

## Asymptotic low-temperature behavior of two-dimensional $\mathbb{R}P^{N-1}$ models

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We investigate the low-temperature behavior of two-dimensional (2D)  $\mathbb{R}P^{N-1}$  models, characterized by a global  $O(N)$  symmetry and a local  $\mathbb{Z}_2$  symmetry. For  $N = 3$  we perform large-scale simulations of four different 2D lattice models: two standard lattice models and two different constrained models. We also consider a constrained mixed  $O(3)$ - $\mathbb{R}P^2$  model for values of the parameters such that vector correlations are always disordered. We find that all these models show the same finite-size scaling (FSS) behavior, and therefore belong to the same universality class. However, these FSS curves differ from those computed in the 2D  $O(3)$   $\sigma$  model, suggesting the existence of a distinct 2D  $\mathbb{R}P^2$  universality class. We also performed simulations for  $N = 4$ , and the corresponding FSS results also support the existence of an  $\mathbb{R}P^3$  universality class, different from the  $O(4)$  one.

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### I. INTRODUCTION

Global and local gauge symmetries play a crucial role in the construction of quantum and statistical field theories, relevant for fundamental interactions [1] and emerging phenomena in condensed matter physics [2]. They determine the main features of the model, such as the phase diagram and the nature of their thermal and quantum phase transitions. The critical behavior arising from the interplay between global and local gauge symmetries has been investigated in several physical contexts. Paradigmatic examples are the finite-temperature transitions in quantum chromodynamics, the theory of strong interactions [3–5], and in the multicomponent Abelian-Higgs model [6,7]. In the case of non-Abelian gauge symmetries, the nature of the phase transitions is mostly determined by the global symmetries, in both three-dimensional (3D) and two-dimensional (2D) models, while the modes associated with the local gauge symmetries play only a marginal role, see, e.g., Refs. [3,7–10]. This is not the case for the Abelian  $U(1)$  gauge theories, in which some features of the gauge group—in particular, the topology of the gauge-field configurations—play an important role. There is now a wide consensus that, in three dimensions, the critical behavior depends on the presence/absence of topological defects like monopoles and hedgehogs and on the compact/noncompact nature of the gauge fields [11–18]. Also in the case of antiferromagnetic models, gauge fields apparently play an important role, and indeed, effective models in which they are integrated out do not describe their critical behavior [19].

$\mathbb{R}P^{N-1}$  models represent a class of systems characterized by the simultaneous presence of a global and a local gauge

symmetry. They are  $N$ -component vector models that are invariant under global  $O(N)$  and local  $\mathbb{Z}_2$  transformations, and they are expected to describe the universal features of the isotropic-nematic transition in liquid crystals [20]. Ferromagnetic  $\mathbb{R}P^{N-1}$  models in three dimensions are not particularly interesting as the finite-temperature transition is of first order [20]. Critical transition are instead observed in 3D antiferromagnetic models [21–23], whose nature, however has not yet been fully clarified for  $N \geq 4$ . Antiferromagnetic models are also relevant (but after an analytic continuation to  $N = -1$ ) in the analysis of the behavior of spanning forests (see Ref. [24] and references therein).

In this work we will study the critical behavior of ferromagnetic 2D  $\mathbb{R}P^{N-1}$  models. Their behavior has been for long controversial, and at present, it is not yet fully understood, see, e.g., Refs. [25–30]. For  $N \geq 3$ , these models are not expected to undergo finite-temperature continuous transitions related to the breaking of the  $O(N)$  symmetry, because of the Mermin-Wagner theorem [31]. *A priori*, transitions with quasilong-range order are possible, but they can be excluded using simple comparison arguments [27]. For finite values of the temperature only first-order transitions are generically allowed, and indeed such transitions are expected for large values of  $N$  [32–34]. Magnetic modes can become critical only in the zero-temperature limit, and in this limit magnetic correlations increase exponentially, similarly to what occurs in 2D  $O(N)$   $\sigma$  models. The nature of such asymptotic low-temperature behavior has been long debated. References [27–29] reported arguments to support the claim that  $\mathbb{R}P^{N-1}$  and  $O(N)$  models belong to the same

universality class, implying the irrelevance of the  $\mathbb{Z}_2$  gauge symmetry in the zero-temperature limit. However, these arguments were never supported by numerical data: in all cases [25,27]  $\text{RP}^{N-1}$  results were in large disagreement with the predictions obtained by assuming the equivalence of the two classes of models. A rigorous argument in favor of the equivalence was put forward in Ref. [27]. However, it was based on models effectively designed to eliminate the topological defects, whose presence is essential to obtain a different low-temperature behavior for  $\text{RP}^{N-1}$  models and  $O(N)$  models.

In this paper we return to the issue of the nature of the low-temperature critical behavior of 2D  $\text{RP}^{N-1}$  models. Indeed, topological defects, even if exponentially rare, can change the asymptotic nonperturbative behavior of the model (this is the case of the compact  $U(1)$  gauge theory in three dimensions, see Ref. [11]). For this purpose we study the finite-size scaling (FSS) behavior of several different  $\text{RP}^{N-1}$  models, with  $N = 3$  and 4. If all these models are in the same  $O(N)$  universality class, we would expect them to have the same FSS behavior as the standard  $O(N)$  model. If discrepancies are present, in this scenario they would be interpreted as scaling corrections that would be therefore nonuniversal, that is they would depend on the model. Therefore, the results corresponding to the different models should either fall on top of the  $O(N)$  FSS curves or should all be different. As we shall see, this does not occur. The  $\text{RP}^{N-1}$  data show universality: all data fall on the same FSS curve, with tiny differences that would be naturally interpreted as scaling corrections. The resulting FSS curve is distinctly different from the corresponding one obtained in the  $O(N)$  vector model. Therefore, the observed universal behavior supports the existence of an  $\text{RP}^{N-1}$  distinct universality class.

The paper is organized as follows. In Sec. II we report the Hamiltonians of the two standard  $\text{RP}^{N-1}$  models we consider. In Sec. III we review the different scenarios for the behavior of  $\text{RP}^{N-1}$  models. In Sec. IV we define our observables and review the FSS methods that are used in the numerical analysis of the data. In Sec. V we present the FSS analyses of the numerical data for the two models introduced in Sec. II. In Sec. VI we consider a class of models introduced in Refs. [27,35], discuss the rigorous arguments of Ref. [27], and present numerical results for this class of models. Our conclusions are reported in Sec. VII.

## II. TWO-DIMENSIONAL $\text{RP}^{N-1}$ MODELS

$\text{RP}^{N-1}$  models are  $N$ -vector models characterized by a global  $O(N)$  symmetry and a local  $\mathbb{Z}_2$  gauge symmetry. A lattice formulation of the  $\text{RP}^{N-1}$  model on a square lattice can be obtained by considering real  $N$ -dimensional vectors  $\mathbf{S}_x$  of unit length defined on the sites of the lattice (they satisfy  $\mathbf{S}_x \cdot \mathbf{S}_x = 1$ ) and the Hamiltonian

$$H = -J \sum_{x,\mu} (\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}})^2. \quad (1)$$

Here  $\hat{\mu} = \hat{1}, \hat{2}$  are unit vectors along the lattice directions and the sum runs over all lattice links. The partition function of the system reads

$$Z = \int [d\mathbf{S}_x] e^{-\beta H}, \quad \beta \equiv 1/T. \quad (2)$$

Alternatively we may consider a lattice model with an explicit  $\mathbb{Z}_2$  gauge variable  $\sigma_{x,\mu} = \pm 1$  associated with each link. The Hamiltonian is in this case

$$H_\sigma = -J \sum_{x,\mu} \mathbf{S}_x \cdot \sigma_{x,\mu} \mathbf{S}_{x+\hat{\mu}} \quad (3)$$

and the partition function reads

$$Z = \int [d\mathbf{S}_x] \sum_{\{\sigma_{x,\mu}\}} e^{-\beta H_\sigma}. \quad (4)$$

The fields  $\sigma_{x,\mu} = \pm 1$  can be trivially integrated out, obtaining the effective model with partition function

$$Z = \int [d\mathbf{S}_x] e^{-\beta H_{\sigma,\text{eff}}}, \quad H_{\sigma,\text{eff}} = -\beta^{-1} \sum_{x,\mu} \ln 2 \cosh(\beta J |\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}|). \quad (5)$$

For  $\beta$  large, the expression of the Hamiltonian can be simplified obtaining

$$H_{\sigma,\text{eff}} = -J \sum_{x,\mu} |\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}|, \quad (6)$$

with corrections that are exponentially small in  $\beta$ . These models are invariant under the global  $O(N)$  rotations of the  $N$ -component spin variables and under the local  $\mathbb{Z}_2$  gauge transformations  $\mathbf{S}_x \rightarrow s_x \mathbf{S}_x$  [supplemented by  $\sigma_{x,\mu} \rightarrow s_x \sigma_{x,\mu} s_{x+\hat{\mu}}$  for model (3)] with  $s_x = \pm 1$ . We set  $J = 1$  for both lattice models without loss of generality.

Due to the  $\mathbb{Z}_2$  gauge symmetry, the critical behavior can be characterized by studying the correlations of the spin-2 gauge-invariant operator

$$Q_x^{ab} = S_x^a S_x^b - \frac{1}{N} \delta^{ab}. \quad (7)$$

In two dimensions, according to the Mermin-Wagner theorem [31], no finite-temperature transition related to the breaking of the  $O(N)$  symmetry can occur. Spins order only in the limit  $\beta \rightarrow \infty$ . The asymptotic zero-temperature behavior can be studied using perturbation theory. It predicts the emergence of long-range correlations

characterized by a length scale that increases exponentially in  $\beta$ , as it also occurs in 2D  $O(N)$   $\sigma$  models.

### III. DIFFERENT SCENARIOS FOR THE CRITICAL BEHAVIOR OF $RP^{N-1}$ MODELS

We wish now to present the different scenarios for the behavior of  $RP^{N-1}$  model that have been proposed in the literature. One possibility is that these models undergo a transition at finite temperature and indeed, a first-order transition is predicted for large values of  $N$  [32–34]. In principle, it is also possible to have a finite-temperature continuous transition, where energy-energy correlations display long-range order, while magnetic modes are non-critical in agreement with the Mermin-Wagner theorem. Such continuous transitions, whose existence was put forward in Ref. [28], were observed in a class of modified  $O(N)$  models. It was proved rigorously that a finite-temperature first-order transition line occurs in a class of  $O(N)$  and  $RP^{N-1}$  models with nonlinear Hamiltonians [36]. The endpoint of the transition line is expected to correspond to a continuous finite-temperature transition in the Ising universality class: this was verified numerically in Ref. [37] for  $N = 3$  and in the large- $N$  limit in Ref. [38]. A similar behavior is expected in mixed  $O(N)$ - $RP^{N-1}$  models for large values of  $N$  [32,38].

*A priori* it is also possible that the system has a continuous magnetic transition without the presence of a magnetized low-temperature phase, as it occurs for  $N = 2$ . As discussed in Ref. [27] this possibility is unlikely. Consider indeed the model with Hamiltonian (3): the role of the  $\sigma$  fields is that of adding additional disorder in the system and thus we expect (and verify numerically in the following) that the magnetic correlation lengths in the  $RP^{N-1}$  model and in the corresponding  $O(N)$  model satisfy the inequality  $\xi_{RP^{N-1}}(\beta) < \xi_{O(N)}(\beta)$ . Since  $\xi_{O(N)}(\beta)$  is always finite for finite  $\beta$ , we can exclude the presence of a finite-temperature transition with a diverging magnetic correlation length.

At present, there is no indication of the presence of a (continuous or first-order) finite-temperature transition for  $N = 3$  [25,26,29]. The only scenario that is consistent with the data is the one in which no transition occurs for finite  $\beta$ : a critical behavior is only observed for  $\beta \rightarrow \infty$ . In this limit the  $\beta$ -dependence of the observables can be computed in perturbation theory. For the Hamiltonian (1) the perturbative behavior, however, can only be observed for large values of  $\beta$ —therefore, for very large correlation lengths—since perturbative corrections are very large [39]. For  $N = 3$  it is practically impossible to verify the perturbative asymptotic scaling [39]. For perturbative considerations, it is much more interesting to consider the gauge Hamiltonian (3). From Eq. (6), it is obvious that any quantity has exactly the same perturbative expansion in the gauge  $RP^{N-1}$  model and in the usual  $O(N)$  model. For instance, if we consider

the infinite-volume correlation length computed from  $Q_x$ , the ratio  $\xi_{\infty,O(N)}/\xi_{\infty,RP^{N-1}}$  should be constant apart from nonperturbative corrections that decay exponentially in  $\beta$ . Moreover, if  $O(N)$  and  $RP^{N-1}$  models have the same asymptotic universal behavior, the ratio should approach one as  $\beta \rightarrow \infty$ .

The nonperturbative behavior is a different issue. As discussed in Ref. [25,26], the question of the equivalence of  $RP^{N-1}$  and  $O(N)$  is directly related to the question of the nature of their lowest-energy excitations. If an  $RP^{N-1}$  universality class exists, one expects the lowest-energy excitations to be associated with the bilinear field  $Q_x$ . On the other hand, if such a universality class does not exist, the lowest-energy excitations are associated with vector modes as in the standard  $O(N)$  model. In Refs. [25,26], the authors considered this possibility unlikely, as the vector correlation function  $\langle \mathbf{S}_x \cdot \mathbf{S}_y \rangle$  is trivial in  $RP^{N-1}$  models. However, in the context of these models, it is probably more appropriate to consider the gauge-invariant correlation function

$$G_P(x, y) = \mathbf{S}_x \cdot \mathbf{S}_y \prod_{\ell \in P} \text{sign}(\mathbf{S} \cdot \mathbf{S})_{\ell}, \quad (8)$$

where  $P$  is a path connecting  $x$  and  $y$ ,  $\ell$  is a link belonging to the path, and  $(\mathbf{S} \cdot \mathbf{S})_{\ell}$  is the scalar product of the two spins at the endpoints of the link. In models in which there is an explicit gauge field  $\sigma_{x,\mu}$ ,  $\text{sign}(\mathbf{S} \cdot \mathbf{S})_{\ell}$  can be replaced by  $\sigma_{\ell}$ . For continuous gauge groups (for instance, in the case of  $CP^{N-1}$  models) this correlation function is not critical, even for  $\beta \rightarrow \infty$  [40–43]. Indeed, local string fluctuations always add up to give rise to an exponential decay  $e^{-aL_P}$ , where  $L_P$  is the length of the path  $P$  and  $a$  is a path-independent constant. In our case, the gauge group is discrete and therefore the behavior of strings of  $\sigma$  fields and of the correlation function  $G_P(x, y)$  is less clear.

The possible presence of two distinct universality classes, the  $O(N)$  and the  $RP^{N-1}$  universality class, is related with the behavior of the effective  $\mathbb{Z}_2$  excitations associated with the field  $\sigma_{x,\mu}$ . In models in which there are no explicit gauge fields, one can equivalently define

$$\sigma_{x,\mu} = \text{sign}(\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}). \quad (9)$$

The relevant variable is the plaquette

$$\Pi_x = \sigma_{x,1}\sigma_{x,2}\sigma_{x+\hat{1},2}\sigma_{x+\hat{2},1}. \quad (10)$$

If  $\Pi_x = 1$  for all sites, in infinite volume (in a finite volume there are some subtleties [27], see below) we can write  $\sigma_{x,\mu} = \tau_x \tau_{x+\hat{\mu}}$ , where  $\tau_x$  is an Ising spin defined on the sites of the lattice. In this case,  $O(N)$  and  $RP^{N-1}$  models are equivalent [27,28]. Thus, the existence of an  $RP^{N-1}$  universality class depends on the density of the plaquettes with  $\Pi_x = -1$  (we will call them topological defects). This

problem has never been addressed quantitatively, although simple calculations show that the outcome may be action dependent [28]. In particular, one can devise  $\text{RP}^{N-1}$  models [27] (we will come back to this issue in Sec. VI) such that one can rigorously prove that defects are absent in the asymptotic regime in which the system orders. Therefore, they have the same critical behavior as the usual  $O(N)$  vector model.

Finally, we mention the scenario proposed by Catterall *et al.* [29]. They considered a variant of the gauge  $\text{RP}^{N-1}$  model obtained by adding a term  $\mu \sum_x \Pi_x$  to the action, where  $\mu$  plays the role of a chemical potential for the defects. For this action they identified numerically a specific renormalization-group trajectory which flows to a ‘‘vorticity’’ fixed point and apparently attracts the renormalization-group trajectories for the standard  $\text{RP}^{N-1}$  gauge model. They conjectured that this specific trajectory is responsible for the observed quasiuniversal behavior [29,44], which is expected to hold only when  $\xi$  is less than a crossover correlation length  $\xi^{\text{cross}}$ . For  $\xi \gtrsim \xi^{\text{cross}}$ ,  $O(N)$  behavior should instead be observed. Such a scenario might explain the observed phenomenology, however it can be hardly tested numerically since the crossover correlation length is enormous,  $\xi^{\text{cross}} \approx 10^9$ . Indeed, nonperturbative differences between  $\text{RP}^{N-1}$  and  $O(N)$  models can only be observed when  $\xi/L \lesssim 1$  (see below), so that any investigation of the nonperturbative behavior for  $\xi \approx \xi^{\text{cross}}$  requires huge systems with  $L \approx 10^9$ .

If  $O(N)$  and  $\text{RP}^{N-1}$  models are equivalent, FSS functions in the two models can be directly related. One should, however, take into account that different boundary conditions should be considered in the two cases. As discussed in Ref. [27], the FSS functions of gauge-invariant quantities for the  $\text{RP}^{N-1}$  model with periodic boundary conditions should be the same as those of the  $O(N)$  model with periodic/antiperiodic boundary conditions. The same argument of Ref. [27] can be used to prove the equivalence of the  $\text{RP}^{N-1}$  FSS functions with those of the  $O(N)$  model with link-fluctuating boundary conditions (LFBC). To define it, consider a cubic lattice with periodic boundary conditions and divide the set of lattice links into two disjoint subsets  $\mathcal{B}$  and  $\mathcal{C}$ . We indicate with  $\mathcal{B}$  the set of boundary links connecting points  $\mathbf{x} = (L, m)$ ,  $\mathbf{y} = (1, m)$ , and points  $\mathbf{x} = (m, L)$ ,  $\mathbf{y} = (m, 1)$  ( $m = 1, \dots, L$ );  $\mathcal{C}$  corresponds to the set of internal links (the lattice links that do not belong to  $\mathcal{B}$ ). The  $O(N)$  model with LFBC is defined by the Hamiltonian

$$H_{\text{lf}} = - \sum_{(x,\mu) \in \mathcal{B}} \mathbf{S}_x \cdot \sigma_{x,\mu} \mathbf{S}_{x+\hat{\mu}} - \sum_{(x,\mu) \in \mathcal{C}} \mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}. \quad (11)$$

We can also consider an equivalent Hamiltonian, which is the analogue of Hamiltonian (1):

$$H_{\text{lf}2} = - \sum_{(x,\mu) \in \mathcal{B}} (\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}})^2 - \sum_{(x,\mu) \in \mathcal{C}} \mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}. \quad (12)$$

#### IV. FINITE-SIZE SCALING IN THE ZERO-TEMPERATURE LIMIT

In this paper we investigate the nature of the asymptotic large- $\beta$  behavior of the lattice  $\text{RP}^{N-1}$  models. For this purpose we consider  $\text{RP}^{N-1}$  models on a square lattice of linear size  $L$  with periodic boundary conditions.

We mostly focus on correlations of the gauge-invariant local variable  $Q_x^{ab}$  defined in Eq. (7), which is a symmetric and traceless matrix. Its two-point correlation function is defined as

$$G(\mathbf{x} - \mathbf{y}) = \langle \text{Tr} Q_x Q_y \rangle, \quad (13)$$

where the translation invariance of the system has been taken into account. The susceptibility and the correlation length are defined as  $\chi = \sum_x G(\mathbf{x})$  and

$$\xi^2 \equiv \frac{1}{4\sin^2(\pi/L)} \frac{\tilde{G}(\mathbf{0}) - \tilde{G}(\mathbf{p}_m)}{\tilde{G}(\mathbf{p}_m)}, \quad (14)$$

where  $\tilde{G}(\mathbf{p}) = \sum_x e^{i\mathbf{p}\cdot\mathbf{x}} G(\mathbf{x})$  is the Fourier transform of  $G(\mathbf{x})$ , and  $\mathbf{p}_m = (2\pi/L, 0)$ . We also consider the Binder parameter defined as

$$U = \frac{\langle \mu_2^2 \rangle}{\langle \mu_2 \rangle^2}, \quad \mu_2 = \frac{1}{V^2} \sum_{x,y} \text{Tr} Q_x Q_y, \quad (15)$$

where  $V = L^2$  is the volume. To determine the universal features of the asymptotic zero-temperature behavior we use a FSS approach [45–49]. At finite-temperature continuous transitions the FSS limit is obtained by taking  $\beta \rightarrow \beta_c$  and  $L \rightarrow \infty$  keeping  $X \equiv (\beta - \beta_c)L^{1/\nu}$  fixed, where  $\beta_c$  is the inverse critical temperature and  $\nu$  is the correlation-length exponent. Any renormalization-group invariant quantity  $R$ , such as the ratio

$$R_\xi \equiv \xi/L \quad (16)$$

and the Binder parameter  $U$ , is expected to asymptotically behave as  $R(\beta, L) = f_R(X) + O(L^{-\omega})$ , where  $\omega$  is a universal exponent. The scaling function  $f_R(X)$  is universal apart from a trivial normalization of its argument; it only depends on the shape of the lattice and on the boundary conditions. Since  $R_\xi$  is generally monotonic, we can also write [46–49],

$$R(\beta, L) = F_R(R_\xi) + O(L^{-\omega}), \quad (17)$$

where  $F_R$  is a universal scaling function. Eq. (17) is particularly convenient, as it allows a direct check of universality, without the need of tuning any parameter. Moreover, it applies directly, without any change, to two-dimensional asymptotically free models [6], in which a critical behavior is only obtained in the limit  $\beta \rightarrow \infty$ , see

Refs. [50–53] and references therein. In this case, scaling corrections decay as  $L^{-2} \log^p L$ , where  $p$  cannot be determined in perturbation theory [see Ref. [53] for a discussion in the  $O(N)$  model].

In the following, we consider the finite-size behavior of the Binder parameter  $U$  as a function of  $R_\xi$ : If two models belong to the same universality class, the Binder parameter  $U$  must satisfy the FSS relation (17) with the same asymptotic curve  $F_U(R_\xi)$ . Universality also implies that all dimensionless renormalization-group invariant quantities have the same asymptotic large- $\beta$  behavior, both in the thermodynamic and in the FSS limit.

## V. NUMERICAL RESULTS FOR THE LATTICE $RP^2$ AND $RP^3$ MODELS

To identify the nature of the universal zero-temperature behavior, we have performed simulations of the lattice  $RP^2$  models (1) and (3) on a wide range of lattice sizes (up to  $L = 640$ ) with periodic boundary conditions and of the lattice  $O(3)$  model (11) with LFBC (up to  $L = 160$ ). We have also performed a limited study of the case  $N = 4$ , considering the lattice  $RP^3$  model (3) and the lattice  $O(4)$  model (11). For both the models (1) and (3) a standard Metropolis and an overrelaxation algorithm were used to update the fields  $S_x$ , while just Metropolis was used to update  $\sigma_{x,\mu}$ . For the case of the model (1) it is however numerically convenient to introduce continuous link fields, and rewrite the Hamiltonian as that of an  $O(N)$  model with annealed gaussian random links with zero average and variance  $\beta/2$ , see e.g., [33].

As explained in Sec. IV, we focus on the FSS behaviors of the Binder parameter  $U$  and the ratio  $R_\xi \equiv \xi/L$ . Figure 1 shows  $U$  versus  $\beta$  for several lattice sizes. The results for the models with Hamiltonians (1) and (3) are similar. The Binder parameter  $U$  varies between the strong-coupling value

$$U = 1 + \frac{4}{(N-1)(N+2)}, \quad (18)$$

thus  $U = 7/5$  for  $N = 3$ , and the weak-coupling value  $U = 1$ . We also note that the datasets corresponding to different lattice sizes do not show any crossing point, confirming the absence of a finite-temperature transition. Analogous results are obtained for the ratio  $R_\xi = \xi/L$ .

As already anticipated in Sec. IV, our FSS analysis is based on the determination of the behavior of  $U$  as a function of  $R_\xi \equiv \xi/L$ . Figure 2 shows the results for models (1) and (3). In both cases the data approach an asymptotic FSS curve as  $L$  increases. Corrections are small, in particular for the model (3). More interestingly, the results show a clear evidence of universality: the data for the two models corresponding to the largest sizes apparently fall onto the same asymptotic curve, see the lower

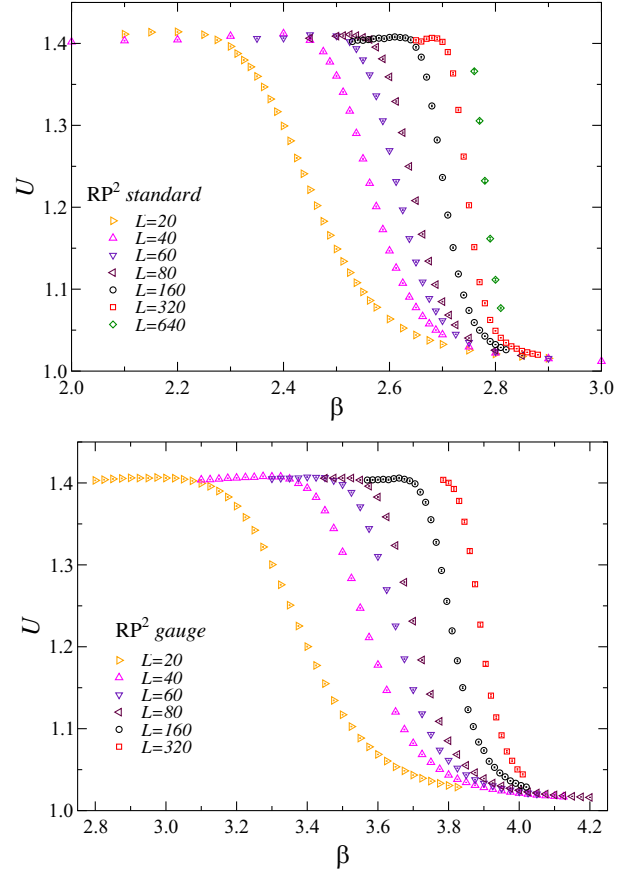


FIG. 1. The Binder parameter  $U$  vs the inverse temperature  $\beta$  for the standard lattice  $RP^2$  model (1) (top panel), and for the lattice  $RP^2$  model (3) with explicit  $Z_2$  gauge link variables (bottom panel).

panel of Fig. 2. Corrections to the zero-temperature critical behavior are expected to decay as  $L^{-2}$  times a function of  $\ln L$ . In the case of the two lattice  $RP^2$  models considered, convergence is roughly consistent with  $L^{-1}$  corrections, likely because the logarithmic corrections mimic a power term, as often observed in  $O(N)$   $\sigma$  models, see, e.g., Ref. [54] for a discussion. Ref. [30] suggested that the critical behavior should be related to that of the  $O(5)$  vector model. We have verified that our curve differs from that computed in the  $O(5)$  model. The  $O(5)$  model may turn out to be more appropriate to describe the behavior of the antiferromagnetic  $RP^2$  model, as discussed at length for the three-dimensional case [21,23].

As we mentioned, if  $O(N)$  and  $RP^{N-1}$  models belong to the same universality class, the  $RP^{N-1}$  scaling functions for gauge-invariant quantities should agree with the corresponding ones for the  $O(N)$  model with LFBC. In the lower panel of Fig. 2, we also report corresponding data [also in the  $O(3)$  model we consider the correlation length and the Binder parameter of the gauge-invariant  $Q_x$  defined in Eq. (7)] for the model with Hamiltonian (11). It is evident that the  $O(3)$  results are very different from those obtained

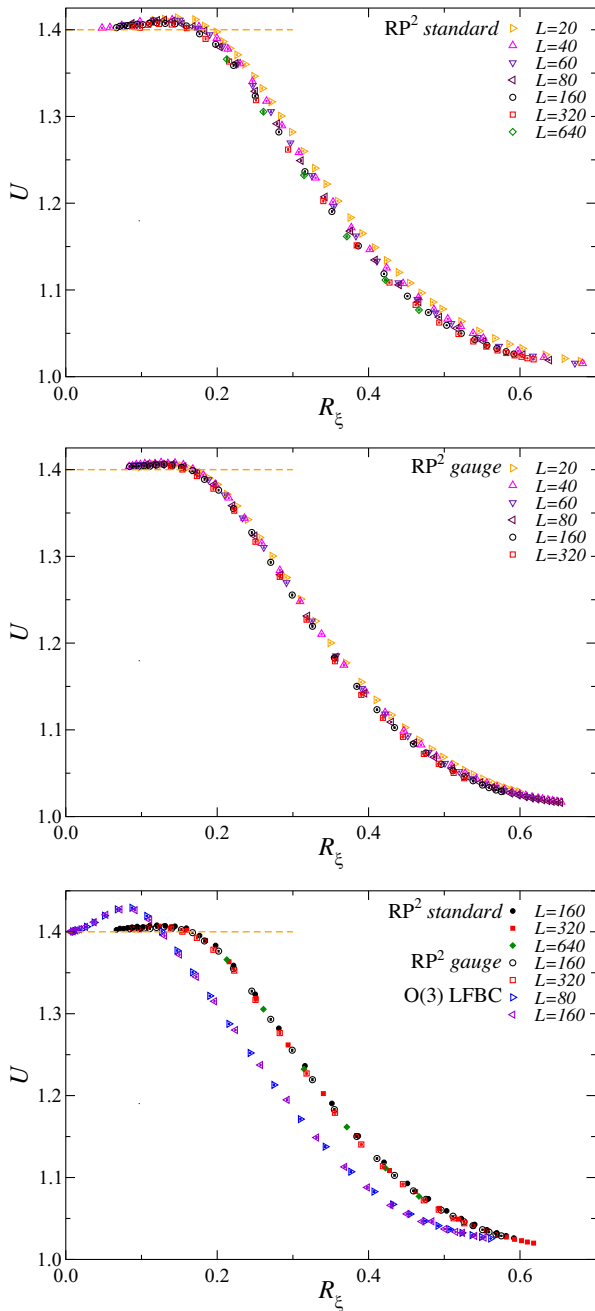


FIG. 2. Plot of  $U$  vs  $R_\xi$  for the standard lattice  $\text{RP}^2$  model (1) (top panel) and for the lattice  $\text{RP}^2$  model (3) with explicit  $\mathbb{Z}_2$  gauge link variables (middle panel). In the lowest panel we report the data of both models (they are labelled “ $\text{RP}^2$  standard” and “ $\text{RP}^2$  gauge”) for the largest lattices available and the data for the  $\text{O}(3)$  model with LFBC (“ $\text{O}(3)$  LFBC”) with Hamiltonian (11). The horizontal dashed line corresponds to the strong-coupling value  $U = 7/5$ .

for the two  $\text{RP}^2$  models. This large disagreement, already noted in Ref. [27] for a different scaling function, naturally raises some doubts on the scenario in which  $\text{O}(N)$  and  $\text{RP}^{N-1}$  models have the same nonperturbative behavior. Note that the differences are only observed for  $R_\xi \lesssim 0.6$ .

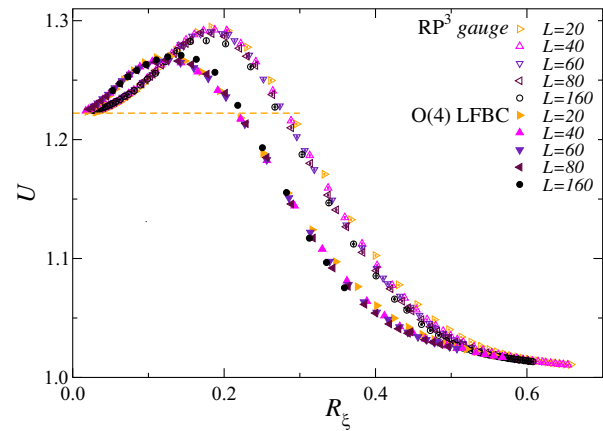


FIG. 3. Plot of  $U$  vs  $R_\xi$  for the lattice  $\text{RP}^3$  model (3) with explicit  $\mathbb{Z}_2$  gauge link variables (“ $\text{RP}^3$  gauge”) and for the  $\text{O}(4)$  model with LFBC (“ $\text{O}(4)$  LFBC”) with Hamiltonian (11). The horizontal dashed line corresponds to the strong-coupling value  $U = 11/9$ .

For larger values of  $R_\xi$ , the  $\text{O}(N)$  and  $\text{RP}^{N-1}$  scaling functions are essentially the same: we are indeed entering the perturbative regime in which the scaling functions can be computed using perturbation theory, which, as we already mentioned, is expected to be the same for the two classes of models.

As an additional check, we consider the values of the correlation length as a function of  $\beta$ . As we mentioned in Sec. III, the correlation lengths computed in the standard  $\text{O}(3)$  model and in the gauge model at the same value of  $\beta$  should be equal, with corrections that decrease exponentially in  $\beta$ , if the two models are asymptotically equivalent. The values of the correlation length in the  $\text{O}(3)$  model can be computed using the four-loop results of Ref. [55] (deviations are small [52] and practically irrelevant for our considerations) and using the estimate [25]  $\xi_V/\xi \approx 3.44$ , where  $\xi_V$  is the correlation length computed from the vector correlation  $\langle \mathbf{S}_x \cdot \mathbf{S}_y \rangle$ . For instance, for the  $\text{O}(3)$  model we obtain  $\xi \approx 1.1 \times 10^7$  at  $\beta = 3.785$  to be compared with the  $\text{RP}^2$  result  $\xi \approx 44$ . Clearly, the correlation length in the  $\text{RP}^2$  model is much smaller than what it should be if the  $\text{O}(3)$  and the gauge model were nonperturbatively equivalent. A similar discrepancy was already noted [56] for the standard action (1), but its significance was not clear because of the presence of very large perturbative corrections decaying as an inverse power of  $\beta$  [39]. Here instead, perturbative corrections are very small (they are the same as in the  $\text{O}(N)$  model, for which the four-loop expression accurately reproduces the data for  $\beta \approx 2-3$  [52]). Therefore, the large discrepancy we observe can be hardly interpreted as a finite- $\beta$  correction.

To close this section we present some data for case  $N = 4$ : in Fig. 3 the FSS of  $U$  as a function of  $R_\xi$  is shown for the lattice  $\text{RP}^3$  model with explicit  $\mathbb{Z}_2$  gauge link variables (3), and for the  $\text{O}(4)$  model with link fluctuating

boundary conditions (11). These results show that also for  $N = 4$  significant differences are observed between the  $\text{RP}^3$  and  $\text{O}(4)$  data, which point to the existence of a  $\text{RP}^3$  fixed point, distinct from that of the  $\text{O}(4)$   $\sigma$  model.

## VI. PATRASCIOIU-SEILER MODEL

In this section, we discuss another class of models introduced by Patrascioiu and Seiler [35] and used in the present context by Hasenbusch [27]. We first consider the constrained  $\text{RP}^{N-1}$  model, whose partition function is

$$Z = \int [d\mathbf{S}_x] \prod_{\langle x\mu \rangle} \Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}| - C], \quad (19)$$

where  $\Theta(x)$  is the usual Heaviside function,  $\Theta(x) = 1, 0$  for  $x > 0$  and  $x < 0$ , respectively, and  $C$  is a free parameter that plays the role of  $\beta$ . The product extends over all lattice links. We will also consider a mixed  $\text{O}(N)$ - $\text{RP}^{N-1}$  model defined by the partition function

$$\begin{aligned} Z = & \int [d\mathbf{S}_x] \prod_{\langle x\mu \in \mathcal{B} \rangle} \Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}| - C] \\ & \times \prod_{\langle x\mu \in \mathcal{C} \rangle} \{p\Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}| - C] \\ & + (1-p)\Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}| - C]\}, \end{aligned} \quad (20)$$

where  $0 \leq p \leq 1$  is a second free parameter,  $\mathcal{B}$  and  $\mathcal{C}$  are the sets of boundary and internal links, respectively, as defined in Sec. III. For  $p = 0$ , we reobtain model (19), while for  $p = 1$  we obtain an  $\text{O}(N)$  model which corresponds to the standard one with LFBC, see Eq. (12). The parameter  $C$  plays the role of temperature. For  $C \rightarrow 1$ , spins order, so that this limit corresponds to the limit  $\beta \rightarrow \infty$  in the standard case. However, models with partition functions (19) and (20) are not amenable to a perturbative treatment, so that perturbative considerations on the equivalence of the different models cannot be used here. Nonetheless, in the  $\text{O}(N)$  case it has been shown quite precisely that these constrained models have the same nonperturbative behavior (same continuum limit) as the standard models [57,58].

In order to have a model in which the geometry of the interactions is different—so far we have only considered models with nearest-neighbor interactions—we also consider a Hamiltonian in which also the spins along the plaquette diagonals interact. The partition function is given by

$$\begin{aligned} Z = & \int [d\mathbf{S}_x] \prod_{x\mu} \Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}}| - C] \\ & \times \prod_{xd} \Theta[|\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{d}}| - C], \end{aligned} \quad (21)$$

where the vectors  $\hat{d}$  are the diagonal vectors (1,1) and (1, -1), the first product is over all lattice links and the second one is over all lattice plaquette diagonals.

The constrained models are particularly interesting because one can prove rigorous results concerning their FSS behavior [27]. For instance, for  $C > C^* = \cos \pi/4$ , the behavior of model (20) is independent of  $p$ . In particular, the  $\text{O}(N)$  model ( $p = 1$ ) with LFBC is equivalent to the  $\text{RP}^{N-1}$  model ( $p = 0$ ) with periodic boundary conditions. This implies that the  $\text{RP}^{N-1}$  model and the  $\text{O}(N)$  model have the same nonperturbative critical behavior. The same is true for model (21): for  $C > \cos \pi/3$ , the  $\text{RP}^{N-1}$  can be exactly mapped onto an  $\text{O}(N)$  model with LFBC. As we shall discuss below, this exact result should not be taken as a proof that all  $\text{RP}^{N-1}$  models are equivalent to  $\text{O}(N)$  models. As the approximate calculations of Ref. [28] show, topological defects may be relevant or irrelevant depending on the explicit form of the Hamiltonian, and therefore it is possible that some  $\text{RP}^{N-1}$  models are not in the attraction domain of the  $\text{RP}^{N-1}$  fixed point, if it exists.

As discussed in Ref. [27], the value  $C^*$  corresponds to very large values of the infinite-volume correlation length. In the region of sizes in which simulations can be done,  $C$  is smaller than  $C^*$ : The data that we will show below for model (19) belong to the interval  $0.5 \leq C \lesssim 0.60$ , to be compared with  $C^* \approx 0.707$ . Thus, for the values of  $C$  we consider, the exact equivalence does not hold. Therefore, it would not be surprising that the FSS functions we determine for model (19) differ somewhat from the corresponding  $\text{O}(N)$  FSS functions. However, if an  $\text{RP}^{N-1}$  universality class does not exist, we would expect these deviations to be different from those observed for the more standard  $\text{RP}^{N-1}$  models discussed in the previous section.

As a first check, we have verified that the constrained  $\text{O}(N)$  model with LFBC (partition function (20) with  $p = 0$ ) is equivalent to the standard  $\text{O}(N)$  model with the same boundary conditions [Hamiltonian (11)]. Results for the Binder parameter versus  $R_\xi$ —both quantities are computed using the spin-2 operator  $Q_x$ , see Eqs. (14) and (15)—are compared in Fig. 4 (all results presented in this section have been obtained using a cluster algorithm [27]). As expected [57], we observe very good scaling, indicating that the two models have the same nonperturbative behavior, although they are not perturbatively related.

We then turn to the analysis of the behavior of the  $\text{RP}^{N-1}$  models. We have performed simulations for the models with partition functions (19) (up to  $L = 400$ ) and (21) (only  $L = 50$ ). The estimates of  $U$  are plotted vs  $R_\xi$  in Fig. 5. The results for the constrained model (19) show very good scaling: all results with  $50 \leq L \leq 400$  fall on the same curve within the statistical errors. Apparently, corrections to scaling are tiny, a feature that this model shares with its  $\text{O}(N)$  counterpart [57,58]. The results are also compared with those of the standard  $\text{RP}^2$  model. We observe a quite good agreement, that again would suggest that all these

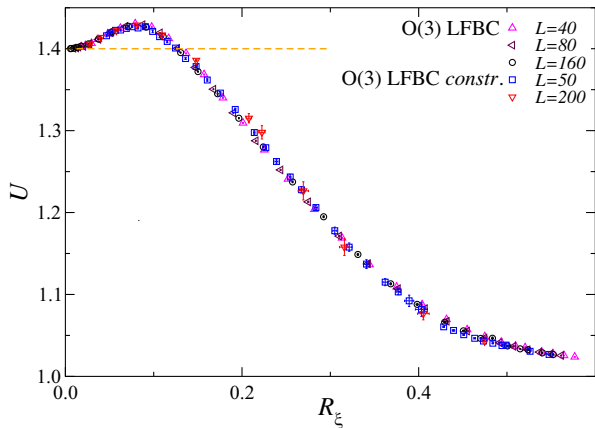


FIG. 4. Data of  $U$  vs  $R_\xi$  for the standard O(3) model with fluctuating boundary conditions [Hamiltonian (11)] (“O(3) LFBC”) and for the model with partition function (20) with  $p = 0$  (“O(3) LFBC constr”).

models have the same asymptotic behavior. Small deviations are observed for  $0.3 \lesssim R_\xi \lesssim 0.5$ , which are of the same order of the deviations observed in the top panel of Fig. 2 for the model with Hamiltonian (1). If an  $\text{RP}^2$  fixed point exists, they may be interpreted as scaling corrections. We also report results for the model with Hamiltonian (21): the data are again consistent with the results for the other  $\text{RP}^2$  models. Note that the data for the two models (19) and (21) correspond to values of  $C$  that are quite different. For  $L = 50$ , the FSS data we show correspond to  $0.50 \leq C \leq 0.60$  in the case of the constrained model with partition function (19) and to  $0.25 \leq C \leq 0.35$  for model (21). Thus, in the two models we consider regions of configuration space that are quite different. In spite of that, the FSS curves are essentially the same.

As an additional check we have considered the mixed O(3)- $\text{RP}^2$  model. In Ref. [25,26] it was suggested that the  $\text{RP}^{N-1}$  universal behavior might be observed also in mixed O( $N$ )- $\text{RP}^{N-1}$ . The idea was that of considering the model with Hamiltonian

$$\beta H = -\beta_V \sum_{x\mu} \mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}} - \beta_T \sum_{x\mu} (\mathbf{S}_x \cdot \mathbf{S}_{x+\hat{\mu}})^2, \quad (22)$$

and take the limit  $\beta_T \rightarrow \infty$  at fixed  $\beta_V$ . In this limit spins order apart from a sign, i.e., we have  $S_x = \hat{n}\tau_x$ , where  $\tau_x$  is an Ising spin. In this limit, one therefore obtains an effective Ising model with  $\beta = \beta_V$ . It was therefore conjectured that the limiting theory is different depending on whether  $\beta_V$  is larger or smaller than  $\beta_{c,I}$ , the 2D Ising inverse critical temperature. For  $\beta_V < \beta_{c,I}$ , the Ising spins are disordered and the behavior is the same as that of the  $\text{RP}^{N-1}$  model. In the opposite case, the Ising spins magnetize and one obtains O( $N$ ) behavior. To verify this conjecture, we have also performed runs with model (20). Also in this case, for

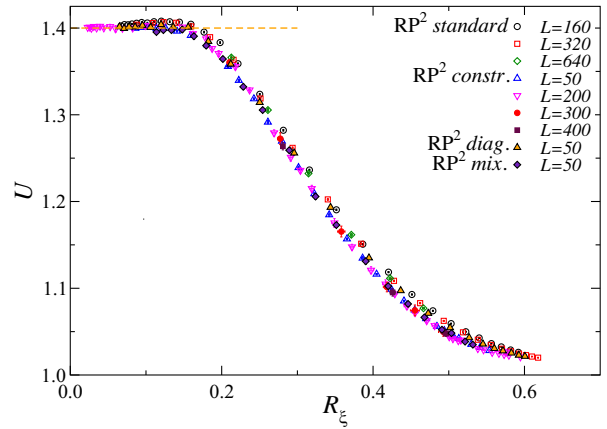


FIG. 5. Data of  $U$  vs  $R_\xi$  for several different  $\text{RP}^2$  models: the model with Hamiltonian (1) (“standard”), the model with partition function (19) (“constr”), and the model with partition function (21) (“diag”). We also include data for the mixed O(3)- $\text{RP}^2$  model with partition function (20) and  $p = 0.2$  (“mix”).

$C \rightarrow 1$ , we obtain an effective Ising model, with inverse temperature  $\beta$  related to  $p$  by  $p = 1 - e^{-2\beta}$ . Thus, for  $p < p_c = 1 - e^{-2\beta_{c,I}} = \sqrt{2} - 1 \approx 0.41$ , we expect to observe a behavior analogous to that observed for the  $\text{RP}^2$  model, if the conjecture holds. Results for  $p = 0.2$  are reported in Fig. 5. They scale on top of the  $\text{RP}^2$  data, apparently confirming the conjecture.

The results obtained for the constrained models are difficult to justify if no  $\text{RP}^2$  fixed point exists. Indeed, if the deviations we observe between the  $\text{RP}^2$  results and O(3) results are nonuniversal corrections, we do not see reasons why the  $\text{RP}^2$  results are consistently the same, given that we consider models that have quite different Hamiltonians and interactions. We believe that the most likely hypothesis is that an  $\text{RP}^{N-1}$  universality class really exists. The  $\text{RP}^{N-1}$  fixed point controls the asymptotic behavior of models (1) and (3) and, moreover, it also controls the apparent scaling behavior we observe in the constrained models. In the renormalization-group language, for values of  $C$  well below  $C^*$ , the system is close to the  $\text{RP}^{N-1}$  fixed point, so that we observe the  $\text{RP}^{N-1}$  FSS functions quite precisely. Of course, as  $C$  increases, the  $\text{RP}^{N-1}$  scaling behavior will eventually cease to hold and a crossover will eventually occur toward the asymptotic O( $N$ ) behavior. But, given that  $C^*$  corresponds to  $\xi \sim 10^9$ , this will occur when  $L$  is much larger than the sizes we consider.

To provide evidence that the behavior in constrained models for the current values of  $C$  is controlled by the putative  $\text{RP}^{N-1}$  fixed point we have analyzed the density of defects

$$\rho = \frac{1}{2}(1 - \langle \Pi_x \rangle). \quad (23)$$



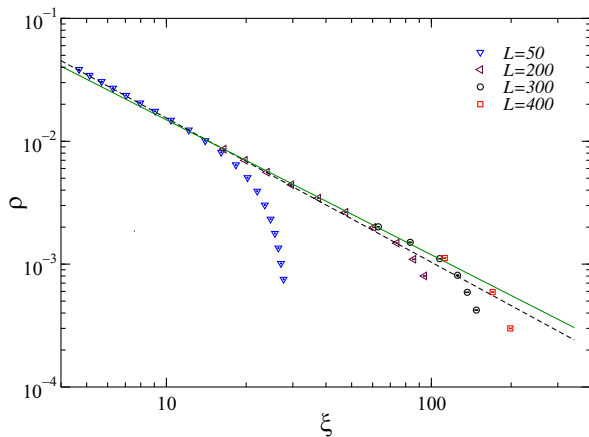


FIG. 6. Estimates of the defect density  $\rho$  versus  $\xi$  for the model with partition function (19). We also report an interpolation of the infinite-volume data with  $L = 50, 200$  (dashed line) ( $\rho = 0.230\xi^{-1.173}$ ) and with  $L = 200, 300$  (continuous line) ( $\rho = 0.187\xi^{-1.098}$ ).

The results are reported in Fig. 6 versus the correlation length  $\xi$ . The infinite-volume data ( $\xi/L \lesssim 0.2$ ) scale approximately as a power of  $\xi$ . A fit of  $\rho$  versus  $\xi^{-p}$  gives  $p \approx 1$ , see Fig. 6. This result shows that in a correlation volume of size  $\xi^2$  the number of defects increases as  $\xi$ . Defects are relevant for the values of  $C$  we are considering.

To conclude this section, it is interesting to discuss the phase structure of the constrained models as a function of  $C$ . Since the density of defects is a nontrivial function of  $C$  for  $C < C^*$  and vanishes identically for  $C > C^*$ , the point  $C = C^*$  is a nonanalyticity point of  $\rho$ . Given the role that  $\rho$  plays in determining the phase behavior, we expect  $C = C^*$  to be a nonanalyticity point also of the free energy: in other words,  $C = C^*$  is a transition point. We do not have information on the order of this transition, but the simplest possibility would be that the transition is of first order. It would separate an approximate  $\text{RP}^{N-1}$  phase, where the behavior would be controlled by the nearby (but unreachable)  $\text{RP}^{N-1}$  fixed point, from an asymptotic  $\text{O}(N)$  phase. The presence of this transition is a peculiarity of the constrained models. If we consider Hamiltonians (1) and (3), we expect  $\rho$  to be nonvanishing for all values of  $\beta$ , allowing us to observe the exact asymptotic  $\text{RP}^{N-1}$  behavior.

## VII. CONCLUSIONS

In this work we analyze the low-temperature behavior of  $\text{RP}^{N-1}$  models, which are invariant under global  $\text{O}(N)$  and local  $\mathbb{Z}_2$  transformations, with the purpose of understanding whether these models have a nonperturbative behavior that is different from that of  $\text{O}(N)$  vector models, in spite of the fact that both models are perturbatively equivalent. The question effectively boils down to the question of the relevance/irrelevance of topological  $\mathbb{Z}_2$  defects.

Their density decreases exponentially in  $\beta$ , but this does not necessarily imply their irrelevance, as also the correlation length depends exponentially on the inverse temperature. The question has been extensively discussed in the 1990s, and several arguments were presented, favoring the existence of a distinct  $\text{RP}^{N-1}$  universality class [25,26], as well as favoring the equivalence of  $\text{RP}^{N-1}$  and  $\text{O}(N)$  models [27–29].

In recent years there has been a widespread interest in the role that topology plays in determining the phase behavior of lattice systems. As an example, we mention here the case of the three-dimensional Abelian-Higgs model (scalar electrodynamics) and of its limiting case, the  $\text{CP}^{N-1}$  model. This model has been extensively studied and there is now a general consensus that topology plays a crucial role: The critical behavior depends on the compact/noncompact nature of the  $\text{U}(1)$  gauge fields or, equivalently, on the presence/absence of monopoles [11,13–18]. In particular, the large- $N$  fixed point predicted by the Abelian-Higgs field theory [59] can only be observed in models in which monopoles are suppressed [17,18]. With these examples in mind, we have decided to revisit the problem, focusing on models with  $N = 3$ . A less detailed analysis has also been performed for  $N = 4$ .

We present results of large-scale MC simulations of several different  $\text{RP}^2$  models. We consider the standard model with Hamiltonian (1) and the one with explicit gauge fields [Hamiltonian (3)] and two models, of the type introduced by Patrascioiu and Seiler [35], that we name constrained models. In such systems there is no perturbative expansion. However, in an appropriate limit spins order as in the usual lattice  $\text{RP}^{N-1}$  models [27]. We consider two variants of the model, differing by the geometry of the interactions. Finally, we also consider a mixed  $\text{O}(3)$ - $\text{RP}^2$  model for a value of the parameters such that it should behave as an  $\text{RP}^2$  model, according to the discussion of Ref. [25,26]. The data obtained from the four different  $\text{RP}^2$  models and the mixed  $\text{O}(3)$ - $\text{RP}^2$  model show a universal FSS behavior. If we plot the Binder parameter  $U$  as a function of  $R_\xi \equiv \xi/L$ , all data fall onto a single curve, with tiny deviations that can be interpreted as scaling corrections. If the  $\text{RP}^2$  model has the same nonperturbative behavior of the  $\text{O}(3)$  model, the FSS data should fall on top of an appropriate FSS curve computed in the  $\text{O}(3)$  model. We have performed this comparison, observing a large discrepancy, that can be hardly explained with the presence of nonuniversal size corrections. On the basis of the numerical data, we thus conclude that an  $\text{RP}^2$  universality class exists, which is distinct from the  $\text{O}(3)$  one. Nonperturbatively, the limiting zero-temperature behaviors for these two classes of models are therefore different. We have repeated the analysis for  $N = 4$ , observing analogously large differences. We should also remark that additional evidence in favor of two distinct universality classes is also provided by the analysis of a class of

non-Abelian gauge theories with  $O(N)$  global symmetry that are predicted to have the same low-temperature behavior as  $RP^{N-1}$  models. The numerical data for these gauge models are perfectly consistent with the results obtained here for  $RP^{N-1}$  models [60].

The presence of a distinct  $RP^{N-1}$  universality class implies that the topological  $\mathbb{Z}_2$  defects are relevant perturbations of the  $O(N)$  fixed point. For  $N = 3$ , we have determined the behavior of the density of these defects in a particular case, verifying that the number of defects in a correlation volume apparently increases as  $\xi$ , which is consistent with a relevant perturbation.

To conclude, let us mention that our results cannot exclude the scenario proposed in Ref. [29], in which the behavior of  $RP^{N-1}$  models for typical box sizes is essentially controlled by a renormalization-group trajectory that

flows to a vorticity fixed point, which, however, is never reached. In this scenario, as soon as  $\xi$  is very large (they estimate  $\xi \approx 10^9$  for the gauge action) the apparent  $RP^{N-1}$  scaling disappears and  $O(N)$  behavior is obtained. Given the very large value of the crossover correlation length, we believe that it is in practice impossible to distinguish between a  $\beta = \infty$  fixed point and a finite- $\beta$  fixed point that is relevant up to a value of  $\beta$  so large that  $\xi \approx 10^9$ .

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