Pentagon OPE resummation in $\mathcal{N}=4$ SYM: Hexagons with one effective particle contribution

L. V. Bork[1,2,*] and A. I. Onishchenko[3,4,†]

[1]Institute for Theoretical and Experimental Physics, Moscow 117218, Russia
[2]The Center for Fundamental and Applied Research, All-Russia Research Institute of Automatics, Moscow 101000, Russia
[3]Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia
[4]Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia

(Received 20 January 2020; accepted 12 June 2020; published 1 July 2020)

We present the technique for resummation of flux tube excitations series arising in pentagon operator expansion program for polygonal Wilson loops in $\mathcal{N}=4$ SYM. Here we restrict ourselves with contributions of one-particle effective states and consider as a particular example NMHV_6 amplitude at one-loop. The presented technique is also applicable at higher loops for one effective particle contributions and has the potential for generalization to contributions with more effective particles.

DOI: 10.1103/PhysRevD.102.026002

I. INTRODUCTION

The discovery of integrability of $\mathcal{N}=4$ SYM in planar limit, see [1,2] for a review, has led to tremendous progress in our ability to compute different observables in general at arbitrary values of $\mathcal{N}=4$ SYM coupling constant. In particular the collinear or pentagon Operator Product Expansion (POPE) approach to null-polygonal Wilson loops thanks to duality between amplitudes and (super)Wilson loops [3–8] gives us means for computing scattering amplitudes both at weak and strong values of coupling constant [9–28]. There is also a similar approach[1] to structure constants [30–32] and correlation functions [30,33–47].

The important problem present within pentagon OPE approach is the problem of resummation of contributions coming from different flux tube excitations. The latter is required if we are going to recover full kinematical dependence of scattering amplitudes computed within POPE approach without restriction to collinear limits. At weak coupling a procedure for resummation of single particle gluon bound states was presented in [48,49], see also [50] for resummation in the context of n-point functions of Bogomol'nyi Prasad Sommerfield (BPS) operators. At strong coupling the procedure for systematic resummation was studied in [24–28], where one should account for resummation of contributions from gluons, scalars, fermions, and mesons. On the other hand a systematic approach for resummation at weak coupling [20,51–53] is tightly connected with the concept of effective particles [18,20]. The latter are formed by fundamental excitations (gluon or its bound states, scalars, and large fermions/antifermions) together with arbitrary number of small fermions/antifermions. The introduction of effective particles allowed us to reconstruct several scattering amplitudes in general kinematics at tree level [20,52,53] and Maximal Helicity Violating (MHV) hexagon amplitude at one-loop level. Here we are going to extend these results and present the technique for resummation of one effective particle contributions to hexagon amplitudes at arbitrary order of perturbation theory. As a particular example we consider NMHV_6 amplitude at one-loop. Within POPE approach the one effective particle contribution for scattering amplitudes is given in terms of multiple series with complex summand. On the other hand we know that the result for scattering amplitudes[2] is expressible in terms of multiple polylogarithms. So, there should be a way to evaluate mentioned series in terms of multiple polylogarithms. The aim of the present paper is to offer a possible solution to precisely this problem. The presented technique has also the potential for generalizations to both higher point scattering amplitudes and contributions with more then one effective particle.

This paper is organized as follows. In Sec. II we give a brief introduction to the collinear pentagon OPE approach

*bork@itep.ru
†onish@theor.jinr.ru
[1]See [29] for introduction.

[2]At least for sufficiently low loop orders.

and the concept of effective particles. Section III contains details of our resummation technique in the case of Next to Maximal Helicity Violating (NMHV) one-loop hexagon together with the prescription for treating one effective particle contributions in the case of hexagons at arbitrary loop order. Finally in Sec. IV we come with our conclusion. The Appendixes contain explanation of notation together with different details of our calculation of NMHV hexagon amplitude at one-loop.

II. HEXAGON PENTAGON OPE: ONE EFFECTIVE PARTICLE STATES

Let us briefly remind the essential for our further discussion ideas and facts about the pentagon OPE (POPE) approach and effective particle concept. For detailed discussion see [13–16] and [17–20]. Using duality between amplitudes and (super)Wilson loops [3–8] one can recast the problem of calculation of finite remainder function $\mathcal{R}_n^{(k)}$ for the N^{k}MHV$_n$ amplitude[3] into the problem of evaluation of the ratios $\mathcal{W}_n$ of vacuum expectation values of polygonal lightlike Wilson loops [54,55]. The latter within pentagon OPE approach are then decomposed into $n-3$ successive fluxes/squares [11–13]. The first essential ingredient for building pentagon OPE expansion is given by the knowledge of color flux tube excitation spectrum, which in $\mathcal{N}=4$ SYM is known thanks to integrability for *arbitrary* values of coupling constant g [56]. The second important ingredient is supplied by transitions from one flux/square to another and described by pentagon operators. The matrix elements of the latter are also known at any coupling from the integrable bootstrap [13], see further development in [14–18]

To be more specific, the renormalized[4] vacuum expectation value of n polygonal super Wilson loop within pentagon OPE approach is given by [17,18]:

$$\mathcal{W}_n=\sum_{\Psi_i}\mathbb{P}(0|\Psi_1)\mathbb{P}(\Psi_1|\Psi_2)\ldots\mathbb{P}(\Psi_{n-6}|\Psi_{n-5})\mathbb{P}(\Psi_{n-5}|0)e^{\sum_j(-E_j\tau_j+ip_j\sigma_j+im_j\phi_j)}, \tag{2.1}$$

where $\{\tau_i,\sigma_i,\phi_i\}$ is a base of conformal ratios, parametrizing propagation of Ψ_i excitation (in general multiparticle) in ith flux/square. The E_i, p_i, and m_i denote energy,[5] momentum and angular momentum (helicity) of ith excitation. The transition probabilities from one flux to another $\mathbb{P}(\Psi_i|\Psi_j)$ are described by matrix elements of charged or super pentagon operators introduced in [17]:

$$\mathbb{P}=P+\chi^AP_A+\chi^A\chi^BP_{AB}+\chi^A\chi^B\chi^CP_{ABC}+\chi^A\chi^B\chi^C\chi^DP_{ABCD}, \tag{2.2}$$

where χ^A is a Grassmann parameter transforming in the fundamental representation of $SU(4)_R$ R-symmetry group. $P_{A_1\ldots A_k}$ or $P^{[k]}$ for short is charged pentagon transition transforming as kth antisymmetric product. Charged[6] pentagon transitions contrary to ordinary uncharged pentagon transitions P used to describe MHV amplitudes via bosonic polygonal Wilson loops may produce states with nonzero R-charge. For example, the creation amplitude $P_{AB}(0|\ldots)$ may produce scalar fields ϕ_{AB} out of the vacuum, as the quantum numbers of the latter match those of pentagon operator.

In a particular case of hexagon the pentagon OPE expansion gives[7] [18,20]:

$$\mathcal{W}_6^{[r_1,r_2]}=\sum_m\frac{1}{S_m}\int\frac{du_1\ldots du_m}{(2\pi)^m}\Pi_{\rm dyn}\times\Pi_{\rm FF}^{[r_1,r_2]}\times\Pi_{\rm mat}^{[r_1,r_2]}, \tag{2.3}$$

where u_k are rapidities of intermediate multiparticle states and r_1, r_2 are $SU(4)_R$ charges of top and bottom pentagons related to the particle content of the $\mathcal{R}_6$ remainder function. In the NMHV case r_1, r_2 are constrained, such that $r_1+r_2=4$ and as a consequence NMHV hexagon has five different POPE components. The multiparticle flux tube excitations are built from *fundamental excitations* given[8] by gluon bound states, fermions, antifermions and scalars [56]: $\{F_b,\psi,\bar{\psi},\phi\}$. The integrand in Eq. (2.3) has a factorized form and consists from coupling dependent *dynamical* Π_{dyn} and *form factor* $\Pi_{\rm FF}^{[r_1,r_2]}$ parts. The *matrix*

[3] See Appendix A for more details.

[4] See [13] for more details.

[5] The energies of excitations are in one to one correspondence with anomalous dimensions of corresponding single trace GKP operators [57] $\mathrm{Tr}(ZD^{S_1}\mathcal{O}D_+^{S_2}Z)$, where Z is one of three complex scalars in $\mathcal{N}=4$ SYM, $D_+=n_+^\mu D_\mu=D_0+D_3$ is the light-cone covariant derivative and $\mathcal{O}$ is some monomial constructed from $\{F_b,\psi,\phi\}$ fields. It is also assumed, that $S_1+S_2\gg1$.

[6] When thought in terms of pentagon Wilson loops the charged pentagons have additional fields insertions at their cusps and edges compared to usual uncharged pentagons.

[7] $1/S_m$ is a symmetry factor.

[8] Here, we suppressed $SU(4)_R$ and projected Lorenz indexes of fields.

part $\Pi_{\rm mat}^{[r_1,r_2]}$, which takes into account $SU(4)_R$ structure of flux excitations, is coupling independent.

The dynamical part has the form

$$\Pi_{\rm dyn}=\prod_j \mu(u_j)e^{-E(u_j)\tau+ip(u_j)\sigma+im_j\phi}\times\prod_{i<j}\frac{1}{|P(u_i|u_j)|^2}, \tag{2.4}$$

where τ, σ and ϕ are real parameters encoding *all* external kinematical dependence (they parametrize three conformal cross ratios u_1, u_2, u_3 on which $\mathcal{W}_6^{[r_1,r_2]}$ depends). They also play the role of flux tube time, space and angle coordinates respectively. In addition, the τ variable parametrizes the measure of collinearity of two adjacent amplitude momenta, such that the limit $\tau\to\infty$ corresponds to collinear configuration [11,13], see Appendix A for more details. $P(u_i|u_j)$ in Eq. (2.4) are uncharged pentagon transitions[9] between different fundamental excitations and $\mu(u_i)$ are corresponding measures. The expressions for $P(u_i|u_j)$ and $\mu(u_i)$ are known for *arbitrary* values of coupling constant and can be found in [13–16].

The form factor part contribution is obtained by expressing charged pentagon transitions in terms of uncharged ones and is nontrivial only for NMHV hexagons. In our case it is given by [18,20]:

$$\Pi_{\rm FF}^{[r_1,r_2]}=g^{\frac{1}{8}r_1(r_1-4)+\frac{1}{8}r_2(r_2-4)}\times\prod_i h(u_i)^{r_1-r_2}, \tag{2.5}$$

where $h(u_i)$ are the so-called form factors, which are also known for arbitrary values of the coupling constant [18].

The matrix part contribution takes into account contraction of $SU(4)_R$ indexes of each pentagon and in our case takes the form of integral over auxiliary roots [58]:

$$\Pi_{\rm mat}^{[r_1,r_2]}=\frac{1}{K_1!K_2!K_3!}\int\prod_{i=1}^{K_1}\frac{dw_i^1}{2\pi}\prod_{i=1}^{K_2}\frac{dw_i^2}{2\pi}\prod_{i=1}^{K_3}\frac{dw_i^3}{2\pi}\times \tag{2.6}$$

$$\times\frac{g(\mathbf{w}^1)g(\mathbf{w}^2)g(\mathbf{w}^3)}{f(\mathbf{w}^1,\mathbf{w}^2)f(\mathbf{w}^2,\mathbf{w}^3)f(\mathbf{w}^1,\mathbf{v})f(\mathbf{w}^2,\mathbf{s})f(\mathbf{w}^3,\bar{\mathbf{v}})}, \tag{2.7}$$

where w_i are auxiliary roots (rapidities) corresponding to three nodes of $SU(4)$ Dynkin diagram and $\{v_i,s_i,\bar{v}_i\}$ are rapidities for fermions, scalars and antifermions correspondingly. In addition, $g(\mathbf{w})=\prod_{i<j}(w_i-w_j)^2[(w_i-w_j)^2+1]$ and $f(\mathbf{w},\mathbf{v})=\prod_{i,j}[(w_i-v_j)^2+\frac{1}{4}]$. The numbers of auxiliary rapidities K_1, K_2, and K_3 are found as the solution of the following system of equations:

$$N_\psi-2K_1+K_2=\delta_{r_1,3} \tag{2.8}$$

$$N_\phi+K_1-2K_2+K_3=\delta_{r_1,2} \tag{2.9}$$

$$N_{\bar\psi}+K_2-2K_3=\delta_{r_1,1}, \tag{2.10}$$

where N_ψ, N_ϕ and $N_{\bar\psi}$ are respectively the number of fermions, scalars and antifermions in multi-particle excitation.

In the weak coupling regime the contributions of different excitations scale as $g^{2l}e^{-\tau N}$, where N is the total twist of corresponding multiparticle state and l is number of loops. For such expansion to be convergent one has to consider only collinear enough configurations with momenta corresponding to $\tau>1$. The coefficients in front of $g^{2l}e^{-\tau N}$ could be compared with independently computed expressions for amplitudes expanded in collinear limit [14,15,48]. They can also serve as predictions for such collinear limits [48,49]. On the other hand, one can try to resum contributions of all possible flux tube excitations contributing at a given loop order l. Together with analytical continuation of the obtained result to $\tau\le 1$ this should allow for a full reconstruction of the whole kinematical dependence [16]. The possibility of such resummation also implies means of getting POPE results for $\mathcal{R}_n$ remainder functions without any reference to $\mathcal{N}=4$ Lagrangian and corresponding Feynman rules or unitarity cuts.

To make such resummation possible one has to understand the hierarchy of flux tube excitations in the weak coupling regime. That is we need to know when and which excitation starts to give contribution to perturbative expansion. The useful hint comes from the structure of fermionic excitations, which are usually separated into *large* ψ and *small* ψ_s fermions. The latter property is due to the fact that in terms of Bethe rapidity the fermionic excitations are defined on two-sheeted Riemann surface [15]. On one Riemann sheet the fermion momentum is large, while on the other it is small. When attached to another particle, small fermions $\psi_s,\bar\psi_s$ act as supersymmetry generators [59]. The action of $\psi_s\bar\psi_s$ pairs (or derivatives D_+) creates $SL(2)$ conformal *descendants* as at weak coupling there is an enhancement of symmetry[10] from $SU(4)$ to $SL(2|4)$ [12,56,60].

The very useful notion for the purposes of pentagon OPE resummation is provided by the concept of effective particles [18,20]. By effective particle we will understand a fundamental excitation together with arbitrary number ("sea") of small fermion (antifermion) excitations N_{ψ_s} ($N_{\bar\psi_s}$). Having more then one fundamental excitation surrounded by the sea of small fermions/antifermions will lead to more then one effective particle state. Integrating out small fermion/antifermion rapidities together with auxiliary $SU(4)_R$ roots leads to the description of effective particles

[9]These functions depend only on types of fundamental excitations, their spectral parameters and coupling constant g.

[10]This symmetry is exact only at one loop level, however the same bookkeeping turns out to be useful also at higher loops.

in terms of Bethe string complexes. In general, the effective particle (excitation) is described by three parameters: the helicity or angular momentum of excitation a, its descendant number n and $SU(4)_R$ representation in which it transforms. One can show that for the NMHV$_6$ amplitude the contribution of one effective particle is sufficient for its reconstruction both at tree and one-loop (LO and NLO) levels. The account for two effective particles is enough to reconstruct two, three and four loops.[11] So, we see that the number of effective particles we should take into account grows rather slowly with loop order.

To demonstrate our resummation technique in the next section we will use $\mathcal{W}_6^{[2,2]}$ NMHV POPE component. In this case, restricting ourselves with one effective particle contributions, we should account for the following effective particles, transforming in the vector representation of $SU(4)_R$ [20]:

$$\Phi^1_{a,n} = F_a\psi_s\psi_s(\bar\psi_s\psi_s)^n, \qquad \Phi^2_{a,n} = \bar F^a\bar\psi_s\bar\psi_s(\bar\psi_s\psi_s)^n,$$
$$\Phi^3_n = \phi(\bar\psi_s\psi_s)^n, \qquad \Phi^4_n = \psi\psi_s(\bar\psi_s\psi_s)^n,$$
$$\Phi^5_n = \bar\psi\bar\psi_s(\bar\psi_s\psi_s)^n. \tag{2.11}$$

In the case $n = 0$ the above states are $SL(2)$ conformal primaries. Taking integrals over small fermion/antifermion rapidities and auxiliary $SU(4)_R$ roots by residues the expression for $\mathcal{W}_6^{[2,2]}$ POPE component takes the form [20]:

$$\mathcal{W}_6^{[2,2]} = \sum_\Phi \int \frac{du}{2\pi} e^{E_\Phi(u)\tau + ip_\Phi(u)\sigma + im_\Phi\phi}\mu_\Phi^{[2,2]}(u) + \ldots, \tag{2.12}$$

where dots denote multiple effective particle contributions and the expressions for energies $E_\Phi(u)$, momenta $p_\Phi(u)$, angular momenta m_Φ and integration measures $\mu_\Phi^{[2,2]}(u)$ of effective particles can be found in Appendix B.

The first steps in the problem of resummation of series in Eq. (2.12) were made in [20] at leading order. However, we found that the method employed there is somewhat hard to generalize to higher orders. So, in the following section we are going to present an algorithm which should allow one to compute series representation for $\mathcal{W}_6^{[r_1,r_2]}$ functions similar to that for $\mathcal{W}_6^{[2,2]}$ (2.12) in terms of multiple polylogarithms [61,62] of kinematical variables at any order of perturbation theory. Presumably, the same algorithm should be also applicable to other cases with $n > 6$ and contributions with more effective particles. As an illustration for our method we will consider LO and NLO contributions to $\mathcal{W}_6^{[2,2]}$ POPE component. In this case it is sufficient to consider one effective particle contribution only. To compare the results of pentagon OPE resummation with results for hexagon amplitudes computed with other methods we should recall that the usual way to package together all helicity amplitudes is to use super Wilson loop [54,55]:

$$\mathbb{W}_6 = W_{6,\mathrm{MHV}} + \eta_i^1\eta_j^2\eta_k^3\eta_l^4 W_{6,\mathrm{NMHV}}^{\langle ijkl\rangle} + \ldots, \tag{2.13}$$

where W_{NMHV} is the NMHV amplitude divided by Parke-Taylor MHV factor. Here, the Grassmann variables η_j^A are Grassmann components of hexagon momentum twistors (see Appendix A for more details) with upper index transforming in the fundamental representation of $SU(4)_R$ and lower index labeling the edge of hexagon. The important thing here is that these Grassmann variables are different from those used within POPE framework (2.2). Nevertheless there is a map from one set of Grassmann variables to another [17]. In particular, it turns out that [17,20]:

$$\mathcal{W}_6^{[2,2]} = -\mathcal{W}_6^{\langle 1144\rangle}, \tag{2.14}$$

where $\mathcal{W}_6^{\langle 1144\rangle}$ is the $W_{6,\mathrm{NMHV}}^{\langle ijkl\rangle}$ component from Eq. (2.13).

III. RESUMMATION TECHNIQUE

Before presenting the general algorithm for treating one effective particle contributions in the case of hexagons, let us first start with the particular example of hexagon NMHV amplitude and then formulate the general prescription for the resummation of one effective particle contributions to hexagon Wilson loops in $\mathcal{N} = 4$ SYM. Up to one loop the expression for $\mathcal{W}_6^{\langle 1144\rangle}$ component takes the form[12]:

$$\mathcal{W}_6^{\langle 1144\rangle} = -\mathcal{W}_6^{[2,2]} = \sum_{a=-\infty}^{\infty}\sum_{n=0}^{\infty}\int\frac{du}{2\pi i}e^{-(|a|+2n+1)\tau + 2u\sigma + ia\phi}(-1)^{a+n}$$
$$\times\frac{\Gamma(\frac{|a|}{2} - u - \frac{1}{2})\Gamma(\frac{|a|}{2} + u + \frac{3}{2} + n)^2}{\Gamma(\frac{|a|}{2} + u + \frac{3}{2})\Gamma(|a| + n + 1)n!}\{1 + g^2 f_{a,n}^{\mathrm{NLO}}(u) + \mathcal{O}(g^4)\}, \tag{3.1}$$

[11]For MHV$_6$ amplitude one effective particle is sufficient for one-loop reconstruction [51] and combination of one and two effective particles will be enough to reconstruct amplitude up to (including) five loops [18,20].

[12]See Appendix B for the expression for $\mathcal{W}_6^{[2,2]}$, which we expand up to one-loop order. We have also made change of variables $u \to -iu$, so that now the integration contour goes along imaginary axis. Also $g \equiv g_{\mathrm{YM}}^2 N_c/(16\pi^2)$.

where

$$
\begin{aligned}
f_{a,n}^{\mathrm{NLO}}(u) = {} & \frac{\pi^2}{3} - \frac{6}{(1-|a|+2u)^2} + \frac{2}{|a|(1-|a|+2u)} + \frac{2}{(1+|a|+2u)^2} - \frac{2}{|a|(1+|a|+2u)} \\
& - 2\tau\left[2\gamma_E + \Psi^{(0)}\left(\frac{|a|+1}{2} - u\right) + \Psi^{(0)}\left(\frac{|a|+1}{2} + u\right)\right] \\
& + 2\sigma\left[\Psi^{(0)}\left(\frac{|a|-1}{2} - u\right) + \Psi^{(0)}\left(\frac{|a|+3}{2} + u\right) - 2\Psi^{(0)}\left(\frac{|a|+3}{2} + n + u\right)\right] \\
& - \frac{1}{2}\left[2\gamma_E + \Psi^{(0)}\left(\frac{|a|+1}{2} - u\right) + \Psi\left(\frac{|a|+1}{2} + u\right)\right]^2 \\
& - \frac{1}{2}\left[\Psi^{(0)}\left(\frac{|a|-1}{2} - u\right) + \Psi^{(0)}\left(\frac{|a|+3}{2} + u\right) - 2\Psi^{(0)}\left(\frac{|a|+3}{2} + n + u\right)\right]^2 \\
& - \Psi^{(1)}\left(\frac{|a|+1}{2} - u\right) - \Psi^{(1)}\left(\frac{|a|+1}{2} + u\right) + 2\Psi^{(1)}\left(\frac{|a|+3}{2} + u\right) - 2\Psi^{(1)}\left(\frac{|a|+3}{2} + n + u\right), \qquad (3.2)
\end{aligned}
$$

Here $\Psi^{(n)}(z)$ are polygamma functions. To evaluate the above expression both at LO and higher we start with taking residues in u-variable. To achieve this we first use reflection identities

$$
\Gamma\left(\frac{|a|-1}{2} - u\right) = \frac{\pi \csc\left(\frac{\pi(|a|-1)}{2} - \pi u\right)}{\Gamma\left(\frac{3-|a|}{2} + u\right)}, \qquad (3.3)
$$

$$
\Psi^{(n)}\left(\frac{|a|+3}{2} - u\right) = (-1)^n \Psi^{(n)}\left(u - \frac{|a|+1}{2}\right) - \pi \frac{\partial^n}{\partial u^n} \cot\left(\frac{\pi(|a|+3)}{2} - \pi u\right) \qquad (3.4)
$$

to isolate singular terms into elementary functions with known Taylor expansions. It is also convenient to transform present polygamma functions to the same argument as far as possible using the following recurrence relation

$$
\Psi^{(n)}(z+1) = \Psi(z) + (-1)^n n! z^{-n-1}. \qquad (3.5)
$$

Note, that it is the general procedure when taking Mellin-Barnes integrals and was used already in the context of collinear OPE in [48,49]. Now, taking residues at $u = \frac{|a|-1}{2} + k$ we get:

$$
\mathcal{W}_6^{\langle 1144\rangle} = \mathcal{W}_{6,m}^{\langle 1144\rangle} + \mathcal{W}_{6,b}^{\langle 1144\rangle}, \qquad (3.6)
$$

where subscripts m and b denote what we will call main and boundary[13] contributions. The latter are given by:

$$
\begin{aligned}
\mathcal{W}_{6,m}^{\langle 1144\rangle} = {} & \sum_{a=-\infty, a\neq 0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{a+n+k}}{k!n!} e^{-(|a|+2n+1)\tau + (|a|+2k-1)\sigma + ia\phi} \\
& \times \frac{(|a|+n+k)!}{(|a|+k)!} \frac{(|a|+n+k)!}{(|a|+n)!} \{1 + g^2 \tilde{f}_{a,n}^{\mathrm{NLO}}(k) + \mathcal{O}(g^4)\} \\
& + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{n+k}}{k!n!} e^{-(2n+1)\tau + (2k-1)\sigma} \frac{(n+k)!}{k!} \frac{(n+k)!}{n!} \{1 + g^2 \tilde{f}_{0,n}^{\mathrm{NLO}}(k) + \mathcal{O}(g^4)\}, \qquad (3.7)
\end{aligned}
$$

and

$$
\mathcal{W}_{6,b}^{\langle 1144\rangle} = \sum_{a=-\infty, a\neq 0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{a+n}}{n!} e^{-(|a|+2n+1)\tau + (|a|-1)\sigma + ia\phi} \frac{(|a|+n)!}{|a|!} \{g^2 \overset{\approx}{f}{}_{a,n}^{\mathrm{NLO}} + \mathcal{O}(g^4)\} \qquad (3.8)
$$

[13] The boundary contributions start contributing from NLO order.

where

$$\tilde{f}_{a,n}^{\mathrm{NLO}}(k) = \frac{\pi^2}{3} - \frac{2}{k^2}(1-\delta_{k,0}) - \frac{2}{(k+|a|)^2} + \frac{2(\sigma+\tau)}{k}(1-\delta_{k,0}) + \frac{2(\sigma+\tau)}{k+|a|} - 4\sigma\tau - 2\Psi^{(1)}(|a|+n+k+1) - 2(\Psi^{(0)}(|a|+n+k+1)+\gamma_E)^2 + 2\left(\frac{1}{k}(1-\delta_{k,0}) + \frac{1}{k+|a|} - 2\sigma - 2\tau\right)(\Psi^{(0)}(|a|+n+k+1)+\gamma_E), \tag{3.9}$$

and

$$\overset{\approx}{f}{}_{a,n}^{\mathrm{NLO}} = 4\sigma\tau - \frac{\pi^2}{6} + \Psi^{(1)}(1+|a|) + 4\tau(\Psi^{(0)}(|a|+n+1)+\gamma_E) - (\Psi^{(0)}(1+|a|)+\gamma_E)^2 + 2(\Psi^{(0)}(|a|+1)+\gamma_E)(\sigma-\tau+\Psi^{(0)}(|a|+n+1)). \tag{3.10}$$

Introducing notations $x=e^{-\tau}$, $y=e^{\sigma}$, $z=e^{i\phi}$ the above expressions take the form

$$\mathcal{W}_{6,m}^{\langle 1144\rangle} = \sum_{a=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{a+n+k}x^{a+2n+1}y^{a+2k-1}(z^a+z^{-a}) \times \binom{a+n+k}{n}\binom{a+n+k}{k}\{1+g^2\tilde{f}_{a,n}^{\mathrm{NLO}}(k)+\mathcal{O}(g^4)\} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}x^{2n+1}y^{2k-1}\binom{n+k}{n}\binom{n+k}{k}\{1+g^2\tilde{f}_{0,n}^{\mathrm{NLO}}(k)+\mathcal{O}(g^4)\} \tag{3.11}$$

and

$$\mathcal{W}_{6,b}^{\langle 1144\rangle} = \sum_{a=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{a+n}x^{a+2n+1}y^{a-1}(z^a+z^{-a})\binom{a+n}{n}\{g^2\overset{\approx}{f}{}_{a,n}^{\mathrm{NLO}}+\mathcal{O}(g^4)\} \tag{3.12}$$

A. Leading order

To evaluate the sums left after taking residues in u it is convenient to introduce the following integral representations for binomial coefficients[14]:

$$\binom{n}{k} = \frac{1}{2\pi i}\int_{|t|=1}(t+1)^n t^{-k-1}dt. \tag{3.13}$$

Then at leading order we have

$$\mathcal{W}_{6}^{\langle 1144\rangle,\mathrm{LO}} = \sum_{a=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{a+n+k}}{(2\pi i)^2}\int_{|t_1|=1}dt_1\int_{|t_2|=1}dt_2 \times x^{a+2n+1}y^{a+2k-1}(z^a+z^{-a})[(t_1+1)(t_2+1)]^{a+n+k}t_1^{-n-1}t_2^{-k-1} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{n+k}}{(2\pi i)^2}\int_{|t_1|=1}dt_1\int_{|t_2|=1}dt_2 x^{2n+1}y^{2k-1}[(t_1+1)(t_2+1)]^{a+n+k}t_1^{-n-1}t_2^{-k-1} \tag{3.14}$$

Now, the series summation is straightforward and we get

$$\mathcal{W}_{6}^{\langle 1144\rangle,\mathrm{LO}} = -\frac{1}{(2\pi i)^2}\int_{|t_1|=1}dt_1\int_{|t_2|=1}dt_2\frac{(1+t_1)(1+t_2)x}{(t_1+(1+t_1)(1+t_2)x^2)(t_2+(1+t_1)(1+t_2)y^2)} \times\left\{\frac{y}{t_2}+\frac{x}{(1+t_1)(1+t_2)xy+z}+\frac{xz}{1+(1+t_1)(1+t_2)xyz}\right\}. \tag{3.15}$$

[14]Here, the integration contour is actually going around $z=0$.

Next, performing partial fractioning in t_2 variable and taking residues at $t_2=0$ and $t_2=-1+\frac{1}{1+(1+t_1)y^2}$ together with subsequent residues in t_1 at $t_1=-\frac{x^2}{1+x^2}$ and $t_1=\frac{-1-x^2-y^2+\sqrt{(1+x^2+y^2)^2-4x^2y^2}}{2y^2}$ we get

$$\mathcal{W}_6^{\langle 1144\rangle,\mathrm{LO}}=\frac{x}{y}\left(\frac{z}{z+(y+xz)(x+yz)}-\frac{1}{1+x^2}\right) \tag{3.16}$$

in agreement with [20]. We would like to clarify the particular choice of points, at which residues over t_1 and t_2 should be taken. First, we know that in the limit $x\to 0$, $y\to 0$ the residue should be taken at the point $t_1=t_2=0$ and so our points at which we took residues should go to this particular point in this limit. And of course we may greatly benefit from numerical checks for some particular values of Mandelstam variables to insure that we actually get the correct expression at the end.

B. Next to leading order

The integration procedure at next to leading order (NLO) and higher goes similar to the LO case. To illustrate our technique let us consider several terms in NLO contribution. The results for the rest of terms could be found in accompanying *Mathematica* notebook. The different terms in the main contribution at NLO can be written as

$$\begin{aligned}\mathcal{W}_{6,m}^{\langle 1144\rangle}[f_{a,n}(k)]=&\sum_{a=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{a+n+k}x^{a+2n+1}y^{a+2k-1}(z^a+z^{-a})\binom{a+n+k}{n}\binom{a+n+k}{k}f_{a,n}(k)\\&+\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{n+k}x^{2n+1}y^{2k-1}\binom{n+k}{n}\binom{n+k}{k}f_{0,n}(k)\end{aligned} \tag{3.17}$$

where $f_{a,n}(k)$ are given by terms in the sum of Eq. (3.9). To calculate the latter it is convenient to express $1/k^n$ and $1/(k+a)^n$ factors in terms of polygamma functions as

$$\frac{1}{z^n}=\frac{(-1)^n}{(n-1)!}[\Psi^{(n-1)}(z)-\Psi^{(n-1)}(z+1)] \tag{3.18}$$

and use for polygamma functions the following integral representations

$$\Psi^{(n)}(z)=\int_0^1\frac{x^{z-1}\log^n x}{x-1}dx\quad \text{if } n>0, \tag{3.19}$$

$$\Psi^{(0)}(z)=\int_0^1\frac{1-x^{z-1}}{1-x}dx-\gamma_E. \tag{3.20}$$

For example following the above prescription for $1/(k+a)$ term, that is rewriting

$$\frac{1}{k+a}=\Psi^{(0)}(k+a+1)-\Psi^{(0)}(k+a)=\int_0^1 dx_1 x_1^{k+a-1} \tag{3.21}$$

and using the same integral representations for binomial coefficients as at LO we may again easily sum the geometric series in a, n, and k and get

$$\begin{aligned}\mathcal{W}_{6,m}^{\langle 1144\rangle}\left[\frac{1}{k+a}\right]=&\int_0^1 dx_1\int_{|t_1|=1}dt_1\int_{|t_2|=1}dt_2\frac{x(1+t_1)(1+t_2)}{(t_1+(1+t_1)(1+t_2)x^2)(t_2+(1+t_1)(1+t_2)x_1y^2)}\\&\times\left\{\frac{y}{t_2}+\frac{x}{(1+t_1)(1+t_2)xyx_1+z}+\frac{xz}{1+(1+t_1)(1+t_2)xyzx_1}\right\}.\end{aligned} \tag{3.22}$$

Now, taking residues in t_2 at $t_2=0$ and $t_2=-1+\frac{1}{1+(1+t_1)y^2x_1}$ together with subsequent residues in t_1 at $t_1=-\frac{x^2}{1+x^2}$ and $t_1=\frac{-1-x^2-y^2x_1+\sqrt{(1+x^2+y^2x_1)^2-4x^2y^2x_1}}{2y^2x_1}$ we have

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}\left[\frac{1}{k+a}\right] = \int_0^1 dx_1 \frac{x}{2yx_1}\left\{\frac{2}{1+x^2} + \frac{xyz(x_1-1)(2xyz(1+x_1)+(1+x^2+y^2x_1)(1+z^2))}{p_1(x,y,x_1)(z+(xz+yx_1)(x+yz))(z+(y+xz)(x+yzx_1))}\right.$$
$$\left. - \frac{z(2zx^2+2z(1+y^2x_1)+xy(1+z^2)(1+x_1))}{(z+(xz+yx_1)(x+yz))(z+(y+xz)(x+yzx_1))}\right\}, \tag{3.23}$$

where $p_1(x,y,x_1) = \sqrt{(1+x^2+y^2x_1)^2 - 4x^2y^2x_1}$. Note, that the points at which residues were taken are deformations of corresponding points we had at LO for $x_1 = 1$. Next, the integral in x_1 could be easily evaluated by rationalizing root in $p_1(x,y,x_1)$ with the following variable substitution

$$x_1 = \frac{2x^2}{y^2-t} - \frac{2}{y^2+t} \tag{3.24}$$

As a result we get

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}\left[\frac{1}{k+a}\right] = \frac{x}{y(1+x^2)}\left\{2\log\left(\frac{2xy^2}{1+x^2}\right) - \log(1+x^2+y^2-p(x,y)) - \log(-1-x^2+y^2+p(x,y))\right\}$$
$$+ \frac{x}{y(1+x^2+xyz)}\left\{-\log\left(\frac{2xy^3z}{1+x^2}\right) + \log(-1-y^2-x(2x+x^3-xy^2+yz+x^2yz-y^3z)\right.$$
$$\left. + (1+x^2+xyz)p(x,y))\right\} + \frac{zx}{y(z+x(y+xz))}\left\{-\log\left(\frac{2xy^3}{1+x^2}\right)\right.$$
$$\left. + \log(xy(-1-x^2+y^2) - (1+x^2)^2z + (-1+x^2)y^2z + (z+x(y+xz))p(x,y))\right\}, \tag{3.25}$$

where $p(x,y) = \sqrt{(1+x^2+y^2)^2 - 4x^2y^2}$.

In the case of $\Psi^{(1)}(n+k+a+1)$ term we proceed essentially the same way. Indeed, using the integral representations for $\Psi^{(1)}$ and binomial coefficients as above, resumming geometric series in n, a, k and taking residues in variables entering integral representations for binomial coefficients we get

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}[\Psi^{(1)}(n+k+a+1)] = \int_0^1 dx_1 \frac{xx_1(x+yz+xz^2)\log x_1}{(x_1-1)(1+x^2x_1)(z+x_1(y+xz)(x+yz))}. \tag{3.26}$$

The left integration over x_1 is straightforward and gives the following expression

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}[\Psi^{(1)}(n+k+a+1)] = \frac{\pi^2 x(x+yz+xz^2)}{6(1+x^2)(x^2z+(1+y^2)z+xy(1+z^2))}$$
$$- \frac{x}{y(1+x^2)}\mathrm{Li}_2(-x^2) + \frac{xz}{y(x^2z+(1+y^2)z+xy(1+z^2))}\mathrm{Li}_2\left(-\frac{xy+x^2z+y^2z+xyz^2}{z}\right). \tag{3.27}$$

As a final example of a term in the main contribution $\mathcal{W}_{6,m}^{\langle 1144\rangle}$ let us consider the case of $(\Psi^{(0)}(n+k+a+1)+\gamma_E)^2$. Again, writing integral representations for polygamma functions and binomial coefficients as above, summing resulting geometric series in n, a, k and taking residues in variables entering integral representations for binomial coefficients we get

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}[(\Psi^{(0)}(n+k+a+1)+\gamma_E)^2]$$
$$= \int_0^1 dx_1 \int_0^1 dx_2 \frac{x(x+yz+xz^2)}{(1-x_1)(1-x_2)}\left\{\frac{1}{(1+x^2)(z+(y+xz)(x+yz))} - \frac{x_1}{(1+x^2x_1)(z+x_1(y+xz)(x+yz))}\right.$$
$$\left. - \frac{x_2}{(1+x^2x_2)(z+x_2(y+xz)(x+yz))} + \frac{x_1x_2}{(1+x^2x_1x_2)(z+x_1x_2(y+xz)(x+yz))}\right\} \tag{3.28}$$

Now, the integrations in x_1 and x_2 are straightforward and we finally obtain

$$\begin{aligned}
&\mathcal{W}_{6,m}^{\langle 1144\rangle}[(\Psi^{(0)}(n+k+a+1)+\gamma_E)^2] \\
&= -\frac{\pi^2 x(x+yz+xz^2)}{6(1+x^2)(x^2z+(1+y^2)z+xy(1+z^2))} + \frac{x\log(1+x^2)(-4\log x+3\log(1+x^2))}{2y(1+x^2)} \\
&\quad -\frac{xz}{y(z+(y+xz)(x+yz))}\log\left(\frac{(y+xz)(x+yz)}{(z+(y+xz)(x+yz))^2}\right)\log\left(\frac{z}{z+(y+xz)(x+yz)}\right) \\
&\quad -\frac{xz}{2y(x^2z+(1+y^2)z+xy(1+z^2))}\log\left(\frac{z}{z+(y+xz)(x+yz)}\right)\log\left(z(z+(y+xz)(x+yz))\right) \\
&\quad +\frac{x}{y(1+x^2)}\mathrm{Li}_2\left(\frac{1}{1+x^2}\right) - \frac{xz}{y(x^2z+(1+y^2)z+xy(1+z^2))}\mathrm{Li}_2\left(\frac{z}{xy(1+z^2)+z(1+x^2+y^2)}\right).
\end{aligned} \tag{3.29}$$

The evaluation of boundary contribution goes similar to the main one. The different terms in the boundary contribution at NLO can be written as

$$\mathcal{W}_{6,b}^{\langle 1144\rangle}[f_{a,n}] = \sum_{a=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{a+n}x^{a+2n+1}y^{a-1}(z^a+z^{-a})\binom{a+n}{n}f_{a,n} \tag{3.30}$$

where $f_{a,n}$ are given by terms in the sum of Eq. (3.10). Take for example the case with $f_{a,n}=\Psi^{(0)}(n+a+1)+\gamma_E$. Using integral representations for polygamma function and binomial coefficient as before, resuming resulting geometric series in a, n variables and taking integral for binomial coefficient by residues we get

$$\begin{aligned}
&\mathcal{W}_{6,b}^{\langle 1144\rangle}[\Psi^{(0)}(n+a+1)+\gamma_E] \\
&= \int_0^1 dx_1 \frac{x^2}{1-x_1}\left\{\frac{1+2xyz+z^2+x^2(1+z^2)}{(1+x^2)(1+x^2+xyz)(z+x(y+xz))} - \frac{x_1(1+2xyzx_1+z^2+x^2(1+z^2)x_1)}{(1+x^2x_1)(z+x(y+xz)x_1)(1+x(x+yz)x_1)}\right\}.
\end{aligned} \tag{3.31}$$

The left integration in x_1 is straightforward and as a result we obtain

$$\mathcal{W}_{6,b}^{\langle 1144\rangle}[\Psi^{(0)}(n+a+1)+\gamma_E] = -\frac{2x\log(1+x^2)}{y(1+x^2)} + \frac{xz}{y(xy+z(1+x^2))}\log\left(\frac{xy+z(1+x^2)}{z}\right) + \frac{x\log(1+x^2+xyz)}{y(1+x^2+xyz)}. \tag{3.32}$$

The results for all other terms both in main and boundary contributions could be found in accompanying *Mathematica* notebook. Gathering all contributions and using symbols[15] to simplify the resulting expression we finally get

$$\begin{aligned}
\mathcal{W}_6^{\langle 1144\rangle,\mathrm{NLO}} &= \frac{x}{y}\left(\frac{1}{1+x^2+xyz}+\frac{z}{z+x(y+xz)}\right) \\
&\quad \times\log\left(\frac{(1+x^2)z}{x^2(z+(y+xz)(x+yz))}\right)\log\left(\frac{(1+x^2)(z+(y+xz)(x+yz))}{y^2z}\right) \\
&\quad +\frac{x}{y(1+x^2)}\left\{-\frac{\pi^2}{6}+\log^2x-\log^2\left(\frac{1+x^2}{xy^2}\right)+4\log^2y+\mathrm{Li}_2\left(\frac{1}{1+x^2}\right)+\mathrm{Li}_2\left(\frac{x^2}{1+x^2}\right)\right\} \\
&\quad +\frac{xz}{y(z+(y+xz)(x+yz))}\left\{\frac{\pi^2}{6}-\log^2x-\log^2\left(\frac{1+x^2}{x}\right)-2\log^2y\right. \\
&\quad \left.+2\log^2\left(\frac{yz}{x(z+(y+xz)(x+yz))}\right)-\mathrm{Li}_2\left(\frac{1}{1+x^2}\right)-\mathrm{Li}_2\left(\frac{x^2}{1+x^2}\right)\right\},
\end{aligned} \tag{3.33}$$

[15]See Appendix C for more details.

which coincides with $(\mathcal{R}_6^{\langle 1144\rangle}\mathcal{W}_6^{\mathrm{BDS}})^{\mathrm{NLO}}$ in agreement with [63,64]. See Appendix A for notation.

C. Prescription for arbitrary order

The LO and NLO resummation for other NMHV hexagon components[16] goes similar to the case of $\mathcal{W}_6^{\langle 1144\rangle}$ component. To see that the technique we used at LO and NLO can be also used for resummation of one effective particle contributions at higher loop orders it is instructive to see the expressions for $\tilde{f}_{a,n}^{\mathrm{NNLO}}(k)$ and $\approx{f}_{a,n}^{\mathrm{NNLO}}$ factors, which could be found in accompanying *Mathematica* notebook. These expressions and those for even higher loop orders can be easily obtained similar to what we did at NLO. What is important to us here is the particular structure of the terms in these factors. They are precisely of the form we discussed before and can be treated along the same lines. The contour integration over variables coming from integral representations for binomial coefficients are taken by residues at points, which are deformations of corresponding points at leading order by variables entering integral representations of polygamma functions and simple fractions. The subsequent series resummation is again the same as for LO and NLO. Then we are left with definite integrals over rational functions in remained variables. What is important is that they are similar to parametric integrals (for example in Feynman parameters) one typically encounters when evaluating Feynman diagrams. There are different ways to proceed now. One is to use direct integration which we used at NLO. It should be possible as we know that the result is expressible in terms of multiple polylogarithms after all. Indeed, consider as a simple example the term $\Psi^{(3)}(n+k+a+1)$. Using the integral representations for $\Psi^{(3)}$ and binomial coefficients as above, resuming geometric series in n, a, k and taking residues in variables entering integral representations for binomial coefficients we get

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}[\Psi^{(3)}(n+k+a+1)] = \int_0^1 dx_1 \frac{x x_1 (x+yz+xz^2)\log^3 x_1}{(x_1-1)(1+x^2x_1)(z+x_1(y+xz)(x+yz))}. \tag{3.34}$$

The left integration over x_1 is straightforward and gives the following expression

$$\mathcal{W}_{6,m}^{\langle 1144\rangle}[\Psi^{(3)}(n+k+a+1)] = \frac{\pi^4 x(x+yz+xz^2)}{15(1+x^2)(x^2z+(1+y^2)z+xy(1+z^2))} - \frac{6x}{y(1+x^2)}\mathrm{Li}_4(-x^2) + \frac{6xz}{y(x^2z+(1+y^2)z+xy(1+z^2))}\mathrm{Li}_4\left(-\frac{xy+x^2z+y^2z+xyz^2}{z}\right). \tag{3.35}$$

Next, it is easy to see, that similar technique is also applicable for one effective particle contribution to MHV hexagon. Indeed, from [51] we have

$$\mathcal{W}_6^{\mathrm{MHV,NLO}} = 2\sum_{n=0}^{\infty}\int\frac{du}{2\pi i}x^{2n+2}y^{2u}\mu_{0,n}^{\mathrm{MHV,non-gluonic}}(u) + \sum_{a=1}^{\infty}\sum_{n=0}^{\infty}\int\frac{du}{2\pi i}x^{2n+a}y^{2u}(z^a+z^{-a})[\mu_{a,n}^{\mathrm{MHV,gluonic}}(u) + x^2\mu_{a,n}^{\mathrm{MHV,non-gluonic}}(u)], \tag{3.36}$$

where

$$\mu_{a,n}^{\mathrm{MHV,gluonic}}(u) = \frac{(-1)^{a+n}\Gamma(\frac{a}{2}-u)}{(\frac{a}{2}-u)(\frac{a}{2}+u)\Gamma(n+1)\Gamma(a+n)}\frac{\Gamma(n+u+\frac{a}{2})^2}{\Gamma(u+\frac{a}{2})}, \tag{3.37}$$

$$\mu_{a,n}^{\mathrm{MHV,non-gluonic}}(u) = \frac{(-1)^{a+n}\Gamma(\frac{a}{2}-u)}{(\frac{a}{2}+u)\Gamma(n+1)\Gamma(a+n+2)}\frac{\Gamma(n+u+\frac{a}{2}+1)^2}{\Gamma(u+\frac{a}{2}+1)}. \tag{3.38}$$

Taking residues in $u=\frac{a}{2}+k$, $k\geq 0$ we get

[16] The starting expressions for our resummation algorithm similar to Eq. (3.1) can be obtained from the results of [20].

$$\mathcal{W}_6^{\mathrm{MHV,NLO}} = 2\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} x^{2n+2}y^{2k}\frac{(-1)^{k+n}}{k(n+1)}\binom{n+k}{k}\binom{n+k}{n} + \sum_{a,k=1}^{\infty}\sum_{n=0}^{\infty} x^{2n+2+a}y^{2k+a}(z^a+z^{-a})(-1)^{a+k+n}\binom{a+n+k}{k}\binom{a+n+k}{n} \times\left[\frac{1}{k(n+1)}+\frac{1}{(k+a)(n+a+1)}\right] + \text{boundary terms} \quad (3.39)$$

Staring from this expression for $\mathcal{W}_6^{\mathrm{MHV,NLO}}$ we may follows the same steps as for NMHV hexagon in previous subsection. Namely, we use integral representation for binomial coefficients, express simple fractions $1/k$, $1/(n+1)$, $1/(k+a)$, $1/(n+a+1)$ in terms of polygamma functions and introduce integral representations for the latter. Now summing geometrical series in a, n and k variables we continue with taking residues in variables entering integral representations for binomial coefficients. The left integrations in variables entering integral representations of polygamma functions are then more or less straightforwardly taken in terms of multiple polylogarithms [61,62]. The same technique should be also applicable for the resummation in the case of polygonal Wilson loops with $n > 6$, see [52,53] for tree level resummation in this case. We also think that the presented technique should be applicable to the resummation of contributions from several effective particles. However, to be on the save side here we state the algorithm for the resummation of one effective particle contributions to hexagons only, but for arbitrary loop order of weak coupling expansion. The necessary steps are given by

(1) Following [20] write down the one effective particle contribution to hexagon POPE component you are interested in and expand it in coupling constant up to required loop order. For example, Appendix B contains corresponding expression in the case of $\mathcal{W}_6^{[2,2]}$ component.
(2) Take residues in rapidity of effective particle. It is convenient to first use reflection identities (3.3) and (3.4) to isolate singular terms with known Taylor expansions. It is also useful to transform present polygamma functions to the same argument as far as possible using Eq. (3.5).
(3) Transform the obtained summand to the form of a product of binomial coefficients with simple fractions. For binomial coefficients write down integral representations as in Eq. (3.13). In the case of simple fractions express the latter in terms of polygamma functions using Eq. (3.18) and eventually write down integral representations for polygamma functions present.
(4) Sum the series present. Now, they are all of geometric progression type and could be easily summed.
(5) Take residues in variables entering integral representations for binomial coefficients.
(6) Take integrals in variables entering integral representations of polygamma functions. These are integrals from rational functions and are frequently encountered in the calculation of multiloop Feynman diagrams. In particular, they appear in the process of direct integration over Feynman parameters. When, the latter integrals satisfy criterion of linear reducibility [65,66] one can come with an algorithmic way of expressing required integrals in terms of multiple polylogarithms [61,62]. In our case there are could be also roots from quadratic polynomials present. The latter however may be rationalized with variable change, see for example [67].

IV. CONCLUSION

In this paper we presented an algorithmic approach for computing one effective particle contributions to hexagon scattering amplitudes applicable at in principle arbitrary order of perturbation theory. The approach reduces the problem of evaluation of integral over effective particle rapidity and sums over effective particle helicity and descendant number to the problem of evaluation of integrals over rational functions, otherwise known as periods, in terms of multiple polylogarithms [61,62]. If the latter integrals satisfy the criterion of linear reducibility [65,66], then there is an algorithmic way for taking such integrals. In the problem at hand, the integrals may also contain roots of quadratic polynomials. The latter however could be also treated in algorithmic way [67]. At the same time it should be noted that there are several different ways to treat mentioned integrals over rational functions at the end. In the present paper we used direct integration The other way is for example to use first integration by parts to reduce these integrals to so called master integrals. The latter could be integrated using a bunch of available techniques such as Mellin-Barnes representation, direct integration, or differential equations. So, what we actually did is turned the problem of evaluating POPE series into the problem of evaluating integrals over rational functions in terms of multiple polylogarithms, for which we have much more experience. The presented approach has also the potential for the generalization both for higher point scattering amplitudes and contributions with more then one effective particle and will be the subject of one of our future papers.

ACKNOWLEDGMENTS

This work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”.

APPENDIX A: THE REMAINDER FOR NMHV$_6$ SUPERAMPLITUDE

The $\mathcal{R}_n^{(k)}$ remainder function is defined to all orders of perturbation theory as the ratio of N^{k}MHV$_n$ and MHV$_n$ amplitudes:

$$\mathcal{R}_n^{(k)} = \frac{A_n^{(k)}}{A_n^{(0)}}. \tag{A1}$$

In NMHV case $k = 1$. From now on we will drop the (k) superscript and stick with NMHV$_6$ case only. Note, that due to universal (independent from particles helicities) structure of IR divergences the remainder function is IR finite. In addition, it is also dual conformal invariant.

Using momentum twistors $\mathcal{Z}_i = (\lambda_i, \mu_i, \eta_i)$ [68] and splitting the $\mathcal{R}_6$ remainder function into *even* and *odd* parts we have [63,64]:

$$\mathcal{R}_6 = \mathcal{R}_6^{\text{even}} + \mathcal{R}_6^{\text{odd}}, \tag{A2}$$

where

$$\mathcal{R}_6^{\text{even}} = \frac{[13456]+[12346]}{2} V(u_1,u_2,u_3) + \frac{[12456]+[12345]}{2} V(u_2,u_3,u_1) + \frac{[23456]+[12356]}{2} V(u_3,u_1,u_2), \tag{A3}$$

and[17]

$$\mathcal{R}_6^{\text{odd}} = ([12346]-[13456])\tilde{V}(u_1,u_2,u_3) + ([12456] - [12345])\tilde{V}(u_2,u_3,u_1) + ([23456] - [12356])\tilde{V}(u_3,u_1,u_2). \tag{A4}$$

V and $\tilde{V}$ are scalar functions, which depend only on (dual) conformal cross ratios and coupling constant g. $[abcde]$ is dual conformal invariant (five-bracket) defined as

$$[ijklm] = \frac{\delta^4(\langle ijkl\rangle\eta_m + \text{cyclic permutation})}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle} \tag{A5}$$

with four-brackets $\langle ijkl\rangle$ being defined through bosonic components of momentum twistors $Z_i = (\lambda_i, \mu_i)$ as

$$\langle ijkl\rangle = \varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D = \det(Z_i Z_j Z_k Z_p). \tag{A6}$$

The expansion of functions V and $\tilde{V}$ in coupling constant reads

$$V(u_1,u_2,u_3) = 1 + \sum_{l=1} (2g^2)^l V^{(l)}(u_1,u_2,u_3) \tag{A7}$$

$$\tilde{V}(u_1,u_2,u_3) = \sum_{l=1} (2g^2)^l \tilde{V}^{(l)}(u_1,u_2,u_3). \tag{A8}$$

All information about helicity content of the remainder function is contained in $[abcde]$ rational functions, which are all loop exact. The coupling constant dependence is through V and $\tilde{V}$ functions only. Note also, that due to the six term identity

$$[23456]-[13456]+[12456]-[12356]+[12346] - [12345] = 0 \tag{A9}$$

at leading order we have

$$\mathcal{R}_6^{\text{LO}} = [12345]+[12356]+[13456], \tag{A10}$$

which is $[1,2\rangle$ BCFW representation of normalized tree level six point amplitude.

Dual conformal cross ratios for six point functions can be conveniently written in terms of dual variables[18] as

$$u_1 \equiv v = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 \equiv w = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \qquad u_3 \equiv u = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2}. \tag{A11}$$

Using the relation $x_{jk}^2 = \frac{\langle j-1,j,k-1,k\rangle}{\langle j-1,j\rangle\langle k-1,k\rangle}$ the latter could be also written in terms of four-brackets

$$u = \frac{\langle 1236\rangle\langle 3456\rangle}{\langle 2356\rangle\langle 1346\rangle}, \qquad v = \frac{\langle 1234\rangle\langle 1456\rangle}{\langle 1245\rangle\langle 1346\rangle}, \qquad w = \frac{\langle 1256\rangle\langle 2345\rangle}{\langle 1245\rangle\langle 2356\rangle}. \tag{A12}$$

[17] It is convenient to define different set of arguments for $\tilde{V}$, which however can be expressed through u_1, u_2, u_3 [64]. Since we are actually will be interested only in V function we will not write them here.

[18] $x_{ij}^2 = (\sum_{k=i}^{j-1} p_k)^2$ with p_i standing for momentum of ith particle and sum being understood in cyclic sense.

At next-to-leading order $V^{(l)}$ function is given by:

$$V^{(1)}(u_1,u_2,u_3)=\frac{1}{2}\left(\sum_{i=1}^{3}\mathrm{Li}_2(u_i)+(\log(u_1)+\log(u_3))\log(u_2)-\log(u_1)\log(u_3)-\frac{\pi^2}{3}\right), \quad (A13)$$

while $\tilde{V}^{(1)}=0$, i.e., there is no contribution to $\mathcal{R}_6^{(1)\mathrm{odd}}$ at NLO.

As an illustration of our summation method we have chosen a particular component $\mathcal{R}_6^{\langle 1144\rangle}$ of $\mathcal{R}_6$ remainder function proportional to $\eta_1\eta_1\eta_4\eta_4$ Grassmann monomial. At NLO it is given by

$$\begin{aligned}2\mathcal{R}_6^{\langle 1144\rangle,\mathrm{NLO}}&=2\mathcal{R}_6^{\mathrm{even,NLO}}|_{\eta_1\eta_1\eta_4\eta_4}\\&=g^2([13456]+[12346])|_{\eta_1\eta_1\eta_4\eta_4}V^{(1)}(v,w,u)\\&\quad+g^2([12456]+[12345])|_{\eta_1\eta_1\eta_4\eta_4}V^{(1)}(w,u,v).\end{aligned} \quad (A14)$$

The coefficients in front of $V^{(1)}(v,w,u)$ and $V^{(1)}(w,u,v)$ are then given by:

$$([13456]+[12346])|_{\eta_1\eta_1\eta_4\eta_4}=\frac{\langle 1356\rangle\langle 3456\rangle}{\langle 1345\rangle\langle 1456\rangle\langle 1346\rangle}+\frac{\langle 1236\rangle\langle 2346\rangle}{\langle 1234\rangle\langle 1346\rangle\langle 1246\rangle}, \quad (A15)$$

$$([12456]+[12345])|_{\eta_1\eta_1\eta_4\eta_4}=\frac{\langle 1256\rangle\langle 2456\rangle}{\langle 1245\rangle\langle 1456\rangle\langle 1246\rangle}+\frac{\langle 1235\rangle\langle 2345\rangle}{\langle 1234\rangle\langle 1345\rangle\langle 1245\rangle}. \quad (A16)$$

At LO this leads to

$$\mathcal{R}_6^{\mathrm{LO}}|_{\eta_1\eta_1\eta_4\eta_4}=\frac{\langle 2345\rangle\langle 1235\rangle}{\langle 1234\rangle\langle 1345\rangle\langle 1245\rangle}+\frac{\langle 3456\rangle\langle 1356\rangle}{\langle 1345\rangle\langle 1456\rangle\langle 1346\rangle}. \quad (A17)$$

In collinear OPE approach the kinematics for six point amplitude is parametrized by three real parameters: τ,σ,ϕ. Dual conformal cross ratios u, v, w as well as all $\langle abcd\rangle$ invariants are then expressed via these parameters using explicit parametrization of hexagon momentum twistors (here we use notation from the main text $x=e^{-\tau}, y=e^{\sigma}, z=e^{i\phi}$):

$$\begin{pmatrix}Z_1\\Z_2\\Z_3\\Z_4\\Z_5\\Z_6\end{pmatrix}=\begin{pmatrix}yz^{-1/2}&0&z^{1/2}x^{-1}&xz^{1/2}\\1&0&0&0\\-1&0&0&1\\0&1&-1&1\\0&1&0&0\\0&y^{-1}z^{-1/2}&x^{-1}z^{1/2}&0\end{pmatrix} \quad (A18)$$

For example for dual conformal cross ratios we get:

$$u=\frac{z}{xy+(1+x^2+y^2)z+xyz^2} \quad (A19)$$

$$v=\frac{y^2z}{(1+x^2)(xy+(1+x^2+y^2)z+xyz^2)} \quad (A20)$$

$$w=\frac{x^2}{1+x^2} \quad (A21)$$

while the coefficients in front of $V^{(1)}(v,w,u)$ and $V^{(1)}(w,u,v)$ functions take the form

$$\begin{aligned}([13456]+[12346])|_{\eta_1\eta_1\eta_4\eta_4}&=\frac{x^3z(x+yz)}{(-xz-x^3z-x^2yz^2)(x^2y+xz+x^3z+xy^2z+x^2yz^2)}\\&\quad+\frac{-xz(xy^2z+x^2yz^2)}{y(xy+z+x^2z)(x^2y+xz+x^3z+xy^2z+x^2yz^2)}\\([12456]+[12345])|_{\eta_1\eta_1\eta_4\eta_4}&=\frac{x^2z}{(-z-x^2z)(xy+z+x^2z)}+\frac{-x^3z^3}{(-z-x^2z)(-xz-x^3z-x^2yz^2)}.\end{aligned} \quad (A22)$$

In this parametrization the limit $x\to 0$ (large τ) describes regime when momenta p_1 and p_6 are becoming collinear.

The LO contribution to the remainder function in terms of collinear OPE variables reads:

$$\mathcal{R}_6^{\mathrm{LO}}|_{\eta_1\eta_1\eta_4\eta_4}=\frac{x}{y}\left(\frac{z}{z+(y+xz)(x+yz)}-\frac{1}{1+x^2}\right). \quad (A23)$$

Within collinear OPE approach one actually computes not the reminder function $\mathcal{R}_6$ itself, but another finite function $\mathcal{W}_6$ of the same dual conformal invariants, which is related to $\mathcal{R}_6$ as

$$\mathcal{R}_6=\frac{\mathcal{W}_6}{\mathcal{W}_6^{\mathrm{MHV}}}, \quad (A24)$$

where $\mathcal{W}_6^{\mathrm{MHV}}=\mathcal{R}_6^{\mathrm{MHV}}\mathcal{W}_6^{\mathrm{BDS}}$. Here $\mathcal{R}_6^{\mathrm{MHV}}$ is MHV_6 remainder function and $\mathcal{W}_6^{\mathrm{BDS}}$ is known function of cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) = 4g^2 - \frac{4\pi^2}{3}g^4 + O(g^6) \tag{A25}$$

and dual conformal invariants u_i:

$$\begin{aligned}\mathcal{W}_6^{\text{BDS}}(u_1,u_2,u_3) &= \exp\left\{\frac{\Gamma_{\text{cusp}}(g)}{4}\left(\text{Li}_2(u_2) - \text{Li}_2(1-u_1) - \text{Li}_2(1-u_3)\right.\right.\\ &\quad + \log^2(1-u_2) - \log(u_1)\log(u_3)\\ &\quad \left.\left. + \log(u_1/u_3)\log(1-u_2) + \frac{\pi^2}{6}\right)\right\}. \end{aligned}\tag{A26}$$

At NLO $\mathcal{R}_6^{\text{MHV}} = 1$ and we are left with the following relation between collinear OPE result and the NMHV remainder function:

$$\mathcal{R}_6^{\langle 1144\rangle,\text{NLO}} = \left(\frac{\mathcal{W}_6^{\langle 1144\rangle}}{\mathcal{W}_6^{\text{BDS}}(u,w,v)}\right)^{\text{NLO}}, \tag{A27}$$

where it is assumed that $\mathcal{W}_6^{\langle 1144\rangle}/\mathcal{W}_6^{\text{BDS}}$ should be expanded up to $O(g^2)$.

APPENDIX B: MEASURES, ENERGIES, AND MOMENTA

The expression for charged pentagon component $\mathcal{W}_6^{[2,2]}$ considered in the main body of the paper written in terms of a sum over effective particles contributions is given by [20]:

$$\begin{aligned}\mathcal{W}_6^{[2,2]} &= \sum_{\Phi}\int\frac{du}{2\pi}e^{E_\Phi(u)\tau + ip_\Phi(u)\sigma + im_\Phi\phi}\mu_\Phi^{[2,2]}(u)\\ &= \sum_{n=0}^{\infty}\sum_{a=-\infty}^{\infty}\int\frac{du}{2\pi}e^{-E_{a,n}^{\text{eff}}(u)\tau + ip_{a,n}^{\text{eff}}(u)\sigma + ia\phi}\mu_{a,n}^{[2,2],\text{eff}}(u),\end{aligned}\tag{B1}$$

where energies and momenta of effective particles have the form

$$\begin{aligned}E_{a,n}^{\text{eff}}(u) &= 2n + 1 + |a| + 4g(\mathbb{QM}\cdot\kappa_{a,n}^{\text{eff}})_1,\\ p_{a,n}^{\text{eff}}(u) &= 2u - 4g(\mathbb{QM}\cdot\tilde{\kappa}_{a,n}^{\text{eff}})_1\end{aligned}\tag{B2}$$

Here, infinite matrices $\mathbb{Q}$ and $\mathbb{M}$ are given by [15]:

$$\begin{aligned}\mathbb{Q}_{ij} &= \delta_{ij}(-1)^{i+1}i, \mathbb{M} = [\mathbb{I} + \mathbb{K}]^{-1},\\ \mathbb{K}_{ij} &= 2j(-1)^{j(i+1)}\int_0^\infty\frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t - 1}.\end{aligned}\tag{B3}$$

Up to NLO we have

$$\mathbb{QM} = \begin{pmatrix} 1 - \frac{g^2\pi^2}{3} & -4g^3\zeta(3)\\ -4g^3\zeta(3) & -2 + \frac{2g^4\pi^4}{15}\end{pmatrix} + O(g^4), \tag{B4}$$

The infinite vectors $\kappa_{a,n}^{\text{eff}}$ and $\tilde{\kappa}_{a,n}^{\text{eff}}$ are build from Bethe string describing effective particle transforming in vector representation of $SU(4)$ and labeled by helicity a and descendant number n. This way we get [20]:

$$\begin{aligned}\kappa_{a,n}^{\text{eff}} &= k_a(u) + \sum_{j=1}^{n+2}\kappa_{\psi_S}\left(u - i\left(\frac{|a|-3}{2} + j\right)\right)\\ &\quad + \sum_{j=1}^{n}\kappa_{\psi_S}\left(u - i\left(\frac{|a|+1}{2} + j\right)\right),\end{aligned}\tag{B5}$$

$$\begin{aligned}\tilde{\kappa}_{a,n}^{\text{eff}} &= \tilde{k}_a(u) + \sum_{j=1}^{n+2}\tilde{\kappa}_{\psi_S}\left(u - i\left(\frac{|a|-3}{2} + j\right)\right)\\ &\quad + \sum_{j=1}^{n}\tilde{\kappa}_{\psi_S}\left(u - i\left(\frac{|a|+1}{2} + j\right)\right),\end{aligned}\tag{B6}$$

where [15]:

$$\begin{aligned}\kappa_a(u) &\equiv (\kappa_{a,1}(u), \kappa_{a,2}(u), \ldots),\\ \kappa_{\psi_S}(u) &\equiv (\kappa_{\psi_S,1}, \kappa_{\psi_S,2}, \ldots),\\ \tilde{\kappa}_a(u) &\equiv (\tilde{\kappa}_{a,1}(u), \tilde{\kappa}_{a,2}(u), \ldots),\\ \tilde{\kappa}_{\psi_S}(u) &\equiv (\tilde{\kappa}_{\psi_S,1}, \tilde{\kappa}_{\psi_S,2}, \ldots)\end{aligned}\tag{B7}$$

with ($J_j(z)$ are Bessel functions)

$$\kappa_{a,j}(u) = \int_0^\infty\frac{dt}{t(e^t - 1)}J_j(2gt)(J_0(2gt) - \cos(ut)e^{f_t(j,a)}), \tag{B8}$$

$$\tilde{\kappa}_{a,j}(u) = (-1)^{j+1}\int_0^\infty\frac{dt}{t(e^t - 1)}J_j(2gt)\sin(ut)e^{f_t(j+1,a)}, \tag{B9}$$

$$\begin{aligned}\kappa_{\psi_S,j}(u) &= \frac{(-1)^{j/2}(1 + (-1)^j)}{4j}\left(\frac{g}{x(u)}\right)^j,\\ \tilde{\kappa}_{\psi_S,j} &= \frac{(-1)^{\frac{j+1}{2}}(1 - (-1)^j)}{4j}\left(\frac{g}{x(u)}\right)^j.\end{aligned}\tag{B10}$$

Here $x(u)$ is Zhukovsky variable $x(u) = \frac{1}{2}(u + \sqrt{u^2 - 4g^2})$ and $f_t(j,a) = t(1 - \frac{|a| - (-1)^j}{2})$

The measures for effective particles are also built on the basis of their Bethe string representations and are given by [20]:

$$\mu_{a,n}^{[2,2],\mathrm{eff}} = g^{-1}\frac{M_{a,n}(u)}{f_{a,0}(u)f_{a,0}(-u)}\exp_{a,n}^{\mathrm{eff}}(u), \tag{B11}$$

where

$$\exp_{a,n}^{\mathrm{eff}}(u) = \exp[-2(\kappa_{a,n}^{\mathrm{eff}})^t\cdot\mathbb{QM}\cdot\kappa_{a,n}^{\mathrm{eff}} + 2(\tilde{\kappa}_{a,n}^{\mathrm{eff}})^t\cdot\mathbb{QM}\cdot\tilde{\kappa}_{a,n}^{\mathrm{eff}}] \tag{B12}$$

$$\log(f_{a,0}(u)) = \int_0^\infty \frac{dt}{t(e^t-1)}(J_0(2gt)-1) \times \left[\frac{1}{2}J_0(2gt)+\frac{1}{2}-e^{\frac{(1-|a|)t}{2}-iut}\right] \tag{B13}$$

and ($x^{[a]} = x(u-ia/2)$)

$$M_{a,n}(u) = \frac{M_{a,0}(u)}{\Gamma(n+1)\Gamma(|a|+n+1)}\prod_{l=1}^{n}(x^{[2l+|a|+1]})^2 \tag{B14}$$

$$M_{a,0}(u) = g(-1)^a\Gamma\left(iu+\frac{|a|+1}{2}\right)\Gamma\left(-iu+\frac{|a|+1}{2}\right) \times \frac{x^{[1+|a|]}}{x^{[1-|a|]}}\frac{x^{[1-|a|]}x^{[1+|a|]}-g^2}{\sqrt{(x^{[1-|a|]})^2-g^2}\sqrt{(x^{[1+|a|]})^2-g^2}} \tag{B15}$$

APPENDIX C: SIMPLIFYING $\mathcal{W}_6^{\langle 1144\rangle}$ WITH SYMBOLS

To compare the result of pentagon OPE resummation[19] for $\mathcal{W}_6^{\langle 1144\rangle}$ with the known results from generalized unitarity and bootstrap [63,64] we need to simplify our expression. The most convenient way to do it to use symbols [69–71], in particular the *Mathematica* package PolyLogTools [72]. In fact, we only need the following two symbols:

$$\mathrm{Li}_2(z) \to -(1-z)\otimes z, \tag{C1}$$

$$\log(x)\log(y) \to x\otimes y + y\otimes x. \tag{C2}$$

[19]It can be found in accompanying *Mathematica* notebook.

Note, that symbol mapping is blind to constants[20] and satisfy the relations

$$a_1\otimes\ldots\otimes a_ia_j\otimes\ldots\otimes a_n = a_1\otimes\ldots\otimes a_i\otimes\ldots\otimes a_n + a_1\otimes\ldots\otimes a_j\otimes\ldots\otimes a_n \tag{C3}$$

$$a_1\otimes\ldots\otimes a_i^n\otimes\ldots\otimes a_n = n(a_1\otimes\ldots\otimes a_i\otimes\ldots\otimes a_n). \tag{C4}$$

To simplify consideration we will consider the simplification of the difference of the our resulting expression with [63,64]. In the case of 1 loop NMHV_6 amplitude contrary to the case of 1 loop MHV amplitude [51] the resulting expressions contain rational factors in front of dilogarithms and logarithms. The latter after partial fraction in x variable are given by

$$p_1=\frac{x}{(1+x^2)y},\quad p_2=\frac{xz}{y(xy+z+zx^2)},$$
$$p_3=\frac{x}{y(1+x^2+xyz)},\quad p_4=\frac{xz}{y(xy+z+x^2z)(z^2-1)},$$
$$p_5=\frac{xz^3}{y(xy+z+x^2z)(z^2-1)},\quad p_6=\frac{x}{y(1+x^2+xyz)(z^2-1)},$$
$$p_7=\frac{xz}{y(xy+z+x^2z+y^2z+xyz^2)}. \tag{C5}$$

The usage of symbol map with PolyLogTools package reduces to the application of just three commands `SymbolMap`, `SymbolExpand` and `SymbolFactor` together with the simplification of symbol entries with *Mathematica* command `FullSimplify`. Using symbol map for the considered difference it easy to show that coefficients in front of p_1 and p_7 rational factors are equal to zero, while the coefficient in front of p_2 equal to the coefficients in front of p_3, p_4 factors and minus coefficient in front of p_5. Taking into account found functional identities and using again partial fractioning in x variable it is easy to see that the coefficient in front of p_6 in the expression for $\mathcal{W}_6^{\langle 1144\rangle}$ also cancels. This finishes the proof of equivalence of our and [63,64] results.

[20]The constants could be fixed by comparing expressions at some fixed kinematical point.

[1] N. Beisert *et al.*, Review of AdS/CFT integrability: An overview, Lett. Math. Phys. **99**, 3 (2012).

[2] D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, S. Negro, I. M. Szécsényi, A. Sfondrini, S. J. van Tongeren, and A. Torrielli, An integrability primer for the gauge-gravity correspondence: An introduction, J. Phys. A **49**, 320301 (2016).

[3] L. F. Alday and J. M. Maldacena, Gluon scattering amplitudes at strong coupling, J. High Energy Phys. 06 (2007) 064.

[4] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, Conformal properties of four-gluon planar amplitudes and Wilson loops, Nucl. Phys. **B795**, 385 (2008).
[5] A. Brandhuber, P. Heslop, and G. Travaglini, MHV amplitudes in $N = 4$ super Yang-Mills and Wilson loops, Nucl. Phys. **B794**, 231 (2008).
[6] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, The two-loop six-Gluon MHV amplitude in maximally supersymmetric Yang-Mills theory, Phys. Rev. D **78**, 045007 (2008).
[7] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Hexagon Wilson loop = six-gluon MHV amplitude, Nucl. Phys. **B815**, 142 (2009).
[8] N. Berkovits and J. Maldacena, Fermionic T-duality, dual superconformal symmetry, and the amplitude/Wilson loop connection, J. High Energy Phys. 09 (2008) 062.
[9] L. F. Alday, D. Gaiotto, and J. Maldacena, Thermodynamic Bubble Ansatz, J. High Energy Phys. 09 (2011) 032.
[10] L. F. Alday, J. Maldacena, A. Sever, and P. Vieira, Y-system for scattering amplitudes, J. Phys. A **43**, 485401 (2010).
[11] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, An operator product expansion for polygonal null Wilson loops, J. High Energy Phys. 04 (2011) 088.
[12] D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, Pulling the straps of polygons, J. High Energy Phys. 12 (2011) 011.
[13] B. Basso, A. Sever, and P. Vieira, Spacetime and Flux Tube S-Matrices at Finite Coupling for $N = 4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **111**, 091602 (2013).
[14] B. Basso, A. Sever, and P. Vieira, Space-time S-matrix and flux tube S-matrix II. Extracting and matching data, J. High Energy Phys. 01 (2014) 008.
[15] B. Basso, A. Sever, and P. Vieira, Space-time S-matrix and flux-tube S-matrix III. The two-particle contributions, J. High Energy Phys. 08 (2014) 085.
[16] B. Basso, A. Sever, and P. Vieira, Space-time S-matrix and Flux-tube S-matrix IV. Gluons and fusion, J. High Energy Phys. 09 (2014) 149.
[17] B. Basso, J. Caetano, L. Cordova, A. Sever, and P. Vieira, OPE for all helicity amplitudes, J. High Energy Phys. 08 (2015) 018.
[18] B. Basso, J. Caetano, L. Cordova, A. Sever, and P. Vieira, OPE for all helicity amplitudes II. Form factors and data analysis, J. High Energy Phys. 12 (2015) 088.
[19] B. Basso, A. Sever, and P. Vieira, Spacetime and Flux Tube S-Matrices at Finite Coupling for $N = 4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **111**, 091602 (2013).
[20] L. Cordova, Hexagon POPE: Effective particles and tree level resummation, J. High Energy Phys. 01 (2017) 051.
[21] A. V. Belitsky, Nonsinglet pentagons and NMHV amplitudes, Nucl. Phys. **B896**, 493 (2015).
[22] A. V. Belitsky, Fermionic pentagons and NMHV hexagon, Nucl. Phys. **B894**, 108 (2015).
[23] A. V. Belitsky, On factorization of multiparticle pentagons, Nucl. Phys. **B897**, 346 (2015).
[24] D. Fioravanti, S. Piscaglia, and M. Rossi, Asymptotic Bethe Ansatz on the GKP vacuum as a defect spin chain: scattering, particles and minimal area Wilson loops, Nucl. Phys. **B898**, 301 (2015).
[25] A. Bonini, D. Fioravanti, S. Piscaglia, and M. Rossi, Strong Wilson polygons from the lodge of free and bound mesons, J. High Energy Phys. 04 (2016) 029.
[26] A. Bonini, D. Fioravanti, S. Piscaglia, and M. Rossi, The contribution of scalars to $\mathcal{N} = 4$ SYM amplitudes, Phys. Rev. D **95**, 041902 (2017).
[27] A. Bonini, D. Fioravanti, S. Piscaglia, and M. Rossi, The contribution of scalars to $\mathcal{N} = 4$ SYM amplitudes II: Young tableaux, asymptotic factorisation and strong coupling, Nucl. Phys. **B931**, 19 (2018).
[28] A. Bonini, D. Fioravanti, S. Piscaglia, and M. Rossi, Fermions and scalars in $\mathcal{N} = 4$ Wilson loops at strong coupling and beyond, Nucl. Phys. **944**, 114644 (2019).
[29] S. Komatsu, Lectures on three-point functions in $N = 4$ supersymmetric Yang-Mills theory, arXiv:1710.03853.
[30] B. Basso, S. Komatsu, and P. Vieira, Structure constants and integrable bootstrap in planar $N = 4$ SYM theory, arXiv: 1505.06745.
[31] B. Basso, V. Goncalves, and S. Komatsu, Structure constants at wrapping order, J. High Energy Phys. 05 (2017) 124.
[32] Y. Jiang, S. Komatsu, I. Kostov, and D. Serban, Clustering and the three-point function, J. Phys. A **49**, 454003 (2016).
[33] B. Eden and A. Sfondrini, Tessellating cushions: Four-point functions in $\mathcal{N} = 4$ SYM, J. High Energy Phys. 10 (2017) 098.
[34] B. Basso, F. Coronado, S. Komatsu, H. T. Lam, P. Vieira, and D.-l. Zhong, Asymptotic four point functions, J. High Energy Phys. 07 (2019) 082.
[35] B. Eden, Y. Jiang, D. le Plat, and A. Sfondrini, Colour-dressed hexagon tessellations for correlation functions and non-planar corrections, J. High Energy Phys. 02 (2018) 170.
[36] T. Bargheer, J. Caetano, T. Fleury, S. Komatsu, and P. Vieira, Handling Handles: Nonplanar Integrability in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **121**, 231602 (2018).
[37] T. Fleury and S. Komatsu, Hexagonalization of correlation functions II: Two-particle contributions, J. High Energy Phys. 02 (2018) 177.
[38] S. Giombi and S. Komatsu, Exact correlators on the Wilson loop in $\mathcal{N} = 4$ SYM: Localization, defect CFT, and integrability, J. High Energy Phys. 05 (2018) 109; Erratum J. High Energy Phys. 11 (2018) 123.
[39] B. Eden, Y. Jiang, M. de Leeuw, T. Meier, D. le Plat, and A. Sfondrini, Positivity of hexagon perturbation theory, J. High Energy Phys. 11 (2018) 097.
[40] F. Coronado, Bootstrapping the Simplest Correlator in Planar $N = 4$ SYM at all Loops, Phys. Rev. Lett. **124**, 171601 (2020).
[41] I. Kostov, V. B. Petkova, and D. Serban, Determinant Formula for the Octagon Form Factor in $N = 4$ Supersymmetric Yang-Mills Theory, Phys. Rev. Lett. **122**, 231601 (2019).
[42] T. Bargheer, F. Coronado, and P. Vieira, Octagons I: Combinatorics and non-planar resummations, J. High Energy Phys. 08 (2019) 162.
[43] I. Kostov, V. B. Petkova, and D. Serban, The octagon as a determinant, J. High Energy Phys. 11 (2019) 178.

[44] Y. Jiang, S. Komatsu, and E. Vescovi, Structure constants in $\mathcal{N}=4$ SYM at finite coupling as Worldsheet g-function, arXiv:1906.07733.

[45] Y. Jiang, S. Komatsu, and E. Vescovi, Exact three-point functions of determinant operators in planar $N=4$ supersymmetric Yang-Mills theory, Phys. Rev. Lett. **123**, 191601 (2019).

[46] B. Basso and D.-L. Zhong, Three-point functions at strong coupling in the BMN limit, J. High Energy Phys. 04 (2020) 076.

[47] T. Bargheer, F. Coronado, and P. Vieira, Octagons II: Strong coupling, arXiv:1909.04077.

[48] G. Papathanasiou, Hexagon Wilson loop OPE and harmonic polylogarithms, J. High Energy Phys. 11 (2013) 150.

[49] J. M. Drummond and G. Papathanasiou, Hexagon OPE resummation and multi-Regge kinematics, J. High Energy Phys. 02 (2016) 185.

[50] M. de Leeuw, B. Eden, D. l. Plat, and T. Meier, Polylogarithms from the bound state S-matrix, arXiv:1907.07014.

[51] H. T. Lam and M. von Hippel, Resumming the POPE at one loop, J. High Energy Phys. 12 (2016) 011.

[52] A. V. Belitsky, Resummed tree heptagon, Nucl. Phys. **B929**, 113 (2018).

[53] A. V. Belitsky, Multichannel conformal blocks for scattering amplitudes, Phys. Lett. B **780**, 66 (2018).

[54] L. J. Mason and D. Skinner, The complete planar S-matrix of $N=4$ SYM as a Wilson loop in twistor space, J. High Energy Phys. 12 (2010) 018.

[55] S. Caron-Huot, Notes on the scattering amplitude/Wilson loop duality, J. High Energy Phys. 07 (2011) 058.

[56] B. Basso, Exciting the GKP string at any coupling, Nucl. Phys. **B857**, 254 (2012).

[57] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, A semiclassical limit of the gauge/string correspondence, Nucl. Phys. **B636**, 99 (2002).

[58] B. Basso, A. Sever, and P. Vieira, Hexagonal Wilson loops in planar $\mathcal{N}=4$ SYM theory at finite coupling, J. Phys. A **49**, 41LT01 (2016).

[59] L. F. Alday and J. M. Maldacena, Comments on operators with large spin, J. High Energy Phys. 11 (2007) 019.

[60] D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, Bootstrapping null polygon Wilson loops, J. High Energy Phys. 03 (2011) 092.

[61] A. B. Goncharov, Multiple Polylogarithms, cyclotomy and modular complexes, Math. Res. Lett. **5**, 497 (1998).

[62] E. Remiddi and J. A. M. Vermaseren, Harmonic polylogarithms, Int. J. Mod. Phys. A **15**, 725 (2000).

[63] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Generalized unitarity for $N=4$ superamplitudes, Nucl. Phys. **B869**, 452 (2013).

[64] L. J. Dixon and M. von Hippel, Bootstrapping an NMHV amplitude through three loops, J. High Energy Phys. 10 (2014) 065.

[65] F. Brown, The massless higher-loop two-point function, Commun. Math. Phys. **287**, 925 (2009).

[66] E. Panzer, Feynman integrals and hyperlogarithms, Ph.D. Thesis, Humboldt University, 2015.

[67] M. Besier, D. Van Straten, and S. Weinzierl, Rationalizing roots: An algorithmic approach, Commun. Numer. Theor. Phys. **13**, 253 (2019).

[68] A. Hodges, Eliminating spurious poles from gauge-theoretic amplitudes, J. High Energy Phys. 05 (2013) 135.

[69] K.-T. Chen, Iterated path integrals, Bull. Am. Math. Soc. **83**, 831 (1977).

[70] F. C. S. Brown, Multiple zeta values and periods of moduli spaces $\mathfrak{m}_{0,n}$, Ann. Sci. Ecole Norm. Sup. **42**, 371 (2009).

[71] A. B. Goncharov, A simple construction of Grassmannian polylogarithms, arXiv:0908.2238.

[72] C. Duhr and F. Dulat, PolyLogTools—Polylogs for the masses, J. High Energy Phys. 08 (2019) 135.