Some nontrivial aspects of Poincaré and *CPT* invariance of flavor vacuum

M. Blasone,[1,*] P. Jizba,[2,3,†] N. E. Mavromatos,[4,‡] and L. Smaldone[5,§]

[1]*Dipartimento di Fisica, Università di Salerno, Via Giovanni Paolo II, 132 84084 Fisciano, Italy & INFN Sezione di Napoli, Gruppo collegato di Salerno, Italy*

[2]*FNSPE, Czech Technical University in Prague, Břehová 7, 115 19 Praha 1, Czech Republic*

[3]*ITP, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany*

[4]*Theoretical Particle Physics and Cosmology Group, Department of Physics, King's College London, Strand WC2R 2LS, United Kingdom*

[5]*Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 18000 Praha 8, Czech Republic*

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We study the explicit form of Poincaré and discrete transformations of flavor states in a two-flavor scalar model, which represents the simplest example of the field mixing. Because of the particular form of the flavor vacuum condensate, we find that the aforementioned symmetries are spontaneously broken. The ensuing vacuum stability group is identified with the Euclidean group $E(3)$. With the help of Fabri–Picasso theorem, we show that flavor vacua with different time labels and in different Lorentz frames are unitarily inequivalent to each other and they constitute a manifold of zero-flavor-charge states. Despite the spontaneous breakdown of Poincaré and *CPT* symmetries that characterizes such vacua, we provide arguments on the absence of Goldstone Bosons. We also prove that the phenomenologically relevant oscillation formula is invariant under these transformations. In particular we prove that flavor oscillation formula on flavor vacuum has the same form in all Lorentz frames, by means of general arguments, valid at all energy scales.

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I. INTRODUCTION

The fundamental particles are usually classified, following Bargmann and Wigner [1], in terms of unitary irreducible representations of Poincaré group [2,3]. According to this classification, particles and ensuing vacuum states are characterized by their mass m and spin s (or helicity, in the case of massless particles). In the case of particles without a sharp value of mass (e.g., unstable particles), such a classification can be regarded, at best, as an approximation [3]. In such cases the concept of sharp mass is substituted with a finite mass-width distribution. Ensuing variance is proportional to the inverse of particle half-life due to time-energy uncertainty relation [4,5]. This picture can also be explained in terms of a nontrivial vacuum structure possessed by such systems [6].

In the context of quantum field theories (QFT) with mixing, it was proposed in [7] that the *physically relevant* Fock space is the one of states with *definite flavor* rather than the one for mass eigenstates. These two spaces are *unitarily inequivalent* to each other, and the *flavor vacuum* is structurally similar to a BCS condensate. Such a proposal, if realized in nature, will have important phenomenological consequences, which have been examined over the years since its development [8], and pertain to both, particle physics, specifically particle oscillations, in particular neutrinos [9–11], and cosmology [12], where it was argued that the flavor vacuum condensate structure can contribute to the dark energy of the Universe. It is worthy of stressing that, as argued in [13], there are significant differences between the respective neutrino oscillation formulas in the flavor and mass eigenstate formalisms that can lead to observable effects in a nonrelativistic energy regime.

An important aspect of the flavor vacuum is its nonstandard Lorentz invariance properties. Indeed, within this approach, preliminary study of the QFT oscillation formula in different Lorentz frames was undertaken [14] and Lorentz violating effects, observable in principle, were identified. We remark at this stage that Lorentz invariance properties of the neutrino oscillation formula were studied in a number of papers, e.g., [14–16]. In Ref. [15], the

*blasone@sa.infn.it
†p.jizba@fjfi.cvut.cz
‡Nikolaos.Mavromatos@kcl.ac.uk
§smaldone@ipnp.mff.cuni.cz

invariance of the standard oscillation formula [17], was explicitly proved but only in the ultrarelativistic case. However, it can be argued [7,8,18–20] that such a formula should be regarded as a quantum-mechanical (QM) approximation of a more fundamental QFT relation. In particular QM behavior cannot grasp the nontrivial vacuum structure whose observable effects are more important at lower energies. It is this point which was deeply analyzed within the framework of the *flavor vacuum* [7], where in principle observable Lorentz-violating effects have been identified, as mentioned above. We remark in passing that low-energy neutrino phenomenology is becoming increasingly important in understanding cosmic neutrino background (CNB) and its potential cosmological implications [21,22], hence in this respect the potential cosmological effects of the flavor vacuum are quite relevant.

Within this QFT treatment of neutrino oscillations, associated deformations of the Lorentz energy dispersion relations were studied in Ref. [16]. Such a modification of the dispersion relations for the flavor states can be understood as an "environmental" effect of quantum-gravitational degrees of freedom in a concrete model of quantum gravity within the framework of string/brane theory, the so-called D-foam model [23]. In this context, the scattering between open strings, representing flavored matter, and $D0$-branes, which are viewed as Poincaré-symmetry-breaking point-like space-time defects, of quantum-gravitational stringy nature, is modeled by an effective theory, which entails the dynamical generation of mixing via flavor vacuum condensates [24,25], in agreement with the generic feature of the flavor vacuum [7], mentioned previously. Such a vacuum structure can also be obtained via algebraic, i.e., nonperturbative methods, in the case of two [26] and three flavor [27] models with $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R \times U(1)_V$ chiral flavor symmetry. This nonperturbative analysis shows that the structure of the flavor vacuum emerges *necessarily* in that context.

In this connection we can also point out that Lorentz violating effects implied by a fundamental string theoretical dynamics characterize also the Standard Model extension (SME) proposed by Colladay and Kostelecky [28]. In the SME, CPT and Lorentz violating terms are explicitly added to the Standard Model (SM) Lagrangian. At this stage, we should also like to recall the work of [29], according to which a violation of CPT *necessarily* implies the violation of Lorentz invariance. Within such a framework the neutrino oscillations were studied in [30] and modified dispersion relations connected with an underlying Planck scale physics were found. Following these developments, many authors dedicated their efforts to an understanding of both the theoretical and the phenomenological implications of SME or SME-like scenarios (see, e.g., [31–33]). It can also be argued [34] that bounds on the parameters of SME can be fixed through generalized uncertainty principle [35].

It was recently pointed out [36] that flavor neutrino states share a common feature with unstable particles, in the sense that only their energy (mass) distribution has a physical meaning and the width of this distribution is related to the inverse of the oscillation length which can be again deduced from time-energy uncertainty relation [36,37]. Furthermore, the latter result was recently generalized, in a quantum mechanical context, to stationary curved space-times [38].

It is worthwhile therefore to clarify the relation between flavor states and unitary representations of Poincaré group. This point was first tackled in Ref. [39], where it was proposed to extend the Poincaré group so as to include an internal SU(3) flavor symmetry in the Standard Model. Because of *Coleman–Mandula theorem* [40], the extended group can only be $T^{3,1} \rtimes O(3,1) \times \mathrm{SU}(3)$.

The aim of this paper is to study Poincaré and discrete symmetries in a simple toy model that describes oscillations of a two flavor (A, B) scalar field doublet with mixing [41,42]. In this context we propose yet another solution to the apparent incompatibility of Poincaré symmetry on flavor states, namely that the Poincaré symmetry is spontaneously broken on *flavor vacuum* [7,8,36,41,42]. So, in particular, the Lagrangian symmetry does not leave vacuum invariant and the residual symmetry is found to be $E(3)$. This spontaneous symmetry breakdown (SSB) is caused by the complicated condensate structure of the flavor vacuum. Here we do not specify the origin of this condensate, which can be motivated by physics beyond SM as is done, e.g., in Refs. [24–26]. This would, in turn, indicate the necessity for a dynamical origin of mixing. The action of the broken charges as symmetry generators on the vacuum, defines a linear manifold of flavor-degenerate states, which represent the *flavor vacuum manifold*. All points on such a vacuum manifold represent unitarily inequivalent Fock spaces. With the same reasoning we prove that CPT symmetry is also spontaneously broken on the flavor vacuum, with the residual symmetry being CP. In view of the theorem in [29], then, the breaking of Lorentz symmetry by the flavor vacuum can be attributed to the (spontaneous) breaking of CPT symmetry in this approach.

As a *main result*, we prove, quite surprisingly, that such a violation does not affect the phenomenologically relevant flavor oscillation formula, which is demonstrated to be Poincaré invariant. In fact, here we employ a wave-packet approach for neutrino oscillations developed in Ref. [13], which permits to treat this issue in a manifestly covariant way. The same result can be derived for continuous time-translations, T and CPT transformations.

The present paper is organized as it follows: in Sec. II we discuss the incompatibility of irreducible representations of the Poincaré group on flavor states. In Sec. III the canonical quantization of flavor (scalar) fields is reviewed [41,42] and we set up convention employed in the rest of the paper. Here, unlike in Refs. [41,42], we use the invariant form of canonical commutation relations, which makes more evident eventual Lorentz violations. In Sec. IV Poincaré group generators are explicitly constructed, in the flavor representations, and SSB of time-translations and Lorentz boosts is shown. Then, in Sec. V, the same procedure is repeated for the case of discrete symmetries, showing that CPT is broken on the flavor vacuum. Finally, in Sec. VI, conclusions and future perspectives are presented. For the reader's convenience we include two Appendixes that complement more technical aspects from the main text.

II. POINCARÉ GROUP REPRESENTATIONS AND FIELD MIXING

In this section we briefly discuss the problem of constructing flavor states in connection with unitarily irreducible representations of Poincaré group. By using the commutation relations (A15)–(A17) one can verify that Poincaré group has two Casimir invariants [2,3]:

$$M^2 \equiv P_\sigma P^\sigma \qquad W^2 = W_\sigma W^\sigma, \tag{1}$$

where

$$W_\sigma = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}\ P^\rho, \tag{2}$$

is the *Pauli–Lubansky* operator.

After Bargmann and Wigner [1], particle states are usually assumed to belong to the unitary irreducible representations of the Poincaré group [2,3,43]. It follows that the two quadratic Casimir invariants act as a multiple of the identity operator, on these states:

$$\begin{aligned} M^2|k^2,s,\sigma\rangle &= m_\sigma^2|k^2,s,\sigma\rangle, \\ W^2|k^2,s,\sigma\rangle &= -m_\sigma^2 s(s+1)|k^2,s,\sigma\rangle, \end{aligned} \tag{3}$$

where σ indicates some quantum number (e.g., flavor), m_σ is the particle mass and s is its spin.[1]

Let us now consider the Lagrange density

$$\mathcal{L}(x) = \partial_\mu \boldsymbol{\varphi}_f^\dagger(x)\partial^\mu \boldsymbol{\varphi}_f(x) - \boldsymbol{\varphi}_f^\dagger(x) M^2 \boldsymbol{\varphi}_f(x), \tag{4}$$

where

$$\boldsymbol{\varphi}_f(x) = \begin{bmatrix} \varphi_A(x) \\ \varphi_B(x) \end{bmatrix}, \qquad M^2 = \begin{bmatrix} m_A^2 & m_{AB}^2 \\ m_{AB}^2 & m_B^2 \end{bmatrix}, \tag{5}$$

which describes the dynamics of two coupled (mixed) scalar fields that we will call *flavor fields*, in a close analogy with the terminology used in quark and neutrino physics. A pressing problem in the study of fundamental aspects of flavor physics is the correct definition of flavor states [7,8,19,20]. However, it is clear that these cannot be taken as elements of irreducible representations of the Poincaré group. This was already noticed, e.g., in Ref. [39]. The argument for this is very simple: if this were not true we should have[2]:

$$M^2|k_\sigma,\sigma\rangle = m_\sigma^2|k_\sigma,\sigma\rangle, \qquad \sigma = A,B. \tag{6}$$

which is clearly false, because flavor states do not have a definite mass.[3]

It thus seems that Poincaré symmetry is not compatible with flavor mixing. One possibility would be to extend the Poincaré group. For instance, in Ref. [39] it was proposed to consider $T^{3,1} \rtimes O(3,1) \times \mathrm{SU}(n)$, where n is the number of flavors involved. In sections to follow we propose and discuss yet another possibility, namely we will quantize flavor fields directly in the flavor space, where the vacuum is manifestly Poincaré noninvariant and show that the Poincaré symmetry is spontaneously broken in the symmetry breaking scheme

$$T^{3,1} \rtimes O(3,1) \to E(3). \tag{7}$$

Here $E(3)$ denotes the three dimensional Euclidean group.

The present analysis does not investigate the actual mechanism that is responsible for this SSB. A simple dynamical model where such a SSB can naturally be encountered is considered in Ref. [26]. There it is shown that a *necessary* condition for dynamical generation of fermion mixing, in models characterized by *chiral flavor symmetry*, is the vacuum condensation of fermion-antifermion pairs, which mix particles with different masses and so, *dynamical mixing generation requires a mixing at level of vacuum*. In that context, the Lorentz symmetry is spontaneously broken by the presence of such exotic condensates, via the SSB scheme:

$$\begin{aligned} &T^{3,1} \rtimes O(3,1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times U(1)_V \\ &\quad \to U(1)_V \times E(3), \end{aligned} \tag{8}$$

where L and R indicate the left and right components of the chiral group, respectively, and V is the vector group. The global $U(1)_V$ invariance is related to the conservation of total flavor charge. Here we believe that it is quite feasible that a similar mechanism drives the SSB of Poincaré symmetry also in the bosonic case. We stress that, in this case, the *physical vacuum* must be necessarily the flavor vacuum: otherwise, we cannot have dynamical

[1]Here, for simplicity, we assume the same spin for each σ.

[2]Here we do not consider the Pauli–Lubansky operator because we limit to the case of scalar (spinless) fields.

[3]Strictly speaking, in QM one can construct an operator of the form (6) but such operator cannot be interpreted as a mass operator. In QFT this is impossible due to unitary inequivalence of flavor and mass representation.

mixing generation. This is a nonperturbative result, whose details can be found in Ref. [26]. This has far reaching phenomenological and theoretical consequences: corrections to standard oscillation formula [13], modifications of Casimir force [44], emergence of a natural contribution to dark energy [12]. Moreover, in the following we show that oscillation formula derived by means of flavor vacuum is Poincaré invariant, whereas in the case of standard oscillation formula this was only proved in the ultrarelativistic limit [15]. The two formulas coincide in that regime.

III. FLAVOR FIELDS QUANTIZATION

Let us now consider a simple scalar model for flavor oscillations described by the Lagrange density (4), which can be diagonalized through the following transformation:

$$\boldsymbol{\varphi}_f(x) = U\boldsymbol{\varphi}_m(x), \qquad U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad (9)$$

where $\tan 2\theta = 2m_{AB}^2/(m_B^2 - m_A^2)$. After this transformation, $\mathcal{L}$ becomes

$$\mathcal{L}(x) = \partial_\mu \boldsymbol{\varphi}_m^\dagger(x)\partial^\mu \boldsymbol{\varphi}_m(x) - \boldsymbol{\varphi}_m^\dagger(x) M_d^2 \boldsymbol{\varphi}_m(x), \quad (10)$$

where

$$\boldsymbol{\varphi}_m(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}, \qquad M_d^2 = \begin{bmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{bmatrix}. \quad (11)$$

The Lagrange density (10) describes two free scalar fields with definite particle masses m_1 and m_2. They can be thus expanded as:

$$\varphi_j(x) = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},j}(2\pi)^3}[a_{\mathbf{k},j}e^{-i\omega_{\mathbf{k},j}t} + b_{-\mathbf{k},j}^\dagger e^{i\omega_{\mathbf{k},j}t}]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad j = 1, 2, \quad (12)$$

where the annihilation and creation operators satisfy the following commutation relations:

$$[a_{\mathbf{k},i}, a_{\mathbf{p},j}^\dagger] = [b_{\mathbf{k},i}, b_{\mathbf{p},j}^\dagger] = 2\omega_{\mathbf{k},i}(2\pi)^3\delta(\mathbf{k}-\mathbf{p})\delta_{ij}, \quad (13)$$

and annihilate the *mass vacuum*:

$$a_{\mathbf{k},j}|0\rangle_{1,2} = b_{\mathbf{k},j}|0\rangle_{1,2} = 0, \quad (14)$$

i.e., the ground state of the system. Note that, in contrast to Refs. [41,42] we use the Lorentz invariant commutation relations (13). We now expand flavor fields in a similar way:

$$\varphi_\sigma(x) = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a_{\mathbf{k},\sigma}(t)e^{-i\omega_{\mathbf{k},\sigma}t} + b_{-\mathbf{k},\sigma}^\dagger(t)e^{i\omega_{\mathbf{k},\sigma}t}]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \sigma = A, B, \quad (15)$$

with $\omega_{\mathbf{k},\sigma} = \sqrt{|\mathbf{k}|^2 + \mu_\sigma^2}$ and μ_σ are mass parameters which have to be specified. From the mixing transformation (9) it follows that[4]:

$$a_{\mathbf{k},A}(t) = \int d^3\mathbf{x}\, e^{i(\omega_{\mathbf{k},A}t - \mathbf{k}\cdot\mathbf{x})} i\overleftrightarrow{\partial}_0(\cos\theta\varphi_1(x) + \sin\theta\varphi_2(x)), \quad (16)$$

and similarly for other operators. Explicitly, we find that

$$\begin{bmatrix} a_{\mathbf{k},A} \\ b_{-\mathbf{k},A}^\dagger \\ a_{\mathbf{k},B} \\ b_{-\mathbf{k},B}^\dagger \end{bmatrix} = \begin{bmatrix} c_\theta\rho_{A1}^{\mathbf{k}} & c_\theta\lambda_{A1}^{\mathbf{k}} & s_\theta\rho_{A2}^{\mathbf{k}} & s_\theta\lambda_{A2}^{\mathbf{k}} \\ c_\theta\lambda_{A1}^{\mathbf{k}*} & c_\theta\rho_{A1}^{\mathbf{k}*} & s_\theta\lambda_{A2}^{\mathbf{k}*} & s_\theta\rho_{A2}^{\mathbf{k}*} \\ -s_\theta\rho_{B1}^{\mathbf{k}} & -s_\theta\lambda_{B1}^{\mathbf{k}} & c_\theta\rho_{B2}^{\mathbf{k}} & c_\theta\lambda_{B2}^{\mathbf{k}} \\ -s_\theta\lambda_{B1}^{\mathbf{k}*} & -s_\theta\rho_{B1}^{\mathbf{k}*} & c_\theta\lambda_{B2}^{\mathbf{k}*} & c_\theta\rho_{B2}^{\mathbf{k}*} \end{bmatrix} \begin{bmatrix} a_{\mathbf{k},1} \\ b_{-\mathbf{k},1}^\dagger \\ a_{\mathbf{k},2} \\ b_{-\mathbf{k},2}^\dagger \end{bmatrix}, \quad (17)$$

where $c_\theta \equiv \cos\theta$, $s_\theta \equiv \sin\theta$, and

$$\rho_{\sigma j}^{\mathbf{k}} = |\rho_{\sigma j}^{\mathbf{k}}|e^{i(\omega_{\mathbf{k},\sigma} - \omega_{\mathbf{k},j})t}, \qquad \lambda_{\sigma j}^{\mathbf{k}} = |\lambda_{\sigma j}^{\mathbf{k}}|e^{i(\omega_{\mathbf{k},\sigma} + \omega_{\mathbf{k},j})t}, \quad (\sigma, j) = (A, 1), (B, 2), \quad (18)$$

where

$$|\rho_{\sigma j}^{\mathbf{k}}| = \frac{1}{2}\left(\frac{\omega_{\mathbf{k},\sigma}}{\omega_{\mathbf{k},j}} + 1\right), \qquad |\lambda_{\sigma j}^{\mathbf{k}}| = \frac{1}{2}\left(\frac{\omega_{\mathbf{k},\sigma}}{\omega_{\mathbf{k},j}} - 1\right). \quad (19)$$

Note that (17) represents a canonical transformation because

$$[a_{\mathbf{k},\sigma}(t), a_{\mathbf{p},\rho}^\dagger(t)] = [b_{\mathbf{k},\sigma}(t), b_{\mathbf{p},\rho}^\dagger(t)] = 2\omega_{\mathbf{k},\sigma}(2\pi)^3\delta(\mathbf{k}-\mathbf{p})\delta_{\sigma\rho}. \quad (20)$$

For future convenience, we write explicitly the inverse transformation as:

$$\boldsymbol{a}_{\mathbf{k},j} = \sum_{\sigma=A,B} J_{j\sigma}^{\mathbf{k}}(t)\boldsymbol{a}_{\mathbf{k},\sigma}(t), \quad (21)$$

where $\boldsymbol{a}_{\mathbf{k},j} = [\,a_{\mathbf{k},j} \quad b_{-\mathbf{k},j}^\dagger\,]^T$, $\boldsymbol{a}_{\mathbf{k},\sigma} = [\,a_{\mathbf{k},\sigma} \quad b_{-\mathbf{k},\sigma}^\dagger\,]^T$, and the matrix $J^{\mathbf{k}}$ has the form

[4]Here the time dependence of creation and annihilation operators indicates that flavor fields are interacting fields. Actually, this interacting model can be solved exactly, without perturbation expansion.

$$J^{\mathbf{k}}(t) \equiv \begin{bmatrix} c_\theta\rho^{\mathbf{k}}_{1A} & c_\theta\lambda^{\mathbf{k}}_{1A} & -s_\theta\rho^{\mathbf{k}}_{1B} & -s_\theta\lambda^{\mathbf{k}}_{1B} \\ c_\theta\lambda^{\mathbf{k}*}_{1A} & c_\theta\rho^{\mathbf{k}*}_{1A} & -s_\theta\lambda^{\mathbf{k}*}_{1B} & -s_\theta\rho^{\mathbf{k}*}_{1B} \\ s_\theta\rho^{\mathbf{k}}_{2A} & s_\theta\lambda^{\mathbf{k}}_{2A} & c_\theta\rho^{\mathbf{k}}_{2B} & c_\theta\lambda^{\mathbf{k}}_{2B} \\ s_\theta\lambda^{\mathbf{k}*}_{2A} & s_\theta\rho^{\mathbf{k}*}_{2A} & c_\theta\lambda^{\mathbf{k}*}_{2B} & c_\theta\rho^{\mathbf{k}*}_{2B} \end{bmatrix} = \begin{bmatrix} J^{\mathbf{k}}_{1A}(t) & J^{\mathbf{k}}_{1B}(t) \\ J^{\mathbf{k}}_{2A}(t) & J^{\mathbf{k}}_{2B}(t) \end{bmatrix}, \tag{22}$$

and $J^{\mathbf{k}}_{j\sigma}$ are 2×2 symmetric matrices.

Let us notice that we have not specified the mass parameters μ_σ. The situation here is similar to the one encountered in QFT in curved spacetime [45] where one has an infinite set of creation and annihilation operators related by a Bogoliubov transformation. In Ref. [44] it was shown that different choices of μ_σ affect the strength of the Casimir force between two plates. Typical choices studied in literature [46] are $\mu_A = m_1$, $\mu_B = m_2$ and $\mu_A = m_A$, $\mu_B = m_B$.

Therefore, one can define the *flavor vacuum* as the state, which is annihilated by flavor annihilation operators at a fixed time t[5]:

$$a_{\mathbf{k},\sigma}(t)|0(t)\rangle_{A,B} = b_{\mathbf{k},\sigma}(t)|0(t)\rangle_{A,B} = 0. \tag{23}$$

This is characterized by a boson-condensate structure in terms of modes with definite mass:

$${}_{A,B}\langle 0(t)|a^\dagger_{\mathbf{k},1}a_{\mathbf{k},1}|0(t)\rangle_{A,B} = {}_{A,B}\langle 0(t)|b^\dagger_{\mathbf{k},1}b_{\mathbf{k},1}|0(t)\rangle_{A,B} = 2(2\pi)^3(\cos^2\theta|\lambda^{\mathbf{k}}_{1A}|^2\omega_{\mathbf{k},A} + \sin^2\theta|\lambda^{\mathbf{k}}_{1B}|^2\omega_{\mathbf{k},B}), \tag{24}$$

$${}_{A,B}\langle 0(t)|a^\dagger_{\mathbf{k},2}a_{\mathbf{k},2}|0(t)\rangle_{A,B} = {}_{A,B}\langle 0(t)|b^\dagger_{\mathbf{k},2}b_{\mathbf{k},2}|0(t)\rangle_{A,B} = 2(2\pi)^3(\sin^2\theta|\lambda^{\mathbf{k}}_{2A}|^2\omega_{\mathbf{k},A} + \cos^2\theta|\lambda^{\mathbf{k}}_{2B}|^2\omega_{\mathbf{k},B}), \tag{25}$$

$${}_{A,B}\langle 0(t)|a^\dagger_{\mathbf{k},1}a_{\mathbf{k},2}|0(t)\rangle_{A,B} = {}_{A,B}\langle 0(t)|b^\dagger_{\mathbf{k},1}b_{\mathbf{k},2}|0(t)\rangle_{A,B} = 2(2\pi)^3\sin 2\theta(\lambda^{\mathbf{k}*}_{1A}\lambda^{\mathbf{k}}_{2A}\omega_{\mathbf{k},A} - \lambda^{\mathbf{k}*}_{1B}\lambda^{\mathbf{k}}_{2B}\omega_{\mathbf{k},B}). \tag{26}$$

Shortly we will see that this structure is responsible for the Poincaré and *CPT* symmetry breaking. In particular, the exotic condensates (26), which mix particles and antiparticles with different masses could represent a signature of a fundamental dynamical symmetry breaking mechanism that spontaneously breaks Poincaré symmetry and at the same time generates mixing (see Refs. [26,27]) in the fermion case. Note that all these condensates vanish for ultrarelativistic modes ($|\mathbf{k} \gg m_\sigma|$ and $|\mathbf{k} \gg m_\sigma|$). In this regime, eventual effects of SSB should vanish. The same is true also for $\theta = 0$.

Flavor states are defined as excitations over the flavor vacuum, i.e.,

$$|a_{\mathbf{k},\sigma}(t)\rangle \equiv a^\dagger_{\mathbf{k},\sigma}(t)|0(t)\rangle_{A,B}, \quad |b_{\mathbf{k},\sigma}(t)\rangle \equiv b^\dagger_{\mathbf{k},\sigma}(t)|0(t)\rangle_{A,B}. \tag{27}$$

The later are eigenstates of flavor charges

$$Q_\sigma(t) = i\int \mathrm{d}^3\mathbf{x} :\varphi^\dagger_\sigma(x)\overset{\leftrightarrow}{\partial}_0\varphi_\sigma(x) := \int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}(a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) - b^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t)), \tag{28}$$

at fixed time[6] t. In particular

$$Q_\sigma(t)|a_{\mathbf{k},\sigma}(t)\rangle = |a_{\mathbf{k},\sigma}(t)\rangle, \quad Q_\sigma(t)|b_{\mathbf{k},\sigma}(t)\rangle = -|b_{\mathbf{k},\sigma}(t)\rangle. \tag{29}$$

Although flavor charges are not conserved one can introduce the *total flavor charge*:

$$Q = \sum_\sigma Q_\sigma(t), \tag{30}$$

which is conserved ($[Q,H] = 0$) and which also satisfies the relation

$$Q|a_{\mathbf{k},\sigma}(t)\rangle = |a_{\mathbf{k},\sigma}(t)\rangle, \quad Q|b_{\mathbf{k},\sigma}(t)\rangle = -|b_{\mathbf{k},\sigma}(t)\rangle. \tag{31}$$

From (28) it is also clear that

$$Q_\sigma(t)|0(t)\rangle_{A,B} = Q|0(t)\rangle_{A,B} = 0. \tag{32}$$

We next proceed to discuss SSB of Poincaré symmetry in this system.

IV. SPONTANEOUS POINCARÉ SYMMETRY BREAKING

A. Spacetime translations

Let us start by considering spacetime translations, i.e., the subgroup $T^{3,1}$ of the Poincaré group. The generator of *space translations* has the usual form:

$$P_i = \sum_{\sigma=A,B}\int \mathrm{d}^3\mathbf{x}(\pi^\dagger_\sigma(x)\partial_i\varphi_\sigma(x) + \pi^\dagger_\sigma(x)\partial_i\varphi^\dagger_\sigma(x)), \quad i = 1,2,3, \tag{33}$$

[5]Here and throughout we work in the Heisenberg representation.

[6]Here normal ordering is taken with respect to $|0(t)\rangle_{A,B}$.

so that

$$T(\mathbf{b}) \equiv \exp(i\mathbf{b}\cdot\mathbf{P}) = \exp(ib^iP_i) = \exp(-ib^iP^i), \quad (34)$$

and

$$T(\mathbf{b})\varphi_\sigma(t,\mathbf{x})T^{-1}(\mathbf{b}) = \varphi_\sigma(t,\mathbf{x}+\mathbf{b}). \quad (35)$$

By using the expansion (15), P_i can be rewritten as:

$$P_i = \sum_{\sigma=A,B}\int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}k_i(a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t)+b^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t)). \quad (36)$$

This is time independent and commutes with the flavor charge, i.e., $[P_i, Q_\sigma(t)] = 0$ at all times. One can also easily check that

$$P_i|0(t)\rangle_{A,B} = 0, \quad (37)$$

and so

$$T(\mathbf{b})|0(t)\rangle_{A,B} = |0(t)\rangle_{A,B}. \quad (38)$$

In other words, *flavor vacuum is invariant under space translations*.

The situation changes if one looks at *time translations*. By using canonical commutation relations one can see that

$$a_{\mathbf{k},\sigma}(t) = \sum_{\rho=A,B}(f^{\mathbf{k}}_{\sigma\rho}(t)a_{\mathbf{k},\rho}(0)+g^{\mathbf{k}}_{\sigma\rho}(t)b^\dagger_{-\mathbf{k},\rho}(0)), \quad (39)$$

$$b^\dagger_{-\mathbf{k},\sigma}(t) = \sum_{\rho=A,B}(-g^{\mathbf{k}*}_{\sigma\rho}(t)a_{\mathbf{k},\rho}(0)+f^{\mathbf{k}*}_{\sigma\rho}(t)b^\dagger_{-\mathbf{k},\rho}(0)), \quad (40)$$

where

$$f^{\mathbf{k}}_{\sigma\rho}(t) = \frac{1}{(2\pi)^3 2\omega_{\mathbf{k},\rho}}[a_{\mathbf{k},\sigma}(t), a^\dagger_{\mathbf{k},\rho}(0)],$$
$$g^{\mathbf{k}}_{\sigma\rho}(t) = \frac{1}{(2\pi)^3 2\omega_{\mathbf{k},\rho}}[b_{-\mathbf{k},\rho}(0), a_{\mathbf{k},\sigma}(t)]. \quad (41)$$

The explicit form of these functions is listed in Appendix B. It is then clear that flavor vacuum is not time-independent. To see this explicitly, let us write the Hamiltonian in the normal-ordered form[7]:

$$H = \int d^3\mathbf{x}(:\boldsymbol{\pi}^\dagger_f(x)\boldsymbol{\pi}_f(x)+\nabla\boldsymbol{\varphi}^\dagger_f(x)\cdot\nabla\boldsymbol{\varphi}_f(x) + \boldsymbol{\varphi}^\dagger_f(x)M^2\boldsymbol{\varphi}_f(x):). \quad (42)$$

[7]Normal ordering is defined with respect to flavor vacuum at $t = 0$.

Because the Hamiltonian is time independent, we can expand it in terms of flavor creation and annihilation operators at $t = 0$:

$$H = \sum_{\sigma,\tau}\int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3} \times [w^{\mathbf{k}}_{\sigma\tau}(a^\dagger_{\mathbf{k},\sigma}(0)a_{\mathbf{k},\tau}(0)+b^\dagger_{\mathbf{k},\sigma}(0)b_{\mathbf{k},\tau}(0)) + y^{\mathbf{k}}_{\sigma\tau}(a^\dagger_{\mathbf{k},\sigma}(0)b^\dagger_{-\mathbf{k},\tau}(0)+b_{-\mathbf{k},\sigma}(0)a_{\mathbf{k},\tau}(0))], \quad (43)$$

where the coefficients are given in Eqs. (B7)–(B11). It is now easy to verify that the Hamiltonian does not annihilate the flavor vacuum, since

$$H|0\rangle_{A,B} = \sum_{\sigma,\tau}\int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}y^{\mathbf{k}}_{\sigma\tau}|a_{\mathbf{k},\sigma}\rangle\otimes|b_{-\mathbf{k},\tau}\rangle \neq 0, \quad (44)$$

where $|0\rangle_{A,B} \equiv |0(t=0)\rangle_{A,B}$. Note, however, that ${}_{A,B}\langle 0|H|0\rangle_{A,B} = 0$ as it should. Therefore, the *symmetry under time translations is spontaneously broken* since the action and ensuing field equations are invariant under time translations. By using Eq. (32) one can explicitly verify that the state (44) carries the zero total charge, i.e.,

$$QH|0\rangle_{A,B} = 0, \quad (45)$$

as we would expect from the conservation of Q. We see, therefore, that flavor vacua at different times form a *flavor vacuum manifold*:

$$|0(t)\rangle_{A,B} = T(t)|0\rangle_{A,B}, \quad (46)$$

where

$$T(t) \equiv \exp(iHt), \quad (47)$$

is the time-evolution operator. The flavor vacuum manifold was introduced in close analogy with *vacuum manifold* defined in the study of SSB in gauge theories. However, here the different vacua are degenerate with respect to total flavor charge and not to energy. In fact, the states representing the flavor vacuum manifold do not posses any sharp value of energy—energy fluctuates (has a nontrivial variance) on each flavor vacuum [36], see also Eq. (50).

From Eqs. (44) and (46) we can also find that for generic t

$$H|0(t)\rangle_{A,B} = \sum_{\sigma,\tau}\int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}y^{\mathbf{k}}_{\sigma\tau}|a_{\mathbf{k},\sigma}(t)\rangle\otimes|b_{-\mathbf{k},\tau}(t)\rangle \neq 0, (48)$$

which completes our proof of the SSB of the time translation symmetry. We have thus proved that spacetime translation symmetry is spontaneously broken on flavor vacuum. The residual vacuum symmetry is then T^3, i.e., the group of spatial translations.

In passing, we can also establish an analogue of the *Fabri–Picasso theorem* [47] for the present situation. Let us consider the square norm of $H|0\rangle_{A,B}$:

$$\begin{aligned}\|H|0\rangle_{A,B}\|^2 &= {}_{A,B}\langle 0|H^2|0\rangle_{A,B} = \int d^3\mathbf{x}\,{}_{A,B}\langle 0|HT^{00}(\mathbf{x})|0\rangle_{A,B}\\ &= \int d^3\mathbf{x}\,{}_{A,B}\langle 0|H\mathcal{H}(\mathbf{x})|0\rangle_{A,B},\end{aligned} \quad (49)$$

where $T^{00}(\mathbf{x})$ and $\mathcal{H}(\mathbf{x})$ are the timetime component of energy momentum tensor and Hamilton density, respectively. Let us regulate H so that for a sufficiently large space domain Ω of volume V we introduce $H_V = \int_\Omega d^3\mathbf{x}\mathcal{H}(\mathbf{x})$. By using the space-translation invariance of the vacuum [cf. Eq. (38)], we find that

$$\|H_V|0\rangle_{A,B}\|^2 = {}_{A,B}\langle 0|H_V^2|0\rangle_{A,B} = V_{A,B}\langle 0|H_V\mathcal{H}(0)|0\rangle_{A,B}, \quad (50)$$

where $V = \int_\Omega d^3\mathbf{x}$. If we now send $V \to \infty$, we see that (50) diverges unless $\lim_{V\to\infty} H_V|0\rangle_{A,B} = H|0\rangle_{A,B} = 0$. This would, however, be in contradiction with the symmetry breaking condition (44). Therefore, the mathematical implementation of these ideas is rather delicate [48]. The finite volume Hamiltonian H_V induces a "finite time translation," $T_V(t) = \exp(itH_V)$, which in turn gives rise to a "shifted ground state," $[|0(t)\rangle_{A,B}]_V = T_V(t)|0\rangle_{A,B}$. However, very much like the limit $\lim_{V\to\infty} H_V$ does not exist, the operator $\exp(itH)$ is not well defined on the flavor Fock space $\mathcal{H}_f(\tau)$ (for any τ). As a consequence [48]:

$$\lim_{V\to\infty} {}_{A,B}\langle 0|[|0(t)\rangle_{A,B}]_V = \lim_{V\to\infty} {}_{A,B}\langle 0|\exp(itH_V)|0\rangle_{A,B} = 0. \quad (51)$$

In other words, flavor Fock spaces at different times are unitarily inequivalent.

The intuitive picture of spontaneous symmetry breaking, based on the observation that a symmetry transformation (44) does not leave the flavor vacuum state intact, suggests high degeneracy of equivalent flavor vacuum states $|0(t)\rangle_{A,B}$. Indeed, since the Hamiltonian H commutes with the charge operator Q, so will a finite symmetry transformation $T(t)$ generated by H. It will therefore transform the one flavor vacuum state into another with the same flavor charge. Since the time-translation symmetry group is continuous, we will find infinitely many degenerate flavor vacuum states. On account of the fact that they are all connected by symmetry transformations, they must be physically equivalent and any one of them can serve as a starting point for the construction of the spectrum of excited flavor states. Let us consider, for example, the flavor oscillation formula [42]:

$$\mathcal{Q}_{\sigma\to\rho}(t, t_0) = {}_{A,B}\langle a_{\mathbf{k},\sigma}(t_0)|Q_\rho(t)|a_{\mathbf{k},\sigma}(t_0)\rangle_{A,B}. \quad (52)$$

One can easily verify that

$$\mathcal{Q}_{\sigma\to\rho}(t, t_0) = \mathcal{Q}_{\sigma\to\rho}(t - t_0), \quad (53)$$

i.e., flavor oscillations are *invariant* under time translations. In fact,

$$\begin{aligned}&{}_{A,B}\langle a_{\mathbf{k},\sigma}(t_0)|Q_\rho(t)|a_{\mathbf{k},\sigma}(t_0)\rangle_{A,B}\\ &= {}_{A,B}\langle a_{\mathbf{k},\sigma}(0)|T(t-t_0)Q_\rho(0)T^{-1}(t-t_0)|a_{\mathbf{k},\sigma}(0)\rangle_{A,B}\\ &= {}_{A,B}\langle a_{\mathbf{k},\sigma}(0)|Q_\rho(t-t_0)|a_{\mathbf{k},\sigma}(0)\rangle_{A,B},\end{aligned} \quad (54)$$

where we used the group property $T^{-1}(t_0)T(t) = T(t - t_0)$. It is thus clear that the choice of time t_0, which we use for the construction of the (Heisenberg representation) state space, is quite immaterial.

It can also be shown that unlike the transformations of physical states, finite symmetry transformations $T_V(t)$ of observables can be defined consistently in the $V \to \infty$ limit in theories that are sufficiently causal [47]. In the following reasoning it will always be implicitly understood that the large-V regulator should be properly employed according to indicated lines whenever expectation values are to be computed.

B. Proper Lorentz group

It is well known that the generator of proper Lorentz algebra $so(3,1)$ can be expressed as [49]

$$J^{\mu\nu} \equiv \int d^3\mathbf{x} :(x^\mu T^{0\nu} - x^\nu T^{0\mu}):, \quad \mu,\nu = 0,\ldots 3. \quad (55)$$

Here $T_{\mu\nu}$ is the energy-momentum tensor.

Let us start from its spatial part:

$$\begin{aligned}J_{ij} = -\sum_\sigma \int d^3\mathbf{x} :&x_i(\pi_\sigma^\dagger(x)\partial_j\varphi_\sigma(x) + \partial_j\varphi_\sigma^\dagger(x)\pi_\sigma(x))\\ &- x_j(\pi_\sigma^\dagger(x)\partial_i\varphi_\sigma(x) + \partial_i\varphi_\sigma^\dagger(x)\pi_\sigma(x)):.\end{aligned} \quad (56)$$

One can equivalently use the angular-momentum operators J^k defined in Eq. (A12)

$$\begin{aligned}\mathbf{L} \equiv \mathbf{J} = -\sum_\sigma \int d^3\mathbf{x}[&\pi_\sigma^\dagger(x)(\mathbf{x}\times\nabla)\varphi_\sigma(x)\\ &+ \varphi_\sigma^\dagger(x)(\mathbf{x}\times\overleftarrow{\nabla})\pi_\sigma(x)],\end{aligned} \quad (57)$$

where we identified $\mathbf{J}$ with the orbital angular-momentum vector $\mathbf{L} = (L^1, L^2, L^3)$ because no extra spin contribution is present for scalar fields. In terms of annihilation and creation operators we have:

$$\mathbf{L} = \int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a^\dagger_{\mathbf{k},\sigma}(t)(\mathbf{k}\times\nabla_\mathbf{k})a_{\mathbf{k},\sigma}(t) + b^\dagger_{\mathbf{k},\sigma}(t)(\mathbf{k}\times\nabla_\mathbf{k})b_{\mathbf{k},\sigma}(t)], \tag{58}$$

One can easily verify that

$$[\mathbf{L}, Q_\sigma(t)] = [\mathbf{L}, H] = 0. \tag{59}$$

It is also not difficult to see that this operator annihilates the flavor vacuum:

$$\mathbf{L}|0(t)\rangle_{A,B} = 0. \tag{60}$$

In fact, we can always perform a unitary canonical transformation which diagonalizes one of the components of the angular momentum.[8] For example, mimicking the case of a free scalar field [50] we can perform the canonical transformation

$$a_{plm,\sigma}(t) \equiv i^l p \int \mathrm{d}\Omega_p Y^*_{lm}(\Omega_p) a_{\mathbf{p},\sigma}(t), \tag{61}$$

$$b_{plm,\sigma}(t) \equiv i^l p \int \mathrm{d}\Omega_p Y^*_{lm}(\Omega_p) b_{\mathbf{p},\sigma}(t), \tag{62}$$

where $p = |\mathbf{p}|$, Y_{lm} are the spherical harmonics and Ω_p is the solid angle at fixed p. In this representation L^3 has a diagonal form:

$$L^3 = \sum_{l,m,\sigma}\int_0^\infty \mathrm{d}p\, m(a^\dagger_{plm,\sigma}(t)a_{plm,\sigma}(t) + b^\dagger_{plm,\sigma}(t)b_{plm,\sigma}(t)). \tag{63}$$

From Eqs. (61), (62) it is evident that:

$$a_{plm,\sigma}(t)|0(t)\rangle_{A,B} = b_{plm,\sigma}(t)|0(t)\rangle_{A,B} = 0. \tag{64}$$

It follows that $L_3|0\rangle = 0$. The same procedure can be repeated for the other components. In the same way, by defining the generator of rotations

$$R(\boldsymbol{\vartheta}) = \exp(-i\boldsymbol{\vartheta}\cdot\mathbf{L}), \tag{65}$$

one can verify that

$$R(\boldsymbol{\vartheta})|0(t)\rangle_{A,B} = |0(t)\rangle_{A,B}, \tag{66}$$

which shows that *flavor vacuum is rotationally invariant*.

[8]These cannot be diagonalized simultaneously, because of the $SO(3)$ commutation relations.

Let us now analyze the transformation properties of the flavor vacuum under the Lorentz boosts.[9] The generator of a boost along the lth axis is

$$K_l = \int \mathrm{d}^3\mathbf{x} :(x^0 T^{0l} - x^l T^{00}):. \tag{67}$$

This can also be written as

$$K_l = x^0 P^l - \int \mathrm{d}^3\mathbf{x}\, x^l \mathcal{H}. \tag{68}$$

In our case:

$$K_l = \Big[x^0 \int \mathrm{d}^3\mathbf{x} :(\boldsymbol{\pi}^\dagger_f(x)\partial^l\boldsymbol{\varphi}_f(x) + \partial^l\boldsymbol{\varphi}^\dagger_f(x)\boldsymbol{\pi}_f(x)): - \int \mathrm{d}^3\mathbf{x}\, x^l :(\boldsymbol{\pi}^\dagger_f(x)\boldsymbol{\pi}_f(x) + \nabla\boldsymbol{\varphi}^\dagger_f(x)\cdot\nabla\boldsymbol{\varphi}_f(x) + \boldsymbol{\varphi}^\dagger_f(x)M^2\boldsymbol{\varphi}_f(x)):\Big]. \tag{69}$$

We can now rewrite (69) in terms of flavor creation and annihilation operators (17). By noticing that in the mass basis this is just the sum of boost generators for the two massive fields φ_1 and φ_2 (cf. e.g., Ref. [52]), we get

$$K_l = -i\sum_{j=1,2}\int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},j}(2\pi)^3} :\boldsymbol{a}^\dagger_{\mathbf{k},j}\Omega^\mathbf{k}_j \frac{\partial}{\partial k^l}\boldsymbol{a}_{\mathbf{k},j}: \tag{70}$$

$$= -i\sum_{\sigma,\rho=A,B}\int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3} :\boldsymbol{a}^\dagger_{\mathbf{k},\sigma}(t)\Omega^\mathbf{k}_{\sigma\rho}(t)\frac{\partial}{\partial k^l}\boldsymbol{a}_{\mathbf{k},\rho}(t) + \boldsymbol{a}^\dagger_{\mathbf{k},\sigma}(t)\Omega^\mathbf{k}_{\sigma\rho,l}(t)\boldsymbol{a}_{\mathbf{k},\rho}(t):, \tag{71}$$

where we have introduced the matrices

$$\Omega^\mathbf{k}_j = \omega_{\mathbf{k},j}\mathbb{I}_2, \qquad \Omega^\mathbf{k}_{\sigma\rho}(t) = \omega_{\mathbf{k},\sigma}\sum_{j=1,2}J^{\mathbf{k}\dagger}_{j\sigma}(t)J^\mathbf{k}_{j\rho}(t),$$
$$\Omega^\mathbf{k}_{\sigma\rho,l}(t) = \omega_{\mathbf{k},\sigma}\sum_{j=1,2}J^{\mathbf{k}\dagger}_{j\sigma}(t)\Omega^\mathbf{k}_j\frac{\partial}{\partial k^l}J^\mathbf{k}_{j\rho}(t), \tag{72}$$

and $\mathbb{I}_2$ is the 2×2 identity matrix. The explicit form of $\Omega^\mathbf{k}_\sigma(t)$ and $\Omega^\mathbf{k}_{\sigma,i}(t)$ is not very illuminating and we do not report it here. We only notice that these are nondiagonal matrices.

[9]Note that here, as in Ref. [51] for unstable particles, flavor states have a definite momentum. This is important to remark, because for states that are not energy eigenstates boost and momentum translation are not equivalent.

A generic boost can be thus expressed in the form:

$$U(L) = \exp\left(-i\sum_{l=1}^{3}\xi^l K_l\right), \tag{73}$$

where $L(\xi)$ indicates the Lorentz boost transformation:

$$x^\mu \to x'^\mu = L^\mu{}_\nu(\xi)x^\nu. \tag{74}$$

Now, for flavor fields we can write

$$U(L)\varphi_\sigma(x)U^{-1}(L) = \varphi_\sigma(x'), \tag{75}$$

i.e., φ_σ behaves as a scalar under Lorentz boost. From Eq. (15) we get:

$$U(L)\varphi_\sigma(x)U^{-1}(L) = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3} \times [a_{\mathbf{k},\sigma}(t')e^{-ikx'} + b^\dagger_{\mathbf{k},\sigma}(t')e^{ikx'}]. \tag{76}$$

Here and in the following we formally use the notation $L\mathbf{k}$ to indicate $L^j_\mu k^\mu$ (j = 1, 2, 3), respectively. This equation should be actually written in the form:

$$U(L)\varphi_\sigma(x)U^{-1}(L) = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta^4(k^2-\mu_\sigma^2)\theta(k_0)[a_{k,\sigma}(t')e^{-ikx'} + b^\dagger_{k,\sigma}(t')e^{ikx'}]. \tag{77}$$

Performing the change of variables [49]: $k \to k' = L^{-1}k$, we have:

$$U(L)\varphi_\sigma(x)U^{-1}(L) = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta^4(k^2-\mu_\sigma^2)\theta(k_0)[a_{Lk,\sigma}(t')e^{-ikx} + b^\dagger_{Lk,\sigma}(t')e^{ikx}]. \tag{78}$$

By integrating over k_0 we find:

$$U(L)\varphi_\sigma(x)U^{-1}(L) = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a_{L\mathbf{k},\sigma}(t')e^{-ikx} + b^\dagger_{L\mathbf{k},\sigma}(t')e^{ikx}]. \tag{79}$$

Therefore, by comparing with Eq. (15) we find

$$U(L)a_{\mathbf{k},\sigma}(t)U^{-1}(L) = a_{L\mathbf{k},\sigma}(t'), \tag{80}$$

$$U(L)b_{\mathbf{k},\sigma}(t)U^{-1}(L) = b_{L\mathbf{k},\sigma}(t'). \tag{81}$$

To find the explicit form of these operators, in terms of the ones at time t, we can employ canonical commutation relations to get:

$$U(L)a_{\mathbf{k},\sigma}(t)U^{-1}(L) = a_{L\mathbf{k},\sigma}(t') = \sum_{\rho=A,B}\frac{1}{2\omega_{L\mathbf{k};\rho}(2\pi)^3}([a_{L\mathbf{k},\sigma}(t'), a^\dagger_{L\mathbf{k},\rho}(t)]a_{L\mathbf{k},\rho}(t) - [a_{L\mathbf{k},\sigma}(t'), b_{-L\mathbf{k},\rho}(t)]b^\dagger_{-L\mathbf{k},\rho}(t)), \tag{82}$$

and similar relations hold also for the other operators.[10] These are analogous to Eqs. (39)–(40). If we now look at flavor-vacuum transformation properties under boosts we have

$$|0(t';\boldsymbol{\xi})\rangle_{A,B} = U(L)|0(t)\rangle_{A,B}$$
$$= \exp\left(-\sum_{l=1}^{3}\xi^l\sum_{\sigma,\rho=A,B}\int\frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}:\boldsymbol{a}^\dagger_{\mathbf{k},\sigma}(t)\Omega^{\mathbf{k}}_{\sigma\rho}(t)\frac{\partial}{\partial k^l}\boldsymbol{a}_{\mathbf{k},\rho}(t) + \boldsymbol{a}^\dagger_{\mathbf{k},\sigma}(t)\Omega^{\mathbf{k}}_{\sigma\rho,l}(t)\boldsymbol{a}_{\mathbf{k},\rho}(t):\right)|0(t)\rangle_{A,B}. \tag{83}$$

We see immediately that $|0(t';\boldsymbol{\xi})\rangle_{A,B} \neq |0(t)\rangle_{A,B}$ and hence the flavor vacuum is changed under the action of Lorentz boosts, while the action and ensuing field equations stay unchanged. In other words, the *Lorentz boosts symmetry is spontaneously broken on flavor vacuum*. By varying $\boldsymbol{\xi}$, we generate a flavor vacuum manifold of unitarily inequivalent states, as in the case of flavor vacua at different times, which we analyzed in Sec. IV A. In fact, since

[10]Note that commutators at different times are c-numbers.

$$[K_l, Q] = 0, \tag{84}$$

all states defined in (83) correspond to zero total flavor charge. Same considerations as in the time-translation case, based on Fabri–Picasso theorem and the large-V regularization, can be repeated here in the same way.

From the above discussion it is clear that only rotational symmetry $SO(3)$, whose generators are given by Eq. (65) is a symmetry of the flavor vacuum. This result, together with the one of the previous section, tells us that the flavor vacuum symmetry group is the Euclidean group $E(3)$, as stated in Sec. II. The quadratic Casimir of this group are [43] $\mathbf{P}^2 \equiv \mathbf{P}\cdot\mathbf{P}$ and $\mathbf{J}\cdot\mathbf{P}$, which now substitute P^2 and W^2. It is worthy of remarking that similar results were derived in the case of unstable particles [6,53], which strengthens even more the analogy between flavor mixing and unstable particles proposed in Ref. [36]. Note that the flavor vacuum manifold has together 4 flavor flat directions (i.e., directions along which the total flavor charge remains zero) corresponding to the number of broken generators. In particular, the flavor vacuum manifold $\mathcal{M} = \{|0(t;\boldsymbol{\xi})\rangle_{A,B}, (t,\boldsymbol{\xi}) \in \mathbb{R}^4\}$ is isomorphic to the quotient space $(T^{3,1} \rtimes O(3,1))/E(3)$. Note that dimension of the quotient space, i.e., $\dim[(T^{3,1} \rtimes O(3,1))/E(3)]$ is correctly $10 - 6 = 4$. Let us also observe that there are no *energy flat directions* on $\mathcal{M}$. Indeed, from the Fabri–Picasso theorem [cf. Eq. (50)] we see that the variance of the energy is infinite at any point on the vacuum manifold $\mathcal{M}$, which in turn prohibits the existence of energy flat directions on $\mathcal{M}$. Note that such a divergence is basically an infrared problem (large-V problem) and it can be controlled by means of an appropriate regularization scheme. This argument indicates that there should be *no Goldstone modes* present in the theory, since these are normally associated with gapless fluctuations along flat energy directions.

So, while the charge $Q_A(t)$ *does not* fluctuate on the state $|0(t;\boldsymbol{\xi})\rangle_{A,B} \in \mathcal{M}$, the fluctuations of E are on the very same state *unbounded*. This complementarity between E and Q_A fluctuations on $\mathcal{M}$ might also be viewed as a direct manifestation of flavor-energy uncertainty relations [36].[11]

[11] In fact, for any label time τ there exists $Q_\sigma(\tau)$ such that $Q_\sigma(\tau)|0(\tau;\boldsymbol{\xi})\rangle_{A,B} = 0$ [cf. Eq. (32)] but $[Q_\sigma(t), H] \neq 0$. Let us now consider the flavor-energy uncertainty relations [36]

$$\Delta E \Delta Q_\sigma(t) \geq \frac{1}{2}\left|\frac{\mathrm{d}\langle Q_\sigma(t)\rangle}{\mathrm{d}t}\right|, \tag{85}$$

where ΔQ_σ and ΔE are standard deviations of charge and energy, respectively evaluated on $|0(\tau;\boldsymbol{\xi})\rangle_{A,B}$ flavor vacuum at the fixed label time (e.g., $\tau = 0$). Because $Q_\sigma(0)|0(0;\boldsymbol{\xi})\rangle_{A,B} = 0$ we have that $\Delta Q_\sigma(t)|_{\tau\to 0} = 0$. The right-hand side (rhs) of (85) equals zero only for $\theta = 0$ or $m_1 = m_2$, i.e., for the nonmixing case. This is, however, trivial situation since in this case $|0\rangle_{A,B} = |0\rangle_{1,2}$ and hence no symmetry breaking is present. On the other hand, for $\theta \neq 0$, the rhs of (85) is nonzero, while on the left-hand side (lhs) $\Delta Q_\sigma(t)|_{t\to 0} = 0$, implying $\Delta E \to \infty$.

As in the case of time-translation, we can now show that different states in the flavor vacuum manifold are physically equivalent. In other words, flavor oscillations can be equivalently described in every Lorentz frame. Let us consider a flavor wave packet:

$$|a_\sigma(y)\rangle \equiv \int \frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3} e^{-iky} f(\mathbf{k}) a_{\mathbf{k},\sigma}(y_0)|0(y_0)\rangle_{A,B}, \tag{86}$$

and suppose that the momentum space distribution $f(\mathbf{k})$ is Lorentz invariant. Therefore:

$$|a'_\sigma(y)\rangle \equiv U(L)|a_\sigma(y)\rangle = |a_\sigma(y')\rangle, \tag{87}$$

as one can derive from Eqs. (80), (81). Covariant oscillation formula should be written as:

$$\mathcal{J}^\mu_{\sigma\to\rho}(x-y) = \langle a_\sigma(y)|J^\mu_\rho(x)|a_\sigma(y)\rangle, \tag{88}$$

where $J^\mu_\rho(x)$ are the flavor currents defined as [42]

$$J^\mu_\rho(x) \equiv i\varphi^\dagger_\rho(x)\overleftrightarrow{\partial^\mu}\varphi_\rho(x). \tag{89}$$

Clearly, Eq. (52) can be obtained by taking $\mu = 0$ and integrating on space variables.

In the primed Lorentz frame Eq. (52) reads

$$\langle a'_\sigma(y)|J^\mu_\rho(x')|a'_\sigma(y)\rangle = \langle a_\sigma(y')|J^\mu_\rho(x')|a_\sigma(y')\rangle = \mathcal{J}^\mu_{\sigma\to\rho}(x'-y'). \tag{90}$$

Therefore, the flavor oscillation formula in the primed Lorentz frame is the same as in the unprimed one. This shows, once more, that Poincaré (and Lorentz) symmetry breaking on the flavor vacuum, which leads to nonzero vector current vacuum expectation values (89), has no direct consequences on flavor oscillations. Thus, Poincaré invariance breaking contributions to QFT oscillation formula as reported in [14] are mere artifacts of the noncovariant formalism (oscillations in time) used in that work.

As we have seen above, another important feature of the Poincarè/Lorentz SSB via the dynamical flavor condensates is the apparent *absence* of any Goldstone bosons, as discussed above. Thus the spectrum of the flavor vacuum remains the same as the mass eigenstate one, and we have no extra massless modes. This situation is to be contrasted with the standard lore of nonflavored QFT. Indeed, it has been suggested in [54], that, in gauge theories with Lorentz SSB, in the sense of a vector gauge boson acquiring a vacuum expectation value, the massless U(1) photon plays the role of such a Goldstone boson. In the current, non-gauge, context, although the flavor currents (89) acquire nonzero vacuum expectation values (90) in terms of the flavor vacuum, nonetheless, as we explained above, they are *not associated with any* Goldstone bosons.

V. DISCRETE SYMMETRIES

Until now we did not consider the discrete symmetries. However, they have to be included in a complete study of Lorentz group properties of flavor operators. Moreover, in the current literature, Lorentz symmetry breaking is often discussed in parallel with *CPT* symmetry breaking [28–30], because the *CPT* theorem strongly depends on the assumption of Lorentz invariance [2].

In this section we study the behavior of flavor annihilation and creation operators under parity, charge conjugation and time reversal. This will be done by considering discrete symmetries both separately and in different relevant combinations. We will see that time reversal is spontaneously broken and as consequence also *CPT* is not a symmetry of the flavor vacuum.

A. Parity

The parity transformation of the flavor scalar fields is given by:

$$P\varphi_\sigma(x)P^{-1} = \eta_{\sigma,P}\varphi_\sigma(\tilde{x}), \tag{91}$$

where P is the unitary parity operator and $\tilde{x} = (t, -\mathbf{x})$. We choose the intrinsic parity to satisfies $|\eta_{\sigma,P}|^2 = 1$. By using the explicit expansion (15), we find:

$$\begin{aligned} P\varphi_\sigma(x)P^{-1} &= \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[Pa_{\mathbf{k},\sigma}(t)P^{-1}e^{-i\omega_{\mathbf{k},\sigma}t} + Pb^\dagger_{-\mathbf{k},\sigma}(t)P^{-1}e^{i\omega_{\mathbf{k},\sigma}t}]e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \eta_{\sigma,P}\int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a_{\mathbf{k},\sigma}(t)e^{-i\omega_{\mathbf{k},\sigma}t} + b^\dagger_{-\mathbf{k},\sigma}(t)e^{i\omega_{\mathbf{k},\sigma}t}]e^{-i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \tag{92}$$

Consequently, transformations of creation and annihilation operators satisfy the following relations:

$$Pa_{\mathbf{k},\sigma}(t)P^{-1} = \eta_{\sigma,P}a_{-\mathbf{k},\sigma}(t), \qquad Pb_{\mathbf{k},\sigma}(t)P^{-1} = \eta^*_{\sigma,P}b_{-\mathbf{k},\sigma}(t), \tag{93}$$

$$Pa^\dagger_{\mathbf{k},\sigma}(t)P^{-1} = \eta^*_{\sigma,P}a^\dagger_{-\mathbf{k},\sigma}(t), \qquad Pb^\dagger_{\mathbf{k},\sigma}(t)P^{-1} = \eta_{\sigma,P}b^\dagger_{-\mathbf{k},\sigma}(t). \tag{94}$$

It can be checked that the explicit form of P satisfying above relations reads (see also Ref. [50])

$$P = \exp\left\{i\frac{\pi}{2}\int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[(a^\dagger_{-\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) + b^\dagger_{-\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t)) - \eta_{\sigma,P}(a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) + b^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t))]\right\}. \tag{95}$$

By inspection we see that the *flavor vacuum is invariant under parity transformation*, i.e., up to an irrelevant phase factor we have

$$P|0(t)\rangle_{A,B} = |0(t)\rangle_{A,B}. \tag{96}$$

As a simple consequence we get that

$$P|a_{\mathbf{k},\sigma}(t)\rangle = |a_{-\mathbf{k},\sigma}(t)\rangle, \tag{97}$$

and flavor charges (28) remain invariant, i.e.,

$$[P, Q_\sigma(t)] = 0. \tag{98}$$

B. Charge conjugation

The charge conjugation transformation of the flavor scalar fields is given by

$$C\varphi_\sigma(x)C^{-1} = \eta_{\sigma,C}\varphi^\dagger_\sigma(x), \tag{99}$$

where C is the unitary charge conjugation operator. Again, our convention is $|\eta_{\sigma,C}|^2 = 1$. Once more, by using the explicit expansion (15), we find:

$$\begin{aligned} C\varphi_\sigma(x)C^{-1} &= \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[Ca_{\mathbf{k},\sigma}(t)C^{-1}e^{-ikx} + Cb^\dagger_{\mathbf{k},\sigma}(t)C^{-1}e^{ikx}] \\ &= \eta_{\sigma,C}\int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a^\dagger_{\mathbf{k},\sigma}(t)e^{ikx} + b_{\mathbf{k},\sigma}(t)e^{-ikx}]. \end{aligned} \tag{100}$$

Transformations of creation and annihilation operators follow:

$$Ca_{\mathbf{k},\sigma}(t)C^{-1} = \eta_{\sigma,C}b_{\mathbf{k},\sigma}(t), \qquad Cb_{\mathbf{k},\sigma}(t)C^{-1} = \eta^*_{\sigma,C}a_{\mathbf{k},\sigma}(t), \tag{101}$$

$$Ca^\dagger_{\mathbf{k},\sigma}(t)C^{-1} = \eta^*_{\sigma,C}b^\dagger_{\mathbf{k},\sigma}(t), \qquad Cb^\dagger_{\mathbf{k},\sigma}(t)C^{-1} = \eta_{\sigma,C}a^\dagger_{\mathbf{k},\sigma}(t). \tag{102}$$

From this, the explicit form of C reads

$$C = \exp\left\{i\frac{\pi}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[(b^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) + a^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t)) - \eta_{\sigma,C}(a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t) + b^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t))]\right\}, \tag{103}$$

which shows that the *flavor vacuum is invariant under charge conjugation*, i.e.,

$$C|0(t)\rangle_{A,B} = |0(t)\rangle_{A,B}. \tag{104}$$

Consequently, a flavor state (27) transforms as

$$C|a_{\mathbf{k},\sigma}(t)\rangle = |b_{\mathbf{k},\sigma}(t)\rangle, \tag{105}$$

while flavor charge (28) reverses its sign

$$CQ_\sigma(t)C^{-1} = \int\frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}(b^\dagger_{\mathbf{k},\sigma}(t)b_{\mathbf{k},\sigma}(t) - a^\dagger_{\mathbf{k},\sigma}(t)a_{\mathbf{k},\sigma}(t)) = -Q_\sigma(t), \tag{106}$$

as expected.

C. Time reversal

The time reversal transformation of the flavor scalar fields is given by:

$$T\varphi_\sigma(x)T^{-1} = \eta_{\sigma,T}\varphi_\sigma(-\tilde{x}), \tag{107}$$

where T is the *antiunitary* time reversal operator. We employ the convention for the phase $|\eta_{\sigma,T}|^2 = 1$. By using the explicit expansion (15), we find:

$$\begin{aligned} T\varphi_\sigma(x)T^{-1} &= \int\frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[Ta_{\mathbf{k},\sigma}(t)T^{-1}e^{i\omega_{\mathbf{k},\sigma}t} + Tb^\dagger_{-\mathbf{k},\sigma}(t)T^{-1}e^{-i\omega_{\mathbf{k},\sigma}t}]e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \eta_{\sigma,T}\int\frac{\mathrm{d}^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}[a_{\mathbf{k},\sigma}(-t)e^{i\omega_{\mathbf{k},\sigma}t} + b^\dagger_{-\mathbf{k},\sigma}(-t)e^{-i\omega_{\mathbf{k},\sigma}t}]e^{i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \tag{108}$$

Transformations of creation and annihilation operators follow:

$$Ta_{\mathbf{k},\sigma}(t)T^{-1} = \eta_{\sigma,T}a_{-\mathbf{k},\sigma}(-t) \qquad Tb_{\mathbf{k},\sigma}(t)T^{-1} = \eta^*_{\sigma,T}b_{-\mathbf{k},\sigma}(-t), \tag{109}$$

$$Ta^\dagger_{\mathbf{k},\sigma}(t)T^{-1} = \eta^*_{\sigma,T}a^\dagger_{-\mathbf{k},\sigma}(-t) \qquad Tb^\dagger_{\mathbf{k},\sigma}(t)T^{-1} = \eta_{\sigma,T}b^\dagger_{-\mathbf{k},\sigma}(-t). \tag{110}$$

Let us note in this connection that for flavor A we can explicitly write

$$\begin{aligned} Ta_{\mathbf{k},A}(t)T^{-1} &= \eta_{\sigma,T}a_{-\mathbf{k},A}(-t) \\ &= \eta_{\sigma,T}\sum_{\rho=A,B}\frac{1}{2\omega_{\mathbf{k},\rho}(2\pi)^3}([a_{-\mathbf{k},\sigma}(-t), a^\dagger_{-\mathbf{k},\rho}(t)]a_{-\mathbf{k},\rho}(t) - [a_{-\mathbf{k},\sigma}(-t), b_{\mathbf{k},\rho}(t)]b^\dagger_{\mathbf{k},\rho}(t)), \end{aligned} \tag{111}$$

where on the second line the result is phrase in terms of operators $a_{-\mathbf{k},\rho}(t)$ and $b^\dagger_{\mathbf{k},\rho}(t)$ at original time t. Commutators involved are just c-numbered functions due to a quadratic nature of our model system. Similar relations hold for the other operators and flavor B. If one now looks at flavor vacuum transformation properties

$$|0(t)\rangle^T_{A,B} = T|0(t)\rangle_{A,B}, \tag{112}$$

one finds that time-reversal symmetry is spontaneously broken. This could also be seen by looking at flavor charge (28) transformation:

$$TQ_\sigma(t)T^{-1} = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}(a^\dagger_{\mathbf{k},\sigma}(-t)a_{\mathbf{k},\sigma}(-t) - b^\dagger_{\mathbf{k},\sigma}(-t)b_{\mathbf{k},\sigma}(-t)) = Q_\sigma(-t), \tag{113}$$

i.e., $[Q_\sigma(t), T] \neq 0$ in a nontrivial way (they neither commute or anticommute). This implies that

$$Q_\sigma(t)T|0(t)\rangle_{A,B} = Q_\sigma(t)|0(t)\rangle^T_{A,B} \neq 0, \tag{114}$$

while $Q_\sigma(t)|0(t)\rangle_{A,B} = 0$. This shows that the *time-reversal symmetry is spontaneously broken*.

Once more, we notice that oscillation formula for our toy-model system is left unchanged by time reversal transformation. In fact, from Eq. (52), we have

$$\mathcal{Q}_{\sigma\to\rho}(-t) = \langle a_{\mathbf{k},\sigma}(0)|Q_\rho(-t)|a_{\mathbf{k},\sigma}(0)\rangle = \langle a_{\mathbf{k},\sigma}(0)|TQ_\rho(t)T^{-1}|a_{\mathbf{k},\sigma}(0)\rangle = \mathcal{Q}_{\sigma\to\rho}(t), \tag{115}$$

where we used that

$$T^{-1}|a_{\mathbf{k},\sigma}(0)\rangle = |a_{\mathbf{k},\sigma}(0)\rangle. \tag{116}$$

D. *CP* and *CPT* symmetry

From the previous considerations it is evident that *CP* is an exact symmetry in the flavor representation[12]:

$$CP|0(t)\rangle_{A,B} = |0(t)\rangle_{A,B}. \tag{117}$$

However, from Eq. (112), it follows that *CPT* symmetry is spontaneously broken on the flavor vacuum:

$$|0(t;\Theta)\rangle_{A,B} = \Theta|0(t)\rangle_{A,B}, \tag{118}$$

where $\Theta \equiv CPT$. This is a consequence of the transformation law of creation and annihilation operators:

$$\Theta a_{\mathbf{k},\sigma}(t)\Theta^{-1} = \eta_\sigma b_{\mathbf{k},\sigma}(-t), \quad \Theta b_{\mathbf{k},\sigma}(t)\Theta^{-1} = \eta^*_\sigma a_{\mathbf{k},\sigma}(-t), \tag{119}$$

$$\Theta a^\dagger_{\mathbf{k},\sigma}(t)\Theta^{-1} = \eta^*_\sigma b^\dagger_{\mathbf{k},\sigma}(-t), \quad \Theta b^\dagger_{\mathbf{k},\sigma}(t)\Theta^{-1} = \eta_\sigma a^\dagger_{\mathbf{k},\sigma}(-t). \tag{120}$$

[12]This is not true for the three flavor case, where *CP* symmetry can be *explicitly* broken because of a complex phase in the mass matrix.

where $\eta_\sigma \equiv \eta_{\sigma,C}\eta_{\sigma,P}\eta_{\sigma,T}$. This implies the charge transformation:

$$\Theta Q_\sigma(t)\Theta^{-1} = \int \frac{d^3\mathbf{k}}{2\omega_{\mathbf{k},\sigma}(2\pi)^3}(b^\dagger_{\mathbf{k},\sigma}(-t)b_{\mathbf{k},\sigma}(-t) - a^\dagger_{\mathbf{k},\sigma}(-t)a_{\mathbf{k},\sigma}(-t)) = -Q_\sigma(-t).$$

By repeating the same reasoning as in Sec. IV we obtain for the flavor current

$$\mathcal{J}^\mu_{\sigma\to\rho}(x) = \mathcal{J}^\mu_{\bar\sigma\to\bar\rho}(-x), \tag{121}$$

i.e., *flavor oscillations are CPT invariant*.

VI. CONCLUSIONS AND OUTLOOK

In this paper, we have studied the nontrivial behavior of flavor states with respect to Poincaré and *C*, *P* and *T* symmetry and we argued that flavor states are not compatible with Poincaré symmetry. Instead of extending Poincaré, as proposed in Ref. [39], we show that the flavor Fock space constructed *à la* Refs. [7,8,41,42], naturally leads to Poincaré SSB, with the residual symmetry of the vacuum state being $E(3)$. This SSB is caused by the nontrivial flavor condensate structure [see Eqs. (24)–(26)], which, however, becomes phenomenologically insignificant for ultrarelativistic modes and also for mixing angle $\theta = 0$.

In order to demonstrate our point, we analyzed the properties of flavor creation and annihilation operators under Poincaré and discrete symmetry transformations, in a toy-model describing a flavor scalar doublet with mixing. Moreover, we have defined *flavor vacuum manifold* as the set of flavor-degenerate states (all with zero-flavor charge). We have provided explicit examples of flavor vacua at label times, and in different Lorentz frames. With the help of the Fabri–Picasso theorem we showed that the respective flavor Fock spaces are unitarily inequivalent. We also proved that time-reversal and *CPT* symmetries are spontaneously broken, while *CP* symmetry is exact, in our two-flavor case, as expected. However, this type of SSB of Poincaré and *CPT* symmetry does not imply the presence of any Goldstone bosons or Poincaré or *CPT* violating effects in the flavor oscillations formula, which is of phenomenological interest. This is, in our knowledge, the first general proof of Lorentz invariance of flavor oscillation formula: Lorentz invariance of standard oscillation formula was only proved in the ultrarelativistic case [15], where it coincides with Eq. (88).

Nonetheless, we should remark at this stage that the flavor-vacuum energy term, associated with the Lorentz- and *CPT*-violating flavor condensate, might have other nontrivial phenomenological consequences, when the model is properly extended to cosmology. Indeed, it is known [12], that the nonperturbative condensate of flavor-vacua leads to novel contributions to dark energy. Our current work points to the fact that such contributions break spontaneously the Lorentz and CPT symmetries of the Universe ground state. It would then be interesting to study the effects of such flavor-induced Lorentz- and CPT-violating effects [cf. the vector vacuum expectation value (89)] on the early Universe, such as their imprint on cosmic microwave background, inflationary perturbations, etc.

It should be stressed that our analysis is related to the problem of dynamical mixing generation [24–27]. In fact, in such a context one can explain the origin of Poincaré and CPT symmetry breaking together with the origin of field mixing. In this direction, another interesting possibility is that such a mechanism, when properly extended to chiral fermions, could lead, through the Lorentz- and CPT-violating flavor-vacuum chiral condensates, to phenomena like the *chiral magnetic effect* [55] in quantum chromodynamics: the Lorentz violating condensate on flavor vacuum can act as a finite temperature background, where a current J is dynamically generated in regions with an external magnetic field. We reserve a further detailed study of such speculative issues for a future work.

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APPENDIX A: BASIC STRUCTURE OF THE POINCARÉ GROUP

In order to fix the notation and the conventions, we briefly review the main features of Lorentz and Poincaré group, following Ref. [43]. Given the Minkowski space $(\mathbb{R}^4, \mathrm{d}s^2)$ where ds^2 is the indefinite quadratic form:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{A1}$$

and $g = \mathrm{diag}(1, -1, -1, -1)$ is the metric tensor.

The *homogeneous Lorentz group* is the set of transformations which leave unchanged the quadratic form (A1). This definition can be expressed from the relation:

$$g_{\mu\nu}\Lambda^\mu_\lambda \Lambda^\nu_\sigma = g_{\lambda\sigma}. \tag{A2}$$

Because of the symmetry of the metric tensor these are 10 independent constraints. Therefore, the Lorentz group has six parameters. If in Eq. (A2) we put $\lambda = \sigma = 0$ we find the condition

$$(\Lambda^0_0)^2 - \sum_{i=1}^{3} (\Lambda^i_0)^2 = 1, \tag{A3}$$

and then, $(\Lambda^0_0)^2 \geq 0$, i.e., $\Lambda^0_0 \geq 0$ or $\Lambda^0_0 \leq 0$. Considering only the transformations continuously connected with the identity we must choose only the first condition. Moreover

$$(\det \Lambda)^2 = 1. \tag{A4}$$

Because we are limiting ourselves to transformations that are continuously connected with the identity, we must consider only the case $\det \Lambda = 1$. These two choices define the *proper orthochronous Lorentz group* $SO^+_\uparrow(3, 1)$. If these restrictions are dropped (e.g., when discrete P and T symmetries are included) one speaks about the *full Lorentz group*.

The spatial part of Eq. (A2) can be rewritten as the condition

$$R^{-1} = R^T, \tag{A5}$$

that defines the group of $O(3)$ matrices. The condition on the determinant is fulfilled by $SO(3)$ matrices which thus define a three parameters subgroup of the proper Lorentz group. A second large (3-parametric) class of Lorentz transformations consists of the so-called *Lorentz boosts* (or special Lorentz transformations). These represent class of rotationfree Lorentz transformation. The Lorentz boosts do not form a group—successive boosts along nonparallel directions do not yield a boost, but the combination of a boost and spatial rotation. For instance, a Lorentz boost along the x axis is of the form:

$$L_1 = \begin{bmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A6}$$

This represents the transformation between an inertial frame and another inertial frame, moving along the x axis with velocity $v = c \tanh\xi$. The parameter ξ is known as *rapidity* and since $-c \leq v \leq c$ one has that $-\infty < \xi < +\infty$, so the full Lorentz group, which is indicated with $SO(3, 1)$, is noncompact. One can also prove that a general Lorentz transformation within $SO^+_\uparrow(3, 1)$ can be decomposed in terms of boosts and rotations as:

$$\Lambda = R(\alpha, \beta, 0)L_3(\xi)R(\phi, \theta, \psi)^{-1}, \tag{A7}$$

where the rotation matrix are written in terms of Euler’s angles.

The *inhomogeneous Lorentz group* or *Poincaré group*, includes also spacetime translations, whose group is indicated with $T^{3,1}$. It can be thus indicated as $T^{3,1} \rtimes O(3,1)$ (or $ISO(3,1) \equiv T^{3,1} \rtimes SO(3,1)$ for transformations continuously connected with the identity). A generic Poincaré transformation can be written as:

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu}. \tag{A8}$$

Therefore the Poincaré group is a ten parameters group.

Let us now consider an infinitesimal transformation, to determine the Lie algebra associated with the Poincaré group $ISO(3,1)$. First we take into account spacetime translations. An infinitesimal translation can be written as

$$T(\delta b) = \mathbb{I} + i\delta b^{\mu} P_{\mu}. \tag{A9}$$

As known P_{μ} is the four momentum operator. An infinitesimal Lorentz transformation can be written as

$$\Lambda(\delta\omega) = \mathbb{I} - \frac{i}{2}\delta\omega_{\mu\nu} J^{\mu\nu}, \tag{A10}$$

where $\delta\omega^{\mu\nu}$ is an antisymmetric matrix (has six independent parameters). We have seen that, considering only the spatial indexes, these transformations coincides with $SO(3)$ elements. An infinitesimal rotation can be written as

$$R(\delta\theta) = \mathbb{I} - i\delta\vartheta_k J^k. \tag{A11}$$

We are then led to do the following identifications:

$$\delta\vartheta_k = \varepsilon_{klm}\delta\omega_{lm}, \quad \varepsilon_{lmk} J^k = -J_{lm}, \quad k,l,m = 1,2,3. \tag{A12}$$

In the same way a Lorentz boost can be written as

$$\Lambda(\delta\xi) = \mathbb{I} - i\delta\xi^k K_k, \tag{A13}$$

identifying

$$\delta\xi^m = \delta\omega_{0m}, \qquad K_m = J^{0m}, \quad m = 1,2,3. \tag{A14}$$

One can thus derive the Poincaré algebra:

$$[P_{\mu}, P_{\lambda}] = 0, \tag{A15}$$

$$[P_{\mu}, J_{\lambda\sigma}] = i(P_{\lambda} g_{\mu\sigma} - P_{\sigma} g_{\mu\lambda}), \tag{A16}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i(J_{\lambda\nu} g_{\mu\sigma} - J_{\sigma\nu} g_{\mu\lambda} + J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda}). \tag{A17}$$

APPENDIX B: TIME EVOLUTION OF FLAVOR LADDER OPERATORS

We here report the explicit form of the functions $f^{\mathbf{k}}_{\sigma\rho}$ and $g^{\mathbf{k}}_{\sigma\rho}$ introduced in Eq. (41). By using Eq. (17) we get:

$$\begin{aligned} f^{\mathbf{k}}_{AA}(t) = {} & \cos^2\theta \frac{\omega_{\mathbf{k},1}}{\omega_{\mathbf{k},A}} (|\rho^{\mathbf{k}}_{A1}|^2 e^{i(\omega_{\mathbf{k},A} - \omega_{\mathbf{k},1})t} - |\lambda^{\mathbf{k}}_{A1}|^2 e^{i(\omega_{\mathbf{k},A} + \omega_{\mathbf{k},1})t}) \\ & + \sin^2\theta \frac{\omega_{\mathbf{k},2}}{\omega_{\mathbf{k},A}} (|\rho^{\mathbf{k}}_{A2}|^2 e^{i(\omega_{\mathbf{k},A} - \omega_{\mathbf{k},2})t} - |\lambda^{\mathbf{k}}_{A2}|^2 e^{i(\omega_{\mathbf{k},A} + \omega_{\mathbf{k},2})t}), \end{aligned} \tag{B1}$$

$$\begin{aligned} f^{\mathbf{k}}_{BB}(t) = {} & \sin^2\theta \frac{\omega_{\mathbf{k},1}}{\omega_{\mathbf{k},B}} (|\rho^{\mathbf{k}}_{B1}|^2 e^{i(\omega_{\mathbf{k},B} - \omega_{\mathbf{k},1})t} - |\lambda^{\mathbf{k}}_{B1}|^2 e^{i(\omega_{\mathbf{k},B} + \omega_{\mathbf{k},1})t}) \\ & + \cos^2\theta \frac{\omega_{\mathbf{k},2}}{\omega_{\mathbf{k},B}} (|\rho^{\mathbf{k}}_{B2}|^2 e^{i(\omega_{\mathbf{k},B} - \omega_{\mathbf{k},2})t} - |\lambda^{\mathbf{k}}_{B2}|^2 e^{i(\omega_{\mathbf{k},B} + \omega_{\mathbf{k},2})t}), \end{aligned} \tag{B2}$$

$$f^{\mathbf{k}}_{\sigma\rho}(t) = \frac{\sin\theta\cos\theta}{\omega_{\mathbf{k},\rho}} \sum_{j=1}^{2} (-1)^j \omega_{\mathbf{k},j} (|\rho^{\mathbf{k}}_{\sigma j}||\rho^{\mathbf{k}}_{\rho j}| e^{i(\omega_{\mathbf{k},\sigma} - \omega_{\mathbf{k},j})t} - |\lambda^{\mathbf{k}}_{\sigma j}||\lambda^{\mathbf{k}}_{\rho j}| e^{i(\omega_{\mathbf{k},\sigma} + \omega_{\mathbf{k},j})t}) \qquad \sigma \neq \rho, \tag{B3}$$

$$g^{\mathbf{k}}_{AA}(t) = 2ie^{i\omega_{\mathbf{k},A}t}\left(\cos^2\theta \frac{\omega_{\mathbf{k},1}}{\omega_{\mathbf{k},A}} |\rho^{\mathbf{k}}_{A1}||\lambda^{\mathbf{k}}_{A1}| \sin(\omega_{\mathbf{k},1}t) + \sin^2\theta \frac{\omega_{\mathbf{k},2}}{\omega_{\mathbf{k},A}} |\rho^{\mathbf{k}}_{A2}||\lambda^{\mathbf{k}}_{A2}| \sin(\omega_{\mathbf{k},2}t)\right), \tag{B4}$$

$$g^{\mathbf{k}}_{BB}(t) = 2ie^{i\omega_{\mathbf{k},B}t}\left(\sin^2\theta \frac{\omega_{\mathbf{k},1}}{\omega_{\mathbf{k},B}} |\rho^{\mathbf{k}}_{B1}||\lambda^{\mathbf{k}}_{B1}| \sin(\omega_{\mathbf{k},1}t) + \cos^2\theta \frac{\omega_{\mathbf{k},2}}{\omega_{\mathbf{k},A}} |\rho^{\mathbf{k}}_{B2}||\lambda^{\mathbf{k}}_{B2}| \sin(\omega_{\mathbf{k},2}t)\right), \tag{B5}$$

$$g^{\mathbf{k}}_{\sigma\rho}(t) = \frac{\sin\theta\cos\theta}{\omega_{\mathbf{k},\rho}} \sum_{j=1}^{2} (-1)^{j+1} \omega_{\mathbf{k},j} (|\rho^{\mathbf{k}}_{\sigma j}||\lambda^{\mathbf{k}}_{\rho j}| e^{i(\omega_{\mathbf{k},\sigma} - \omega_{\mathbf{k},j})t} - |\lambda^{\mathbf{k}}_{\sigma j}||\rho^{\mathbf{k}}_{\rho j}| e^{i(\omega_{\mathbf{k},\sigma} + \omega_{\mathbf{k},j})t}) \qquad \sigma \neq \rho. \tag{B6}$$

At $t = 0$ we have $f_{\sigma\rho}(0) = \delta_{\sigma\rho}$ and $g_{\sigma\rho}(0) = 0$ as we would expect. Moreover, the functions $w^{\mathbf{k}}_{\sigma\rho}$ and $y^{\mathbf{k}}_{\sigma\rho}$ introduced in Eq. (43), read:

$$w_{AA}^{\mathbf{k}} = \frac{\omega_{\mathbf{k},1}^2 + \omega_{\mathbf{k},2}^2 + 2\omega_{\mathbf{k},A}^2 + \cos 2\theta(\omega_{\mathbf{k},1}^2 - \omega_{\mathbf{k},2}^2)}{4\omega_{\mathbf{k},A}}, \tag{B7}$$

$$w_{BB}^{\mathbf{k}} = \frac{\omega_{\mathbf{k},1}^2 + \omega_{\mathbf{k},2}^2 + 2\omega_{\mathbf{k},B}^2 - \cos 2\theta(\omega_{\mathbf{k},1}^2 - \omega_{\mathbf{k},2}^2)}{4\omega_{\mathbf{k},B}}, \tag{B8}$$

$$w_{\sigma\rho}^{\mathbf{k}} = y_{\rho\sigma}^{\mathbf{k}} = \frac{\sin 2\theta(\omega_{\mathbf{k},2}^2 - \omega_{\mathbf{k},1}^2)}{4\omega_{\mathbf{k},\rho}} \quad \sigma \neq \rho, \tag{B9}$$

$$y_{AA}^{\mathbf{k}} = \frac{\omega_{\mathbf{k},1}^2 + \omega_{\mathbf{k},2}^2 - 2\omega_{\mathbf{k},A}^2 + \cos 2\theta(\omega_{\mathbf{k},1}^2 - \omega_{\mathbf{k},2}^2)}{4\omega_{\mathbf{k},A}}, \tag{B10}$$

$$y_{BB}^{\mathbf{k}} = \frac{\omega_{\mathbf{k},1}^2 + \omega_{\mathbf{k},2}^2 - 2\omega_{\mathbf{k},B}^2 - \cos 2\theta(\omega_{\mathbf{k},1}^2 - \omega_{\mathbf{k},2}^2)}{4\omega_{\mathbf{k},B}}. \tag{B11}$$

Note that when there is no mixing $w_{\sigma\sigma}^{\mathbf{k}} = \omega_{\mathbf{k},\sigma}$ and the other coefficients go to zero, as expected.

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