# Supersymmetric nonlinear sigma models as anomalous gauge theories

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We revisit supersymmetric nonlinear sigma models on the target manifold $CP^{N-1}$ and $SO(N)/SO(N-2)\times U(1)$ in four dimensions. These models are formulated as gauged linear models, but it is indicated that the Wess-Zumino term should be added to the linear model since the hidden local symmetry is anomalous. Applying a procedure used for the quantization of anomalous gauge theories to the nonlinear models, we determine the form of the Wess-Zumino term, by which a global symmetry in the linear model becomes smaller in the action than the conventional one. Moreover, we analyze the resulting linear model in the $1/N$ leading order. Consequently, we find that the model has a critical coupling constant similar to bosonic models. In the weak coupling regime, the $U(1)$ local symmetry is broken but supersymmetry is never broken. In contrast to the bosonic case, it is impossible to find stable vacua in the strong coupling regime as far as in the $1/N$ leading order. These results are straightforwardly generalized to the case of the Hermitian symmetric space.

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## I. INTRODUCTION

A nonlinear sigma model is regarded as a low-energy effective field theory, where the relevant degrees of freedoms are massless Nambu-Goldstone (NG) bosons associated with broken global symmetries. Interestingly, any nonlinear sigma model based on the coset manifold is gauge equivalent to a linear model with a so-called hidden local symmetry (see Ref. [1] and references cited therein). Although the gauge fields for the hidden local symmetry are redundant variables, dynamical vector bosons may be generated by quantum corrections even in four dimensions.

In supersymmetric field theories, Zumino first recognized that the scalar fields of nonlinear models take their values in a Kähler manifold and gave an explicit form of the action for the Grassmann manifold [2]. More general nonlinear realization for more general coset spaces was extensively studied in Refs. [3–10], and general methods to construct a nonlinear Lagrangian are provided. The characteristic feature is that massless fermions appear as supersymmetric partners of NG bosons. These NG bosons and their fermionic partners are described by chiral superfields in four dimensions with $N=1$ supersymmetry. Then, the target space must be the Kähler manifold, since chiral superfields are complex.



Supersymmetric nonlinear sigma models with hidden local symmetries were studied on some Kähler manifolds in Refs. [11–14], and then were generalized by Higashijima-Nitta about twenty years ago [15]. They showed that a supersymmetric nonlinear sigma model is formulated as a linear gauge theory, if its target manifold is the Hermitian symmetric space. However, importantly, this is a classical correspondence between both models.

Supersymmetric nonlinear sigma models were studied in quantum field theories, and many interesting results have been revealed in two dimensions [16–19]. However, nonlinear sigma models are nonrenormalizable in four dimensions. So, they are defined by the theory with ultraviolet momentum cutoff as well as the Nambu–Jona-Lasinio (NJL) model [20], or by some other nonperturbative methods. Although supersymmetry increases difficulties in analyzing the quantum dynamics, they seem not to be physical but rather technical, similar to an ambiguity of subtraction in the NJL model, and so they are a relatively tractable problem.

Most crucially, a hidden local symmetry is generically anomalous in supersymmetric nonlinear models in four dimensions, since the symmetry acts on chiral superfields. For example, let us consider the following Kähler potential as a gauged linear model:

$$K(\phi,\phi^\dagger)=\phi^\dagger e^{2V}\phi-\frac{2}{g^2}V,$$

where $\phi_i(i=1,\ldots,N)$ is a chiral superfield and $V$ is a $U(1)$ gauge vector superfield. The last term is a Fayet-Illiopoulos (FI) term with a coupling constant $g$. The model has the global symmetry $SU(N)$ and the local one $U(1)$. In order to see this model as equivalent to the $CP^{N-1}$ model, it has been thought that one has only to take $\phi_N = 1$ as a gauge fixing condition [14,15]. Eliminating $V$ by the equation of motion, one may find the Kähler potential of the $CP^{N-1}$ model, the target manifold of which is parametrized by the remaining chiral superfields. However, the important point is that the anomalous hidden local symmetry does not allow us to take an arbitrary gauge fixing condition. In this example, $U(1)$ is anomalous, and so it is impossible to transform to the $CP^{N-1}$ model.

For one thing, we can avoid the anomaly problem by considering nonanomalous hidden local symmetries in the gauged linear model. Alternatively, one can add additional chiral superfields coupled to the vector superfield in order to cancel the anomaly. However, both methods are not helpful for formulating the nonlinear sigma model based on the Hermitian symmetric space.

In this paper, we will start with the supersymmetric nonlinear sigma model, which includes only the chiral superfields and so is a well-defined theory without the anomaly. Then, we will rewrite the model by introducing an auxiliary vector superfield and performing a Legendre transformation. At this stage, the vector superfield is not a gauge field, since the original Lagrangian is not gauge invariant, and the path integral measure is not divided by the gauge volume. Next, we will insert the Fadeev-Popov determinant to the partition function by following the technique used for the quantization of anomalous gauge theories in Ref. [21], which is an extension of the method of Ref. [22]. As a result, we obtain the gauged linear model with a Wess-Zumino term which is equivalent to the original nonlinear sigma model.

We should comment that the conceptual setting of the above procedure is not new, because it is almost the same strategy described by de Wit and Grisaru more than thirty years ago [23]. In the case of the $CP^{N-1}$ model, the chiral superfields $\phi^i$ include a compensating field. They showed that the anomaly can always be eliminated by adding local counterterms constructed by using the compensator. However, an advantage of our procedure is that it is obvious which field is a compensator, while there are various options in their arguments. Consequently, a Wess-Zumino term can be uniquely determined in our procedure.

We will explicitly deal with $CP^{N-1}$ and $SO(N)/SO(N-1)\times U(1)$ models, but our results can be generalized straightforwardly to other target manifolds, because these models capture typical features of the models without or with F-term constraint [15]. Both nonlinear models will be formulated as anomalous gauged linear models. Importantly, the symmetry of the action in the gauged linear model is different from a conventional symmetry due to the effect of the Wess-Zumino term. For instance, we will show that the action of the gauged linear model for the $CP^{N-1}$ model has the symmetry $SU(N-1)_{\text{global}}\times U(1)_{\text{local}}$, which is smaller than the conventional symmetry $SU(N)_{\text{global}}\times U(1)_{\text{local}}$. This is essentially the same result as pointed out by de Wit and Grisaru in the discussion of anomalies and compensators [23].

This paper is organized as follows: First, we will show the details about the supersymmetric $CP^{N-1}$ model. In Sec. II A, we will explain the quantum equivalence between this model and an anomalous gauged linear model with a Wess-Zumino term, which is derived from the Jacobian factor for chiral superfields. In Sec. II C, we will calculate a three-point vertex function given by triangle diagrams and exactly determine the form of the Wess-Zumino term in the theory including the momentum cutoff $\Lambda$. For renormalizable theories, the Feynman integral for the triangle diagram is expanded by the powers of $1/\Lambda$, and only finite terms for $\Lambda\to\infty$ contribute to the anomaly [24]. Here, we will provide an exact anomalous term depending on $\Lambda$, which includes higher-power terms of $1/\Lambda$. In Sec. II B, we will discuss the fact that our model is defined on the whole $CP^{N-1}$ manifold. In Sec. II D, we will analyze the effective potential of the linear model in the $1/N$ leading order. We find that the model has the critical coupling, below which the $U(1)_{\text{local}}$ symmetry is broken and supersymmetry is unbroken. Remarkably, in contrast to the bosonic $CP^{N-1}$ model [1], we will show that there is no stable vacuum beyond the critical coupling in the $1/N$ leading order. In Sec. II E, we will discuss the vector supermultiplet which is dynamically generated but unstable as similar to the bosonic case [1]. Interestingly, we observe that, when approaching the critical point, the vector multiplet tends to become massless. This behavior suggests the possibility that the $U(1)_{\text{local}}$ symmetry is restored at the critical coupling. Next, we will consider the $SO(N)/SO(N-2)\times U(1)$ model in Secs. III A and III B as an example of the nonlinear model with F-term constraint. Although an F-term is added to the model, the qualitative features are unchanged. Finally, we will give concluding remarks in Sec. IV. In the Appendix, we present details of the calculation of Feynman integrals in the cutoff theory.

## II. SUPERSYMMETRIC $CP^{N-1}$ MODEL

### A. Anomalous gauged linear models

The supersymmetric $CP^{N-1}$ model is defined by the Lagrangian

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} K_0(\varphi,\varphi^\dagger), \tag{2.1}$$

where $\varphi_i (i = 1, \ldots, N-1)$ are chiral superfields and $K_0$ is the Kähler potential given by

$$K_0(\varphi,\varphi^\dagger) = \frac{1}{g^2}\log\left(\frac{1}{g^2}+\varphi^\dagger\varphi\right). \tag{2.2}$$

As is well known, this Kähler potential provides the Fubini-Study metric for the $CP^{N-1}$ manifold, which is parametrized by the complex fields $\varphi_i$, $\varphi_i^*$. The parameter $g$ is a coupling constant with the dimension of inverse mass. The Kähler potential can be expanded at $\varphi = 0$ as

$$K_0(\varphi, \varphi^\dagger) = \frac{1}{g^2}\log\frac{1}{g^2} + \varphi^\dagger\varphi - \frac{g^2}{2}(\varphi^\dagger\varphi)^2 + \cdots, \quad (2.3)$$

where the first term has no effect on the Lagrangian, and so we find that the chiral field $\varphi$ is canonically normalized in Eq. (2.2).

By introducing an auxiliary vector superfield $V$, we can change the Kähler potential into

$$K_0'(\varphi, \varphi^\dagger, V) = e^{2V}\left(\frac{1}{g^2} + \varphi^\dagger\varphi\right) - \frac{2}{g^2}V, \quad (2.4)$$

where the last term is a FI D-term. The equation of motion of $V$ leads to

$$\frac{\delta K_0'}{\delta V} = 2e^{2V}\left(\frac{1}{g^2} + \varphi^\dagger\varphi\right) - \frac{2}{g^2} = 0$$
$$\Rightarrow -2V = \log\frac{1/g^2 + \varphi^\dagger\varphi}{1/g^2}. \quad (2.5)$$

Substituting this back into Eq. (2.4), we obtain the same Kähler potential (2.2) for the $CP^{N-1}$ model up to irrelevant constant terms.

In Eq. (2.4), we perform a change of variables:

$$2V \to 2V - i(\lambda - \bar{\lambda}), \quad (2.6)$$

$$\varphi_i \to e^{i\lambda}\varphi_i, \quad (2.7)$$

$$\bar{\varphi}_i \to e^{-i\bar{\lambda}}\bar{\varphi}_i, \quad (2.8)$$

where $\lambda$ is a chiral superfield. Then, we find the Kähler potential to become

$$K(\phi, \phi^\dagger, V) = \phi^\dagger e^{2V}\phi - \frac{2}{g^2}V, \quad (2.9)$$

where $\phi_i (i = 1, \ldots, N)$ are chiral superfields:

$$\phi_i = \varphi_i (i = 1, \ldots, N-1), \qquad \phi_N = \frac{1}{g}e^{-i\lambda}. \quad (2.10)$$

This Kähler potential gives a gauged linear model with the global symmetry $SU(N)$ and the local symmetry $U(1)_{\text{local}}$. If we take $\phi_N = 1/g$ as a gauge fixing condition for $U(1)_{\text{local}}$, the Kähler potential (2.9) reproduces the expression (2.4), and then the first one [Eq. (2.2)], by eliminating $V$. Hence, it was claimed that the supersymmetric $CP^{N-1}$ model can be obtained from a gauged linear model.

However, it should be noticed that $U(1)_{\text{local}}$ is an anomalous symmetry, and this anomaly is an obstruction in proving the equivalence between the two models. In order to include the anomaly, we have to deal with contributions from path integral measures. The idea is basically the same as the quantization of anomalous gauge theory [21], although the original Lagrangian (2.1) is not gauge invariant in our case.

At first, we introduce the auxiliary vector superfield $V$ to the partition function of the $CP^{N-1}$ model:

$$Z = \int d\varphi d\varphi^\dagger \exp\left(i\int d^8z K_0(\varphi, \varphi^\dagger)\right)$$
$$= \int d\varphi d\varphi^\dagger dV \exp\left(i\int d^8z K_0'(\varphi, \varphi^\dagger, V)\right), \quad (2.11)$$

where the superspace coordinate is denoted by $z = (x, \theta, \bar{\theta})$, and integration measures denoted by $d^8z = d^4xd^2\theta d^2\bar{\theta}$. In general, the $V$ integration leads not only to $K_0$ as a saddle point, but also to higher-order quantum corrections. However, in supersymmetric theories, we have no quantum corrections, as proved by Higashijima-Nitta [25], and so this is an exact rewriting.

Let us define the Fadeev-Popov determinant $\Delta_f[V]$ for the gauge fixing condition $f[V] = 0$:

$$\Delta_f[V]\int d\lambda d\bar{\lambda}\delta(f[V^{(\lambda,\bar{\lambda})}]) = 1, \quad (2.12)$$

where $d\lambda d\bar{\lambda}$ is a gauge invariant measure and $V^{(\lambda,\bar{\lambda})}$ is a gauge transformation of $V$:

$$2V^{(\lambda,\bar{\lambda})} = 2V + i(\lambda - \bar{\lambda}). \quad (2.13)$$

Inserting Eq. (2.12) into Eq. (2.11) and changing an integration variable as $V \to V^{(-\lambda,-\bar{\lambda})}$, the partition function (2.11) is expressed in terms of the functional integral over $\lambda$, $\bar{\lambda}$ and the original fields:

$$Z = \int d\varphi d\varphi^\dagger \mathcal{D}V d\lambda d\bar{\lambda} \exp\left(i\int d^8z K'(\varphi, \varphi^\dagger, \lambda, \bar{\lambda}, V)\right), \quad (2.14)$$

$$\mathcal{D}V \equiv dV\Delta_f[V]\delta(f[V]), \quad (2.15)$$

where $dV$ is assumed to be gauge invariant, and so $\mathcal{D}V$ corresponds to a gauge invariant measure divided by the gauge volume. The Kähler potential $K'$ is given by

$$K'(\varphi, \varphi^\dagger, \lambda, \bar{\lambda}, V) = e^{2V}\left\{\frac{1}{g^2}e^{i\bar{\lambda}}e^{-i\lambda} + (\varphi^\dagger e^{i\bar{\lambda}})(e^{-i\lambda}\varphi)\right\}$$
$$- \frac{2}{g^2}V. \quad (2.16)$$

If we take the chiral superfields $\varphi' = e^{-i\lambda}\varphi$ as integration variables, the functional measure produces the Jacobian factor derived from the relation [26,27]

$$\frac{\delta\varphi'_j(z)}{\delta\varphi_k(z')} = \delta^k_j e^{-i\lambda(z)} \frac{-\bar{D}^2}{4} \delta^8(z - z'). \tag{2.17}$$

Moreover, we change the variable from $\lambda$ to $\phi_N = e^{-i\lambda}/g$. Since $\lambda$ is a chiral field, we have a similar relation to Eq. (2.17):

$$\frac{\delta\phi_N(z)}{\delta\lambda(z')} = -i\frac{1}{g} e^{-i\lambda(z)} \frac{-\bar{D}^2}{4} \delta^8(z - z'). \tag{2.18}$$

So, in the partition function integrated over the new variables, we have the Wess-Zumino term with the factor $N$, in which $N-1$ and 1 are coming from the measures of $\varphi_i$ and $\lambda$, respectively. Finally, we can rewrite the partition function of the $CP^{N-1}$ model as follows:

$$Z = \int d\phi d\phi^\dagger \mathcal{D}V \exp \times \left( i \int d^8 z K(\phi, \bar{\phi}, V) + i\alpha[V, \phi_N, \bar{\phi}_N] \right), \tag{2.19}$$

$$\alpha[V, \phi_N, \bar{\phi}_N] = -\frac{N}{16\pi^2} \int d^4x d^2\theta \log(g\phi_N) W^\alpha W_\alpha + \text{H.c.} + O(1/\Lambda^2), \tag{2.20}$$

where the Kähler potential is given by Eq. (2.9). $\alpha[V, \phi_N, \bar{\phi}_N]$ is the anomalous term generated by the Jacobian factor. $\Lambda$ is the ultraviolet cutoff parameter to regularize the functional measure [26,27], in which the leading term is given by the Wess-Zumino term for $U(1)_{\text{local}}$.

Consequently, we show that the supersymmetric $CP^{N-1}$ model is quantumly equivalent to the theory given by the Kähler potential [Eq. (2.9)] and the F-term [Eq. (2.20)]. This F-term reduces the flavor symmetry to $SU(N-1)$, and so the action of this gauged linear model has the symmetry $SU(N-1) \times U(1)_{\text{local}}$.

### B. Global structure and inhomogeneous coordinates

We have started from the action (2.2) of the $CP^{N-1}$ model and then have rewritten its partition function as that of the linear model (2.19). In the action (2.2), $\varphi_i$'s denote local affine coordinates of the $CP^{N-1}$ manifold, and so this coordinate patch does not cover $CP^{N-1}$.

First, let us reconfirm that the partition function given by the Kähler potential (2.2) is defined on the whole manifold, while the action is represented by the local coordinates. For simplicity, the coupling constant is set to be 1. $\varphi_i$ are local coordinates in a patch, which is expressed by $U_0$. In the case of $\varphi_k \neq 0$, we can introduce an affine coordinate system in the coordinate patch $U_k$:

$$\varphi'_k = \frac{1}{\varphi_k}, \qquad \varphi'_i = \frac{\varphi_i}{\varphi_k} \quad (i \neq k). \tag{2.21}$$

Importantly, the $N$ coordinate patches $U_i (i = 0, \ldots, N-1)$ cover the $CP^{N-1}$ manifold.

Under the coordinate change (2.21), the Kähler potential (2.2) is transformed to

$$K_0(\varphi, \varphi^\dagger) = K_0(\varphi', \varphi'^\dagger) + f(\varphi') + f^*(\varphi'^*), \tag{2.22}$$

where $f(\varphi)$ is the holomorphic function $f(\varphi') = -\log\varphi'_k$. Since both holomorphic and antiholomorphic terms vanish in the action after supercoordinate integration, the action has the same expression with respect to the coordinates $\varphi'_i$. Accordingly, the partition function given by Eq. (2.2) can be defined on the whole of the $CP^{N-1}$ manifold, if the measure is invariant under the coordinate change.

Thus, it is clear that the nonlinear model is defined on the whole manifold by using inhomogeneous coordinates. Let us remember that, in the linear model, $\phi_i (i = 1, \ldots, N-1)$ are related to the coordinates $\varphi_i$, and then $\phi_N$ is given by the gauge transformation parameter $\lambda$. According to Eq. (2.21), to move from $U_0$ to $U_k$, we have only to transform the superfields as

$$\phi'_k = \frac{\phi_N^2}{\phi_k}, \qquad \phi'_i = \frac{\phi_N \phi_i}{\phi_k} \quad (i \neq k). \tag{2.23}$$

It is easily seen that the action is unchanged under this transformation. Consequently, the linear model is also defined on the whole $CP^{N-1}$ manifold.

Here, it should be emphasized that $\phi_N$ is merely a redundant field, or in other words a compensating field [23], which is irrelevant to a coordinate system for $CP^{N-1}$: $\phi_N = e^{-i\lambda}/g$. If there is no anomalous term in Eq. (2.19), $\phi_i (i = 1, \ldots, N)$ may be interpreted as homogeneous coordinates for $CP^{N-1}$, and $\phi_N = 0$ may represent a hyperplane at infinity in the $CP^{N-1}$. In the present case, $\phi_N$ is not equal to zero due to a logarithmic singularity of the anomalous term (2.20). However, this is not a problem for including the hyperplane at infinity in the model, because the transformation (2.23) makes it possible for us to change coordinate patches and to include the whole manifold. It is noted that, on the contrary, the coordinate transformation (2.23) is a breakdown for $\phi_N = 0$.

### C. Exact anomalous terms in cutoff theories

The $CP^{N-1}$ model in four dimensions is nonrenormalizable, and it is regarded as a low-energy effective field theory with an ultraviolet cutoff. So, we have to evaluate the anomalous contribution in the gauged linear model by keeping the cutoff finite. In this section, we consider the cutoff dependence of the anomalous term by calculating the triangle diagram.

First, we consider the vacuum functional:

$$e^{i\Gamma[V]} = \int d\phi d\phi^\dagger \exp\left(i\int d^8 z K(\phi,\bar{\phi},V)\right). \tag{2.24}$$

Since $U(1)_{\mathrm{local}}$ is anomalous, $\Gamma[V]$ is not gauge invariant due to the triangle diagram. On the other hand, since the partition function (2.19) is gauge invariant, the anomaly from the gauge transformation of $\Gamma[V]$ is canceled by the gauge transformation of $\alpha[V,\phi_N,\bar{\phi}_N]$:

$$\delta\alpha[V,\phi_N,\bar{\phi}_N] = -\delta\Gamma[V]. \tag{2.25}$$

Therefore, $\alpha[V,\phi_N,\bar{\phi}_N]$ can be determined by solving this equation for given $\delta\Gamma[V]$.

Here let us explain in detail the calculation of $\delta\Gamma[V]$ in the cutoff theory. The Lagrangian for the chiral spinor is given by

$$\int d^2\theta d^2\bar{\theta}\phi^\dagger e^{2V}\phi = i\bar{\Psi}\partial\!\!\!/ P_R\Psi + v_\mu\bar{\Psi}\gamma^\mu P_R\Psi + \cdots, \tag{2.26}$$

where we have used four-component notation for the spinor, and $v_\mu$ denotes a vector field in $V$. $P_R$ is a projection operator on the right-handed fermion field: $P_R = (1+\gamma_5)/2$. The famous two triangle diagrams contribute to the three-point vertex function of $v^\mu$ [24]:

$$\Gamma^{(3)}_{\mu\nu\rho}(k_1,k_2) \equiv -N\int\frac{d^4k}{i(2\pi)^4}\left\{\mathrm{tr}\left[\frac{1+\gamma_5}{2}\frac{1}{-k\!\!\!/-a\!\!\!/}\gamma_\mu\frac{1}{-k\!\!\!/-a\!\!\!/+k\!\!\!/_1}\gamma_\nu\frac{1}{-k\!\!\!/-a\!\!\!/-k\!\!\!/2}\gamma_\rho\right]\right.$$
$$\left.+\,\mathrm{tr}\left[\frac{1+\gamma_5}{2}\frac{1}{-k\!\!\!/+a\!\!\!/}\gamma_\rho\frac{1}{-k\!\!\!/+a\!\!\!/+k\!\!\!/_2}\gamma_\nu\frac{1}{-k\!\!\!/+a\!\!\!/-k\!\!\!/_1}\gamma_\mu\right]\right\}, \tag{2.27}$$

where $N$ component fermions yield the factor $N$. As in the NJL model, this integral is divergent, and so we introduce the ultraviolet cutoff $\Lambda$ after Wick rotation. It is noted that the cutoff is different from the previous one in Eq. (2.20), and there is no simple relation between them. The four-vector $a^\mu$ is introduced due to arbitrariness of the momenta carried by internal lines.

More precisely, we can introduce two four-vectors $a^\mu$ and $b^\mu$ independently to each triangle diagram. In this case, we have to choose $a^\mu = -b^\mu$ to avoid nonchiral anomalies for all three currents as explained in Ref. [24]. Actually, the charge conjugation matrix $C$ satisfies $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$, and we have

$$\mathrm{tr}\left[\frac{1}{-k\!\!\!/-a\!\!\!/}\gamma_\mu\frac{1}{-k\!\!\!/-a\!\!\!/+k\!\!\!/_1}\gamma_\nu\frac{1}{-k\!\!\!/-a\!\!\!/-k\!\!\!/2}\gamma_\rho\right] = -\mathrm{tr}\left[\frac{1}{k\!\!\!/+a\!\!\!/}\gamma_\rho\frac{1}{k\!\!\!/+a\!\!\!/+k\!\!\!/_2}\gamma_\nu\frac{1}{k\!\!\!/+a\!\!\!/-k\!\!\!/_1}\gamma_\mu\right].$$

So, the traces which contain no $\gamma_5$ in Eq. (2.27) cancel to each other if a momentum variable is flipped in one diagram: $k^\mu \to -k^\mu$. Therefore, only the traces involving $\gamma_5$ are left, and this justifies a choice of $a^\mu = -b^\mu$.

Now, we evaluate the anomaly term $\delta\Gamma[V]$, which corresponds to the Fourier transformation of the divergence of Eq. (2.27)[1]:

$$(k_1+k_2)^\nu\Gamma^{(3)}_{\mu\nu\rho}(k_1,k_2) = 4Ni\epsilon_{\nu\mu\lambda\rho}\int_{k^2\le\Lambda^2}\frac{d^4k}{(2\pi)^4}\left\{\frac{(k+a)^\nu {k_2}^\lambda}{(k+a)^2(k+a+k_2)^2} - \frac{-(k+a)^\nu {k_1}^\lambda}{(k+a)^2(k+a-k_1)^2}\right\}. \tag{2.28}$$

These integrals can be calculated straightforwardly by picking up antisymmetric parts on the two indices $\nu$, $\lambda$. Combining the denominator by the Feynman parameter technique, we perform the $k$ integration by using the formula in the Appendix. Then, if one rotates back to the Minkowski space, the resulting function is given by

$$i(k_1+k_2)^\nu\Gamma^{(3)}_{\mu\nu\rho}(k_1,k_2) = -\frac{N}{8\pi^2}\epsilon_{\nu\mu\lambda\rho}\int_0^1 dx\{a^\nu {k_2}^\lambda g(-(a+xk_2)^2,-a^2-2xa\cdot k_2 - xk_2^2)$$
$$+\,a^\nu {k_1}^\lambda g(-(a-xk_1)^2,-a^2+2xa\cdot k_1 - xk_1^2)\}, \tag{2.29}$$

[1]In general, a simple momentum cutoff breaks gauge invariance, and this is a well-known problem, for example, as seen in dealing with vector mesons in the NJL model [20]. In the NJL model, a conventional gauge invariant form of vertex functions was used to avoid an ambiguity of mass subtraction. There are many other prescriptions proposed to deal with gauge invariance in cutoff theories. Here, we use arbitrariness in the choice of the momentum shift in the loop integral in order to ensure gauge invariance. As an alternative, you may define the model in the gauge invariant way by higher-derivative kinetic terms as in Ref. [28]. In any case, qualitative features are unchanged.

where $g(p^2, m^2)$ is defined by Eq. (A6). This is the exact result for the anomalous vertex function in the cutoff theory.

Suppose that the currents for the $\mu$, $\rho$ directions are conserved; we then have to choose $a = k_1 - k_2$, as explained in Ref. [24]:

$$i(k_1 + k_2)^\nu \Gamma^{(3)}_{\mu\nu\rho}(k_1, k_2) = -\frac{N}{4\pi^2}\epsilon_{\nu\mu\lambda\rho}{k_1}^\nu {k_2}^\lambda f(k_1, k_2), \tag{2.30}$$

where $f(k_1, k_2)$ is given by

$$\begin{aligned} f(k_1,k_2) = \frac{1}{2}\int_0^1 dx\{&g(-(k_1-(1-x)k_2)^2, \\ &-k_1^2 + 2(1-x)k_1\cdot k_2 - (1-x)k_2^2) \\ &+ g(-((1-x)k_1 - k_2)^2, \\ &-(1-x){k_1}^2 + 2(1-x)k_1\cdot k_2 - k_2^2)\}. \end{aligned} \tag{2.31}$$

This result is expressed in terms of the chiral current $J^\mu \equiv \bar\psi\bar\sigma^\mu\psi = \bar\Psi\gamma^\mu P_R \Psi$:

$$\partial_\nu \langle J^\nu(x)\rangle = -\frac{N}{32\pi^2}\epsilon_{\nu\mu\lambda\rho}F^{\nu\mu} f\left(-i\frac{\overleftarrow{\partial}}{\partial x}, -i\frac{\overrightarrow{\partial}}{\partial x}\right)F^{\lambda\rho}. \tag{2.32}$$

The expansion in powers of $1/\Lambda$ is evaluated as

$$\begin{aligned} \partial_\nu\langle J^\nu(x)\rangle = &-\frac{N}{32\pi^2}\epsilon_{\nu\mu\lambda\rho}F^{\nu\mu}F^{\lambda\rho} + \frac{N}{96\pi^2\Lambda^2}\epsilon_{\nu\mu\lambda\rho}F^{\nu\mu}\Box F^{\lambda\rho} \\ &+ O(1/\Lambda^4), \end{aligned} \tag{2.33}$$

where the first term agrees with the conventional chiral anomaly.[2]

Since the operator $f$ consists of space-time derivatives, we can easily provide $\delta\Gamma[V]$ in the supersymmetric model. Finally, from $\delta\Gamma[V]$ and Eq. (2.25), the resulting anomalous term can be obtained as

$$\begin{aligned} \alpha[V, \phi_N, \bar\phi_N] = -\frac{N}{16\pi^2}\int d^4x d^2\theta \log(g\phi_N) W^\alpha f \\ \times\left(-i\frac{\overleftarrow{\partial}}{\partial x}, -i\frac{\overrightarrow{\partial}}{\partial x}\right)W_\alpha + \text{H.c.} \end{aligned} \tag{2.34}$$

[2]According to calculations in Sec. 22 of Ref. [24], correction terms of order $1/\Lambda^2$ naturally appear in the anomalous term as long as we keep the cutoff finite. ($\Lambda$ should be regarded as a radius $P$ of a large three-sphere in Ref. [24].) Also, by applying the Fujikawa method [26], it is easily seen that the correction term appears in a Jacobian factor for a finite cutoff. Then, it is interesting to understand how to deal with the index theorem in cutoff theories, but this is out of the scope of this paper.

This is an exact result for Eq. (2.20) including all orders of $\Lambda$.

### D. Effective potentials in the $1/N$ leading order

Now that the $CP^{N-1}$ model is formulated as the consistent linear model, we can consider the effective potential of this model in the $1/N$ expansion. In the Wess-Zumino gauge, the scalar components are the D-term $-D$ of the vector superfield $V$ and the first component of $\phi_N$. As in Ref. [29], we take negative sign convention for the D-term of $V$. The F-term of $\phi_N$ is irrelevant to the effective potential.

In order to perform the $1/N$ expansion, we define the coupling $g^2$ by

$$g^2 \equiv \frac{G}{N}, \tag{2.35}$$

and we study the limit of large $N$ with fixed $G$. This is a conventional choice used in the $CP^{N-1}$ model. Moreover, since $g\phi_N$ should be of order 1 for the anomalous term to be leading order, the vacuum expectation value of $\phi_N$ should be defined as

$$\langle\phi_N\rangle \equiv \sqrt{N}z, \tag{2.36}$$

where $z$ is a fixed complex number in the $1/N$ expansion.

Substituting these component fields into Eq. (2.34), we can calculate an anomalous contribution to the effective action:

$$\begin{aligned} \alpha[V, \phi_N, \bar\phi_N] = -\frac{N}{16\pi^2}\int d^4x \log(G|z|^2) D f \\ \times\left(-i\frac{\overleftarrow{\partial}}{\partial x}, -i\frac{\overrightarrow{\partial}}{\partial x}\right)D. \end{aligned} \tag{2.37}$$

For constant $D$, the operator $f$ becomes 1, and so a quadratic term of $D$ is generated in the effective potential.

We notice that for constant $W_\alpha$, higher-order correction terms may arise from other diagrams (square, pentagon, and so on) in the superpotential as

$$\log(g\phi_N)\Lambda^3 F\left(\frac{W^\alpha W_\alpha}{\Lambda^3}\right), \tag{2.38}$$

where $F(\cdots)$ denotes a certain function. If we expand it in the power series of $W^\alpha W_\alpha/\Lambda^3$, since the constant fields are included as $W^\alpha W_\alpha = \theta\theta D^2 + \cdots$ and $\log(g\phi_N) = \log(Gz) + \cdots$, the quadratic and higher powers do not contribute to the effective potential. So, Eq. (2.37) leads to an exact result of the anomalous effective potential.

Consequently, we can provide the effective potential in the leading order in the $1/N$ expansion:

$$\begin{aligned}
&\frac{1}{N}V(z,D)\\
&=-\frac{1}{G}D+D|z|^2+\frac{1}{16\pi^2}D^2\log(G|z|^2)\\
&\quad+\frac{1}{32\pi^2}\left[\Lambda^4\log\left(1+\frac{D}{\Lambda^2}\right)-D^2\log\left(1+\frac{\Lambda^2}{D}\right)+D\Lambda^2\right].
\end{aligned}\tag{2.39}$$

Here, the first and second terms arise from the tree-level action, where we note again the negative sign convention of the D-term. The third term is the anomalous potential from Eq. (2.37). The fourth term is given by one-loop calculation, which is performed in a supersymmetric NJL model in Ref. [29]. In the calculation, $D$ is a mass-squared parameter for the scalar component of $\phi$, and so $D$ must be positive for a consistent vacuum.

The stationarity condition with respect to $z$ is

$$\frac{\delta V}{\delta z}=\frac{D}{z}\left(\frac{1}{16\pi^2}D+|z|^2\right)=0. \tag{2.40}$$

Then, we conclude that $D=0$, and so supersymmetry is never broken in the leading order.

Another stationarity condition leads to

$$\begin{aligned}
\frac{\delta V}{\delta D}=0\Rightarrow-\frac{1}{G}+|z|^2+\frac{1}{32\pi^2}\Bigg[&2\Lambda^2-2D\log\left(1+\frac{\Lambda^2}{D}\right)\\
&+4D\log(G|z|^2)\Bigg]=0.
\end{aligned}\tag{2.41}$$

Substituting $D=0$ into the above, we find

$$|z|^2=\frac{1}{G}-\frac{\Lambda^2}{16\pi^2}. \tag{2.42}$$

The model becomes inconsistent if $G$ is larger than $G_{\rm cr}=16\pi^2/\Lambda^2$.

Accordingly, we conclude that, in the $1/N$ leading order, the model has a stable vacuum only for the weak coupling $G<G_{\rm cr}$, and supersymmetry is unbroken in this vacuum.

Here it should be noted that the anomalous potential play an important role in the robustness of supersymmetry. If we naively quantize the gauged linear model without the anomalous term, the stationarity condition with respect to $z$ becomes $Dz^*=0$ instead of Eq. (2.40), and so we have $D=0$ or $z=0$. The stationarity condition with respect to $D$ implies the gap equation

$$|z|^2-\int\frac{d^4k}{(2\pi)^4}\left(\frac{1}{k^2}-\frac{1}{k^2+D}\right)=\frac{1}{G}-\frac{1}{G_{\rm cr}}, \tag{2.43}$$

which is the same as that of a bosonic $CP^{N-1}$ model [1]. Then, we might have two phases: (i) $G<G_{\rm cr}$, $|z|\neq0$, $D=0$; and (ii) $G>G_{\rm cr}$, $|z|=0$, $D\neq0$. While the first case corresponds to the above supersymmetric model, the second appears as a new phase. If there were no anomaly, $D$ would acquire a vacuum expectation value in the strong coupling region, and so supersymmetry would be spontaneously broken. But this is not the case, and so it is regarded that the anomalous term keeps supersymmetry unbroken.

We note that, although there is no vacuum in the strong coupling region in the $1/N$ leading order, there still remains a possibility of finding a vacuum in higher order or by considering some nonperturbative effects.

### E. Dynamical vector supermultiplets

We showed that $z$ has the vacuum expectation value given by Eq. (2.42) in the weak coupling region. On this vacuum, the anomalous term in Eq. (2.44) induces the kinetic term for the vector superfield:

$$-\frac{N}{16\pi^2}\int d^4xd^2\theta\log(\sqrt{G}z)W^\alpha f\left(-i\frac{\overleftarrow{\partial}}{\partial x},-i\frac{\overrightarrow{\partial}}{\partial x}\right)W_\alpha+\text{H.c.} \tag{2.44}$$

It is well known that, in general, vector bosons are dynamically generated in the model with hidden local symmetries [1]. Also in this model, loop diagrams of components of $\phi$ generate the kinetic term for a vector boson. In addition, the anomalous term [Eq. (2.44)] supplies the kinetic term, which, however, enhances the possibility of the wrong sign due to the logarithmic function. If the logarithmic function is positive, the anomalous term encourages the appearance of negative metric states.

Fortunately, it can be easily seen that the large-$N$ dynamics prohibits such a negative metric state. For the vacuum expectation value [Eq. (2.42)], we find

$$G|z|^2=1-\frac{G}{G_{\rm cr}}<1\quad(G<G_{\rm cr}). \tag{2.45}$$

Therefore, the kinetic term of the vector superfield is well behaved, since the logarithmic function becomes negative for $G<G_{\rm cr}$. Then, the anomalous term [Eq. (2.44)] leads to the vertex function of the vector field:

$$\Gamma^{A(2)}_{\mu\nu}(p)=(p^2\eta_{\mu\nu}-p_\mu p_\nu)\frac{N}{32\pi^2}\log\left(1-\frac{G}{G_{\rm cr}}\right)f(p,-p), \tag{2.46}$$

where $f(p,-p)$ can be evaluated explicitly from Eq. (2.31). For $p^2>0$, we find

$$f(p,-p)=\frac{1+7p^2/3\Lambda^2}{1+2p^2/\Lambda^2}. \tag{2.47}$$

Now, we calculate all of the two-point vertex function of the vector field for the timelike momentum. For loop integrations with a cutoff, we have the freedom to choose a momentum shift carried by internal lines, as well as the anomaly calculation given in Sec. II C. Here, by adopting a symmetric momentum shift ($a^\mu = -p^\mu/2$), the vertex function $\Gamma'^{(2)}_{\mu\nu}(p)$ for the vector component is given by

$$\Gamma'^{(2)}_{\mu\nu}(p) = \Gamma^{f(2)}_{\mu\nu}(p) + \Gamma^{b(2)}_{\mu\nu}(p), \tag{2.48}$$

$$\Gamma^{f(2)}_{\mu\nu}(p) = -N\int_0^1 dx \int_{k^2\leq\Lambda^2} \frac{d^4k}{(2\pi)^4} \times \frac{4k_\mu k_\nu - p_\mu p_\nu - 2(k^2 - p^2/4)\eta_{\mu\nu}}{\{k^2 + 2(1/2 - x)p\cdot k + p^2/4\}^2}, \tag{2.49}$$

$$\Gamma^{b(2)}_{\mu\nu}(p) = N\int_0^1 dx \int_{k^2\leq\Lambda^2} \frac{d^4k}{(2\pi)^4} \times \frac{4k_\mu k_\nu - 2(k^2 + p^2/4)\eta_{\mu\nu}}{\{k^2 + 2(1/2 - x)p\cdot k + p^2/4\}^2}, \tag{2.50}$$

where $\Gamma^f$ and $\Gamma^b$ are coming from fermion and boson one-loop diagrams, respectively. After the $k$ integration by using the formula in the Appendix, we find that each vertex function includes a quadratic term of $\Lambda$, which corresponds to the vector self-energy. It implies that gauge symmetry is broken by introducing the cutoff parameter. However, the quadratic terms cancel to each other in the total vertex function owing to supersymmetry. As a result, the vertex function is expressed in the conventional gauge invariant form: for $0 < p^2 < 4\Lambda^2$ in the Minkowski space,

$$\Gamma'^{(2)}_{\mu\nu}(p) = -(p^2\eta_{\mu\nu} - p_\mu p_\nu)\frac{N}{16\pi^2} \times \left(1 + \log\frac{4\Lambda^2 - p^2}{4p^2} + i\pi\right). \tag{2.51}$$

The integral is calculated as a real number in the Euclidean space, but the imaginary part appears in the Minkowski space due to the logarithm function.

Combining these results with tree-level terms, the resulting vertex function for the timelike momentum is given by

$$\Gamma^{(2)}_{\mu\nu}(p) = -(p^2\eta_{\mu\nu} - p_\mu p_\nu)F(p^2) + m^2\eta_{\mu\nu}, \tag{2.52}$$

$$F(p^2) = \frac{N}{16\pi^2}\left\{1 + \log\frac{4 - p^2/\Lambda^2}{4p^2/\Lambda^2} - \frac{1 + 7p^2/3\Lambda^2}{2(1 + 2p^2/\Lambda^2)}\log\left(1 - \frac{G}{G_{\rm cr}}\right)\right\} + i\frac{N}{16\pi}, \tag{2.53}$$

$$m^2 = N\left(\frac{2}{G} - \frac{2}{G_{\rm cr}}\right). \tag{2.54}$$

From this vertex function, we could expect that a massive vector particle appears dynamically; however, it includes the nonzero imaginary part, and so the "would-be" vector particle is unstable. Actually, the vector particle has couplings with the scalar and spinor components of $\phi$, which remain massless in the $1/N$ leading order, and so the vector state decays into these massless particles.

Finally, we elucidate the behavior of the unstable vector state in terms of the spectral function. The propagator can be derived from the vertex function (2.46):

$$\Delta_{\mu\nu}(p) = i\Delta'(p)\left\{\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}F(p^2)\right\}, \quad \Delta'(p) = \frac{1/F(p^2)}{m^2/F(p^2) - p^2}. \tag{2.55}$$

Here we forget for a moment that $F(p^2)$ is divergent for $\Lambda^2 \to \infty$ as in Ref. [20]. If so, the spectral function $\rho(\sigma^2)$ is given by the imaginary part of $\Delta'(p)$, and then $\Delta'(p)$ is expressed by $\rho(\sigma^2)$:

$$\Delta'(p) = \int_0^{\Lambda^2} d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - p^2 - i\epsilon}, \tag{2.56}$$

where a new cutoff is introduced as in Ref. [20], although there is no simple relation between both cutoffs. By using Eqs. (2.53) and (2.54), we can evaluate $\rho(\sigma^2)$ numerically, and the resulting plots are depicted in Fig. 1. We note that $\rho(\sigma^2)$ is given by order $1/N$.

From these plots, we find a peak in the region $\sigma^2 \lesssim \Lambda^2$ for the coupling $G \gtrsim 0.5G_{\rm cr}$, but the width is large and the peak is hard to distinguish for $0.7G_{\rm cr} \gtrsim G \gtrsim 0.5G_{\rm cr}$. Near the critical coupling, the position of the peak approaches $\sigma^2 \sim 0$, and the width becomes gradually narrower.

The position of the peak can be evaluated numerically by using the numerical results of $\rho(\sigma^2)$. The resulting plots are shown in Fig. 2. We find that the "mass" of the unstable vector state decreases to zero for the coupling $G$ approaching $G_{\rm cr}$. Since supersymmetry is not broken in this vacuum, the vector supermultiplet is dynamically generated for $G \gtrsim 0.7G_{\rm cr}$, but it is unstable.

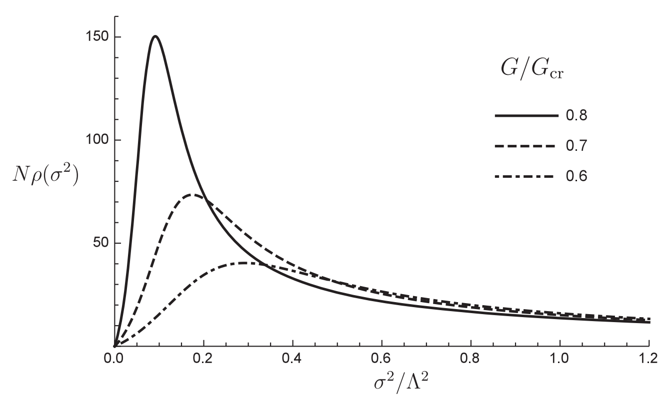


FIG. 1. The plots of the spectral function for the unstable vector state.

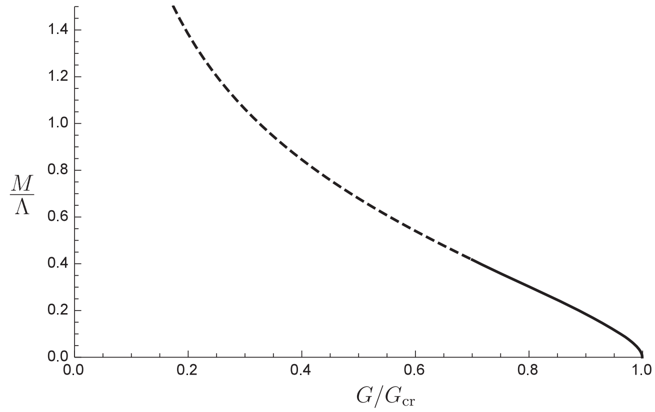


FIG. 2. The plots of the position of the peak of $\rho(\sigma^2)$, $\sigma = M$. It corresponds to the mass of the unstable vector multiplet.

Most interestingly, we find that the spectral function rapidly approaches a delta function for $G \to G_{\rm cr}$, namely $\rho(\sigma^2) \to Z\delta(\sigma^2)$. This behavior suggests that a massless vector supermultiplet is dynamically generated and the $U(1)$ gauge symmetry is restored at the critical coupling. Unfortunately, the analysis just at $G = G_{\rm cr}$ seems to be subtle in the leading order, because $|z|^2$ becomes zero, and so the logarithmic term in the effective potential diverges.

## III. NONLINEAR SIGMA MODELS WITH F-TERM CONSTRAINT

### A. $SO(N)/SO(N-2)\times U(1)$ model

We consider a supersymmetric nonlinear sigma model based on the manifold $SO(N)/SO(N-2)\times U(1)$ [15]. The model is formulated by a gauged linear sigma model as well as the $CP^{N-1}$ model. We introduce the chiral superfields $\phi_i (i = 1,\ldots,N)$, and the Kähler potential is the same as in Eq. (2.4). In addition, the linear model has the superpotential by using an extra chiral superfield:

$$W(\phi_0,\phi) = \frac{1}{2}\phi_0\phi^2, \tag{3.1}$$

where the chiral superfield $\phi_0$ corresponds to a Lagrange multiplier, and then it induces the constraint $\phi^2 = 0$. For the $U(1)$ symmetry, $\phi$ and $\phi_0$ have the charges $+1$ and $-2$, respectively.

In order to transform back to the nonlinear model, we have to fix the gauge of the $U(1)$ symmetry as $\phi_N = 1/g$, similar to the case of the $CP^{N-1}$ model. Here, we should notice that this rewriting also suffers from the anomaly. Since the total $U(1)$ charge for $\phi_0$ and $\phi_i$ equals $N-2$, the anomalous term turns out to be given by[3]

$$\alpha[V,\phi_N,\bar{\phi}_N] = -\frac{N-2}{16\pi^2}\int d^4x d^2\theta \log(g\phi_N) W^\alpha f \times\left(-i\frac{\overleftarrow{\partial}}{\partial x}, -i\frac{\overrightarrow{\partial}}{\partial x}\right) W_\alpha + \text{H.c.} \tag{3.2}$$

As a result, the symmetry of the action is reduced to $SO(N-1)\times U(1)_{\rm local}$, while the Kähler potential has the symmetry $SO(N)\times U(1)_{\rm local}$.

In the background $\langle\phi_0\rangle = [w,0,h]$, the part of the Lagrangian derived from Eq. (3.1) is expanded by the component fields $\phi^i = [A^i,\psi^i,F^i]$ as

$$\int d^2\theta W(\phi_0,\phi) + \text{H.c.} = wF^iA^i + \frac{1}{2}hA^iA^i - w\psi^i\psi^i + \text{H.c.} \tag{3.3}$$

Eliminating the auxiliary fields $F_i$ by the equations of motion $F^{i*} + wA^i = 0$, Eq. (3.3) yields mass terms for component fields. By including the contribution from the Kähler potential, the mass terms in this background are given as

$$\mathcal{L}_{\rm mass} = -(D+|w|^2)A^{i\dagger}A^i + \left(\frac{1}{2}hA^iA^i - w\psi^i\psi^i + \text{H.c.}\right). \tag{3.4}$$

### B. Effective potentials including F-terms

The mass term [Eq. (3.4)] is essentially the same as that of the supersymmetric NJL model analyzed in Ref. [29]. For the scalar, the mass-squared eigenvalues are given by $D + |w|^2 \pm |h|$. According to Ref. [29], the effective potential in the $1/N$ leading order can be calculated as

$$\begin{aligned}\frac{1}{N}V(z,D,w,h) = &-\frac{1}{g^2}D + N(D+|w|^2-|h|\cos\theta)|z|^2 \\ &+\frac{1}{16\pi^2}D^2\log(G|z|^2) \\ &+\frac{1}{16\pi^2}\{F(D+|w|^2+|h|) \\ &+F(D+|w|^2-|h|) - 2F(|w|^2)\},\end{aligned} \tag{3.5}$$

where $\theta$ is the phase of $hA^iA^i$ and the function $F(x)$ is defined by

$$F(x) = \frac{1}{2}\left[\log(1+x) - x^2\log\left(1+\frac{1}{x}\right) + x\right]. \tag{3.6}$$

We set the cutoff $\Lambda$ equal to 1 for simplicity. The potential (3.5) reduces to a similar expression to the previous one [Eq. (2.39)] if taking the limit $h,w \to 0$. We note that the

[3]It is noted that, as in the $CP^{N-1}$ model, $\phi_i (i = 1,\ldots,N-1)$ are related to local coordinates of the manifold. $\phi_0$ and $\phi_N$ are irrelevant to local coordinates. So, we can use coordinate transformations to cover the whole of the manifold.

factor of the anomalous term $N-2$ is approximated as $N$ for large $N$.

Differentiating the potential (3.5), the stationarity conditions are given by

$$\frac{\delta V}{\delta \theta} = 0 \Rightarrow |h||z|^2 \sin\theta = 0, \tag{3.7}$$

$$\frac{\delta V}{\delta |h|} = 0 \Rightarrow I(D + |w|^2 + |h|) - I(D + |w|^2 - |h|) = 16\pi^2 |z|^2 \cos\theta, \tag{3.8}$$

$$\frac{\delta V}{\delta w} = 0 \Rightarrow w^* \{I(D + |w|^2 + |h|) + I(D + |w|^2 - |h|)\} = -16\pi^2 w^* |z|^2, \tag{3.9}$$

$$\frac{\delta V}{\delta z} = 0 \Rightarrow \frac{1}{z}\left(\frac{1}{16\pi^2} D^2 + (D - |h|\cos\theta)|z|^2\right) = 0, \tag{3.10}$$

$$\frac{\delta V}{\delta D} = 0 \Rightarrow -\frac{1}{G} + |z|^2 + \frac{1}{8\pi^2} D \log(G|z|^2) + \frac{1}{16\pi^2}\{I(D + |w|^2 + |h|) + I(D + |w|^2 - |h|)\} = 0, \tag{3.11}$$

where $I(x)$ is defined by

$$I(x) \equiv F'(x) = 1 - x \log\left(1 + \frac{1}{x}\right). \tag{3.12}$$

The stationarity condition (3.7) implies that $\theta = 0$ or $\pi$, or $|h| = 0$. Note that $|z|$ must not be zero, since the potential includes $\log|z|$. Since $I(x)$ is a monotonically decreasing function [29], we find, if $|h| \neq 0$,

$$I(D + |w|^2 + |h|) - I(D + |w|^2 - |h|) < 0. \tag{3.13}$$

So, from Eq. (3.8), it follows that $\theta = \pi$ if $|h| \neq 0$. However, these values do not satisfy the stationarity condition (3.11), and so $|h|$ must be zero. Then, from Eqs. (3.8) and (3.10), it follows that $\theta$ must be $\pi/2$ and $D$ must be zero. At this stage, we conclude that supersymmetry is unbroken in this model, since $D = 0$ and $h = 0$.

From Eqs. (3.9) and (3.11), we find that if $w \neq 0$,

$$-\frac{1}{G} + \frac{1}{8\pi^2} D \log(G|z|^2) = 0. \tag{3.14}$$

It is inconsistent for $D = 0$, and so $w$ must be zero.

After all, $D$, $h$, and $w$ are zero, and $|z|$ is given by the same expression of Eq. (2.42). At this vacuum, the effective action is essentially the same as that of the $CP^{N-1}$ model in the $1/N$ leading order. Therefore, the analysis of the vector boson is also the same, and so one massive vector particle appears in this model, but it decays to massless components.

## IV. CONCLUDING REMARKS

We have shown that the supersymmetric $CP^{N-1}$ and $SO(N)/SO(N-1) \times U(1)$ models are formulated as anomalous gauge theories. By the anomalous term, the gauged linear models have smaller symmetries of the action than conventional ones: the remaining symmetry is $SU(N-1)_{\mathrm{global}} \times U(1)_{\mathrm{local}}$ for $CP^{N-1}$, and $SO(N-1)_{\mathrm{global}} \times U(1)_{\mathrm{local}}$ for $SO(N)/SO(N-2)_{\mathrm{global}} \times U(1)_{\mathrm{local}}$.

In the $1/N$ leading order, the linear model has a vacuum for $G < G_{\mathrm{cr}}$, where the $U(1)_{\mathrm{local}}$ symmetry is broken but supersymmetry is unbroken. It is a remarkable feature of both models that there is no stable vacuum for $G > G_{\mathrm{cr}}$ in the $1/N$ leading order.

From the analysis of the spectral function, we expect that the dynamical gauge boson becomes massless at the critical coupling, and so the $U(1)_{\mathrm{local}}$ symmetry is restored. To show this, it is necessary to study the models in the strong coupling regime by other methods than the $1/N$ leading order. In particular, it is interesting to clarify the fate of supersymmetry for $G > G_{\mathrm{cr}}$.

It has been shown that all supersymmetric nonlinear sigma models for the Hermitian symmetric space are formulated as gauge theories, although the anomaly is not included in Ref. [15]. In this paper, we deal with the two models for the Hermitian symmetric space and show that the anomaly should be taken into account in the models. Then, it is natural to ask whether the anomalous term is required for analyzing the model for other Hermitian symmetric spaces.

In the case of the Grassmann manifold $G_{M,N}$, the linear model is described by a chiral superfield of the $(N, \bar{M})$ representation of $U(N)_L \times U(M)_R$, and the model has no F-term constraint. Since $U(M)_R$ is gauged in this model, the anomalous term should be added in the nonlinear sigma model for $G_{M,N}$.

For $Sp(N)/U(N)$ and $SO(2N)/U(N)$, we similarly have a chiral superfield $\phi$ and an additional chiral field $\phi_0$ to impose the F-term constraint. Although the gauge symmetry is non-Abelian, it can be easily seen that the anomalous term is required also in this case by considering $U(1)_{\mathrm{D}}$, which is a subgroup of $U(N)$ [15]. For $U(1)_{\mathrm{D}}$, $\phi$ and $\phi_0$ have 1 and $-2$ charges, respectively. Counting the total charge, the anomalous factor for $U(1)_{\mathrm{D}}$ is given by $N(N+1)$ for $Sp(N)/U(N)$, and $N(N-1)$ for $SO(2N)/U(N)$. Since these factors are nonzero, we should include the anomalous term in the linear model for these target manifolds.

Similarly, we can deal with $E_6/SO(10) \times U(1)$ and $E_7/E_6 \times U(1)$ in terms of the $U(1)_{\mathrm{D}}$ charge. In the case of $E_6/SO(10) \times U(1)$, there are two chiral superfields of the 27 representations of $E_6$, and they have 1 and $-2$

charges. So, we need the anomalous term in the linear model. For $E_7/E_6 \times U(1)$, we have two chiral superfields of the 56 representations of $E_7$, which have 1 and $-3$ charges for $U(1)_D$, and so the anomalous term is required. Consequently, we conclude that it is necessary to include the anomalous term in all linear models corresponding to the nonlinear sigma model whose target manifold is the Hermitian symmetric space.

Finally, we comment on a supersymmetric NJL model proposed by Cheng, Dai, Faisei, and Kong[30,31]. The model is given by the Kähler-potential-truncating higher-order terms of Eq. (2.3). One analysis of the model was performed in Ref. [29] by introducing an auxiliary vector superfield and calculating an effective potential in the $1/N$ leading order. Relating to an auxiliary vector superfield, the model has hidden $U(1)$ local symmetry with the anomaly, as well as in the $CP^{N-1}$ model. However, the anomalous term was not included in the effective potential in the previous analysis. The result including the anomaly will be reported in the near future [32].

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## APPENDIX: FEYNMAN INTEGRALS IN CUTOFF THEORIES

First, let us consider the Feynman integral

$$I = \int_{k^2 \le \Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + 2k\cdot p + m^2}, \tag{A1}$$

where $k^\mu$ and $p^\mu$ are Euclidean momenta. The dot product for the two momenta is written by $k\cdot p = |k||p|\cos\theta$, where $\theta$ is the angle between the two vectors and $|k|$ is the norm. Writing $k = |k|$ and $p = |p|$, the Feynman integral is expressed as

$$I = \frac{4\pi}{16\pi^4} \int_0^\Lambda dk k^3 \int_0^\pi d\theta \frac{\sin^2\theta}{k^2 + m^2 + 2kp\cos\theta}, \tag{A2}$$

where we have used $d^4k = dkd\theta 4\pi k^3 \sin^2\theta$ in four dimensions.

Here, the $\theta$ integration can be performed by the formula

$$\int_0^\pi d\theta \frac{\sin^2\theta}{a + 2b\cos\theta} = \frac{\pi}{4b^2}\left(a - \sqrt{(a+2b)(a-2b)}\right) \quad (a > 2|b|, b \neq 0). \tag{A3}$$

In the case of $m > p$, we have $k^2 + m^2 > 2kp$, and so the Feynman integral becomes

$$I = \frac{1}{16\pi^2} \int_0^\Lambda dk \frac{k}{p^2} \left\{k^2 + m^2 - \sqrt{(k^2 + m^2 + 2kp)(k^2 + m^2 - 2kp)}\right\}. \tag{A4}$$

Then, the $k$ integration can be easily performed. The resulting integral is

$$I = \frac{1}{16\pi^2} \left\{ \frac{\Lambda^4 + \Lambda^2 m^2 - \Lambda^2 p^2}{\Lambda^2 + m^2} + \frac{p^2}{2}\left(1 - \frac{2p^2}{\Lambda^2 + m^2}\right) g(p^2, m^2) + (p^2 - m^2) h(p^2, m^2) \right\}, \tag{A5}$$

where $h(p^2, m^2)$ and $g(p^2, m^2)$ are defined by

$$g(p^2, m^2) = \frac{\Lambda^4}{2p^4}\left(1 + \frac{m^2}{\Lambda^2}\right)\left\{1 + \frac{m^2}{\Lambda^2} - \sqrt{\left(1 + \frac{m^2}{\Lambda^2}\right)^2 - \frac{4p^2}{\Lambda^2}} - \frac{2p^2}{\Lambda^2 + m^2}\right\}, \tag{A6}$$

$$h(p^2, m^2) = \log \frac{\Lambda^2 + m^2 - 2p^2 + \sqrt{(\Lambda^2 + m^2)^2 - 4\Lambda^2 p^2}}{2(m^2 - p^2)}. \tag{A7}$$

Next, we illustrate the integration with a momentum in the numerator of the integrand:

$$\begin{aligned} \int_{k^2 \le \Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{(k^2 + 2k\cdot p + m^2)^2} &= -\frac{1}{2}\frac{\partial}{\partial p^\mu} \int_{k^2 \le \Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + 2k\cdot p + m^2} \\ &= \frac{4\pi}{16\pi^4} \int_0^\Lambda dk k^3 \int_0^\pi d\theta \frac{k \sin^2\theta \cos\theta}{(k^2 + m^2 + 2kp\cos\theta)^2} \frac{p_\mu}{p}. \end{aligned} \tag{A8}$$

By using the formula

$$\int_0^\pi d\theta \frac{\sin^2\theta\cos\theta}{a+2b\cos\theta} = \frac{\pi(-a^2+2b^2)}{8b^3} + \frac{\pi a}{8b^3}\sqrt{(a+2b)(a-2b)}, \tag{A9}$$

the $\theta$ integration is performed, and then we find that the result of the $k$ integration is given by

$$\int_{k^2\le\Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{(k^2+2k\cdot p+m^2)^2} = \frac{p_\mu}{16\pi^2}\left\{\frac{\Lambda^2}{\Lambda^2+m^2} + \frac{1}{2}\left(1+\frac{2p^2}{\Lambda^2+m^2}\right)g(p^2,m^2) - h(p^2,m^2)\right\}. \tag{A10}$$

Other Feynman integrals can be calculated by similar procedures. We give the results of the calculation of other Feynman integrals used in this paper:

$$\int_{k^2\le\Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+2k\cdot p+m^2)^2} = \frac{1}{16\pi^2}\left\{-\frac{\Lambda^2}{\Lambda^2+m^2} - \frac{p^2}{\Lambda^2+m^2}g(p^2,m^2) + h(p^2,m^2)\right\}, \tag{A11}$$

$$\begin{aligned}\int_{k^2\le\Lambda^2} \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2+2k\cdot p+m^2)^2} = {} & \frac{1}{16\pi^2}\frac{p_\mu p_\nu}{p^2}\left\{\frac{\Lambda^2(\Lambda^2+m^2-3p^2)}{2(\Lambda^2+m^2)} - \frac{1}{4}\left(\Lambda^2+m^2+p^2+\frac{6p^4}{\Lambda^2+m^2}\right)g(p^2,m^2)\right. \\ & \left. + \frac{3p^2-m^2}{2}h(p^2,m^2)\right\} \\ & + \frac{1}{16\pi^2}\frac{-1}{2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left\{-\frac{\Lambda^2(\Lambda^2+3m^2-3p^2)}{3(\Lambda^2+m^2)}\right. \\ & \left. - \frac{1}{6}\left(\Lambda^2+m^2-p^2+\frac{(4m^2-6p^2)p^2}{\Lambda^2+m^2}\right)g(p^2,m^2) - (p^2-m^2)h(p^2,m^2)\right\}.\end{aligned} \tag{A12}$$

It is noted that the consistency of Eqs. (A5), (A10), (A11), and (A12) can be checked by the relation

$$\frac{1}{k^2+2k\cdot p+m^2} = \delta^{\mu\nu}\frac{k_\mu k_\nu}{(k^2+2k\cdot p+m^2)^2} + 2p^\mu\frac{k_\mu}{(k^2+2k\cdot p+m^2)^2} + m^2\frac{1}{(k^2+2k\cdot p+m^2)^2}. \tag{A13}$$


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