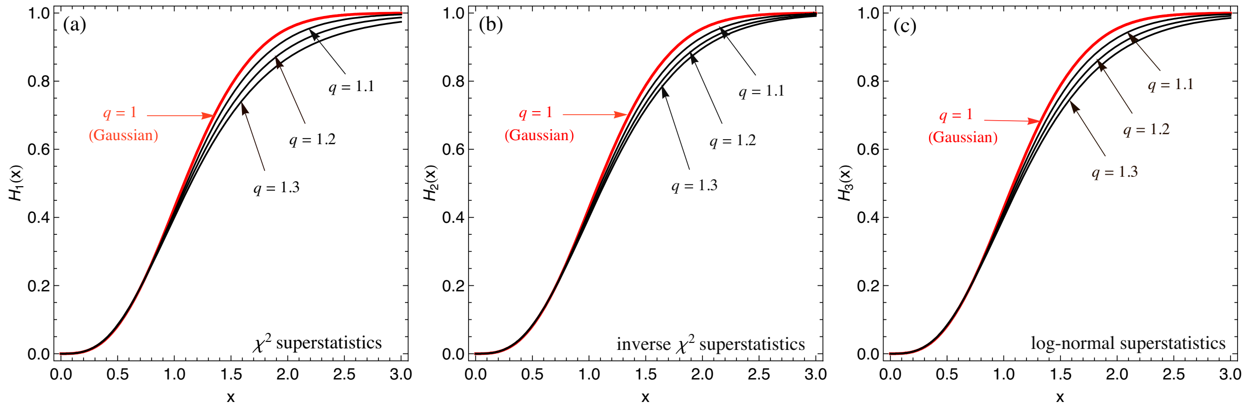

FIG. 5. The functions $H_i(x)$ corresponding to (a) χ^2 superstatistics [Eq. (58)], (b) inverse χ^2 superstatistics [Eq. (59)], and (c) log-normal superstatistics [Eq. (61), computed numerically], for different values of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$.

Above, ${}_2F_1(a,b;c;z)$ denotes the Gaussian hypergeometric function. For the third class of superstatistics, i.e., for log-normal superstatistics, one has

$$H_3(x) = \frac{1}{\sqrt{2\pi \ln(q)}} \int_0^\infty \frac{d\beta}{\beta} e^{-\frac{\ln(\beta\sqrt{q}/\beta_0)^2}{2\ln(q)}} \Bigg[\mathrm{erf}(x\sqrt{\beta/\beta_0}) - \frac{2x}{\sqrt{\pi}} \sqrt{\frac{\beta}{\beta_0}} e^{-\beta x^2/\beta_0} \Bigg], \quad (61)$$

for which there is no closed-form expression, so it will be treated numerically. Figure 5 shows the functions $H_i(x)$ $(i = 1, 2, 3)$ corresponding to the three universality classes, i.e., Eqs. (58), (59), and (61), for different values of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$. One may see that the three universality classes have qualitatively the same effect on $H_i(x)$. In the limit $q = 1$ (thermal equilibrium), the three functions reduce to

$$\lim_{q\to 1} H_i(x) = \mathrm{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2} \quad (i = 1, 2, 3), \quad (62)$$

and Eq. (57) reduces to the standard result [84].

To illustrate the physical effect that emerge from this modification, let us discuss in some detail the case of a GC orbiting through the galaxy field. Because of its interaction with the stellar medium, the GC experiences DF; it loses energy and spiral toward the galaxy center. The time required for the cluster to reach the galaxy center is extremely sensitive to the stellar velocity distribution $f(v)$ through the function $H(x)$ and scales as approximately $1/H(1)$ [25].

In light of Fig. 5, one may conclude that the superstatistical distributions tend to increase this timescale. More precisely and to provide a quantitative analysis, we consider the case of a GC, initially on a circular orbit of radius r_i, and following Ref. [25], we assume a mass density distribution of the galaxy given by

$$\rho(r) = \frac{1}{4\pi G} \left(\frac{v_c}{r}\right)^2, \quad (63)$$

where v_c stands for the circular speed. The frictional force felt by the GC moving through the stellar field with speed v_c reads as [25]

$$F = -G \ln\Lambda \left(\frac{M}{r}\right)^2 H_i(1), \quad (64)$$

where $H_i(x)$ $(i = 1, 2, 3)$ are given in Eqs. (58), (59), and (61), for the three universality classes. The dragging force being tangential to the cluster orbit, the GC gradually loses angular momentum per unit mass L at a rate $dL/dt = Fr/M$. Since $L = rv_c$, Eq. (64) gives

$$r\frac{dr}{dt} = -\left(\frac{GM}{v_c}\right) \ln\Lambda H_q(1). \quad (65)$$

The latter equation can be easily solved, with the initial condition $r(0) = r_i$, to give the time by which the GC reaches the galaxy center as follows:

$$t_i^{(q)} = \frac{v_c r_i^2}{2GM \ln\Lambda H_q(1)}. \quad (66)$$

In the limit $q \to 1$, the latter reduces to the standard result [84], corresponding to the equilibrium MB distribution. In Fig. 6, we show the relative deviation

$$\delta t_i(q) \equiv \frac{t_i(q) - t_0}{t_0}, \quad (67)$$

where t_0 is the DF timescale corresponding to a Maxwellian distribution, i.e., $t_0 = \lim_{q\to 1} t_i(q)$ $(i = 1, 2, 3)$, as a function of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$, for the three universality classes of superstatistics. One may see that, for the three universality classes, as q departs from the equilibrium value $q = 1$, the time by which the GC reaches the galaxy center increases. The superstatistics scenario appears therefore (as recently discussed for Tsallis statistics [25]) as a possible remedy to the problem of the large timescales [85] derived

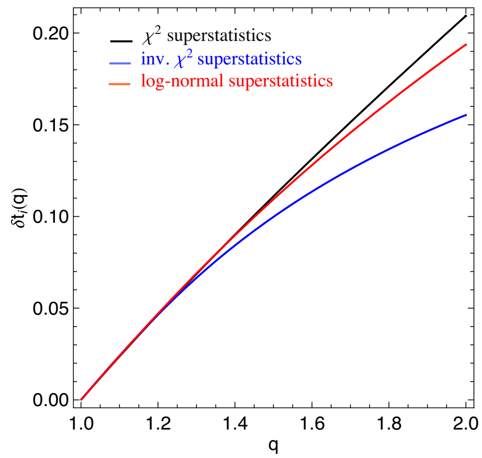

FIG. 6. The relative deviation of the DF timescale $\delta t_i(q)$ [cf. Eq. (67)], as a function of $q \coloneqq \langle\beta^2\rangle/\beta_0^2$, for the three universality classes of superstatistics.

through numerical simulations or semianalytical models that do not concur with the Chandrasekhar model.

Before closing this section, a remark is in order. Note that, although in the special case of χ^2 superstatistics the velocity distribution corresponds to the Tsallis distribution arising in NSM, our prediction for the DF timescale of GC in this case differs from the one obtained in Ref. [25] in the context of NSM. Our results agree *qualitatively*, but the timescale obtained therein increases much more rapidly as the distribution departs from the Maxwellian distribution. This is because, to map the velocity distribution corresponding to the χ^2 class [cf. Eq. (3)] onto the q Gaussian used in NSM, one has not only to reparametrize the parameter q but also the inverse temperature that is redefined as $\tilde{\beta} \coloneqq \beta_0(n+d)/n$ (see Sec. II). Note in this regard that the temperature definition in the framework of NSM (whether it should be identified with the inverse of the Lagrange multiplier β or not) is to date a matter of debate [86,87], and one may make different predictions according to the definition adopted. The superstatistical picture dispels this ambiguity by regarding such distributions as a manifestation of a nonequilibrium situation. In a nonequilibrium state, attributing a single temperature to the whole system is elusive. Nonetheless, the stationary distributions in this case depend only on the mean temperature T_0, that is clearly defined, and the parameter q, measuring the temperature dispersion around its mean value.

VI. CONCLUSIONS

In this analysis, we were dealing quite generally with self-gravitating systems that, due to the long-range nature of the gravitational interaction, remain trapped in a nonequilibrium stationary state. The system is understood as a collection of small regions or subsystems that remain infinitely close to equilibrium. Such subsystems represent regions from where thermalization spreads later over the entire system. At the level of these regions, the whole machinery of equilibrium statistical mechanics holds, but each subsystem has a different temperature assigned to it, according to some probability density. The statistical properties of the system follow as a superposition of statistics, i.e., superstatistics. This methodological attitude allows reproducing the distributions arising in the context of nonextensive statistical mechanics (NSM), that have been widely explored in self-gravitating systems and in fitting observational data [9–15], only as a special case of a more general family of distributions, while the method relies on conventional statistical mechanics and does not introduce any free parameter.

We focus here on the three universality classes of superstatistics, namely, χ^2, inverse-χ^2, and log-normal superstatistics. These classes arise as universal limit statistics in the majority of known superstatistical systems and can be derived from purely probabilistic arguments [41] or from the maximum entropy principle [42]. The velocity distributions corresponding to these classes cover the asymptotic behavior of the main families of distributions encountered in nature, that is, power laws, truncated power laws, and exponential decays. We worked out the velocity distributions corresponding to the three universality classes and confronted them with independent numerical simulations of self-gravitating systems. Then, we explored the consequences of this picture in typical phenomena arising in self-gravitating systems.

We discussed the Jeans instability in this picture, in the classical regime, considering a static and an expanding universe and extended our approach to the quantum regime by considering the Wigner-Moyal procedure and the effect of quantum statistics. In both cases, our results show that quasi-stationary self-gravitating systems are generally more stable than equilibrium systems, indicating that a larger mass is needed to initiate the gravitational collapse, in agreement with previous studies [17,22] using the NSM distributions in the superthermal regime. This result is particularly relevant in the case of Bok globules because their mass and their corresponding Jeans mass are of the same order; hence, a small deviation from equilibrium may lead to a different prediction for their stability. We also examined the Chandrasekhar dynamical friction in a quasiequilibrium medium and analyzed the consequences on the decay of globular orbits. Our results show that the time by which globular clusters reach the galaxy center increases in the superstatistical scenario, offering a potential solution to the problem of the large timescale [85] shown by numerical N-body simulations and semianalytical models.

Our main result is that the typical behavior, usually attributed to the special class of Tsallis distributions emerging in the framework of NSM, is rather a general feature of nonequilibrium systems in a stationary state. Thus, the superstatistical picture presents itself as a promising alternative to handle such systems inasmuch as it is able to produce a wider class of distributions, while overcoming potential problematic aspects and open problems appearing in the NSM context, such as the definition of a physical temperature [86,87], the stability of the averaging schemes [88,89], and possible inconsistencies associated with the nonadditive entropy maximization [90,91].

The present study naturally opens up new prospects for future research. First, a closer examination of the ability of the superstatistical distributions to account for results of N-body simulations of self-gravitating systems and observational data will be welcome, especially in the cases that seem to go beyond the scope of Tsallis distributions [28,29]. This would pave the way for a more complete understanding of the emergence of these distributions in astrophysical situations. Furthermore, the use of the superstatistical kinetic approach presented here, especially in the quantum regime, can be found simple and meaningful in many relevant physical situations. As shown recently [78], the applicability domain of the Schrödinger-Newton equation goes far beyond the scenario of a self-gravitating system discussed here and covers a large spectrum of systems that can be described by a formally identical equation. These systems include quantum plasmas, Bose-Einstein condensates with or without long-range dipolar interactions, and atomic molasses in magneto-optical traps. The method worked out here remains applicable in such different contexts and allows to account for fluctuations of the temperature or the chemical potential (see, for instance, Ref. [92] for a superstatistical approach to a fluctuating chemical potential in the context of quark matter). Besides, as these systems manifest similar elementary excitations, e.g., electron oscillations in a quantum plasma (plasmons), hybrid-phonon modes, or Bogoliubov excitations, the superstatistical approach presented here may allow identifying the signature of nonequilibrium stationary states in the corresponding dispersion relations.

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