Partial decay widths of $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state

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In the present work, the partial decay widths of $P_c(4312)$ to $\eta_c p$ and $J/\psi p$ are investigated with the QCD sum rule method under the assumption that $P_c(4312)$ is a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$. In the analysis, the pole residue of $P_c(4312)$, one of the input parameters for the calculations of the strong decay constants, is calculated first. With the numerical values of the strong decay constants, the partial decay widths to $\eta_c p$ and $J/\psi p$ are estimated to be $\Gamma(P_c(4312) \to \eta_c p) = 5.54^{+0.75}_{-0.5}$ MeV and $\Gamma(P_c(4312) \to J/\psi p) = 1.67^{+0.92}_{-0.56}$ MeV, respectively, which are compatible with the measured total width of $P_c(4312)$. The results suggest that it is reasonable to assign $P_c(4312)$ to be a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$.

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I. INTRODUCTION

Multiquark states with quark substructures $qq\bar{q}\bar{q}$, $qqqq\bar{q}$, and so on, are allowed both in the conventional quark model and quantum chromodynamics (QCD), the correct theory of the strong interaction. They provide a good platform for studying the nonperturbative behavior of QCD. Many physicists have focused on this topic since the observation of $X(3872)$ in 2003 by the Belle Collaboration [1], and there have been many theoretical and experimental progresses on the theme in the last decade (see review articles [2] for details).

The pentaquark states, a typical kind of multiquark states, are the focus of research on the nonconventional hadrons, especially after the discoveries of the $P_c(4380)$ and $P_c(4450)$ states in 2015 by the LHCb Collaboration [3]. These studies based on different assumptions about the quark configurations of the hadrons, including meson-baryon molecules [4–10], diquark-diquark-antiquark pentaquarks [11–14], compact diquark-triquark penta-quarks [15,16], the topological soliton model [17], genuine

multiquark states other than molecules [18], and kinematical effects related to the triangle singularity [19–21], etc.

Recently, a new pentaquark state $P_c(4312)$ with mass $m_{P_c(4312)} = 4311.9 \pm 0.7^{+6.8}_{-0.6}$ MeV and total width $\Gamma_{P_c(4312)} = 9.8 \pm 2.7^{+3.7}_{-4.5}$ MeV was discovered by the LHCb Collaboration in the $J/\psi p$ invariant mass spectrum of the $\Lambda_b \to J/\psi pK$ decay [22]. Triggered by this observation, there are many theoretical investigations on the properties of this state through different approaches, such as QCD sum rule method [23–26], potential models [27–31], and so on [32–37]. However, the concrete nature and substructure of this state are not determined yet. More experimental and theoretical investigations are necessary to understand its properties. For example, studying its possible decay channels may provide valuable insights in this respect.

In this paper, we study the strong decay property of $P_c(4312)$ viewed as a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$ in the QCD sum rule method [38]. First, we calculate the pole residue of $P_c(4312)$, one of the input parameters when computing the strong decay constants. Then we turn to the strong decay constants of $P_c(4312) \to \eta_c p$ and $P_c(4312) \to J/\psi p$. With the above results, we give the partial decay widths, $\Gamma(P_c(4312) \to \eta_c p) = 5.54^{+0.75}_{-0.5}$ MeV and $\Gamma(P_c(4312) \to J/\psi p) = 1.67^{+0.92}_{-0.56}$ MeV. The basic idea of the QCD sum rule method is that the correlation function of interpolating currents of hadrons can be represented in terms of hadronic parameters (the so-called hadronic side) and calculated at quark-gluon level by

operator product expansion (OPE) (the so-called QCD side), and then by matching the two expressions we can extract the physical quantities of the considered hadron. The QCD sum rule method has extensively been used to investigate the X, Y, Z states which are candidates for the multiquark states; for a review, see Ref. [39]. It is reliable for us to investigate the ground pentaquark states using this method before more exact experiments are presented. In fact, there are some related works about the pentaquark states with the QCD sum rule method [14,25,40–44].

The rest of the paper is organized as follows. In Sec. II, we give the sum rules for the pole residue of $P_c(4312)$ and the strong decay constants of $P_c(4312) \to \eta_c p$ and $P_c(4312) \to J/\psi p$. Section III is devoted to the numerical analysis, and a short summary is given in Sec. IV. In Appendix B, the spectral densities are shown.

II. THE DERIVATION OF THE SUM RULES

In this section, the sum rules for the pole residue of $P_c(4312)$ and the strong decay constants of $P_c(4312) \to \eta_c p$ and $P_c(4312) \to J/\psi p$ are given.

A. The pole residue

To estimate the pole residue needed when calculating the strong decay constants, we start with the following two-point correlation function:

$$
\Pi(p) = i \int d^4 x e^{ipx} \langle 0|T[J^{P_c}(x)\bar{J}^{P_c}(0)]|0\rangle
$$
$$
= \not{p}\, \Pi_1(p^2) + \Pi_2(p^2), \tag{1}
$$

where $J^{P_c}(x)$ is the interpolating current of $P_c(4312)$ considered as a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$ in the present work. According to Ref. [42], $J^{P_c}(x)$ can take the form

$$
J^{P_c}(x) = [\bar{c}(x)i\gamma_5 d(x)][\epsilon^{abc}(u_a^T(x)C\gamma_\mu u_b(x))\gamma^\mu\gamma_5 c_c(x)], \tag{2}
$$

where T denotes the matrix transposition of the Dirac spinor indices, C means charge conjugation matrix, and a, b, c are color indices.

There are three main steps in the QCD sum rule calculation which are as follows:

(i) Presenting the correlation function in terms of hadronic parameters.

(ii) Calculating the correlator via OPE at the quark-gluon level.

(iii) Matching the two expressions with the help of quark-hadron duality and extracting the needed quantities.

In the last step, Borel transform is introduced to suppress the higher and continuum states' contributions and improve the convergence of the OPE series.

In order to express the two-point correlation function (1) physically, we insert a complete set of relevant states with the same quantum numbers as $J^{P_c}(x)$ between the two interpolating currents, isolate the ground-state term, and finally get

$$
\Pi^{phe}(p) = \lambda_{P_c}^2 \frac{\not{p} + m_{P_c}}{m_{P_c}^2 - p^2} + \text{higher resonances}, \tag{3}
$$

where m_{P_c} is the hadronic mass, λ_{P_c} is the pole residue of $P_c(4312)$ defined as $\langle 0|J^{P_c}(0)|P_c(p,s)\rangle = \lambda_{P_c} u(p,s)$.

On the other hand, $\Pi(p)$ can be calculated theoretically via OPE method at the quark-gluon level. To this end, one can insert the interpolating current $J^{P_c}(x)$ (2) into the correlation function (1), contract the relevant quark fields by Wick's theorem, and find

$$
\Pi^{\text{OPE}}(p) = -2i\epsilon_{abc}\epsilon_{a'b'c'} \int d^4 x e^{ipx}\gamma^\mu\gamma_5 S_{cc'}^{(c)}(x)\gamma^\nu\gamma_5
$$
$$
\times \text{Tr}[(i\gamma_5)S_{dd'}^{(d)}(x)(i\gamma_5)S_{d'd}^{(c)}(-x)]
$$
$$
\times \text{Tr}[\gamma_\mu S_{bb'}^{(u)}(x)\gamma_\nu C S_{aa'}^{(u)T}(x)C], \tag{4}
$$

where $S^{(c)}(x)$ and $S^{(q)}(x), q = u, d$ are the full charm- and up (down)-quark propagators, whose expressions are given in Appendix A. Through dispersion relation, $\Pi^{\text{OPE}}(p)$ can be written as

$$
\Pi^{\text{OPE}}(p) = \not{p} \int_{4m_c^2}^{\infty} ds \frac{\rho_1(s)}{s - p^2} + \int_{4m_c^2}^{\infty} ds \frac{\rho_2(s)}{s - p^2}, \tag{5}
$$

where $\rho_i(s) = \frac{1}{\pi}\text{Im}\Pi_i^{\text{OPE}}(s), i = 1, 2$ are the spectral densities. The spectral density $\rho_1(s)$ is given in Appendix B.

Finally, we match the phenomenological side (3) and the QCD representation (5) for the Lorentz structure $\not{p}$,

$$
\frac{\lambda_{P_c}^2}{m_{P_c}^2 - p^2} + \text{higher resonances} = \int_{4m_c^2}^{\infty} ds \frac{\rho_1(s)}{s - p^2}. \tag{6}
$$

According to quark-hadron duality, the excited and continuum states' spectral density can be approximated by the QCD spectral density above some effective threshold $s_0^{P_c}$, whose value will be determined in Sec. III,

$$
\frac{\lambda_{P_c}^2}{m_{P_c}^2 - p^2} + \int_{s_0^{P_c}}^{\infty} ds \frac{\rho_1(s)}{s - p^2} + \text{subtractions} = \int_{4m_c^2}^{\infty} ds \frac{\rho_1(s)}{s - p^2}. \tag{7}
$$

Subtracting the contributions of the excited and continuum states, one gets

$$\frac{\lambda_{P_c}^2}{m_{P_c}^2 - p^2} + \text{subtractions} = \int_{4m_c^2}^{s_0^{P_c}} ds \frac{\rho_1(s)}{s - p^2}. \qquad (8)$$

In order to eliminate the subtraction terms, it is necessary to make a Borel transform which can also improve the convergence of the OPE series and suppress the contributions from the excited and continuum states. As a result, we have

$$\lambda_{P_c}^2 e^{-\frac{m_{P_c}^2}{M_B^2}} = \int_{4m_c^2}^{s_0^{P_c}} ds \rho_1(s) e^{-\frac{s}{M_B^2}}, \qquad (9)$$

where M_B^2 is the Borel parameter. To get the sum rules for the mass and the pole residue λ_{P_c}, we take derivative of Eq. (9) with respect to $-\frac{1}{M_B^2}$ and divide it by the original expression. The final result is

$$m_{P_c}^2 = \left(\frac{d}{d(-\frac{1}{M_B^2})} \int_{4m_c^2}^{s_0^{P_c}} ds \rho_1(s) e^{-\frac{s}{M_B^2}}\right) / \int_{4m_c^2}^{s_0} ds \rho_1(s) e^{-\frac{s}{M_B^2}}. \qquad (10)$$

Substituting the obtained mass value into Eq. (9), we can give the sum rule of the pole residue λ_{P_c}. However, in the present case, the mass of $P_c(4312)$ is given by experiment. In order to improve the precision, we can substitute the experimental value of the mass in Eq. (9) to obtain the sum rule for the pole residue λ_{P_c}.

B. The strong decay constants

In the previous subsection, the sum rule of the pole residue of $P_c(4312)$ is given. We now turn to the calculation of the strong decay constants of $P_c(4312) \to \eta_c p$ and $P_c(4312) \to J/\psi p$. To this end, we begin with the following three-point correlation functions:

$$\Gamma(p, p', q) = i^2 \int d^4 x d^4 y e^{ip'x + iqy}$$
$$\times \langle 0|T[J^N(x)J^{\eta_c}(y)\bar{J}^{P_c}(0)]|0\rangle,$$

$$\Gamma_\mu(p, p', q) = i^2 \int d^4 x d^4 y e^{ip'x + iqy}$$
$$\times \langle 0|T[J^N(x)J_\mu^{J/\psi}(y)\bar{J}^{P_c}(0)]|0\rangle, \qquad (11)$$

where $p = p' + q$, $J^{P_c}(x)$ is the interpolating current of $P_c(4312)$ defined in (2), $J^N(x)$, $J^{\eta_c}(x)$ and $J_\mu^{J/\psi}(x)$ are the interpolating currents of the proton, η_c and J/ψ, respectively. The interpolating currents take the following form:

$$J^N(x) = \epsilon_{abc}[u_a^T(x)C\gamma_\mu u_b(x)]\gamma_5\gamma^\mu d_c(x),$$
$$J^{\eta_c}(x) = \bar{c}(x)i\gamma_5 c(x),$$
$$J_\mu^{J/\psi}(x) = \bar{c}(x)\gamma_\mu c(x), \qquad (12)$$

where T denotes the matrix transposition of the Dirac spinor indices, C means charge conjugation, and a, b, c are color indices.

Following the same procedures done above, we calculate the three-point correlators both phenomenologically and theoretically and extract the needed sum rules by matching the two representations of the correlation functions.

In order to get the physical representation of the three-point correlation functions (11), we insert complete sets of states having the same quantum numbers as the interpolating currents into the three-point correlation functions and define the following matrix elements:

$$\langle 0|J^N|N(p')\rangle = \lambda_N u^N(p'),$$
$$\langle 0|J_\mu^{J/\psi}|J/\psi(q)\rangle = f_{J/\psi} m_{J/\psi} \epsilon_\mu(q),$$
$$\langle 0|J^{\eta_c}|\eta_c(q)\rangle = \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c^2}, \qquad (13)$$

$$\langle N(p')\eta_c(q)|P_c(p)\rangle = ig\bar{u}^N(p')u^{P_c}(p),$$
$$\langle N(p')J/\psi(q)|P_c(p)\rangle = \epsilon_\mu^*(q)\bar{u}^N(p')$$
$$\times \left(f_1\gamma^\mu - if_2\frac{\sigma^{\mu\nu}q_\nu}{m_N + m_{P_c}}\right)$$
$$\times \gamma_5 u^{P_c}(p), \qquad (14)$$

where f_{η_c} and m_{η_c} are the decay constant and mass of the η_c state, $m_{J/\psi}$, $f_{J/\psi}$, and $\epsilon_\mu(q)$ are the mass, decay constant, and polarization vector of the J/ψ state, λ_N and $u^N(p')$ are the residue and spinor of the proton, and g, f_1, and f_2 are the strong decay constants, respectively. After algebraic calculations, we reach the phenomenological side of the sum rules as follows:

$$\Gamma(p, p', q) = \left[\frac{g\lambda_N\lambda_{P_c}f_{\eta_c}m_{\eta_c}^2(m_N + m_{P_c})}{2m_c(m_{P_c}^2 - p^2)(m_{\eta_c}^2 - q^2)(m_N^2 - p'^2)} + \frac{a}{(m_{\eta_c}^2 - q^2)(m_N^2 - p'^2)}\right]\not{p}' + \cdots, \qquad (15)$$

$$\Gamma_\mu(p, p', q) = \left[-\frac{\lambda_N\lambda_{P_c}f_{J/\psi}m_{J/\psi}}{(m_{P_c}^2 - p^2)(m_N^2 - p'^2)(m_{J/\psi}^2 - q^2)}\left(f_1\frac{m_N + m_{P_c}}{m_{J/\psi}^2} - f_2\frac{1}{m_N + m_{P_c}}\right) + \frac{a_1}{(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)}\right]\not{p}'\not{q}\gamma_5 q_\mu$$
$$+ \left\{-\frac{\lambda_N\lambda_{P_c}f_{J/\psi}m_{J/\psi}}{(m_{P_c}^2 - p^2)(m_N^2 - p'^2)(m_{J/\psi}^2 - q^2)}\left[-f_1(m_N + m_{P_c}) + f_2\frac{q^2 - p'^2}{m_N + m_{P_c}}\right] + \frac{a_2}{(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)}\right\}\not{p}'\gamma_\mu\gamma_5$$
$$+ \cdots, \qquad (16)$$

where only the Lorentz structures $p\!\!\!/'$, $p\!\!\!/' q\!\!\!/ \gamma_5 q_\mu$, and $p\!\!\!/' \gamma_\mu \gamma_5$ we are interested in, remained, and a, a_1, and a_2 are constant parameters introduced to parametrize the transitions between the ground states and the excited states similar to Ref. [45].

On the theoretical side, by inserting the interpolating currents (2) and (13) into the three-point correlation function (11) and contracting the quark fields, we obtain the following representation of the correlation functions:

$$\Gamma(p, p', q) = i^2 2\epsilon_{abc}\epsilon_{a'b'c'} \int d^4x d^4y e^{ip'x+iqy} \gamma_5 \gamma^\alpha S_{cd'}^{(d)}(x) i\gamma_5 S_{d'd}^{(c)}(-y)(i\gamma_5) S_{dc'}^{(c)}(y)\gamma^\beta \gamma_5$$

$$\text{Tr}[\gamma_\alpha S_{bb'}^{(u)}(x)\gamma_\beta C S_{aa'}^{(u)T}(x)C] = \Gamma(p^2, p'^2, q^2)p\!\!\!/' + \cdots, \tag{17}$$

$$\Gamma_\mu(p, p', q) = i^2 2\epsilon_{abc}\epsilon_{a'b'c'} \int d^4x d^4y e^{ip'x+iqy} \gamma_5 \gamma^\alpha S_{cd'}^{(d)}(x) i\gamma_5 S_{d'd}^{(c)}(-y)\gamma_\mu S_{dc'}^{(c)}(y)\gamma^\beta \gamma_5$$

$$\text{Tr}[\gamma_\alpha S_{bb'}^{(u)}(x)\gamma_\beta C S_{aa'}^{(u)T}(x)C] = \Gamma_1(p^2, p'^2, q^2)p\!\!\!/' q\!\!\!/ \gamma_5 q_\mu + \Gamma_2(p^2, p'^2, q^2)p\!\!\!/' \gamma_\mu \gamma_5 + \cdots, \tag{18}$$

where the coefficients $\Gamma(p^2, p'^2, q^2)$, $\Gamma_1(p^2, p'^2, q^2)$, and $\Gamma_2(p^2, p'^2, q^2)$ can be represented as by the dispersion relation

$$\Gamma_i(p^2, p'^2, q^2) = \int_{4m_c^2}^\infty ds \int_0^\infty du \frac{\rho_i^{(3)}(p^2, s, u)}{(s - q^2)(u - p'^2)}, \tag{19}$$

where $\Gamma_i(p^2, p'^2, q^2)$ stand for $\Gamma(p^2, p'^2, q^2)$, $\Gamma_1(p^2, p'^2, q^2)$, and $\Gamma_2(p^2, p'^2, q^2)$, and $\rho_i^{(3)}(p^2, s, u)$ are the corresponding spectral densities which are given in Appendix B.

Matching the hadronic representations (15), (16) with the QCD representations (19) for the corresponding Lorentz structures and using the quark-hadron duality, one has

$$\frac{g\lambda_N \lambda_{P_c} f_{\eta_c} m_{\eta_c}^2 (m_N + m_{P_c})}{2m_c(m_{P_c}^2 - p^2)(m_{\eta_c}^2 - q^2)(m_N^2 - p'^2)} + \frac{a}{(m_{\eta_c}^2 - q^2)(m_N^2 - p'^2)}$$

$$= \int_{4m_c^2}^{s_{\eta_c}} ds \int_0^{u_N} du \frac{\rho^{(3)}(p^2, s, u)}{(s - q^2)(u - p'^2)},$$

$$-\frac{\lambda_N \lambda_{P_c} f_{J/\psi} m_{J/\psi}}{(m_{P_c}^2 - p^2)(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)}\left(f_1 \frac{m_N + m_{P_c}}{m_{J/\psi}^2} - f_2 \frac{1}{m_N + m_{P_c}}\right)$$

$$+\frac{a_1}{(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)} = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du \frac{\rho_1^{(3)}(p^2, s, u)}{(s - q^2)(u - p'^2)},$$

$$-\frac{\lambda_N \lambda_{P_c} f_{J/\psi} m_{J/\psi}}{(m_{P_c}^2 - p^2)(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)}\left[-f_1(m_N + m_{P_c}) + f_2 \frac{q^2 - p'^2}{m_N + m_{P_c}}\right]$$

$$+\frac{a_2}{(m_{J/\psi}^2 - q^2)(m_N^2 - p'^2)} = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du \frac{\rho_2^{(3)}(p^2, s, u)}{(s - q^2)(u - p'^2)}, \tag{20}$$

where s_{η_c}, $s_{J/\psi}$, and u_N are the threshold parameters corresponding to the η_c, J/ψ, and proton channels, respectively, whose values will be determined in Sec. III.

Setting $p^2 = q^2$ and doing double Borel transform $p^2 \to M_{B_1}^2$ and $p'^2 \to M_{B_2}^2$, we get the following equations:

$$g\lambda_N \lambda_{P_c}(m_N + m_{P_c}) \frac{f_{\eta_c} m_{\eta_c}^2}{2m_c} \frac{(e^{-m_{P_c}^2/M_{B_1}^2} - e^{-m_{\eta_c}^2/M_{B_1}^2})}{m_{\eta_c}^2 - m_{P_c}^2} e^{-m_N^2/M_{B_2}^2}$$

$$+ ae^{-m_{\eta_c}^2/M_{B_1}^2} e^{-m_N^2/M_{B_2}^2} = \int_{4m_c^2}^{s_{\eta_c}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \rho^{(3)}(s,u),$$

$$- \lambda_N \lambda_{P_c} f_{J/\psi} m_{J/\psi} \left[f_1 \frac{m_N + m_{P_c}}{m_{J/\psi}^2} \frac{e^{-m_{P_c}^2/M_{B_1}^2} - e^{-m_{J/\psi}^2/M_{B_1}^2}}{m_{J/\psi}^2 - m_{P_c}^2} e^{-m_N^2/M_{B_2}^2} \right.$$

$$\left. - f_2 \frac{1}{m_N + m_{P_c}} \frac{e^{-m_{P_c}^2/M_{B_1}^2} - e^{-m_{J/\psi}^2/M_{B_1}^2}}{m_{J/\psi}^2 - m_{P_c}^2} e^{-m_N^2/M_{B_2}^2} \right] + a_1 e^{-m_{J/\psi}^2/M_{B_1}^2} e^{-m_N^2/M_{B_2}^2}$$

$$= \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \rho_1^{(3)}(s,u),$$

$$- \lambda_N \lambda_{P_c} f_{J/\psi} m_{J/\psi} \left[-f_1(m_N + m_{P_c}) \frac{(e^{-m_{P_c}^2/M_{B_1}^2} - e^{-m_{J/\psi}^2/M_{B_1}^2})}{m_{J/\psi}^2 - m_{P_c}^2} e^{-m_N^2/M_{B_2}^2} \right.$$

$$\left. + \frac{f_2}{m_N + m_{P_c}} \left(-e^{-m_{P_c}^2/M_{B_1}^2} e^{-m_N^2/M_{B_2}^2} + \frac{m_N^2 + m_{J/\psi}^2}{m_{J/\psi}^2 - m_{P_c}^2} (e^{-m_{P_c}^2/M_{B_1}^2} - e^{-m_{J/\psi}^2/M_{B_1}^2}) e^{-m_N^2/M_{B_2}^2} \right) \right]$$

$$+ a_2 e^{-m_{J/\psi}^2/M_{B_1}^2} e^{-m_N^2/M_{B_2}^2}$$

$$= \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \rho_2^{(3)}(s,u). \tag{21}$$

Taking derivative of the above equations with respect to $-1/M_{B_1}^2$ and solving related equations, we obtain the sum rules of the strong decay constants as follows:

$$g = \frac{2m_c}{\lambda_N \lambda_{P_c} f_{\eta_c} m_{\eta_c}^2 (m_N + m_{P_c})} e^{m_{P_c}^2/M_{B_1}^2} e^{m_N^2/M_{B_2}^2} A(M_{B_1}^2, M_{B_2}^2, s_{\eta_c}, u_N), \tag{22}$$

$$f_1 = \frac{m_{J/\psi}}{\lambda_N \lambda_{P_c} f_{J/\psi}(m_{P_c} + m_N)(m_{J/\psi}^2 + m_N^2 - m_{P_c}^2)} e^{m_{P_c}^2/M_{B_1}^2} e^{m_N^2/M_{B_2}^2}$$
$$[(m_{P_c}^2 - m_N^2) A_1(M_{B_1}^2, M_{B_2}^2, s_{J\psi}, u_N) + A_2(M_{B_1}^2, M_{B_2}^2, s_{J\psi}, u_N)], \tag{23}$$

$$f_2 = \frac{m_N + m_{P_c}}{\lambda_N \lambda_{P_c} f_{J/\psi} m_{J/\psi}(m_{J/\psi}^2 + m_N^2 - m_{P_c}^2)} e^{m_{P_c}^2/M_{B_1}^2} e^{m_N^2/M_{B_2}^2}$$
$$[m_{J/\psi}^2 A_1(M_{B_1}^2, M_{B_2}^2, s_{J\psi}, u_N) + A_2(M_{B_1}^2, M_{B_2}^2, s_{J\psi}, u_N)], \tag{24}$$

with

$$A(M_{B_1}^2, M_{B_2}^2, s_{\eta_c}, u_N) = \int_{4m_c^2}^{s_{\eta_c}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \left[m_{\eta_c}^2 \rho^{(3)}(s,u) - s\rho^{(3)}(s,u) - \frac{\partial \rho^{(3)}(s,u)}{\partial(-1/M_{B_1}^2)} \right],$$

$$A_1(M_{B_1}^2, M_{B_2}^2, s_{J/\psi}, u_N) = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \left[m_{J/\psi}^2 \rho_1^{(3)}(s,u) - s\rho_1^{(3)}(s,u) - \frac{\partial \rho_1^{(3)}(s,u)}{\partial(-1/M_{B_1}^2)} \right],$$

$$A_2(M_{B_1}^2, M_{B_2}^2, s_{J/\psi}, u_N) = \int_{4m_c^2}^{s_{J/\psi}} ds \int_0^{u_N} du e^{-s/M_{B_1}^2} e^{u/M_{B_2}^2} \left[m_{J/\psi}^2 \rho_2^{(3)}(s,u) - s\rho_2^{(3)}(s,u) - \frac{\partial \rho_2^{(3)}(s,u)}{\partial(-1/M_{B_1}^2)} \right]. \tag{25}$$

TABLE I. Some input parameters needed in the calculations.

Parameter	Value
$\langle \bar{q}q \rangle$	$-(0.24 \pm 0.01)^3$ GeV^3
$\langle g_s \bar{q}\sigma Gq \rangle$	$(0.8 \pm 0.1)\langle \bar{q}q \rangle$ GeV^2
$\langle g_s^2 GG \rangle$	0.88 ± 0.25 GeV^4
m_c	$1.275^{+0.025}_{-0.035}$ GeV [46]
$m_{J/\psi}$	3096.900 ± 0.006 MeV [46]
m_N	938.272081 ± 0.000006 MeV [46]
m_{η_c}	2.9839 ± 0.5 MeV [46]
λ_N^2	0.0011 ± 0.0005 GeV^6 [47]
$f_{J/\psi}$	481 ± 36 MeV [48]
f_{η_c}	0.387 ± 0.007 GeV [49]

III. NUMERICAL ANALYSIS AND THE PARTIAL DECAY WIDTHS

The QCD sum rules for the pole residue and the strong decay constants contain some fundamental inputs which are presented in Table I. Besides these parameters, there are a few auxiliary parameters introduced during the calculations: the continuum thresholds and the Borel parameters. They are not physical quantities; hence, the physical observables should be approximately insensitive to them. Therefore, we look for working regions of these parameters such that the dependence of the physical quantities on these parameters is weak. The continuum thresholds are related to the square of the first exited states having the same quantum numbers as the interpolating currents, while the Borel parameters are determined by demanding that both the contributions of the higher states and continuum are sufficiently suppressed and the contributions coming from higher dimensional operators are small.
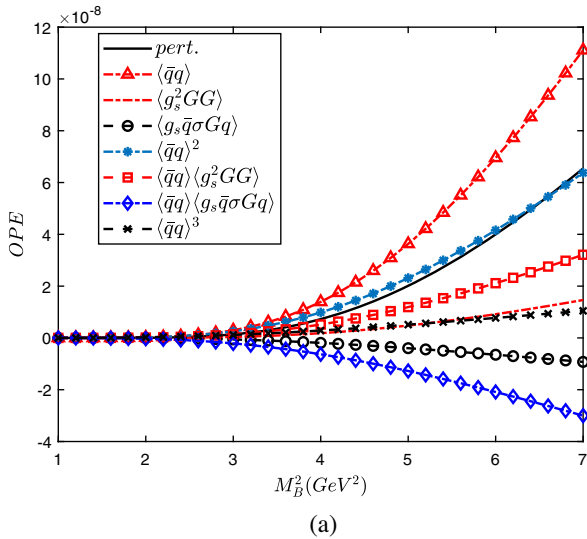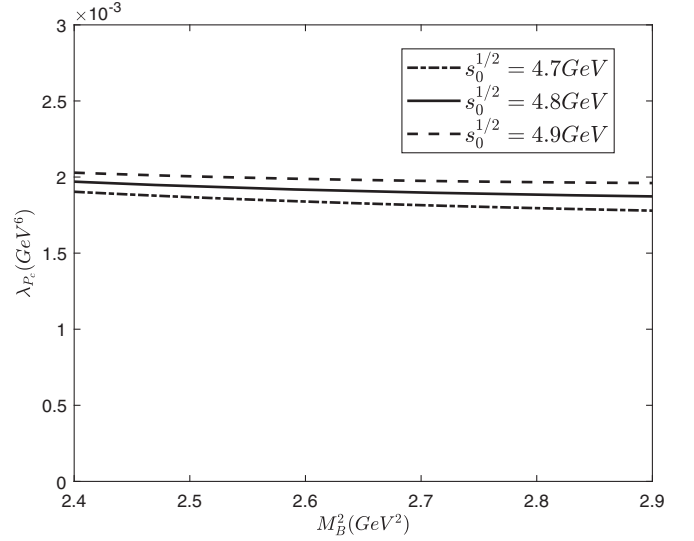

FIG. 2. The figure shows the dependence of the pole residue λ_{P_c} on the Borel parameter M_B^2 in the determined interval at three different values of $s_0^{P_c}$.

We define two quantities, the ratio of the pole contribution to the total contribution (RP) and the ratio of the highest-dimensional term in the OPE series to the total OPE series (RH), as follows:

$$RP \equiv \frac{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_1(s) e^{-\frac{s}{M_B^2}}}{\int_{4m_c^2}^{\infty} ds \rho_1(s) e^{-\frac{s}{M_B^2}}},$$

$$RH \equiv \frac{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_1^{\langle \bar{q}q \rangle^3}(s) e^{-\frac{s}{M_B^2}}}{\int_{4m_c^2}^{s_0^{P_c}} ds \rho_1(s) e^{-\frac{s}{M_B^2}}} \tag{26}$$

(a)
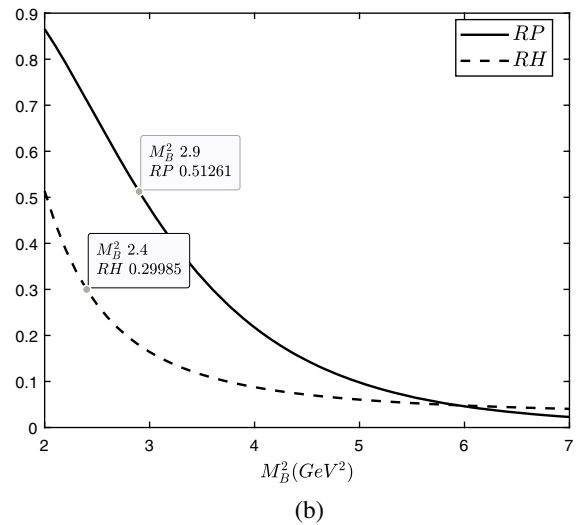
(b)

FIG. 1. (a) denotes the various OPE contributions as functions of M_B^2 with $\sqrt{s_0^{P_c}} = 4.8$ GeV and (b) represents RP and RH varying with M_B^2 at $\sqrt{s_0^{P_c}} = 4.8$ GeV.

for the two-point correlation function and similar quantities for the three-point correlation functions.

We first analyze the pole residue λ_{P_c}. In Fig. 1(a), we compare the various OPE contributions as functions of M_B^2 with $\sqrt{s_0^{P_c}} = 4.8$ GeV. From it, one can see that the quark condensate $\langle \bar{q}q \rangle$, the quark-gluon mixed condensate $\langle g_s \bar{q}\sigma Gq \rangle$, the four-quark condensate $\langle \bar{q}q \rangle^2$, and the dimension-8 term $\langle \bar{q}q \rangle g_s \bar{q}\sigma Gq$ play an important role in the OPE series, but they have opposite sign and cancel each other. As a result, the perturbative part still dominates the OPE series. Indeed, the highest-dimensional term $\langle \bar{q}q \rangle^3$ in our OPE is small relative to others. In other words, the OPE series is under control. Figure 1(b) shows RP and RH

varying with M_B^2 at $\sqrt{s_0^{P_c}} = 4.8$ GeV. The figure shows that it is needed to limit M_B^2 from 2.4 to 2.9 GeV^2 in order to simultaneously satisfy the requirements of pole dominance at the phenomenological side (the pole contribution is bigger than the continuum contribution) and convergence of the operator product expansion (the contribution from the highest-dimensional term is about 30 percent of the total OPE series).

With the obtained interval of M_B^2 and the experimental value of the mass $m_{P_c(4312)} = 4311.9 \pm 0.7^{+6.8}_{-0.6}$ MeV, the pole residue can be estimated. The result is represented in Fig. 2, from which it is obvious that the pole residue varies weakly with the parameters $s_0^{P_c}$ and M_B^2 in the interval
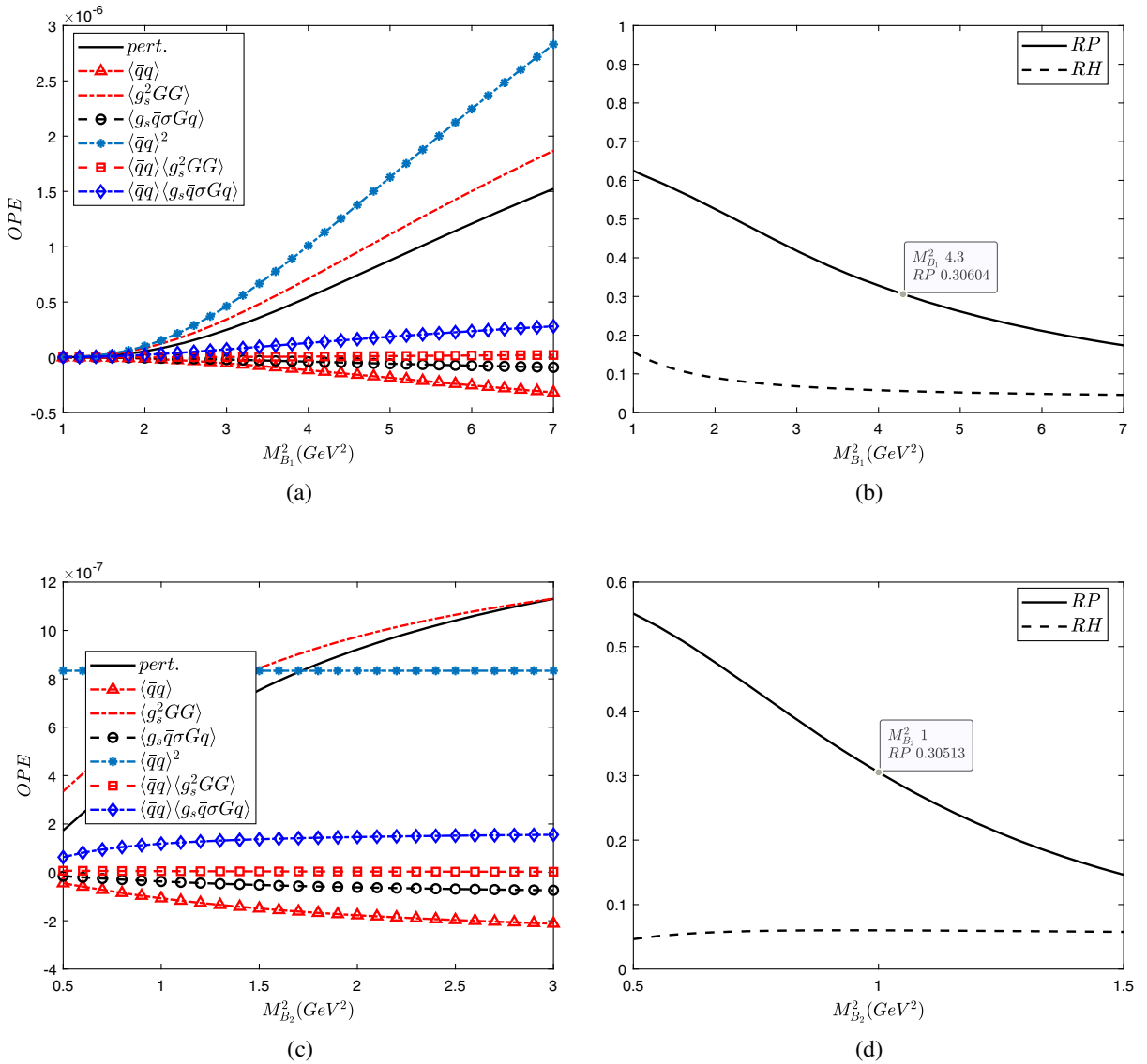

FIG. 3.   The coefficients of the Lorentz structure $\not{p}'$ of the correlation function $\Gamma(p, p', q)$, RP and RH, as functions of the Borel parameters $M_{B_1}^2$ with $M_{B_2}^2 = 0.9$ GeV^2 are showed in (a) and (b) respectively. (c) and (d) represent the same quantities as functions of the Borel parameters $M_{B_2}^2$ with $M_{B_1}^2 = 3.7$ GeV^2.

determined above. As a result, we can reliably read the value of the pole residue, $\lambda_{P_c} = 1.91^{+0.12}_{-0.13} \times 10^{-3}$ GeV^6.

Now, it is time to study the strong decay constants g of the strong decay $P_c(4312) \to \eta_c p$, f_1 and f_2 of the strong decay $P_c(4312) \to J/\psi p$. Similar to above, we determine first the allowed ranges of the Borel parameters $M^2_{B_1}$ and $M^2_{B_2}$. To this end, we show the various OPE contributions of the Lorentz structure $\not{p}'$ of the correlation function $\Gamma(p, p', q)$ in Fig. 3(a), RP and RH in Fig. 3(b) as functions of $M^2_{B_1}$ with $M^2_{B_2} = 0.9$ GeV^2. Figures 3(c) and 3(d) depict the same quantities as functions of $M^2_{B_2}$ at $M^2_{B_1} = 3.7$ GeV^2. In the case of three-point correlation functions, as stated in Ref. [50], the contributions of the pole-continuum

transition terms may be larger than or the same order as the pole contribution and should not be neglected. In the present case, if we require the contribution from the pole larger than the continuum contribution, it is impossible to obtain suitable intervals of the Borel parameters. Therefore, we require that the pole term accounts for 30% of the total contribution. Besides the above requirement, the Borel parameters are also constrained by the criterion that the physical quantities should be independent on the Borel parameters. Finally, the results are shown in Fig. 4, from which we can see that the strong decay constant g varies weakly with the Borel parameters and we can read the value of g: $g = -0.419^{+0.019}_{-0.028}$. In the above analysis, we take $s_{\eta_c} = 3.5^2$ GeV^2 and $u_N = 1.7$ GeV^2.
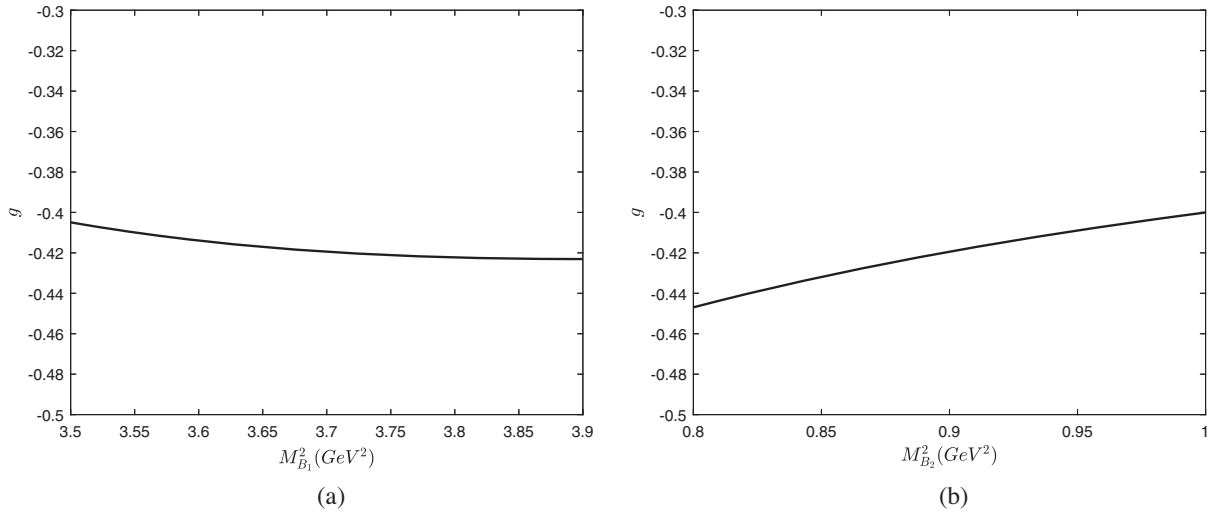

FIG. 4. (a) and (b) show the strong decay constants of the decay $P_c(4312) \to \eta_c p$ in the allowed intervals of the Borel parameters $M^2_{B_1}$ and $M^2_{B_2}$, respectively.
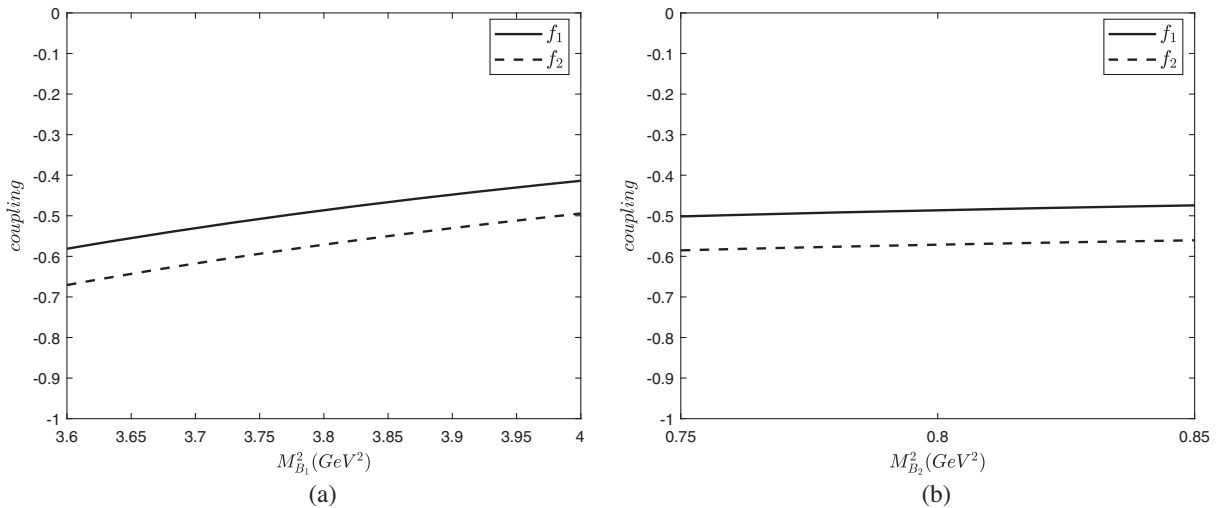

FIG. 5. (a) and (b) shows the dependence of the sum rules for the strong decay constants f_1 and f_2 on the Borel parameters $M^2_{B_1}$ and $M^2_{B_2}$, respectively.

For the strong decay constants f_1 and f_2 of the decay $P_c(4312) \rightarrow J/\psi p$, similar analysis can be done and Figs. 5(a) and 5(b) exhibit the results with $M_{B_2}^2 = 0.8$ GeV^2 and $M_{B_1}^2 = 3.8$ GeV^2, respectively, both at $s_{J/\psi} = 3.6^2$ GeV^2 and $u_N = 1.7$ GeV^2. From Fig. 5, we get the values of f_1 and f_2: $f_1 = -0.486_{-0.095}^{+0.076}$ and $f_2 = -0.571_{-0.1}^{+0.077}$. We list our values of the strong decay constants in Table II.

With all of the above parameters, the decay widths of $P_c(4312) \rightarrow \eta_c p$ and $P_c(4312) \rightarrow J/\psi p$ can be obtained.

TABLE II. Values of the strong decay constants.

Strong decay constant	Value
g	$-0.419_{-0.028}^{+0.019}$
f_1	$-0.486_{-0.095}^{+0.076}$
f_2	$-0.571_{-0.1}^{+0.077}$

Using the transition matrix elements defined in Eq. (14) and following the standard method, one has

$$\Gamma(P_c(4312) \rightarrow \eta_c p) = \frac{g^2[(m_{P_c} + m_N)^2 - m_{\eta_c}^2]}{16\pi m_{P_c}^3} \sqrt{(m_{P_c}^2 + m_{\eta_c}^2 - m_N^2)^2 - 4m_{P_c}^2 m_{\eta_c}^2}, \tag{27}$$

$$\Gamma(P_c(4312) \rightarrow J/\psi p) = \frac{(m_{P_c} + m_N)^2 - m_{J/\psi}^2}{16\pi m_{P_c}^3 m_{J/\psi}^2 (m_{P_c} + m_N)^2} [f_1^2(m_{P_c} + m_N)^2(2m_{J/\psi}^2 + (m_{P_c} - m_N)^2) \\ - 6f_1 f_2 m_{J/\psi}^2(m_{P_c}^2 - m_N^2) + f_2^2 m_{J/\psi}^2(m_{J/\psi}^2 + 2(m_{P_c} - m_N)^2)] \\ \times \sqrt{(m_{P_c}^2 + m_N^2 - m_{J/\psi}^2)^2 - 4m_{P_c}^2 m_N^2}. \tag{28}$$

Substituting the values of the parameters involved in the above formulas, we find

$$\Gamma(P_c(4312) \rightarrow \eta_c p) = 5.54_{-0.5}^{+0.75} \text{ MeV},$$
$$\Gamma(P_c(4312) \rightarrow J/\psi p) = 1.67_{-0.56}^{+0.92} \text{ MeV}, \tag{29}$$

from which one has

$$R \equiv \frac{\Gamma(P_c(4312) \rightarrow \eta_c p)}{\Gamma(P_c(4312) \rightarrow J/\psi p)} = 3.32,$$

$$\Gamma(P_c(4312) \rightarrow \eta_c p) + \Gamma(P_c(4312) \rightarrow J/\psi p)$$
$$= 7.21_{-1.06}^{+1.67} \text{ MeV}. \tag{30}$$

In Refs. [51,52], it was predicted that R is 3 based on the heavy quark spin symmetry. Obviously, our result is agreement with theirs taking into account the uncertainties. The sum of our partial decay widths is large, but still smaller than the total width of $P_c(4312)$, $\Gamma_{P_c(4312)} = 9.8 \pm 2.7_{-4.5}^{+3.7}$ MeV reported by LHCb Collaboration [22].

IV. CONCLUSION

In the present work, the partial decay widths of $P_c(4312) \rightarrow \eta_c p$ and $P_c(4312) \rightarrow J/\psi p$ are studied via the method of QCD sum rule. As a starting point of our investigation, we assume the $P_c(4312)$ as a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$, which is reflected in the molecule-type interpolating current (2).

The pole residue λ_{P_c} of $P_c(4312)$ is an important parameter, which can be used as input parameter in the analyses of the electromagnetic properties and strong decays of $P_c(4312)$. Therefore, we firstly calculate the pole residue λ_{P_c} by using two-point correlation function and get $\lambda_{P_c} = 1.91_{-0.13}^{+0.12} \times 10^{-3}$ GeV^6. Then the strong decay constants are given by using three-point correlation functions and their values are $g = -0.419_{-0.028}^{+0.019}$, $f_1 = -0.486_{-0.095}^{+0.076}$, and $f_2 = -0.571_{-0.1}^{+0.077}$. With the numerical values of the strong decay constants, the partial decay widths to $\eta_c p$ and $J/\psi p$ are estimated to be $\Gamma(P_c(4312) \rightarrow \eta_c p) = 5.54_{-0.5}^{+0.75}$ MeV and $\Gamma(P_c(4312) \rightarrow J/\psi p) = 1.67_{-0.56}^{+0.92}$ MeV, which are compatible with the total width of $P_c(4312)$ measured by LHCb Collaboration: $\Gamma_{P_c(4312)} = 9.8 \pm 2.7_{-4.5}^{+3.7}$ MeV. We also give the ratio R of the decay width of $P_c(4312) \rightarrow \eta_c p$ to that of $P_c(4312) \rightarrow J/\psi p$, $R = 3.32$, which is agreement with the values of Refs. [51,52]. In summary, it is reasonable to assign $P_c(4312)$ to be a $\bar{D}\Sigma_c$ molecular state with $J^P = \frac{1}{2}^-$.

APPENDIX A: THE QUARK PROPAGATORS

The full quark propagators are

$$S_{ij}^q(x) = \frac{i\slashed{x}}{2\pi^2 x^4}\delta_{ij} - \frac{m_q}{4\pi^2 x^2}\delta_{ij} - \frac{\langle \bar{q}q \rangle}{12}\delta_{ij} + i\frac{\langle \bar{q}q \rangle}{48}m_q\slashed{x}\delta_{ij} \\ - \frac{x^2}{192}\langle g_s\bar{q}\sigma Gq \rangle\delta_{ij} + i\frac{x^2\slashed{x}}{1152}m_q\langle g_s\bar{q}\sigma Gq \rangle\delta_{ij} \\ - i\frac{g_s t_{ij}^a G_{\mu\nu}^a}{32\pi^2 x^2}(\slashed{x}\sigma^{\mu\nu} + \sigma^{\mu\nu}\slashed{x}) + \cdots \tag{A1}$$

for light quark, and

$$S_{ij}^Q(x) = i \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \left[\frac{\slashed{k} + m_Q}{k^2 - m_Q^2} \delta_{ij} - \frac{g_s t_{ij}^a G_{\mu\nu}^a}{4} \frac{\sigma^{\mu\nu}(\slashed{k} + m_Q) + (\slashed{k} + m_Q)\sigma^{\mu\nu}}{(k^2 - m_Q^2)^2} + \frac{\langle g_s^2 GG \rangle}{12} \delta_{ij} m_Q \frac{k^2 + m_Q \slashed{k}}{(k^2 - m_Q^2)^4} + \cdots \right] \quad \text{(A2)}$$

for heavy quark. In these expressions, $t^a = \frac{\lambda^a}{2}$ and λ^a are the Gell-Mann matrices, g_s is the strong interaction coupling constant, and i, j are color indices.

APPENDIX B: THE SPECTRAL DENSITIES

In this Appendix, the spectral densities are given.

First, up to dimension-9 and α_s order, the spectral density $\rho_1(s)$ is

$$\rho_1(s) = \rho_1^0(s) + \rho_1^{\langle \bar{q}q \rangle}(s) + \rho_1^{\langle g_s^2 GG \rangle}(s) + \rho_1^{\langle g_s \bar{q} \sigma Gq \rangle}(s) + \rho_1^{\langle \bar{q}q \rangle^2}(s) + \rho_1^{\langle \bar{q}q \rangle \langle g_s^2 GG \rangle}(s) + \rho_1^{\langle \bar{q}q \rangle \langle g_s \bar{q} \sigma Gq \rangle}(s) + \rho_1^{\langle \bar{q}q \rangle^3}(s), \quad \text{(B1)}$$

with

$$\rho_1^0(s) = -\frac{1}{20480\pi^8} \int_{a_{\min}}^{a_{\max}} \frac{da}{a^4} \int_{b_{\min}}^{1-a} \frac{db}{b^4} (1 - a - b)^3 ((a+b)m_c^2 - abs)^5, \quad \text{(B2)}$$

$$\rho_1^{\langle \bar{q}q \rangle}(s) = \frac{\langle \bar{q}q \rangle}{256\pi^6} m_c \int_{a_{\min}}^{a_{\max}} \frac{da}{a^3} \int_{b_{\min}}^{1-a} \frac{db}{b^2} (1 - a - b)^2 ((a+b)m_c^2 - abs)^3, \quad \text{(B3)}$$

$$\rho_1^{\langle g_s^2 GG \rangle}(s) = -\frac{\langle g_s^2 GG \rangle}{24576\pi^8} m_c^2 \int_{a_{\min}}^{a_{\max}} \frac{da}{a^4} \int_{b_{\min}}^{1-a} \frac{db}{b^4} (a^3 + b^3)(1 - a - b)^3 ((a+b)m_c^2 - abs)^2$$
$$- \frac{\langle g_s^2 GG \rangle}{16384\pi^8} \int_{a_{\min}}^{a_{\max}} \frac{da}{a^3} \int_{b_{\min}}^{1-a} \frac{db}{b^3} (2a + b)(1 - a - b)^2 ((a+b)m_c^2 - abs)^3, \quad \text{(B4)}$$

$$\rho_1^{\langle g_s \bar{q} \sigma Gq \rangle}(s) = \frac{3\langle g_s \bar{q} \sigma Gq \rangle}{512\pi^6} m_c \int_{a_{\min}}^{a_{\max}} \frac{da}{a} \int_{b_{\min}}^{1-a} \frac{db}{b^2} (1 - a - b)((a+b)m_c^2 - abs)^2$$
$$- \frac{3\langle g_s \bar{q} \sigma Gq \rangle}{512\pi^6} m_c \int_{a_{\min}}^{a_{\max}} \frac{da}{a} \int_{b_{\min}}^{1-a} \frac{db}{b^3} (1 - a - b)^2 ((a+b)m_c^2 - abs)^2, \quad \text{(B5)}$$

$$\rho_1^{\langle \bar{q}q \rangle^2}(s) = \frac{\langle \bar{q}q \rangle^2}{64\pi^4} \int_{a_{\min}}^{a_{\max}} \frac{da}{a} \int_{b_{\min}}^{1-a} \frac{db}{b} ((a+b)m_c^2 - abs)^2, \quad \text{(B6)}$$

$$\rho_1^{\langle \bar{q}q \rangle \langle g_s^2 GG \rangle}(s) = \frac{\langle \bar{q}q \rangle \langle g_s^2 GG \rangle}{3072\pi^6} m_c^3 \int_{a_{\min}}^{a_{\max}} \frac{da}{a^2} \int_{b_{\min}}^{1-a} \frac{db}{b^3} (a^3 + b^3)(1 - a - b)^2$$
$$+ \frac{\langle \bar{q}q \rangle \langle g_s^2 GG \rangle}{1024\pi^6} m_c \int_{a_{\min}}^{a_{\max}} da \int_{b_{\min}}^{1-a} \frac{db}{b^3} (1 - a - b)^2 ((a+b)m_c^2 - abs)$$
$$+ \frac{\langle \bar{q}q \rangle \langle g_s^2 GG \rangle}{1024\pi^6} m_c \int_{a_{\min}}^{a_{\max}} \frac{da}{a} \int_{b_{\min}}^{1-a} \frac{db}{b} (1 - a - b)((a+b)m_c^2 - abs), \quad \text{(B7)}$$

$$\rho_1^{\langle \bar{q}q \rangle \langle g_s \bar{q} \sigma Gq \rangle}(s) = -\frac{\langle \bar{q}q \rangle \langle g_s \bar{q} \sigma Gq \rangle}{128\pi^4} \int_{a_{\min}}^{a_{\max}} \frac{da}{a} \int_{b_{\min}}^{1-a} db ((a+b)m_c^2 - abs)$$
$$+ \frac{\langle \bar{q}q \rangle \langle g_s \bar{q} \sigma Gq \rangle}{64\pi^4} \int_{a_{\min}}^{a_{\max}} da (m_c^2 - a(1-a)s), \quad \text{(B8)}$$

$$\rho_1^{\langle \bar{q}q \rangle^3}(s) = -\frac{\langle \bar{q}q \rangle^3}{24\pi^2} m_c \int_{a_{\min}}^{a_{\max}} da a, \tag{B9}$$

where $a_{\max} = \frac{1 + \sqrt{1 - \frac{4m_c^2}{s}}}{2}$, $a_{\min} = \frac{1 - \sqrt{1 - \frac{4m_c^2}{s}}}{2}$, and $b_{\min} = \frac{am_c^2}{as - m_c^2}$.

Up to dimension-8 and α_s order, the explicit expressions of the spectral densities $\rho^{(3)}(s, u)$, $\rho_1^{(3)}(s, u)$, and $\rho_2^{(3)}(s, u)$ are

$$
\begin{aligned}
\rho^{(3)}(s, u) = {} & \frac{u^2 \sqrt{s(s - 4m_c^2)}}{2048\pi^6} + \langle \bar{q}q \rangle \frac{m_c u \sqrt{s(s - 4m_c^2)}}{192\pi^4} \\
& + \langle g_s^2 GG \rangle \frac{su^2(3m_c^2 - s)}{18432\pi^6 M_{B_1}^4 \sqrt{s(s - 4m_c^2)}} + \langle g_s^2 GG \rangle \frac{u^2(m_c^2 + s)}{12288\pi^6 M_{B_1}^2 \sqrt{s(s - 4m_c^2)}} \\
& + \langle g_s^2 GG \rangle \frac{m_c^2 u(34s + 3u)}{24576\pi^6 s \sqrt{s(s - 4m_c^2)}} + \langle g_s^2 GG \rangle \frac{\sqrt{s(s - 4m^2)}(s + 2u)}{4096\pi^6 s} \\
& + \langle g_s \bar{q}Gq \rangle \frac{m_c u}{384\pi^4 \sqrt{s(s - 4m_c^2)}} + \langle \bar{q}q \rangle^2 \frac{\sqrt{s(s - 4m_c^2)}\delta(u)}{48\pi^2} \\
& + \langle g_s^2 GG \rangle \langle \bar{q}q \rangle \frac{su(3m_c^2 - s)}{6912\pi^4 M_{B_1}^4 mc \sqrt{s(s - 4m_c^2)}} + \langle g_s^2 GG \rangle \langle \bar{q}q \rangle \frac{u(s - 2m_c^2)}{2304\pi^4 M_{B_1}^2 m_c \sqrt{s(s - 4m_c^2)}} \\
& - \langle g_s^2 GG \rangle \langle \bar{q}q \rangle \frac{m_c \sqrt{s(s - 4m_c^2)}\delta(u)}{4608\pi^4 s} - \langle \bar{q}q \rangle \langle g_s \bar{q}Gq \rangle \frac{\sqrt{s(s - 4m_c^2)}\delta(u)}{192\pi^2 M_{B_2}^2} \\
& + \langle \bar{q}q \rangle \langle g_s \bar{q}Gq \rangle \frac{m_c^2 \delta(u)}{36\pi^2 \sqrt{s(s - 4m_c^2)}} + \langle \bar{q}q \rangle \langle g_s \bar{q}Gq \rangle \frac{\sqrt{s(s - 4m_c^2)}\delta(u)}{96\pi^2 s},
\end{aligned}
\tag{B10}
$$

$$
\begin{aligned}
\rho_1^{(3)}(s, u) = {} & \frac{u^2(s + 2m_c^2)\sqrt{s(s - 4m_c^2)}}{6144\pi^6 s^2} - \langle g_s^2 GG \rangle \frac{u^2(s - 3m_c^2)}{36864\pi^6 M_{B_1}^4 \sqrt{s(s - 4m_c^2)}} \\
& + \langle g_s^2 GG \rangle \frac{m_c^2 u^2}{36864\pi^6 M_{B_1}^2 s \sqrt{s(s - 4m_c^2)}} + \langle g_s^2 GG \rangle \frac{19 m_c^2 u}{73728\pi^6 s \sqrt{s(s - 4m_c^2)}} \\
& + \langle g_s^2 GG \rangle \frac{(s + 2m_c^2)\sqrt{s(s - 4m_c^2)}}{12288\pi^6 s^2} + \langle \bar{q}q \rangle \langle g_s \bar{q}Gq \rangle \frac{(s + 2m_c^2)\sqrt{s(s - 4m_c^2)}\delta(u)}{864\pi^2 M_{B_2}^2 s^2} \\
& + \langle \bar{q}q \rangle \langle g_s \bar{q}Gq \rangle \frac{m_c^2 \delta(u)}{144\pi^2 s \sqrt{s(s - 4m_c^2)}},
\end{aligned}
\tag{B11}
$$

and

$$\rho_2^{(3)}(s,u) = -\frac{u^2\sqrt{s(s-4m_c^2)}(2m_c^2+s)}{6144\pi^6 s} - \langle\bar{q}q\rangle\frac{m_c u\sqrt{s(s-4m_c^2)}}{192\pi^4 s}$$
$$+ \langle g_s^2 GG\rangle\frac{su^2(s-3mc_c^2)}{36864\pi^6 M_{B_1}^4\sqrt{s(s-4m_c^2)}} + \langle g_s^2 GG\rangle\frac{u^2(5m_c^2-2s)}{36864\pi^6 M_{B_1}^2\sqrt{s(s-4m^2)}}$$
$$+ \langle g_s^2 GG\rangle\frac{m_c^2 u(u-38s)}{147456\pi^6 s\sqrt{s(s-4m_c^2)}} - \langle g_s^2 GG\rangle\frac{\sqrt{s(s-4m_c^2)}(2m_c^2+s)}{12288\pi^6 s}$$
$$+ \langle g_s\bar{q}Gq\rangle\frac{m_c u(m_c^2-s)}{576\pi^4 s\sqrt{s(s-4m^2)}} + \langle g_s^2 GG\rangle\langle\bar{q}q\rangle\frac{su(s-3m_c^2)}{6912\pi^4 M_{B_1}^4 m_c\sqrt{s(s-4m_c^2)}}$$
$$+ \langle g_s^2 GG\rangle\langle\bar{q}q\rangle\frac{u(10m_c^2-3s)}{6912\pi^4 M_{B_1}^2 m_c\sqrt{s(s-4m_c^2)}} + \langle g_s^2 GG\rangle\langle\bar{q}q\rangle\frac{m_c\sqrt{s(s-4m_c^2)}\delta(u)}{4608\pi^4 s}$$
$$- \langle\bar{q}q\rangle\langle g_s\bar{q}Gq\rangle\frac{\sqrt{s(s-4m_c^2)}(2m_c^2+s)\delta(u)}{864\pi^2 M_{B_2}^2 s} - \langle\bar{q}q\rangle\langle g_s\bar{q}Gq\rangle\frac{m_c^2\delta(u)}{144\pi^2\sqrt{s(s-4m_c^2)}}, \tag{B12}$$

where $\delta(u)$ is the Dirac δ-function.

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