Landau-Hall states and Berezin-Toeplitz quantization of matrix algebras

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We argue that the Landau-Hall states provide a suitable framework for formulating the Berezin-Toeplitz quantization of classical functions on a Kähler phase space. We derive the star-products for such functions in this framework and generalize the Berezin-Toeplitz quantization to matrix-valued classical functions. We also comment on how this is related to different calculations of the effective action for Hall systems.

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I. INTRODUCTION

In constructing a physical theory, as the basic postulate we must start with the full quantum theory, obtaining the classical theory as a suitable approximation for certain regimes of parameters. But the *a priori* deduction of the quantum version of the theory from experimental data is quite difficult, mainly because we obtain data using classical apparatus and hence our intuition is largely based on classical physics. So we have the process of quantization whereby classical observables, i.e., functions on a phase space, are mapped to self-adjoint operators on a Hilbert space. There are many quantization approaches developed to deal with the inherent ambiguities associated with this mapping due to operator ordering issues, self-adjointness problems, etc. These include various correspondence principles, geometric quantization, deformation quantization, etc. The Berezin-Toeplitz (BT) quantization [1–4] is a procedure for mapping a classical function $A(z,\bar{z})$ to an operator by defining the matrix elements A_{ij} of the corresponding operator $\hat{A}$ as

$$A_{ij} = \int_{\mathcal{M}} dV\, \Psi_i^* A(z,\bar{z}) \Psi_j. \tag{1}$$

Here $\mathcal{M}$ is a complex Kähler manifold with complex coordinates z^α, $\bar{z}^\alpha$. Ψ_i are a complete set of coherent state wave functions on $\mathcal{M}$ satisfying a holomorphicity condition, and dV is the volume element for $\mathcal{M}$. The function $A(z,\bar{z})$ is known as the contravariant symbol for the operator $\hat{A}$. As an example, if we consider a Kähler manifold of the coset type, i.e., $\mathcal{M} = G/H$ for a compact Lie group G, with H being a suitable subgroup, the coherent states are of the form

$$\Psi_k = \sqrt{\dim J}\langle J,k|g|J,w\rangle \tag{2}$$

where $\langle J,k|g|J,l\rangle$ denotes the (k,l)-matrix element of the group element g in a representation denoted as J. The state $|J,w\rangle$ is to be chosen as a state (or set of states) carrying a specific representation of H.

Equation (1) describes the transition from a classical function to an operator. The converse question is to obtain a classical function given an operator. This is done by the covariant symbol, which can be defined as

$$(A) = \mathcal{C}\sum_{k,l} \Psi_k A_{kl} \Psi_l^* \tag{3}$$

where $\mathcal{C}$ is a factor depending on normalizations. For our example of $\mathcal{M} = G/H$, $\mathcal{C} = (1/\dim J)$.

The contravariant and covariant symbols are not exact inverses in the sense that if we start from $A(z,\bar{z})$, construct A_{kl} using (1) and then use (2), the (A) so obtained is not $A(z,\bar{z})$. The exact inverse process would be to identify a function $A(z,\bar{z})$ such that (1) holds where we are given A_{kl} as the input information. The answer to this is the diagonal coherent state representation [5].

There are a couple of questions which arise naturally given this layout of BT quantization. The first is: How do we define a star product which realizes the operator algebra at the level of the contravariant symbols? Explicit formulas for the star product have been obtained before (see [4] for a review), but we will argue that an easier approach involves considering the coherent states as corresponding to the lowest Landau levels of a quantum Hall problem [6–8]. This embedding of the problem in the larger framework

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transforms it to a field theory problem and gives a simple way to write the star product.[1]

The second question we might ask is about the BT quantization of matrix-valued classical functions. There are situations where such functions, with a noncommutative matrix algebra, can arise already at the classical level. (Defining a classical field theory with nonabelian symmetries on a noncommutative space would be one example.) Framing this question as a Landau problem, we can give a definition and construct the corresponding star product. The latter reduces to the matrix algebra at the lowest order, as expected.

There is another somewhat nuanced issue on the physics side of things which is clarified by this work. For the quantum Hall states one can define an effective action in terms of the external gauge fields which is obtained by integrating out the fermion fields [10]. This is standard procedure in the field theory and involves virtual transitions between the lowest Landau level and the higher levels [11]. On the other hand, one can just consider the subspace of states in the lowest Landau level and calculate an effective action [6]. This would involve the use of covariant symbols for operators. The embedding of BT quantization in the framework of the Landau problem shows that the first procedure is identical to the use of the contravariant symbol, clarifying the relation between these two approaches.

What is outlined in the previous three paragraphs summarize the key results of this paper. As for the rest of this paper, in Sec. II, we consider the case of S^2, construct the star products and show consistency with the expected asymptotic behavior. In Sec. III, we do the analysis for matrix-valued functions. In the discussion, we make more specific comments on the relevance to the calculation of the effective action for the Hall problem.

II. LANDAU PROBLEM ON S^2

We consider the two-sphere S^2 as a complex manifold $\mathbb{CP}^1$. It can also be considered as $SU(2)/U(1)$. This tells us that the Riemann curvature tensor of S^2 takes values in the Lie algebra of $U(1)$ and that it is constant in a suitable choice of frames. One can then consider an additional background $U(1)$ field which is proportional to the curvature and hence is constant on S^2. Such a field would be like the magnetic field of a magnetic monopole sitting at the center if we consider the S^2 as embedded in $\mathbb{R}^3$ in the usual way. The Landau problem refers to the dynamics of a charged particle in such a background field [6,12]. It is described by the Hamiltonian

$$H = -\frac{\mathcal{D}^2}{2m} \tag{4}$$

where $\mathcal{D}_i$ is the covariant derivative on the sphere in the background of the constant magnetic field, $\mathcal{D}^2$ being the covariant Laplacian. (m is the mass of the particle.) This can be phrased in terms of the generators of the group $SU(2)$. Translation operators on the sphere in complex coordinates correspond to the group generators $R_\pm$, which obey the $SU(2)$ Lie algebra relations

$$[R_+, R_-] = 2R_3, \qquad [R_3, R_\pm] = \pm R_\pm \tag{5}$$

where the third generator R_3 corresponds to the $U(1)$ generator. The Hamiltonian takes the form

$$H = \frac{R_+R_- + R_-R_+}{4mr^2} \tag{6}$$

the identification being $\mathcal{D}_\pm = iR_\pm/r$. Here r is a scale factor corresponding to the radius of the sphere. Since $[D_+, D_-] = 2B$, where B is the magnetic field, we see, by comparing this to the commutator for $R_\pm$, that we need $R_3 = -n/2$ on the states of interest, where $n = 2Br^2$. The fact that n must be an integer to obtain unitary representations of $SU(2)$ is just the standard Dirac quantization condition.

The eigenfunctions of the Hamiltonian are then easy to construct. Let g denote an element of the group $SU(2)$ in the fundamental representation, as a 2×2 matrix. Explicitly, on a coordinate patch, we can parametrize g as

$$g = \frac{1}{\sqrt{1+\bar{z}z}}\begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix}\begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \tag{7}$$

corresponding to complex coordinates z, $\bar{z}$ for one coordinate patch on S^2, they are the coordinates defined by a stereographic projection of S^2; φ is the angular parameter for which R_3 is the translation operator.

We can define the left and right action of the group generators on g by

$$L_a g = t_a g, \qquad R_a g = g t_a, \quad a = 1,2,3, \tag{8}$$

where $t_a = \frac{1}{2}\sigma_a$, σ_a being the Pauli matrices. The eigenfunctions of the Hamiltonian can be obtained in terms of the representative of g in an arbitrary representation where the eigenvalue for the action of R_3 is fixed to be $-\frac{n}{2}$. Explicitly, they are given by

$$\Psi_k^{(q)} = \sqrt{n+2q+1}\, D^{(q)}_{k,-\frac{n}{2}}$$
$$D^{(q)}_{k,-\frac{n}{2}} = \left\langle \frac{n}{2}+q, k|\hat{g}|\frac{n}{2}+q, -\frac{n}{2}\right\rangle \tag{9}$$

[1]The Landau-Hall framework is close to the formulation of fuzzy spaces. For a discussion of BT quantization as applied to matrix models relevant for M-theory, see [9].

in terms of the standard notation for $SU(2)$ eigenstates as $|j,m\rangle$. Here $j=\frac{n}{2}+q$; q is a positive semidefinite integer taking values 0,1,2, etc., corresponding to the various energy levels. These wave functions are L^2-normalized with the standard Haar measure on the group,

$$\int d\mu \Psi_k^{(q)*}\Psi_l^{(q)}=\delta_{kl} \tag{10}$$

where $d\mu$ is the volume element for the group. For these states, it is also easy to see that

$$R_+R_-\Psi_k^{(q)}=(R^2-R_3^2+R_3)\Psi_k^{(q)}=(qn+q(q+1))\Psi_k^{(q)}. \tag{11}$$

Notice that the group generators L_a for the left action defined by (8) commute with the Hamiltonian and so the states in (9) are degenerate for all values of k for a given $j=\frac{n}{2}+q$. The degeneracy is therefore $2j+1=n+2q+1$, for a given q. We may also note that $\Psi_k^{(q)}$ are not, properly speaking, functions on S^2, since they have a nontrivial transformation under R_3. They are sections of a $U(1)$-bundle over S^2.

It is useful to look at the lowest states in some detail. These correspond to $q=0$, so $-\frac{n}{2}$ is the lowest possible value for m, in the $|j,m\rangle$ notation, since $j=\frac{n}{2}$. We thus have

$$R_-\Psi_k^{(0)}=0. \tag{12}$$

This is a holomorphicity condition on the lowest set of eigenfunctions, corresponding to $q=0$. We may regard $\Psi_k^{(0)}$ as wave functions corresponding to coherent states for the two-sphere. They can be obtained by straightforward geometric quantization of the canonical structure [13]

$$\Omega=n\omega=\frac{n}{2}[-i\mathrm{Tr}(\sigma_3 g^{-1}dgg^{-1}dg)]. \tag{13}$$

Here ω is the Kähler two-form on $S^2=SU(2)/U(1)$. Thus $\Psi_k^{(0)}$ correspond to sections of the nth power of the canonical line bundle of $SU(2)/U(1)$. These lowest levels could be obtained, via geometric quantization of (13), without the need for the full set of eigenfunctions for the Hamiltonian (6). They form an orthonormal basis for a Hilbert space $\mathcal{H}_0$ of dimension $n+1$.

The Hilbert space $\mathcal{H}_0$ is the space of interest for us. Given a function $A(z,\bar{z})$ on the classical phase space S^2, we can define an operator $\hat{A}$ acting on $\mathcal{H}_0$ by its matrix elements as

$$A_{kl}\equiv\langle k|\hat{A}|l\rangle=\int d\mu \Psi_k^{(0)*}A(z,\bar{z})\Psi_l^{(0)} \tag{14}$$

This correspondence of assigning an operator to the function $A(z,\bar{z})$ is the Berezin-Toeplitz (BT) quantization of $A(z,\bar{z})$. If $\hat{A}$ is taken as the given quantity, the function $A(z,\bar{z})$ which leads to it via (14) is referred to as the contravariant symbol for $\hat{A}$. Starting from the operator, one can also define a covariant symbol which is a function on the phase space S^2, given by

$$(A)=D^{(0)}_{k,-\frac{n}{2}}A_{kl}D^{(0)*}_{l,-\frac{n}{2}}. \tag{15}$$

Notice that the group elements are used in this definition, not the correctly normalized wave functions.

The contravariant and covariant symbols are converses of each other albeit in a qualified sense: In the first case we start with the function $A(z,\bar{z})$ on S^2 and define the associated operator via (14). In the second case, we start with the matrix elements of the operator and define a function, namely (A), on S^2. If we start from $A(z,\bar{z})$ and define the operator and take its covariant symbol, in general, we do not get back the function $A(z,\bar{z})$. The map $A(z,\bar{z})\to(A)$ is the Berezin transform [4]. We will see shortly that as $n\to\infty$, (A) becomes the same as $A(z,\bar{z})$.

The construction of star products for the covariant symbol is fairly simple [6,8]. The symbol for the product of two operators, say A and B, is, by definition,

$$(AB)=D^{(0)}_{k,-\frac{n}{2}}A_{kl}B_{lp}D^{(0)*}_{p,-\frac{n}{2}}. \tag{16}$$

We can insert $\delta_{ml}=D^{(0)*}_{m,r}D^{(0)}_{l,r}$ into this expression and simplify it as follows.

$$\begin{aligned}(AB)&=D^{(0)}_{k,-\frac{n}{2}}A_{km}D^{(0)*}_{m,r}D^{(0)}_{l,r}B_{lp}D^{(0)*}_{p,-\frac{n}{2}}\\&=D^{(0)}_{k,-\frac{n}{2}}A_{km}D^{(0)*}_{m,-\frac{n}{2}}D^{(0)}_{l,-\frac{n}{2}}B_{lp}D^{(0)*}_{p,-\frac{n}{2}}\\&\quad+D^{(0)}_{k,-\frac{n}{2}}A_{km}D^{(0)*}_{m,-\frac{n}{2}+1}D^{(0)}_{l,-\frac{n}{2}+1}B_{lp}D^{(0)*}_{p,-\frac{n}{2}}+\cdots\\&=(A)(B)-\frac{1}{n}D^{(0)}_{k,-\frac{n}{2}}A_{km}R_-D^{(0)*}_{m,-\frac{n}{2}}R_+D^{(0)}_{l,-\frac{n}{2}}B_{lp}D^{(0)*}_{p,-\frac{n}{2}}+\cdots\\&=(A)(B)-\frac{1}{n}R_-(A)R_+(B)+\cdots\\&=(A)*(B)\end{aligned} \tag{17}$$

where we have used

$$D^{(0)}_{l,-\frac{n}{2}+1}=\frac{1}{\sqrt{n}}R_+D^{(0)}_{l,-\frac{n}{2}},\qquad D^{(0)*}_{m,-\frac{n}{2}+1}=-\frac{1}{\sqrt{n}}R_-D^{(0)*}_{m,-\frac{n}{2}}. \tag{18}$$

We have also used the fact that, since $R_- D^{(0)}_{k,-\frac{n}{2}} = 0$, we can write

$$D^{(0)}_{k,-\frac{n}{2}} A_{km} R_- D^{(0)*}_{m,-\frac{n}{2}} = R_-\left(D^{(0)}_{k,-\frac{n}{2}} A_{km} D^{(0)*}_{m,-\frac{n}{2}} \right) = R_-(A) \tag{19}$$

with a similar simplification for the symbol for B. It is clear that the higher terms in (17) can be simplified in a similar way and written in terms of $R^s_-(A)R^s_+(B)$ for $s > 1$. For the covariant symbol as we have defined it, the series terminates, for finite n.

The construction of a star product for the contravariant symbol is more involved. Here the question is to find a product of the classical functions $A(z,\bar{z})$ and $B(z,\bar{z})$ such that the BT quantization of the product gives the product of the operators. The product of the BT quantized operators is given by

$$\begin{aligned} A_{kl}B_{lm} &= \int d\mu \Psi^{(0)*}_k A(z,\bar{z}) \Psi^{(0)}_l \int d\mu' \Psi^{(0)*}_l B(z',\bar{z}') \Psi^{(0)}_m \\ &= \int d\mu d\mu' \Psi^{(0)*}_k A(z,\bar{z}) P^{(0)}(g,g') B(z',\bar{z}') \Psi^{(0)}_m \end{aligned} \tag{20}$$

where $P^{(0)}(g,g')$ is the projection operator for the lowest Landau level,

$$P^{(0)}(g,g') = \sum_l \Psi^{(0)}_l(g) \Psi^{(0)*}_l(g'). \tag{21}$$

We denote the arguments as g and g' for brevity, although $P^{(0)}$ is defined on the coset $SU(2)/U(1)$, i.e., independent of the $U(1)$ angle φ. If we consider similar projection operators to the higher levels, we have, by completeness of all the eigenstates of the Hamiltonian (4),

$$\begin{aligned} \delta(g,g') &= \sum \Psi^{(q)}_k(g) \Psi^{(q)*}_k(g') \\ &= \sum_q P^{(q)}(g,g') \\ &= P^{(0)}(g,g') + P^{(1)}(g,g') + \cdots \end{aligned} \tag{22}$$

Our strategy will be to write $P^{(0)}(g,g')$ in terms of $\delta(g,g')$ and derivatives acting on $\delta(g,g')$. Notice that the action of R_+R_- on $P^{(0)}(g,g')$ is zero, due to (12). We also have the result

$$R_+R_- \Psi^{(q)}_k(g) = (qn + q(q+1)) \Psi^{(q)}_k(g). \tag{23}$$

Using this result, we can eliminate $P^{(1)}(g,g')$ from (22) and write

$$\begin{aligned} &\delta(g,g') - \frac{1}{(n+2)} R_+R_-\delta(g,g') \\ &\quad = P^{(0)}(g,g') + \sum_{q=2}^{\infty} \left[1 - \frac{qn + q(q+1)}{(n+2)} \right] P^{(q)}(g,g') \end{aligned} \tag{24}$$

If we use this result for $P^{(0)}(g,g')$ in (20) we can write $A(z,\bar{z})P^{(0)}(g,g')B(z',\bar{z}')$ in terms of $A(z,\bar{z})B(z',\bar{z}')$ and products of derivatives of these functions, as we would expect for a star product. This is the basic idea. To carry this out to higher orders, we need to eliminate $P^{(q)}$, $q \geq 2$, at least recursively. Therefore, more generally, we start by writing $P^{(0)}$ as

$$P^{(0)}(g,g') = \left[1 + \sum_s c_s R^s_+ R^s_- \right] \delta(g,g') \tag{25}$$

for some constant coefficients c_s. The key property we need is that $R^s_- \Psi^{(q)}_k(g) = 0$ for $s > q$, so we can recursively define the coefficients c_s to eliminate the contribution of the higher levels in $\delta(g,g') = \sum \Psi^{(q)}_k(g) \Psi^{(q)*}_k(g')$. The conditions we need are

$$\left[1 + \sum_{s=1}^{q} c_s R^s_+ R^s_- \right] \Psi^{(q)}_k(g) = 0. \tag{26}$$

It is easy to work out the action of $R^s_+R^s_-$ on the wave functions,

$$\begin{aligned} R^s_+R^s_- \Psi^{(q)}_k(g) &= \left[\prod_1^s f(q,s) \right] \Psi^{(q)}_k(g) \\ f(q,s) &= [n(q-s+1) + q(q+1) - s(s-1)]. \end{aligned} \tag{27}$$

One can recursively calculate c_s. The lowest two coefficients are

$$\begin{aligned} c_1 &= -\frac{1}{f(1,1)} = -\frac{1}{(n+2)}, \\ c_2 &= \frac{1}{f(2,2)} \left(\frac{1}{f(1,1)} - \frac{1}{f(2,1)} \right) = \frac{1}{2(n+2)(n+3)}. \end{aligned} \tag{28}$$

Using (25) in (20), we get

$$
\begin{aligned}
A_{kl}B_{lm} &= \int d\mu d\mu' \Psi_k^{(0)*} A(z,\bar{z}) P^{(0)}(g,g') B(z',\bar{z}') \Psi_m^{(0)} \\
&= \int d\mu \Psi_k^{(0)*} A(z,\bar{z}) \left[1 + \sum_s c_s R_+^s R_-^s\right] B(z',\bar{z}') \Psi_m^{(0)} \\
&= \int d\mu \Psi_k^{(0)*} A(z,\bar{z}) B(z,\bar{z}) \Psi_m^{(0)} + \int d\mu \sum_s (-1)^s c_s (R_+^s \Psi_k^{(0)*} A(z,\bar{z}))(R_-^s B(z',\bar{z}') \Psi_m^{(0)}) \\
&= \int d\mu \Psi_k^{(0)*} A(z,\bar{z}) B(z,\bar{z}) \Psi_m^{(0)} + \int d\mu \sum_s (-1)^s c_s \Psi_k^{(0)*} (R_+^s A(z,\bar{z}))(R_-^s B(z',\bar{z}')) \Psi_m^{(0)} \\
&= \int d\mu \Psi_k^{(0)*} [A(z,\bar{z}) * B(z,\bar{z})] \Psi_m^{(0)}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
&A(z,\bar{z}) * B(z,\bar{z}) \\
&\quad = A(z,\bar{z}) B(z,\bar{z}) + \sum_{s=1}^{\infty} (-1)^s c_s (R_+^s A(z,\bar{z}))(R_-^s B(z',\bar{z}')).
\end{aligned} \tag{30}
$$

In (29), in the second step, we did integration by parts to move R_+^s to act on $\Psi_k^{(0)*} A(z,\bar{z})$ and then used the fact that $R_+ \Psi_k^{(0)*} = 0$. In this way we are able to isolate the product of functions which, upon BT quantization, reproduces the operator product. Notice that this is an infinite series even for finite n, unlike the star product we defined on the covariant symbols. The first two terms of this star product can be written out, using (28), as

$$
\begin{aligned}
A * B = AB &+ \frac{1}{(n+2)} (R_+ A)(R_- B) \\
&+ \frac{1}{2(n+2)(n+3)} (R_+^2 A)(R_-^2 B) + \cdots
\end{aligned} \tag{31}
$$

This star product is different from what we found for the covariant symbol. This is to be expected since the covariant symbol is different from the contravariant symbol. However, notice that, in the large n limit,

$$
\begin{aligned}
(A) * (B) - (B) * (A) &= \frac{1}{n} [R_+(A) R_-(B) - R_-(A).R_+(B)] \\
&\quad + \mathcal{O}(1/n^2) \\
&= A * B - B * A + \mathcal{O}(1/n^2).
\end{aligned} \tag{32}
$$

The Poisson bracket is recovered for the star commutator for both cases in the large n limit.

We now return to the relation between the contravariant and covariant symbols. The Berezin transform for A is given by

$$
\begin{aligned}
(A) &= D^{(0)}_{k,-\frac{n}{2}} \left[\int d\mu' \Psi^{(0)*}(g') A(z',\bar{z}') \Psi_l^{(0)}(g')\right] D^{(0)*}_{l,-\frac{n}{2}} \\
&= \int d\mu' P^{(0)}(g,g') A(z',\bar{z}') D^{(0)}_{k,-\frac{n}{2}}(g') D^{(0)*}_{k,-\frac{n}{2}}(g) \\
&= A(z,\bar{z}) + \sum_s c_s R_+^s R_-^s \left(A D^{(0)}_{k,-\frac{n}{2}}(g)\right) D^{(0)*}_{k,-\frac{n}{2}}(g) \\
&= A(z,\bar{z}) + \sum_s c_s R_+^s \left((R_-^s A) D^{(0)}_{k,-\frac{n}{2}}(g)\right) D^{(0)*}_{k,-\frac{n}{2}}(g) \\
&= A(z,\bar{z}) + \sum_s c_s (R_+^s R_-^s A) D^{(0)}_{k,-\frac{n}{2}}(g) D^{(0)*}_{k,-\frac{n}{2}}(g) \\
&= A(z,\bar{z}) + \sum_s c_s (R_+^s R_-^s A).
\end{aligned} \tag{33}
$$

We have used (25) to obtain the third line of this equation, and $R_- D^{(0)}_{k,-\frac{n}{2}}(g) = 0$ to move to the fourth line. Further, we have $D^{(0)}_{k,-\frac{n}{2}}(g) D^{(0)*}_{k,-\frac{n}{2}}(g) = 1$ by the group property and

$$
\begin{aligned}
\sum_k \left[R_+^s D^{(0)}_{k,-\frac{n}{2}}(g)\right] D^{(0)*}_{k,-\frac{n}{2}}(g) &\sim D^{(0)}_{k,-\frac{n}{2}+s}(g) D^{(0)*}_{k,-\frac{n}{2}}(g) \\
&\sim \left\langle -\frac{n}{2} | -\frac{n}{2} + s \right\rangle = 0.
\end{aligned} \tag{34}
$$

This was used in the last two steps. We see that, in the large n limit, the two symbols will coincide. Further,

$$
\begin{aligned}
(A) &= A - \frac{1}{(n+2)} R_+ R_- A + \cdots \\
&= A + \frac{1}{(n+2)} \Delta A + \cdots
\end{aligned} \tag{35}
$$

where Δ is the Laplace operator on S^2. This result is consistent with the theorem of Karabegov and Schlichenmaier, quoted as theorem 7.2 in [4], although our derivation is very different.

III. BT QUANTIZATION FOR MATRIX-VALUED FUNCTIONS

In the last section, the wave functions we used were those for the Landau problem on S^2 with an Abelian $U(1)$ background field. In other words, they were sections of an appropriate line bundle. The lowest set of such wave functions, which were sections of a holomorphic line bundle, were then used to define the BT quantization of a function on S^2. Generalizing, we can use the sections of a suitably chosen vector bundle to define the BT quantization of a matrix-valued function. Naturally, this will mean using a background field corresponding to a nonabelian group H, the wave functions being those of a particle transforming nontrivially according to some representation of H [6]. We will now work out an example of how this can be done using $\mathbb{CP}^2 = SU(3)/U(2)$ as the manifold of interest. The group translation operators can be separated into R_a, $a = 1, 2, 3$, which are generators of $SU(2) \in U(2) \in SU(3)$, R_8 which is the generator for $U(1) \in U(2)$ and the remaining coset generators R_{+i}, R_{-i}, $i = 1, 2$. The last set $R_{\pm i}$ are the translation generators for $\mathbb{CP}^2$, with the Hamiltonian taken as

$$H = \frac{R_{+i}R_{-i}}{2mr^2}. \tag{36}$$

The eigenfunctions are given by

$$\Psi_{A;a} = C\langle A|\hat{g}|\alpha\rangle. \tag{37}$$

C is a normalization constant, we will discuss this below. The index A labels states in the $SU(3)$ representation to be specified by the choice of $|\alpha\rangle$. We will consider a combination of $U(1)$ and $SU(2)$ background fields. Thus the state $|\alpha\rangle$ must be chosen so that it transforms according to the required representation of $SU(2)$ (equivalent to specifying the $SU(2)$ charges) and also carries the required $U(1)$ charge. This means

$$\begin{aligned} R_a|\alpha\rangle &= (T_a)_{\alpha\beta}|\beta\rangle \\ R_8|\alpha\rangle &= -\frac{n}{\sqrt{3}}|\alpha\rangle. \end{aligned} \tag{38}$$

$(T_a)_{\alpha\beta}$ are the charge matrices for the coupling of the particle to the constant $SU(2)$ background field. In the tensor notation T^P_Q for $SU(3)$ representations with P up-indices and Q down-indices (where each index can take values 1,2,3), the possible choices for the states $|\alpha\rangle$ are then of the form

$$|\alpha\rangle = |P;Q\rangle = |33\cdots3;33\ldots3;i_1\cdots i_l\rangle \tag{39}$$

This corresponds to q down-indices all set to 3, with an additional l indices, each of which can take values 1,2 corresponding to the $SU(2)$ spinors. There are also $n - (l/2) + q$ up-indices all set to 3. Thus this state transforms as the spin-$l/2$ representation of $SU(2)$, i.e., $(T_a)_{\alpha\beta}$ are $(l+1)\times(l+1)$ matrices, and has the R_8 value

$$\begin{aligned} &R_8|33\cdots3;33\ldots3;i_1\cdots i_l\rangle \\ &\quad = -\frac{1}{\sqrt{3}}\left(n - \frac{1}{2}l + q\right) + \frac{1}{\sqrt{3}}q - \frac{l}{2\sqrt{3}} = -\frac{n}{\sqrt{3}} \end{aligned} \tag{40}$$

as required in (38). Notice that l should be an even integer, to get an integer number of up-indices, so that we must have integer spin representations for $SU(2)$. (Ultimately, this is related to the fact that $\mathbb{CP}^2$ does not admit a spin structure.) The eigenvalues of $R_{+i}R_{-i}$ are given by

$$\begin{aligned} R_{+i}R_{-i}\Psi^{(q)}_{A;\alpha} &= [C_2(SU(3)) - C_2(SU(2) - R_8^2 + \sqrt{3}R_8]\Psi^{(q)}_{A;\alpha} \\ &= \left[qn + q\left(q + 2 + \frac{1}{2}l\right)\right]\Psi^{(q)}_{A;\alpha}. \end{aligned} \tag{41}$$

The lowest state corresponding to $q = 0$ obeys the holomorphicity condition $R_{-i}\Psi^{(0)}_{A;\alpha} = 0$. The degeneracy of the states as given by the dimension of the representation is

$$N = \frac{1}{2}\left(n + q + 1 - \frac{l}{2}\right)(q + l + 1)\left(n + 2q + \frac{l}{2} + 2\right). \tag{42}$$

We now consider the normalization of these states. Wave functions such as (37) arise for fields which carry the spin-$\frac{l}{2}$ representation of $SU(2)$, so that it can be coupled to the appropriate $SU(2)$ background. Denoting such a field by ϕ_α, the action for the field may be taken as

$$S = \int dt d\mu \left[i\phi^*_\alpha \frac{\partial\phi_\alpha}{\partial t} - \phi_\alpha H_{\alpha\beta}\phi_\beta\right]. \tag{43}$$

The appropriate normalization for the one-particle wave functions should thus be

$$\int d\mu \Psi^{(q)*}_{A;\alpha}\Psi^{(q')}_{B;\alpha} = \delta_{AB}\delta^{qq'}. \tag{44}$$

(There is summation over α in this equation.) With the standard orthogonality relation for group elements in arbitrary representations, the normalized wave functions are

$$\Psi^{(q)}_{A;\alpha} = \sqrt{\frac{N}{l+1}}\langle A|\hat{g}|\alpha\rangle. \tag{45}$$

The completeness relation for these states is

$$\sum_{q,A}\Psi^{(q)}_{A;\alpha}(g)\Psi^{(q)*}_{A;\beta}(g') = \delta(g,g')\delta_{\alpha\beta}. \tag{46}$$

We are now in a position to write down the BT quantization of a matrix-valued function $A_{\alpha\beta}$ as

$$A_{CD} = \int d\mu \Psi^{(0)*}_{C;\alpha}(g) A_{\alpha\beta}\Psi^{(0)}_{D;\beta}(g). \tag{47}$$

The integrand should be a function defined on $\mathbb{CP}^2$ and hence invariant under the $U(2)$ subgroup for the integral to be nonzero. This means that the nontrivial transformation of $\Psi^{(0)*}_{C;\alpha}(g)$ and $\Psi^{(0)}_{D;\beta}(g)$ should be compensated by a suitable transformation of $A_{\alpha\beta}$. Thus strictly speaking, already at the classical level, we are not considering matrix-valued functions, but sections of a suitable vector bundle.

The product of two operators defined as in (47) is given by

$$\begin{aligned}(AB)_{CF} = A_{CD}B_{DF} &= \int d\mu d\mu' \Psi^{(0)*}_{C;\alpha}(g)A_{\alpha\beta}(g)\Psi^{(0)}_{D;\beta}(g)\Psi^{(0)*}_{D;\gamma}(g')B_{\gamma\delta}(g')\Psi^{(0)}_{F;\delta}(g') \\ &= \int d\mu d\mu' \Psi^{(0)*}_{C;\alpha}(g)A_{\alpha\beta}(g)P^{(0)}_{\beta;\gamma}(g,g')B_{\gamma\delta}(g')\Psi^{(0)}_{F;\delta}(g') \end{aligned} \tag{48}$$

$$P^{(0)}_{\beta;\gamma}(g,g') = \sum_D \Psi^{(0)}_{D;\beta}(g)\Psi^{(0)*}_{D;\gamma}(g'). \tag{49}$$

As in the case of S^2, we will now write this projection operator in terms of the Dirac δ-function as

$$P^{(0)}_{\beta;\gamma}(g,g') = \left[\delta_{\beta\gamma} + \sum_s c_s (R_{+i_1}R_{+i_2}\cdots R_{+i_s}R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1})_{\beta\gamma}\right]\delta(g,g'). \tag{50}$$

Acting on $\Psi^{(q)}_{B;\gamma}(g')$ and integrating over g', we see that c_s should obey the conditions

$$\Psi^{(q)}_{B;\beta}(g) + \sum_{s=1}^{q} c_s (R_{+i_1}R_{+i_2}\cdots R_{+i_s}R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1})_{\beta\gamma}\Psi^{(q)}_{B;\gamma}(g) = 0. \tag{51}$$

To make this more concrete, we need to evaluate the coefficient of c_s. It can be done recursively, with the result

$$\sum_s c_s (R_{+i_1}R_{+i_2}\cdots R_{+i_s}R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1})_{\beta\gamma}\Psi^{(q)}_{B;\gamma}(g) = \prod_{k=1}^{s} h(q,k)\Psi^{(q)}_{B;\beta}(g) \tag{52}$$

where

$$h(q,k) = (q-k+1)(n+q+k+1+(l/2)). \tag{53}$$

Thus the coefficients c_s are determined by

$$1 + \sum_{s=1}^{q} c_s \prod_{k=1}^{s} h(q,k) = 0. \tag{54}$$

With the expansion (50) for the projection operator, we find

$$\begin{aligned}(AB)_{CF} &= \int d\mu d\mu' \Psi^{(0)*}_{C;\alpha}(g)A_{\alpha\beta}(g)\left[\delta_{\beta\gamma}\delta(g,g') + \sum_s c_s (R_{+i_1}R_{+i_2}\cdots R_{+i_s}R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1})_{\beta\gamma}\delta(g,g')\right]B_{\gamma\delta}(g')\Psi^{(0)}_{F;\delta}(g') \\ &= \int d\mu \Psi^{(0)*}_{C;\alpha}(g)[A*B]_{\alpha\beta}\Psi^{(0)}_{F;\beta}(g). \end{aligned} \tag{55}$$

We can now read off the star-product from this equation as

$$[A*B]_{\alpha\beta} = A_{\alpha\gamma}B_{\gamma\beta} + \sum_s (-1)^s c_s (R_{+i_s}R_{+i_{s-1}}\cdots R_{+i_1}A)_{\alpha\gamma} \times (R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1}B)_{\gamma\beta} \tag{56}$$

As before we have done integrations by parts to move the R_+ s to act on A, and used the fact that $R_{-i}\Psi^{(0)}_{F;\delta} = 0$, $R_{+i}\Psi^{(0)*}_{C;\alpha} = 0$. Since A and B are defined on $G = SU(3)$, i.e., they carry nonzero charges under $U(2)$, the action of $R_{\pm i}$ should be considered as covariant derivatives.

It is straightforward to solve (54) using (53) to write the coefficients c_s for some low values of s, as we did for the two-sphere. For $q = 1, 2$, we can write out (54) as

$$\begin{aligned} 1 + c_1 h(1,1) &= 0. \\ 1 + c_1 h(2,1) + c_2 h(2,1)h(2,2) &= 0. \end{aligned} \tag{57}$$

where the relevant $h(q,k)$ are given as

$$\begin{aligned} h(1,1) &= \left(n+3+\frac{1}{2}l\right), \qquad h(2,1) = 2\left(n+4+\frac{1}{2}l\right), \\ h(2,2) &= \left(n+5+\frac{1}{2}l\right). \end{aligned} \tag{58}$$

The star-product from(56) can thus be written more explicitly as

$$\begin{aligned} [A*B]_{\alpha\beta} = A_{\alpha\gamma}B_{\gamma\beta} &+ \frac{1}{(n+3+\frac{1}{2}l)}(R_{+i}A)_{\alpha\gamma}(R_{-i}B)_{\gamma\beta} \\ &+ \frac{1}{2(n+3+\frac{1}{2}l)(n+4+\frac{1}{2}l)} \\ &\times (R_{+i}R_{+j}A)_{\alpha\gamma}(R_{-j}R_{-i}B)_{\gamma\beta} + \cdots. \end{aligned} \tag{59}$$

Once again, it is useful to see how this differs from what is obtained for the covariant symbol. The latter is defined from the matrix elements of the operator as

$$(A)_{\alpha\beta} = \sum_{C,D} D^{(0)}_{C;\alpha}(g) A_{CD} D^{(0)*}_{D;\alpha}(g). \tag{60}$$

Using (47), we can now write

$$\begin{aligned} (A)_{\alpha\beta} &= \sum_{C,D} D^{(0)}_{C;\alpha}(g)\left[\int d\mu' \Psi^{(0)}_{C;\gamma}(g')A_{\gamma\delta}(g')\Psi^{(0)}_{D;\delta}(g')\right] D^{(0)*}_{D;\alpha}(g) \\ &= \int d\mu' P^{(0)}_{\alpha;\gamma}(g,g')A_{\gamma\delta}(g')D^{(0)}_{D;\delta}(g')D^{(0)*}_{D;\alpha}(g) \\ &= A_{\alpha\beta}(g) + \sum_s c_s (R_{+i_1}R_{+i_2}\cdots R_{+i_s}R_{-i_s}R_{-i_{s-1}}\cdots R_{-i_1})_{\alpha\gamma}A_{\gamma\beta} \\ &= A_{\alpha\beta}(g) - \frac{1}{(n+3+\frac{1}{2}l)}(R_{+i}R_{-i}A)_{\alpha\beta} + \cdots \end{aligned} \tag{61}$$

where we have used (50) and the results

$$\begin{aligned} R_{-i}D^{(0)}_{D;\delta}(g) &= 0, \\ \left[R_{+i_1}R_{+i_2}\cdots R_{+i_s}D^{(0)}_{D;\delta}(g)\right] D^{(0)*}_{D;\alpha}(g) &= 0 \end{aligned} \tag{62}$$

for reasons similar to what was stated after (34). We see that the covariant and contravariant symbols agree in the large n limit. Equation (61) is our matrix generalization of the theorem 7.2 in [4].

The star-product for the covariant symbols was obtained in [6] as

$$(A)_{\alpha\gamma} * (B)_{\gamma\beta} = \left((A)_{\alpha\gamma}(B)_{\gamma\beta} - \frac{1}{n}R_{-i}(A)_{\alpha\gamma}R_{+i}(B)_{\gamma\beta}\right) + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{63}$$

The star-commutators for the contravariant symbols can be obtained from (59) as

$$[A*B - B*A]_{\alpha\beta} = [AB-BA]_{\alpha\beta} + \frac{1}{n}[(R_{+i}A)_{\alpha\gamma}(R_{-i}B)_{\gamma\beta} - (R_{+i}B)_{\alpha\gamma}(R_{-i}A)_{\gamma\beta}] + \cdots \tag{64}$$

The analogous commutator for the covariant symbols is given by (63) as

$$[(A)*(B)-(B)*(A)]_{\alpha\beta}=[(A)(B)-(B)(A)]_{\alpha\beta}-\frac{1}{n}[R_{-i}(A)_{\alpha\gamma}R_{+i}(B)_{\gamma\beta}-R_{-i}(B)_{\alpha\gamma}R_{+i}(A)_{\gamma\beta}]+\cdots. \tag{65}$$

There is a difference of the ordering of the matrix products in the second term, so these are not identical even to first order in $1/n$.

IV. THE ONE-PARTICLE FIELD THEORY

The BT quantization emerges in a completely natural way when we try to quantize a single-particle problem in the language of quantum field theory and then restrict to one, say the lowest, level. Degeneracy of this level is important to obtain the matrix structure of the operator, and holomorphicity of the one-particle wave functions for this level is important for constructing the star-products. Thus effectively a suitable version of the Landau problem becomes the paradigm for BT quantization. We have already mentioned the action for the field theory for this case, namely, (43). The field operators may be expanded in terms of eigenfunctions of the Hamiltonian as

$$\begin{aligned}\phi_\alpha&=\sum_C a_C^{(0)}\Psi_{C;\alpha}^{(0)}+\sum_{C';q=1}a_{C'}^{(q)}\Psi_{C';\alpha}^{(q)}\\ \phi_\alpha^\dagger&=\sum_C a_C^{(0)\dagger}\Psi_{C;\alpha}^{(0)*}+\sum_{C';q=1}a_{C'}^{(q)\dagger}\Psi_{C';\alpha}^{(q)*}\end{aligned} \tag{66}$$

where the second set of terms refers to the higher energy levels. $a_C^{(q)}$, $a_C^{(q)\dagger}$ are the annihilation and creation operators for the particle represented by ϕ_α.

Now consider a one-particle operator $\tilde{B}$, which can be lifted to the field theory as

$$\hat{B}=\int dV\phi^\dagger\tilde{B}\phi. \tag{67}$$

$\tilde{B}$ is, in general, a function of the coordinates, but it may contain derivatives as well. Using the mode expansion for the fields, the operator $\hat{B}$ takes the form

$$\begin{aligned}\hat{B}&=\sum_{k,l}a_C^{(0)\dagger}B_{CD}a_D^{(0)}\\ &\quad+\sum_{(q,q')\neq(0,0);C,D}\int dV\Psi_C^{(q)*}\tilde{B}\Psi_D^{(q')}a_C^{(q)\dagger}a_D^{(q')}\\ B_{CD}&=\int dV\Psi_C^{(0)*}\tilde{B}\Psi_D^{(0)}\end{aligned} \tag{68}$$

(Here we are using the simpler case of an Abelian background field to illustrate the main point.) If we consider dynamics restricted to the lowest level, i.e., higher levels with $(q,q')\neq(0,0)$ are unoccupied, $a_C^{(q)}$ will annihilate the relevant set of states and $\hat{B}$ effectively reduces to the first term. Further, if the wave functions $\Psi_C^{(0)}$ for the lowest level define coherent states, derivatives appearing in $\tilde{B}$ can be replaced in terms of conjugates of the holomorphic coordinates. Thus B_{kl} takes the form

$$B_{CD}=\int dV\Psi_C^{(0)*}B(z,\bar{z})\Psi_D^{(0)} \tag{69}$$

for some *function* $B(z,\bar{z})$ on $\mathcal{M}$. We now see that, for one-particle states restricted to the lowest level, the matrix elements of the field theory operator $\hat{B}$ are given by B_{kl} as in (69) or as in (1). Thus we get to the BT quantization of a function $B(z,\bar{z})$ on $\mathcal{M}$ from the field theory defined by (43). Although we illustrated this relation with the Abelian background, it is clear that a similar result will hold for a matrix-valued one-particle operator $\tilde{B}_{\alpha\beta}$ and a corresponding matrix-valued function $B_{\alpha\beta}(z,\bar{z})$.

The one-particle quantum mechanical calculation of the effective action in [11] uses the single-particle sector of the field theory we have outlined. From what is said above, this is equivalent to using the BT quantization for operators restricted to the lowest Landau level, with the star-product as in (31). The calculations in [11] are done for the plane, rather than S^2, and the notation is different, so it may be useful to outline the points of contact in some more detail. In the second reference of [11], the authors give a definition of an operator $\hat{A}$ for the lowest Landau level as

$$\hat{A}=\int\frac{d^2z}{\pi}e^{-\bar{z}z}e^{\bar{z}a^\dagger}|0\rangle A(z,\bar{z})\langle 0|e^{za} \tag{70}$$

where a, $a^\dagger$ are the standard creation and annihilation operators. (The notation has been changed to agree with our conventions.) Notice that this is equivalent to

$$A_{pq}=\int\frac{d^2z}{\pi}\Psi_p^*A(z,\bar{z})\Psi_q,\qquad \Psi_q=\frac{z^q}{\sqrt{q!}}e^{-\frac{1}{2}\bar{z}z} \tag{71}$$

Since Ψ_q is the coherent state wave function for the lowest Landau level on the plane, we see that this is equivalent to using the contravariant symbol $A(z,\bar{z})$ for the operator. The star-product is then worked out and used to obtain the effective action for perturbations. Another, closely related, way to write this definition of the operator, which is used in the first reference in [11], is

$$\hat{A}=\sum_{m,n}\frac{(-i)^n i^m}{m!n!}(\pi^\dagger)^n\pi^m\partial_{\bar{z}}^n\partial_z^m A(z,\bar{z})|_{z=a,\bar{z}=a^\dagger} \tag{72}$$

where $\pi^\dagger$ is the operator raising the Landau level by one and the ordering in the last expression is to keep $a^\dagger$'s to the right. Notice that, the matrix elements of this in the lowest Landau level will only involve the $m = n = 0$ term. ($\pi^\dagger$ and π are the analogs of $R_\pm$ in our formulas.) A straightforward projection of a product to the LLL can be made as $\langle p|\hat{A}\,\hat{B}\,|q\rangle$. This will need simplification of products like $\pi^m(\pi^\dagger)^k$ in the middle, a procedure which obviously corresponds to projecting out the higher Landau levels in terms of derivatives with respect to z, $\bar{z}$. The final results are similar to ours, albeit for the plane.

By contrast, the calculations in [6,7] use the covariant symbol and the star-product as in (17). The starting point is the effective action

$$\begin{aligned} S_{\rm eff} &= \int dt \mathrm{Tr}\left[P_0\left(iU^\dagger\frac{\partial U}{\partial t} - U^\dagger H_{\rm int} U\right)\right] \\ &= \int dt \mathrm{tr}\left[(P_0) * \left(i(U^\dagger) * \frac{\partial(U)}{\partial t}\right.\right. \\ &\quad \left.\left. - (U^\dagger) * (H_{\rm int}) * (U)\right)\right]. \end{aligned} \tag{73}$$

Here the entire action is written in terms of the LLL. P_0 is the occupancy matrix equal to 1 for occupied states and zero for the others; eventually P_0 can be set to 1 for all states in the LLL, for $\nu = 1$. U denotes a unitary matrix (which is $(n+1)\times(n+1)$ for S^2). Such a unitary transformation is the most general form of time-evolution keeping entirely within the LLL. $H_{\rm int}$ is the interaction part of the Hamiltonian, and the unperturbed Hamiltonian has been set to zero for the LLL. In the second line of (73) we have written the expression using the covariant symbols. It is now straightforward to use the star-product as given in (17) to simplify the action, keeping derivatives up to any desired order. In the case of a non-Abelian background field (for higher dimensional manifolds), the symbols in (73) are still matrices, of a (smaller) dimension determined by the representation of the subgroup H, say $(l+1)\times(l+1)$ for the example of $\mathbb{CP}^2$. Standard matrix products for the symbols are taken as understood with a final trace as well (denoted by tr). The star-products can still be simplified using a generalization of (17). For more details, we refer the reader to [6]. Needless to say, in all these cases, the action for perturbations is the same, namely, an appropriate Chern-Simons action, for large n.

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