Super Chern-Simons theory: Batalin-Vilkovisky formalism and A_∞ algebras

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This is a companion paper of a long work appeared in [C. Cremonini and P. Grassi, Pictures from super Chern-Simons theory, J. High Energy Phys. 03 (2020) 043] discussing the super-Chern-Simons theory on supermanifolds. Here, it is emphasized that the Batalin-Vilkovisky formalism is naturally formulated using integral forms for any supersymmetric and supergravity models and we show how to deal with A_∞ algebras emerging from supermanifold structures.

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I. INTRODUCTION

The Batalin-Vilkovisky (BV) formalism and supergeometry have been extensively studied during the last years. It has been shown the naturalness of the BV formalism in the supergeometry approach (QP manifolds, odd-symplectic structures, BV bracket) because all fields have their own (classical) opposite-statistic partner leading to a BV-symplectic two-form corresponding to an odd-symplectic structure (see [1–3] and [4]). The application of the BV formalism was ubiquitous in quantum field theory and string theory, but in our opinion the BV formalism for supersymmetric theories has never been deeply explored from the supergeometric point of view and this is the aim of the present work.

As has been shown some years ago by several authors [5–8], any supersymmetric model can be reformulated on a given supermanifold by constructing a p-form Lagrangian $\mathcal{L}^{(p)}$ (the rheonomic Lagrangian, defined according to the rules given in [9]). The action functional is obtained by multiplying the Lagrangian by a PCO $\mathbb{Y}$ (also known as Poincaré dual of the immersion of the bosonic submanifold into the supermanifold) which converts the p-form Lagrangian $\mathcal{L}^{(p)}$ into an integral form $\mathcal{L}^{(p)} \wedge \mathbb{Y}$ which can be integrated on the supermanifold. By choosing $\mathbb{Y}$, one can obtain any superspace representation of the same supersymmetric action.

In [8], that procedure has been applied to super-Chern-Simons theory $D = 3$, $N = 1$, and the details of the construction have been discussed. It has been pointed out that there might be another way to describe Chern-Simons theory using pseudoforms (by pseudoforms, we intend those forms with a nonzero number of delta functions less than the maximal one or, differently stated in our case, with picture number equal to 1). In that case, there are some caveats. Indeed, in order to write the interactions, one needs to insert some PCOs (lowering the number of picture) which are potential sources of ambiguities and difficulties. In [10], it is shown how to deal with those issues by introducing a suitable set of multiproducts leading to an A_∞ algebra.

What is left to study is the BV formalism in the framework of integral forms. For that reason, we again use Chern-Simons theory to pave the ground for more complicated models [11]. We show that the natural way to introduce the antifields in the game is by using the supermanifold version of Serre's duality. Then, when working with the theory at 0 picture number, the natural set of antifields lies into the integral forms complex instead of the usual superforms complex. That automatically takes into account the correct number of degrees of freedom needed to use the BV formalism. In the previous works [8,10], the closure of the Lagrangian form is imposed by consistency with the entire construction in order to have the full off-shell supersymmetry and to allow any change of the PCO interpolating among any superspace realizations. We show that the closure of the antifield part of the Lagrangian form is easily achieved and this is consistent with the

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antifield formalism. We also show that the antifield part of the Lagrangian can be written in a form that is similar to the gauge field part of the action with the PCOs. In particular, this allows us to check that, after specifying suitable choices of PCOs, we get the known superspace or component formulations.

The last part of the present work concerns the discussion of the BV formalism in the language of pseudoforms. In that case, the antifields are introduced as pseudoforms and we get that again the multiproducts are needed also in the antifield sector generating the A_∞ structure. We show that the BV formalism easily adapts to the present framework and the result leads to Chern-Simons (CS) action where the picture number 1 gauge field $A^{(1|1)}$ is replaced by a generic pseudoform $\mathcal{A}$ with any form degree (in particular, it can be negative, exactly as in string field theory [12]) and picture number fixed to 1. The action is automatically invariant under superdiffeomorphisms and it has the standard gauge symmetry. The couplings are obtained by multiproducts, where we have inserted the generic pseudoform $\mathcal{A}$. We show that the action cannot be redefined into a trivial one (i.e., without multiproducts), and we show how to retrieve the supersymmetry in the present context.

II. SUPERFORMS, INTEGRAL FORMS, AND PSEUDOFROMS

The space of differential forms has to be extended in order to define a meaningful integration theory. Given a supermanifold $\mathcal{SM}^{(m|n)}$ of superdimension $(m|n)$, we define $\Omega^{(\bullet|\bullet)}(\mathcal{SM})$ as the complete complex of forms; they are graded with respect to two gradings as $\Omega^{(\bullet|\bullet)} = \bigoplus_{p,q} \Omega^{(p|q)}$, where $q = 0, \ldots, m$, $p \leq n$ if $q = m$, $p \geq 0$ if $q = 0$, and $p \in \mathbb{Z}$ if $q \neq 0, m$. The wedge product for form multiplication is used in the paper, with suitable adjustments due to the picture number as shown in what follows.

Locally, a $(p|q)$-form ω formally reads

$$\omega = \sum_{l,h} \omega_{[a_1 \ldots a_l](\alpha_1 \ldots \alpha_h)[\beta_1 \ldots \beta_q]} dx^{a_1} \ldots dx^{a_l} d\theta^{\alpha_1} \ldots \times d\theta^{\alpha_h} \delta^{g(\beta_1)}(d\theta^{\beta_1}) \wedge \ldots \wedge \delta^{g(\beta_q)}(d\theta^{\beta_q}), \tag{2.1}$$

where $g(x)$ denotes the differentiation degree of the Dirac delta function corresponding to the one-form $d\theta^x$. The *picture number* q counts the number of Dirac delta functions, while p counts the form number. The three indices l, h, and q satisfy the relation

$$l + h - \sum_{k=1}^{q} g(\beta_k) = p, \qquad \alpha_l \neq \{\beta_1, \ldots, \beta_q\} \quad \forall \; l = 1, \ldots, h. \tag{2.2}$$

Each α_l in the above summation should be different from any β_k; otherwise, the degree of the differentiation of the Dirac delta function could be reduced and the corresponding one-form $d\theta^{\alpha_k}$ removed from the basis. The components $\omega_{[i_1 \ldots i_l](\alpha_1 \ldots \alpha_m)[\beta_1 \ldots \beta_r]}$ of ω are superfields.

Once the integral forms are defined, we have to clarify how the integration is performed: given a *top form* $\omega^{(m|n)}$, i.e., a form with either maximum picture number n or maximum form number m, we write

$$I(\omega^{(m|n)}) = \int_{\mathcal{SM}^{(m|n)}} \omega^{(m|n)}. \tag{2.3}$$

The integral is performed by first integrating over dx's, which amounts to selecting the top form, then we use the Berezin integral over θ's, and the integration over $d\theta$, viewed as algebraic bosonic variables [6,13], is performed as a formal algebraic integration using the distributional properties of $\delta(d\theta)$'s. The final expression needs a usual Riemann/Lebesgue integral on x's. The order of integration is not relevant in the flat case, while when dealing with curved supermanifolds one has to recall that the *supervielbeins* $V^\mu = V^\mu_a(x,\theta) dx^a + V^\mu_\alpha(x,\theta) d\theta^\alpha$, $\psi^\lambda = \psi^\lambda_a(x,\theta) dx^a + \psi^\lambda_\alpha(x,\theta) d\theta^\alpha$ are generically expressed as functions of x, θ.

The elements of $\Omega^{(p|0)}$ are denoted by *superforms* and are represented as polynomials of dx's and $d\theta$'s; the forms of the spaces $\Omega^{(p|n)}$ are denoted by *integral forms* and are represented as polynomials of dx's and the product $\delta(d\theta^1) \ldots \delta(d\theta^n)$ and derivatives of the Dirac deltas; finally, $\Omega^{(p|q)}$, $0 < q < n$, are denoted as *pseudoforms*.[1] At a given form number, $\Omega^{(p|0)}$ and $\Omega^{(p|n)}$ are finite-dimensional spaces, while $\Omega^{(p|q)}$ are infinite-dimensional spaces.

Besides the wedge product, we recall that the spaces of forms $\Omega^{(p|q)}$ admit a differential d acting as an antiderivation on each single space. In particular, we have

$$d: \Omega^{(p|q)} \to \Omega^{(p+1|q)} \qquad \omega^{(p|q)} \mapsto d\omega^{(p|q)} = dx^\mu \partial_\mu \omega^{(p|q)} + d\theta^\alpha \partial_\alpha \omega^{(p|q)}. \tag{2.4}$$

In particular, if we consider ω as in (2.1), we have

$$d\omega = \sum_{l,h,q} \big(dx^a \partial_a \omega_{[a_1 \ldots a_l](\alpha_1 \ldots \alpha_h)[\beta_1 \ldots \beta_q]} dx^{a_1} \ldots dx^{a_l} d\theta^{\alpha_1} \ldots d\theta^{\alpha_h} \delta^{g(\beta_1)}(d\theta^{\beta_1}) \wedge \ldots \wedge \delta^{g(\beta_q)}(d\theta^{\beta_q}) + d\theta^\alpha \partial_\alpha \omega_{[a_1 \ldots a_l](\alpha_1 \ldots \alpha_h)[\beta_1 \ldots \beta_q]} dx^{a_1} \ldots dx^{a_l} d\theta^{\alpha_1} \ldots d\theta^{\alpha_h} \delta^{g(\beta_1)}(d\theta^{\beta_1}) \wedge \ldots \wedge \delta^{g(\beta_q)}(d\theta^{\beta_q}) \big).$$

[1] Notice that the definition of pseudoforms in Voronov *et al.* is slightly different [14,15].

Here we recall the distributional property, extensively discussed in [10], $d\theta^\alpha \delta^{(r)}(d\theta^\alpha) = -r\delta^{(r-1)}(d\theta^\alpha)$, to be used when manipulating integral forms. Analogously, we can define the contraction operator ι_X, where X is a vector field. Notice that if X is an odd vector field, ι_X is a commuting derivation. There are several new operators, $(Z, \mathbb{Y}, \Theta, \eta)$ acting on forms, inspired by string theory, which modify also the picture number. In particular, we will consider the *picture raising operator* $\mathbb{Y}^{(0|s)}$ which acts on a $(p|q)$ form $\omega^{(p|q)} \in \Omega^{(p|q)}$ as follows:

$$\mathbb{Y}^{(0|1)}: \Omega^{(p|q)} \to \Omega^{(p|q+1)}$$
$$\omega^{(p|q)} \mapsto \omega^{(p|q)} \wedge \mathbb{Y}^{(0|1)}, \tag{2.5}$$

i.e., it raises the picture number by 1. When $s = n$, the PCO $\mathbb{Y}^{(0|n)}$ corresponds to the Poincaré dual form of the embedding of the bosonic submanifold into the supermanifold $\mathcal{M} \hookrightarrow \mathcal{SM}$.

Again, given a $(p|q)$-form $\omega^{(p|q)} \in \Omega^{(p|q)}$, we define the *picture lowering operator* Z_D as

$$Z_v: \Omega^{(p|q)} \to \Omega^{(p|q-1)},$$
$$\omega^{(p|q)} \mapsto Z_v(\omega^{(p|q)}) = [d, -i\Theta(\iota_D)]\omega^{(p|q)},$$

where $[\cdot,\cdot]$ denotes as usual a graded commutator and the action of the operator $\Theta(\iota_v)$ is defined by the Fourier-like relation of the Heaviside step function

$$\Theta(\iota_v)\omega^{(p|q)}(d\theta^\alpha) = -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dt}{t+i\epsilon}e^{it\iota_v}\omega^{(p|q)}(d\theta^\alpha)$$
$$= -i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dt}{t+i\epsilon}\omega^{(p|q)}(d\theta^\alpha + itv^\alpha), \tag{2.6}$$

where we have used the fact that $e^{it\iota_v}$ is a translation operator. Hence, the operator $\Theta(\iota_v)$ maps

$$\Omega^{p|q} \to \Omega^{p-1|q-1},$$

i.e., it lowers either the form degree or the picture degree. As has been shown in [10] this operator does not give a pseudoform as a result, but rather an *inverse form*, i.e., an expression containing negative powers of $d\theta$ [16]. The computation techniques and results are contained in [10]. There, the $\eta(\iota_v)$ operator has been described as well. The latter is crucial to define and build the higher products of the A_∞ algebra. For the sake of clarity, let us consider the easy example of application of Z_v on the $(0|1)$ form $\omega = f(x,\theta)\delta(d\theta)$. First, let us calculate $\Theta(\iota_v)\omega$,

$$\Theta(\iota_v)f(x,\theta)\delta(d\theta)$$
$$= -i(-1)^{|f|}f(x,\theta)\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{dt}{t+i\epsilon}e^{-it\iota_v}\delta(d\theta)$$
$$= \frac{-i}{iv}(-1)^{|f|}f(x,\theta)\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\frac{dt}{t+i\epsilon}\delta\left(t + i\frac{d\theta}{v}\right)$$
$$= \frac{i(-1)^{|f|}f(x,\theta)}{d\theta}. \tag{2.7}$$

Hence, we have

$$Z_v\omega = -i(d\Theta(\iota_v) + \Theta(\iota_v)d)\omega = (-1)^{|f|}\partial_\theta f(x,\theta)$$
$$+ \frac{(-1)^{|f|}dx\partial_x f(x,\theta)}{d\theta} - \frac{(-1)^{|f|}dx\partial_x f(x,\theta)}{d\theta}$$
$$= (-1)^{|f|}\partial_\theta f(x,\theta), \tag{2.8}$$

where when calculating $d\omega$ we have used the fact that $d\theta\delta(d\theta) = 0$.

III. SUPER CHERN-SIMONS ACTIONS

In this section, we briefly resume super Chern-Simons (SCS) theory on supermanifolds when dealing with picture-0 gauge field $A^{(1|0)}$ (subsection III A) or picture-1 gauge field $A^{(1|1)}$ (Secs. III A, III B, and III C). The full treatment at picture 0 and 1 can be found in [8,10], respectively. In Sec. III D, we consider the emergence of the L_∞ gauge algebra in a slightly different way from [10] by introducing the BRST (Becchi-Rouet-Stora-Tyutin) differential $d + s$.

A. At picture 0

We briefly recall some basic facts about $D = 3$, $\mathcal{N} = 1$ Super Chern-Simons theory (the notation we use is described in the Appendix). That model serves as a simple playground for more sophisticated examples. We start from a $(1|0)$-superform $A^{(1|0)} = A_a V^a + A_\alpha \psi^\alpha$ (where the superfields $A_a(x,\theta)$ and $A_\alpha(x,\theta)$ take values in the adjoint representation of the gauge group), and we define the field strength

$$F^{(2|0)} = dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)} = F_{[ab]}V^a \wedge V^b$$
$$+ F_{a\alpha}V^a \wedge \psi^\alpha + F_{(\alpha\beta)}\psi^\alpha \wedge \psi^\beta. \tag{3.1}$$

In order to reduce the redundancy of degrees of freedom of A_a and A_α of the $(1|0)$-form $A^{(1|0)}$, one imposes *a priori* the *conventional constraint*

$$\iota_\alpha \iota_\beta F^{(2|0)} = 0 \quad \Leftrightarrow$$
$$F_{(\alpha\beta)} = D_{(\alpha}A_{\beta)} + \gamma^a_{\alpha\beta}A_a + \{A_\alpha, A_\beta\} = 0, \tag{3.2}$$

from which it follows that $F_{a\alpha} = \gamma_{a,\alpha\beta}W^\beta$ with $W^\alpha = \nabla^\beta\nabla^\alpha A_\beta$ and $\nabla_\alpha W^\alpha = 0$. The gaugino field strength W^α (a $(0|0)$ form) is gauge invariant under the non-Abelian transformations $\delta A_\alpha = \nabla_\alpha \Lambda$. These gauge transformations descend from the gauge transformations of $A^{(1|0)}$, $\delta A^{(1|0)} = \nabla\Lambda$ where Λ is a $(0|0)$ form.

In order to express the action as an integral on a supermanifold, we use the Poincaré dual form (as known as PCO) $\mathbb{Y}^{(0|2)}$ dual to the immersion of $\mathcal{M}^{(3)}$ into $\mathcal{SM}^{(3|2)}$. The Poincaré dual form $\mathbb{Y}^{(0|2)}$ is closed; it is not exact and any of its variation is d exact. The action can now be written on the full supermanifold as

$$S[A] = \int_{\mathcal{SM}^{(3|2)}} \mathcal{L}^{(3|0)}(A, dA) \wedge \mathbb{Y}^{(0|2)}. \tag{3.3}$$

Any variation of the embedding yields $\delta\mathbb{Y}^{(0|2)} = d\Lambda^{(-1|2)}$ and leaves the action invariant if the Lagragian is closed. The rheonomic Lagrangian $\mathcal{L}^{(3|0)}(A, dA)$ reads

$$\mathcal{L}^{(3|0)}(A, dA) = \mathrm{Tr}\Big(A^{(1|0)} \wedge dA^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} + W^{(0|0)\alpha}\epsilon_{\alpha\beta}W^{(0|0)\beta}V^3\Big) \wedge \mathbb{Y}^{(0|2)}, \tag{3.4}$$

which is a $(3|2)$ form, $V^3 = \frac{1}{3!}\epsilon_{abc}V^a \wedge V^b \wedge V^c$.

This is the most general action that can be built with the rheonomic rules, and the closure of $\mathcal{L}^{(3|0)}$ implies that any gauge-invariant and supersymmetric action can be built by choosing a PCO $\mathbb{Y}^{(0|2)}$ inside the same cohomology class. Therefore, starting from the rheonomic action, one can choose a different "gauge"—or better said a different immersion of the submanifold $\mathcal{M}^{(3)}$ inside the supermanifold $\mathcal{SM}^{(3|2)}$—leading to different form of the action with the same physical content.

B. At picture 1

Any PCO $\mathbb{Y}^{(0|2)}$ can be decomposed into the product of two PCO's $\mathbb{Y}^{(0|1)}$ as follows:

$$\mathbb{Y}^{(0|2)} = \mathbb{Y}_v^{(0|1)} \wedge \mathbb{Y}_w^{(0|1)} + d\Omega^{(-1|2)}, \tag{3.5}$$

where v and w are two independent spinors $\mathrm{Det}(v, w) = v^\alpha\epsilon_{\alpha\beta}w^\beta \neq 0$. Let us analyze the action with the new choice of PCO,

$$S_{\mathrm{SCS}} = \int_{\mathcal{SM}^{(3|2)}} \Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A + W^\alpha W_\alpha V^3\Big) \wedge \mathbb{Y}_v^{(0|1)} \wedge \mathbb{Y}_w^{(0|1)}, \tag{3.6}$$

where the Ω term is dropped by integration by parts. Let us put aside the interaction term, to be discussed later, and let us distribute the two $\mathbb{Y}$'s on the two pieces of the action as follows:

$$S_{\mathrm{SCS}}^{\mathrm{quad}} = \int_{\mathcal{SM}^{(3|2)}} (A \wedge dA \wedge \mathbb{Y}_v^{(0|1)}\mathbb{Y}_w^{(0|1)} + W^\alpha W_\alpha \mathbb{Y}_v^{(0|1)}\mathbb{Y}_w^{(0|1)} \wedge V^3). \tag{3.7}$$

Since the PCOs are closed, we can also bring them after each connection term $A^{(1|0)}$ and after the spinorial $W^{(0|0)}$ forms as

$$S_{\mathrm{SCS}}^{\mathrm{quad}} = \int_{\mathcal{SM}^{(3|2)}} ((A_\wedge \mathbb{Y}_v^{(0|1)}) \wedge d(A_\wedge \mathbb{Y}_w^{(0|1)}) + (W^\alpha{}_\wedge \mathbb{Y}_v^{(0|1)}) \wedge (W_{\alpha\wedge}\mathbb{Y}_w^{(0|1)}) \wedge V^3), \tag{3.8}$$

converting the gauge connection to a $(1|1)$ form as

$$A^{(1|0)} \to A^{(1|1)} \equiv A^{(1|0)}{}_\wedge \mathbb{Y}_v^{(0|1)}. \tag{3.9}$$

In the same way, the $(0|0)$-form W^α is converted into a $(0|1)$ pseudoform. Passing from $A^{(1|0)}$, which has a finite number of components, to $A^{(1|1)}$, we have moved to an infinite-dimensional space. Therefore, we now take into account the more generic action

$$S_{\mathrm{SCS}} = \int_{\mathcal{SM}^{(3|2)}} (A^{(1|1)} \wedge dA^{(1|1)} + W^{(0|1),\alpha}\epsilon_{\alpha\beta} \wedge W^{(0|1),\beta} \wedge V^3). \tag{3.10}$$

The wedge product is taken in the space of pseudoforms and we use the convention that two $(0|1)$ forms must be multiplied with the wedge product. We observe that (3.10) reduces to (the free part of) (3.4) if we consider the factorized gauge field $A^{(1|1)} = A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}$.

Now, we relax the condition (3.9) and we consider the action in terms of the picture-1 fields $A^{(1|1)}$. That opens up to a completely new theory with several implications still to be fully explored. The strategy is to relax the factorization properties and consider all possible terms into a CS type of action. First, we briefly review the result of our previous paper [10] and then we build the BV formalism for this new theory.

Decomposing the pseudoform $A^{(1|1)} = A_0 + A_1 + A_2 + A_3$, where the subscript denotes the number of dx's in the expression, we have

$$A_0 = \sum_{p=0}^{\infty} A_{\alpha\beta}^{(p)}(d\theta^\alpha)^{p+1}\delta^{(p)}(d\theta^\beta),$$

$$A_1 = \sum_{p=0}^{\infty} dx^m A_{m\alpha\beta}^{(p)}(d\theta^\alpha)^p\delta^{(p)}(d\theta^\beta),$$

$$A_2 = \sum_{p=0}^{\infty} dx^m dx^n A_{[mn]\alpha\beta}^{(p)}(d\theta^\alpha)^p\delta^{(p+1)}(d\theta^\beta),$$

$$A_3 = \sum_{p=0}^{\infty} dx^m dx^n dx^r A_{[mnr]\alpha\beta}^{(p)}(d\theta^\alpha)^p\delta^{(p+2)}(d\theta^\beta). \tag{3.11}$$

We can compute the Lagrangian,

$$\begin{aligned}\mathcal{L}^{(3|2)} &= A^{(1|1)} dA^{(1|1)} \\ &= \sum_{p=0}^{\infty} [-p!(p+1)! A^{(p)}_{\alpha\beta} (\partial_{[r} A^{(p)}_{mn]\beta\alpha} - \partial_\beta A^{(p-1)}_{[mnr]\beta\alpha} + (p+2)\partial_\alpha A^{(p)}_{[mnr]\beta\alpha}) + \\ &\quad - p!p! A^{(p)}_{[m\alpha\beta} (\partial_r A^{(p)}_{n]\beta\alpha} - \partial_\beta A^{(p-1)}_{nr]\beta\alpha} + (p+1)\partial_\alpha A^{(p)}_{nr]\beta\alpha}) + \\ &\quad - p!(p+1)! A^{(p)}_{[mn\alpha\beta} (\partial_{r]} A^{(p)}_{\beta\alpha} - \partial_\beta A^{(p)}_{r]\beta\alpha} + (p+1)\partial_\alpha A^{(p+1)}_{r]\beta\alpha}) + \\ &\quad - p!(p+2)! A^{(p)}_{[mnr]\alpha\beta} (-\partial_\beta A^{(p)}_{\beta\alpha} + (p+1)\partial_\alpha A^{(p+1)}_{\beta\alpha})] dx^m \wedge dx^n \wedge dx^r \delta(d\theta^\beta)\delta(d\theta^\alpha).\end{aligned}$$

We obtain the equations of motion by varying the action with respect to the fields $A^{(p)}_{\alpha\beta}$, $A^{(p)}_{m\alpha\beta}$, $A^{(p)}_{[mn]\alpha\beta}$, and $A^{(p)}_{[mnr]\alpha\beta}$; the resulting equations are

$$\begin{aligned} -\partial_\beta A^{(p)}_{\beta\alpha} + (p+1)\partial_\alpha A^{(p+1)}_{\beta\alpha} &= 0, \quad \forall\ p\in\mathbb{N}, \\ \partial_r A^{(p)}_{\beta\alpha} - \partial_\beta A^{(p)}_{r\beta\alpha} + (p+1)\partial_\alpha A^{(p+1)}_{r\beta\alpha} &= 0 \quad \forall\ p\in\mathbb{N}, \\ \partial_{[r} A^{(p)}_{n]\beta\alpha} - \partial_\beta A^{(p-1)}_{[nr]\beta\alpha} + (p+1)\partial_\alpha A^{(p)}_{[nr]\beta\alpha} &= 0 \quad \forall\ p\in\mathbb{N}, \\ \partial_{[r} A^{(p)}_{mn]\beta\alpha} - \partial_\beta A^{(p-1)}_{[mnr]\beta\alpha} + (p+2)\partial_\alpha A^{(p)}_{[mnr]\beta\alpha} &= 0 \quad \forall\ p\in\mathbb{N}, \end{aligned} \tag{3.12}$$

where we stress that in (3.12) and in (3.12) if $p=0$ the fields $A^{(-1)}_{[nr]\beta\alpha}$ and $A^{(-1)}_{[mnr]\beta\alpha}$ are both defined to be zero. The equations of motion for $W^{(0|1),\alpha}$ implies that this is set to zero algebraically, and this is automatically achieved also in the present framework.

To show that this complicated set of linear equations coincides with the usual CS equations of motion, one needs to remove the infinite redundancy by algebraic (i.e., θ dependent) gauge transformations. As a result, we find that a representative of the cohomology class is

$$A^{1|1} = dx^m \theta^\beta \tilde{B}^{(0)}_{m\alpha\beta}(x)\delta^{(0)}(d\theta^\beta), \tag{3.13}$$

and the relative equation of motion is

$$\partial_{[n}\tilde{B}^{(0)}_{m]\alpha\beta}(x) = 0. \tag{3.14}$$

Remarkably, notice that even if we started from an SCS Lagrangian with an infinite number of fields, we have shown that there is only one physical field. All the other fields are d-exact θ-dependent terms.

Moreover, we have shown that starting from the free SCS action with a general $A^{1|1}$ pseudoform, we obtain the factorization

$$A^{(1|1)} = A^{(1|0)} \wedge \mathbb{Y}^{(0|1)}, \quad \text{s.t. } \mathbb{Y}^{(0|1)} = \theta^\beta \delta(d\theta^\beta) + d\Omega^{(-1|1)}. \tag{3.15}$$

Thus, we have recovered a factorized form from a non-factorized Lagrangian.

C. Interaction terms

We now define an interaction term which can be integrated on a supermanifold. Apparently, a problem arises. In order to define an interaction term, we need three gauge fields $A^{(1|1)}$, but the wedge product of three fields vanishes by the anticommutativity of the three Dirac delta functions of $d\theta_1$ or $d\theta_2$.

In [10], we propose an action where Z is inserted into the product of three gauge fields; however, as discussed in the paper, we have to consider all the possible places where to put the PCO. Therefore, following [17,18] (see also the proceeding [19]), we are led to define the two-product with picture degree -1 as

$$\begin{aligned} &\mathfrak{M}_2 : \Omega^{(1|1)} \times \Omega^{(1|1)} \to \Omega^{(2|1)} \\ &(A,A) \mapsto \mathfrak{M}_2(A,A) \\ &\quad = \frac{1}{3}[Z_v(A\wedge A) + Z_v(A)\wedge A + A\wedge Z_v(A)]. \end{aligned} \tag{3.16}$$

Observe that this product has form degree 0, i.e., it does not change the form number and it decreases the picture number by 1. The products of the various fields involve also the matrix multiplication of the generator in the adjoint representation.

In an analogous way, we can define a product with form degree -1 as

$$\tilde{m}_2^{(-1)}: \Omega^{(1|1)} \times \Omega^{(1|1)} \to \Omega^{(1|1)}$$
$$(A, A) \mapsto \tilde{m}_2^{(-1)}(A, A) = \frac{1}{3}[-i\Theta(\iota_v)(A \wedge A) - i\Theta(\iota_v)(A) \wedge A - (-1)^{|A|} A \wedge i\Theta(\iota_v)(A)]. \tag{3.17}$$

This product is needed as an intermediate product to define higher order product as explained in the literature (see, e.g., [18]). It maps the integral forms into the space of inverse forms which do not have a physical interpretation (it is analogous to the so-called *large Hilbert space*). From the definition (3.16), it follows that

$$\mathfrak{M}_2 = [d, \tilde{m}_2^{(-1)}], \tag{3.18}$$

where $[\cdot,\cdot]$ denotes as usual the graded commutator.

We can compute the first interaction term of the Lagrangian $A^{(1|1)} \wedge \mathfrak{M}_2(A^{(1|1)}, A^{(1|1)})$, which explicitly reads

$$\mathcal{L}_{\mathrm{INT}}^{(3|2)} = 2\mathrm{tr}\left\{ \sum_{p,q=0}^{\infty} (-1)^p p! q! q! \{(q+1)[A^{(q)}_{mn\alpha\beta}, A^{(q)}_{\beta\alpha}] + A^{(q)}_{m\alpha\beta} A^{(q)}_{n\beta\alpha}\} \right.$$
$$\left. \times \left\{ \left(\frac{v^\alpha}{v^\beta}\right)^p \left(\frac{v^\alpha}{v^\beta}\partial_\alpha + \partial_\beta\right) A^{(p)}_{r\alpha\beta} + \left(\frac{v^\beta}{v^\alpha}\right)^p \left(\frac{v^\beta}{v^\alpha}\partial_\beta + \partial_\alpha\right) A^{(p)}_{r\beta\alpha} \right\} \right\} \epsilon^{mnr} \epsilon^{\alpha\beta} d^3x \delta^2(d\theta). \tag{3.19}$$

Notice that the interaction term depends on the constant vector v^α through $\frac{v^1}{v^2}$, namely, their relative phase. That resembles the usual frame dependence of superstring field theory actions.

D. Gauge invariance and the emergence of the A_∞ algebra

The nonassociativity of the product $\mathfrak{M}_2$ breaks the gauge invariance of Chern-Simons action; furthermore, the algebra of gauge transformations does not close. To overcome these two problems, we need additional terms in the action and we need to change the gauge transformations.

We now proceed by constructing explicitly the first multiproduct of the A_∞ algebra. Let us consider the action discussed so far,

$$S_A = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge \mathfrak{M}_2(A, A)\right). \tag{3.20}$$

We will assume that the gauge field A is a $(1|1)$ pseudoform and we neglect the WW term for the moment (then we denote this part of the action as S_A). Assuming the cyclicity of the trace, we can compute the field strength from the variation of (3.20)

$$F = dA + \mathfrak{M}_2(A, A). \tag{3.21}$$

The field strength is a $(2|1)$ form as dA; indeed, $\mathfrak{M}_2$ consistently reduces the picture by 1. Upon applying the exterior derivative d, which is a derivation of $\mathfrak{M}_2$ (since $[Z_v, d] = 0$), we get

$$dF = d\mathfrak{M}_2(A, A) = \mathfrak{M}_2(dA, A) - \mathfrak{M}_2(A, dA). \tag{3.22}$$

We can now use (3.21) to substitute the expression for dA in (3.22) and we get

$$dF = \mathfrak{M}_2(F, A) - \mathfrak{M}_2(A, F) - \mathfrak{M}_2(\mathfrak{M}_2(A, A), A)$$
$$+ \mathfrak{M}_2(A, \mathfrak{M}_2(A, A)), \tag{3.23}$$

where, as expected, it appears the extra term given by the associator of $\mathfrak{M}_2$. In order to get rid of this term, we add an extra term to the action such that

$$F' = dA + \mathfrak{M}_2(A, A) + \mathfrak{M}_3(A, A, A) = F + \mathfrak{M}_3(A, A, A),$$
$$dF' = \mathfrak{M}_2(F, A) - \mathfrak{M}_2(A, F) - \mathfrak{M}_3(dA, A, A)$$
$$+ \mathfrak{M}_3(A, dA, A) - \mathfrak{M}_3(A, A, dA), \tag{3.24}$$

which implies that

$$d\mathfrak{M}_3(A, A, A) + \mathfrak{M}_3(dA, A, A) - \mathfrak{M}_3(A, dA, A)$$
$$+ \mathfrak{M}_3(A, A, dA) - \mathfrak{M}_2(\mathfrak{M}_2(A, A), A)$$
$$+ \mathfrak{M}_2(A, \mathfrak{M}_2(A, A)) = 0. \tag{3.25}$$

This is the third A_∞ relation (e.g., see [20–23]). Notice that when we apply d to $\mathfrak{M}_3$, since *a priori* it is not a derivation of $\mathfrak{M}_3$, we also need three additional terms in the first line of (3.25). The explicit form of $\mathfrak{M}_3$ is given in [10], and we do not repeat here its construction: it is based on $\tilde{m}_2$ which is built from $\Theta(\iota_v)$ instead of Z.

Clearly, the Bianchi identity for F' does not hold. In particular, from (3.24), we have

$$\begin{aligned}
dF' &= -\mathfrak{M}_2(F,A) + \mathfrak{M}_2(A,F) - \mathfrak{M}_3(dA,A,A) + \mathfrak{M}_3(A,dA,A) - \mathfrak{M}_3(A,A,dA) \\
&= -\mathfrak{M}_2(F' - \mathfrak{M}_3(A,A,A),A) + \mathfrak{M}_2(A,F' - \mathfrak{M}_3(A,A,A)) \\
&\quad - \mathfrak{M}_3(F' - \mathfrak{M}_2(A,A) - \mathfrak{M}_3(A,A,A),A,A) + \mathfrak{M}_3(A,F' - \mathfrak{M}_2(A,A) - \mathfrak{M}_3(A,A,A),A) \\
&\quad - \mathfrak{M}_3(A,A,F' - \mathfrak{M}_2(A,A) - \mathfrak{M}_3(A,A,A)) \\
&= \mathfrak{M}_2(F',A) - \mathfrak{M}_2(A,F') - \mathfrak{M}_3(F',A,A) + \mathfrak{M}_3(A,F',A) - \mathfrak{M}_3(A,A,F') \\
&\quad - \mathfrak{M}_2(\mathfrak{M}_3(A,A,A),A) + \mathfrak{M}_2(A,\mathfrak{M}_3(A,A,A)) + \mathfrak{M}_3(\mathfrak{M}_2(A,A),A,A) \\
&\quad - \mathfrak{M}_3(A,\mathfrak{M}_2(A,A),A) + \mathfrak{M}_3(A,A,\mathfrak{M}_2(A,A)) + \mathcal{O}(A^5),
\end{aligned} \tag{3.26}$$

where we have neglected the five-gauge field terms $\mathcal{O}(A^5)$. Expressing the field strength F in terms of the corrected one $F' = dA + \mathfrak{M}_2(A,A) + \mathfrak{M}_3(A,A,A) + \mathcal{O}(A^4)$, the equation can be rewritten as

$$\begin{aligned}
&dF' - \mathfrak{M}_2(F',A) + \mathfrak{M}_2(A,F') + \mathfrak{M}_3(F',A,A) - \mathfrak{M}_3(A,F',A) + \mathfrak{M}_3(A,A,F') \\
&\quad = -\mathfrak{M}_2(\mathfrak{M}_3(A,A,A),A) + \mathfrak{M}_2(A,\mathfrak{M}_3(A,A,A)) - \mathfrak{M}_3(\mathfrak{M}_2(A,A),A,A) \\
&\quad\quad - \mathfrak{M}_3(A,\mathfrak{M}_2(A,A),A) + \mathfrak{M}_3(A,A,\mathfrak{M}_2(A,A)) + \mathcal{O}(A^5),
\end{aligned} \tag{3.27}$$

where in the first line we have the Bianchi identities, broken by the right-hand side of the equation, which contains four-gauge field terms expressing the nonassociativity of the $\mathfrak{M}_2$ and $\mathfrak{M}_3$ products. Again, we have an extra term breaking the Bianchi identity. By following the prescription described above, we add to the action an extra term $\mathfrak{M}_4(A,A,A,A)$ for the field strength and correspondently for the action. Proceeding in the same way, we have a new action of the form

$$S_A = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\left(\frac{1}{2}AdA + \sum_{n=2}^{\infty}\frac{1}{n+1}A\wedge\mathfrak{M}_n(A,\ldots,A)\right), \tag{3.28}$$

yielding the equations of motion

$$dA + \sum_n \mathfrak{M}_n(A,\ldots,A) = 0, \tag{3.29}$$

consistent with the Bianchi identities because of the A_∞ relations among the various multiproducts $\mathfrak{M}_n(A,\ldots,A)$. Notice that according to [10] every multiproduct reduces the form degree from n (the total degree of the product of n gauge fields A) to 2 and the picture number from n (the total picture of a formal product of n gauge fields) down to 1 as required for building an integral form. Notice that the form of the action is the conventional homotopy Mauer-Cartan action, also discussed in [24] for general L_∞ algebras (see also [25] for application to string field theory).

Let us now study the gauge symmetry. Previously we have seen that since the product $m_2^{(-1)}$ is not associative, the gauge algebra does not close. We now show that in order to close the algebra, we have to modify the gauge transformation law by introducing multiproducts induced by the A_∞ algebra discussed so far, but then it emerges that one necessarily needs the BV formalism to deal with the gauge symmetries (see [24] for a general discussion and examples). We derive the BRST symmetry by shifting the gauge field A with $A + c$. Notice that we trade the form number with the ghost number, but we have not changed the picture number, since both A and c have the same picture. The second shift is d into $d + s$, introducing the BRST differential. Then, we impose the equation

$$\begin{aligned}
&(d+s)(A+c) + \mathfrak{M}_2(A+c,A+c) \\
&\quad + \mathfrak{M}_3(A+c,A+c,A+c) + \mathcal{O}(A^4,\ldots,c^4) = F',
\end{aligned} \tag{3.30}$$

from which we get the following relations:

$$\begin{aligned}
F' &= dA + \mathfrak{M}_2(A,A) + \mathfrak{M}_3(A,A,A) + \mathcal{O}(A^4), \\
0 &= sA + dc + \mathfrak{M}_2(A,c) + \mathfrak{M}_2(c,A) + \mathfrak{M}_3(A,A,c) \\
&\quad + \mathfrak{M}_3(A,c,A) + \mathfrak{M}_3(c,A,A) + \mathcal{O}(A^3), \\
0 &= sc + \mathfrak{M}_2(c,c) + \mathfrak{M}_3(A,c,c) + \mathfrak{M}_3(c,A,c) \\
&\quad + \mathfrak{M}_3(c,c,A) + \mathcal{O}(A^2), \\
0 &= \mathfrak{M}_3(c,c,c) + \mathcal{O}(A).
\end{aligned} \tag{3.31}$$

The last equation is consistent with the definition of the $\mathfrak{M}_3$ product since it decreases the form number, but c has form degree equal to zero. The first line reproduces the definition of F' given in (3.24). The second line gives the gauge transformation of the gauge field

$$sA = -dc - \mathfrak{L}_2(c,A) - \frac{1}{2}\mathfrak{L}_3(c,A,A) + \mathcal{O}(A^3), \tag{3.32}$$

where the Lie-algebra-like symbols (for anticommuting quantities)

$$\begin{aligned} \mathfrak{L}_2(A,c) &= \mathfrak{M}_2(A,c) + \mathfrak{M}_2(c,A), \\ \mathfrak{L}_3(A,c,c) &= 2\mathfrak{M}_3(A,c,c) + 2\mathfrak{M}_3(c,A,c) + 2\mathfrak{M}_3(c,c,A), \\ \mathfrak{L}_2(X,Y) &= \mathfrak{M}_2(X,Y) - (-1)^{|X||Y|}\mathfrak{M}_2(Y,X), \\ \mathfrak{L}_3(X_1,X_2,X_3) &= \mathfrak{M}_3(X_1,X_2,X_3) - (-1)^{|X_2||X_3|}\mathfrak{M}_3(X_1,X_3,X_2) \\ &\quad + \mathfrak{M}_3(X_2,X_3,X_1) - (-1)^{|X_3||X_1|}\mathfrak{M}_3(X_2,X_1,X_3) \\ &\quad + \mathfrak{M}_3(X_3,X_1,X_2) - (-1)^{|X_1||X_2|}\mathfrak{M}_3(X_3,X_2,X_1) \end{aligned} \tag{3.33}$$

have been introduced. Note that $\mathfrak{L}_2(c,c) = 2\mathfrak{M}_2(c,c)$. In addition, $\mathfrak{L}_2(A,c)$ has form number $= +1$, ghost number $= +1$, and picture $= +1$. $\mathfrak{L}_3(A,c,c)$ has form number $= 0$, ghost number $= +2$, and picture $= +1$. The third line of (3.31) gives us the BRST transformation of the ghost field

$$sc = -\frac{1}{2}\mathfrak{L}_2(c,c) - \frac{1}{2}\mathfrak{L}_3(A,c,c) + \mathcal{O}(A^2). \tag{3.34}$$

In order to study the nilpotency of s, it is useful to verify the compatibility of the multiproducts with s. For example, starting from the second A_∞ relation, we have

$$\begin{aligned} &(s+d)\mathfrak{M}_2(A+c,A+c) - \mathfrak{M}_2((s+d)(A+c),A+c) \\ &\quad + \mathfrak{M}_2(A+c,(s+d)(A+c)) = 0, \end{aligned} \tag{3.35}$$

which implies the following relations:

$$\begin{aligned} s\mathfrak{M}_2(A,A) - \mathfrak{M}_2(sA,A) + \mathfrak{M}_2(A,sA) &= 0, \\ s\mathfrak{L}_2(A,c) - \mathfrak{L}_2(sA,c) - \mathfrak{L}_2(sc,A) &= 0, \\ s\mathfrak{L}_2(c,c) - \mathfrak{L}_2(sc,c) + \mathfrak{L}_2(c,sc) &= 0, \end{aligned} \tag{3.36}$$

where $\mathfrak{L}_2(sA,c) = \mathfrak{M}_2(sA,c) - \mathfrak{M}_2(c,sA)$ since c is fermionic and sA is bosonic. This implies that $\mathfrak{L}_2(sA,c) = -\mathfrak{L}_2(c,sA)$ and $\mathfrak{L}_2(sc,A) = -\mathfrak{L}_2(A,sc)$, leading to

$$s\mathfrak{L}_2(A,c) = \mathfrak{L}_2(sA,c) - \mathfrak{L}_2(A,sc), \tag{3.37}$$

which expressed the Leibniz rule of s with respect to $\mathfrak{L}_2$.

Let us study the nilpotency of s on the ghost field. Acting with s on (3.34), we get

$$\begin{aligned} s^2c &= -\frac{1}{2}s\mathfrak{L}_2(c,c) - \frac{1}{2}s\mathfrak{L}_3(A,c,c) + \mathcal{O}(A^2) \\ &= -\mathfrak{L}_2(sc,c) - \frac{1}{2}\mathfrak{L}_3(sA,c,c) + \mathfrak{L}_3(A,sc,c) \\ &\quad - \mathfrak{L}(A,c,sc) + \mathcal{O}(A^2). \end{aligned} \tag{3.38}$$

Inserting the BRST transformations of c, using the A_∞ relations, it yields

$$s\left(sc - \frac{1}{4}\mathfrak{L}_4(A,A,c,c)\right) = \frac{1}{3}\mathfrak{L}_4(F,c,c,c) + \mathcal{O}(A^2), \tag{3.39}$$

where on the left-hand side we have reabsorbed $\mathfrak{L}_4(A,A,c,c)$ in the definition of the BRST transformation of the ghost c, and on the right-hand side we finally found that the algebra is not closed, but it closes on the field strength F of the gauge field A. This is crucial, since this implies that we need the antifield formalism to deal with it since the field strength is just the variation of the antifield A^* of the gauge field. As it will be shown in the last section, it is precisely the integral form formalism that gives the correct quantum number for A^*. It should be a $(2|1)$ form.

IV. BV ACTION

A. Picture-0 gauge fields

To build the *BV action*, we need to include into the previous action the antifields as the generators of the BRST transformations (the superspace formulation has been studied in [26]); at the moment we assume these antifields to be superforms. For example, the term corresponding to the BRST transformation of the gauge field reads

$$\int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}[A^{*(2|0)} \wedge \nabla c^{(0|0)}] \wedge \mathbb{Y}^{(0|2)}, \tag{4.1}$$

where $A^{*(2|0)}$ is the $(2|0)$ superform

$$A^* = V^a V^b A^*_{ab} + V^a \psi^\alpha A^*_{a\alpha} + \psi^\alpha \psi^\beta A^*_{\alpha\beta}. \tag{4.2}$$

The PCO $\mathbb{Y}^{(0|2)}$ is used to convert it into an integral form. Comparing with the component formalism, we see that there are too many independent components in A^* and, therefore, we need a convenient constraint to reduce them systematically. Hence, we set

$$\begin{aligned} &\nabla A^* = 0, \qquad A^*_{\alpha\beta} = 0 \quad \Rightarrow \\ &A^* = V^a V^b A^*_{ab} + V^a \psi^\alpha \gamma_{a\alpha\beta} W^{*\beta} \end{aligned} \tag{4.3}$$

as for the field strength $F^{(2|0)}$ [see Eq. (3.1)]. A^*_{ab} serves as the antifield for the gauge field, while $W^{*\alpha}$ for the gaugino. The vanishing of the covariant derivative of A^* is needed to require the closure of the 0-picture factor of (4.1). Here, this choice seems to be too adapted to the specific example, then the quest for a more natural setting. Nevertheless, by taking the susy PCO, i.e., $\mathbb{Y}^{(0|2)} = V^a V^b \epsilon_{abc} \iota_\alpha \gamma^{c\alpha\beta} \iota_\beta \delta^2(\psi)$, Eq. (4.1) reads

$$\int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}[(V^a V^b A^*_{ab} + V^a \psi^\alpha \gamma_{a\alpha\beta} W^{*\beta}) \wedge (V^c \nabla_c c + \psi^\gamma \nabla_\gamma c)] \wedge \mathbb{Y}^{(0|2)}_{\mathrm{susy}} = \int_{x,\theta} \mathrm{Tr}(W^{*\alpha} \nabla_\alpha c). \tag{4.4}$$

This result matches with the superspace CS BV action.

The integral form formalism provides a more natural way to obtain the correct BV terms. We observe that the *Serre's* dual to a $(1|0)$ superform is a $(2|2)$ integral form (see also [5]),

$$A^{*(2|2)} = V^a V^b \epsilon_{abc} A^{*c} \delta^2(\psi) + V^a V^b V^c \epsilon_{abc} W^{*\alpha} \iota_\alpha \delta^2(\psi). \tag{4.5}$$

We do not impose any constraint on $A^{*(2|2)}$, since this integral form already contains the correct number of fields. Analogously, the BRST symmetry of the ghost c is coupled to a $(3|2)$-integral form c^*, representing its antifield

$$c^{*(3|2)} = c^* V^a V^b V^c \epsilon_{abc} \delta^2(\psi). \tag{4.6}$$

Therefore, the action reads

$$S = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\left[\left(AdA + \frac{2}{3}A^3 + \frac{1}{2}W^2 V^3\right) \wedge \mathbb{Y} + A^* \nabla c + \frac{1}{2} c^*[c,c]\right] = S_{\mathrm{SCS}} + S_{\mathrm{BV}}. \tag{4.7}$$

To compare (4.7) with the component or with the superspace actions, it is convenient to rewrite it into a factorized form $\mathcal{L}^{(3|0)} \wedge \mathbb{Y}^{(0|2)}$. For that, we have

$$\begin{aligned} \mathrm{Tr}\left(A^* \nabla c + \frac{1}{2} c^*[c,c]\right) &= Z\mathrm{Tr}\left(A^* \nabla c + \frac{1}{2} c^*[c,c]\right) \wedge \mathbb{Y} \\ &= \mathrm{Tr}\left[Z(A^* \nabla c) + Z\left(\frac{1}{2} c^*[c,c]\right)\right] \wedge \mathbb{Y} = \mathcal{L}_{\mathrm{BV}} \wedge \mathbb{Y}, \end{aligned} \tag{4.8}$$

where the formal inverse Z of $\mathbb{Y}$ and the linearity have been used.[2] It follows that

$$\begin{aligned} d\mathcal{L}_{\mathrm{BV}} &= d\mathrm{Tr}\left[Z(A^* \nabla c) + Z\left(\frac{1}{2} c^*[c,c]\right)\right] = dZ\mathrm{Tr}(A^* \nabla c) + dZ\mathrm{Tr}\left(\frac{1}{2} c^*[c,c]\right) \\ &= Z\mathrm{Tr}\left[d(A^* \nabla c) + d\left(\frac{1}{2} c^*[c,c]\right)\right] = 0, \end{aligned} \tag{4.9}$$

using $[d, Z] = 0$ and noting that d acts on a top integral form. The closure of the Lagrangian suggests that we do not need any other terms. In particular, we do not need any further antifield since all the needed d.o.f. are already present. To verify this, we compute (4.8) for two choices of PCO, namely, the supersymmetric PCO and the component PCO. Let us start from the supersymmetric case. It is easy to verify that

$$\begin{aligned} \mathrm{Tr}A^* \nabla c &= \mathrm{Tr}[[V^a V^b \epsilon_{abc} A^{*c} \delta^2(\psi) + V^a V^b V^c \epsilon_{abc} W^{*\alpha} \iota_\alpha \delta^2(\psi)] \wedge [V^d \nabla_d c + \psi^\gamma \nabla_\gamma c]] \\ &= \mathrm{Tr}[[A^{*a} \psi \gamma_a \psi + V^c W^* \gamma_c \psi] \wedge [V^d \nabla_d c + \psi^\gamma \nabla_\gamma c]] \wedge \mathbb{Y}^{(0|2)}_{\mathrm{susy}} \\ &= \mathrm{Tr}[A^{*a} \psi \gamma_a \psi V^d \nabla_d c + V^c W^* \gamma_c \psi \psi^\gamma \nabla_\gamma c] \wedge \mathbb{Y}^{(0|2)}_{\mathrm{susy}}. \end{aligned} \tag{4.10}$$

We see that the first term contains the antifield relative to the gauge field A^a, while the second term contains the antifield of the gaugino. Equation (4.10) coincides with the result in (4.4) up to a rescaling: recall that $\nabla_a = \frac{1}{2}\gamma_a^{\alpha\beta}\{\nabla_\alpha, \nabla_\beta\}$, then

[2] Notice that by Z we mean the product of two PCOs Z_v and Z_w, where v and w represent two independent directions.

$$\begin{aligned}\mathrm{Tr}[A^{*a}\psi\gamma_a\psi V^d\nabla_d c + V^c W^*\gamma_c\psi\psi^\gamma\nabla_\gamma c]\wedge\mathbb{Y}^{(0|2)}_{\rm susy} &= \mathrm{Tr}[A^{*a}\psi\gamma_a\psi V^b\gamma_b^{\alpha\beta}\nabla_\alpha\nabla_\beta c + V^c W^*\gamma_c\psi\psi^\beta\nabla_\beta c]\wedge\mathbb{Y}^{(0|2)}_{\rm susy}\\ &= \mathrm{Tr}[V^c[-\nabla_\alpha A^{*a}\psi\gamma_a\psi\gamma_c^{\alpha\beta} + W^*\gamma_c\psi\psi^\beta]\nabla_\beta c]\wedge\mathbb{Y}^{(0|2)}_{\rm susy} \qquad (4.11)\end{aligned}$$

up to exact terms.

The same analysis can be repeated for the component PCO, namely $\theta^2\delta^2(d\theta)$,

$$\begin{aligned}\mathrm{Tr}A^*\nabla c &= V^aV^bV^c\epsilon_{abc}\mathrm{Tr}[A^{*d}\nabla_d c - W^{*\alpha}\nabla_\alpha c]\delta^2(\psi)\\ &= dx^a dx^b dx^c\epsilon_{abc}\mathrm{Tr}[A^{*d}\nabla_d c - W^{*\alpha}\nabla_\alpha c]\delta^2(d\theta), \qquad (4.12)\end{aligned}$$

passing from the vielbein basis to the component one given by dx's and $d\theta$'s. We can now apply the PCO Z and by recalling that $Z(f\delta^2(d\theta)) = \partial_{\theta^1}\partial_{\theta^2}f$, we get

$$\begin{aligned}Z[dx^a dx^b dx^c\epsilon_{abc}\mathrm{Tr}(A^{*d}\nabla_d c - W^{*\alpha}\nabla_\alpha c)\delta^2(d\theta)] &= dx^a dx^b dx^c\epsilon_{abc}\mathrm{Tr}(\partial^2_\theta(A^{*d}\nabla_d c - W^{*\alpha}\nabla_\alpha c))\\ &= dx^a dx^b dx^c\epsilon_{abc}\mathrm{Tr}(\partial^2_\theta[(-\nabla_\alpha A^{*d}\gamma_d^{\alpha\beta} - W^{*\beta})\nabla_\beta c]). \qquad (4.13)\end{aligned}$$

Again the gaugino antifield emerges and it couples correctly to the BRST variation of the fields.

B. Picture-1 gauge fields

We now want to study the BV formalism in the context of picture-1 fields, namely, pseudoforms. Let us start by considering the action (3.28). We can substitute the gauge field $A^{(1|1)}$ with a form $\mathcal{A}$ with general form number and picture number 1,

$$S_{\rm SCS-BV} = \int_{\mathcal{SM}^{(3|2)}}\mathrm{Tr}\left[\frac{1}{2}\mathcal{A}\wedge d\mathcal{A} + \mathcal{A}\wedge\sum_{i=2}^{\infty}\frac{1}{i+1}\mathfrak{M}_i(\mathcal{A},\mathcal{A},\ldots)\right]. \qquad (4.14)$$

The gauge transformations are obtained by applying the rules described in the previous sections: we replace the exterior derivative d with $d+s$ in the e.o.m., and the $(1|1)$ gauge field $A^{(1|1)}$ with a sum of $(p|1)$ forms, $p\in\mathbb{Z}$,[3] namely,

$$d\to d+s,\qquad A^{(1|1)}\to\mathcal{A} = \sum_{p=-\infty}^{\infty}A^{(p|1)} \Rightarrow$$
$$\Rightarrow (d+s)\mathcal{A} + \sum_{i=2}^{\infty}\mathfrak{M}_i(\mathcal{A},\mathcal{A},\ldots) = 0. \qquad (4.15)$$

The action written in these terms contains an infinite number of terms with all powers of any $A^{(p|1)}$. See also [24] where the general construction has been worked out in several examples. Here, we want to study the possible field redefinitions.

The BV terms look quite redundant compared to the previous results with a two-form and a three-form only; at picture number 1, we first need $A^{(2|1)}$ and $A^{(3|1)}$. These two fields are the natural antifields for $A^{(1|1)}$ and for $A^{(0|1)}\equiv c^{(0|1)}$, respectively. Serre's duality at picture 1 involves only picture-1 forms. Then, we denote them as

$$A^{(2|1)}\equiv A^{*(2|1)},\qquad A^{(3|1)}\equiv c^{*(3|1)}. \qquad (4.16)$$

Notice that *a priori* we may have pseudoforms with different form numbers as well; for example, in the action, there can be a term as

$$A^{(-1|1)}\wedge\mathfrak{M}_2(A^{*(2|1)},A^{*(2|1)}). \qquad (4.17)$$

In order to justify these structures, we proceed into two steps. First, we show that there is always a field redefinition which can reexpress the entire set of forms into those with $0\le p\le 3$, but the BV action and the BV symplectic form are not compatible with that field redefinition (see [27] for a detailed description of compatible field redefinition for A_∞ BV string field theory action).[4] In particular, we show that this field redefinition implies nontrivial constraints on the set of fields.

[3]Notice that this formula is analogous to the string field in bosonic or super string field theory where the form number is replaced by the ghost number and the physical fields are those with ghost number equal to 1; the components with nonpositive form numbers are interpreted as the ghosts and ghosts-for-ghosts of several generations, and those with form number greater than 1 are interpreted as the antifields of the gauge field and of the entire set of ghosts.

[4]In [27], the authors show that in the case of tensor construction, the "gauge group" d.o.f.—which in principle could be described by a A_∞ algebra with an even symplectic form—can be suitably redefined, under adequate hypotheses, and the only nontrivial structure is an associative algebra.

First, we show that we need to include all possible BV fields, in particular those with negative and positive (greater than three) form number fields by studying the structure of multiproducts. For that, we observe that we can rewrite the fields by starting from the expansion of a general picture-1 form on a basis of forms

$$\mathcal{A}^{(\bullet|1)} = \sum_{p=-\infty}^{\infty} A^{(p|1)} = \sum_{i=1,2}\sum_{n=0}^{3}\sum_{q=-\infty}^{\infty} A^{i}_{n,q} e^{q|1}_{n,i} \quad \in \bigoplus_{r=-\infty}^{\infty} \Omega^{(r|1)}. \tag{4.18}$$

This notation means that the bosonic form number is n, the fermionic form number is q while the index i indicates the argument of the delta function, i.e., $d\theta_i$. In particular, we have

$$e^{q|1}_{n,i} = \underbrace{dx\wedge\ldots\wedge dx}_{n} \sum_{p=0}^{\infty} (d\theta_j)^p (\iota_i)^p \times \begin{cases} (d\theta_j)^q \delta(d\theta_i), & \text{if } q\geq 0, \\ \iota_i^q \delta(d\theta_i), & \text{if } q \leq 0, \end{cases} \quad i\neq j. \tag{4.19}$$

We single out a box defined by the conditions $0\leq n\leq 3$ and $0\leq n+q\leq 3$,

$$e^{-q|1}_{n,i} \quad \begin{matrix} e^{0|1}_{0,i} & e^{1|1}_{0,i} & e^{2|1}_{0,i} & e^{3|1}_{0,i} \\ e^{-1|1}_{1,i} & e^{0|1}_{1,i} & e^{1|1}_{1,i} & e^{2|1}_{1,i} \\ e^{-2|1}_{2,i} & e^{-1|1}_{2,i} & e^{0|1}_{2,i} & e^{1|1}_{2,i} \\ e^{-3|1}_{3,i} & e^{-2|1}_{3,i} & e^{-1|1}_{3,i} & e^{0|1}_{3,i} \end{matrix} \quad e^{q|1}_{n,i}\, q\geq 0. \tag{4.20}$$

As a consequence of (4.19), we can write every element of the full basis as elements of the diagonal of the cohomological box (we call it "cohomological box" since the whole cohomology is contained in those spaces only; see, e.g., [10,18]) as follows:

$$e^{q|1}_{n,i} = \underbrace{dx\wedge\ldots\wedge dx}_{n} \sum_{p=0}^{\infty} (d\theta_j)^p (\iota_i)^p \times \begin{cases} (d\theta_j)^q \delta(d\theta_i) = (d\theta_j)^q e^{0|1}_{n,i}, & \text{if } q\geq 0, \\ \iota_i^q \delta(d\theta_i) = \iota_i^q e^{0|1}_{n,i}, & \text{if } q \leq 0, \end{cases} \quad i\neq j. \tag{4.21}$$

The elements of the box in (4.20) have a natural interpretation: they provide a basis for the ghost c, the gauge field A, the antifield A^*, and the antighost c^*,

$$\begin{aligned} c^{(0|1)} &= A^0_0 + A^{-1}_1 + A^{-2}_2 + A^{-3}_3, \\ A^{(1|1)} &= A^1_0 + A^0_1 + A^{-1}_2 + A^{-2}_3, \\ A^{*(2|1)} &= A^{*2}_0 + A^{*1}_1 + A^{*0}_2 + A^{*-1}_3, \\ c^{*(3|1)} &= c^{*3}_0 + c^{*2}_1 + c^{*1}_2 + c^{*0}_3. \end{aligned} \tag{4.22}$$

Let us first focus on the gauge field only. In [10], we have explicitly shown that, on-shell, the complete tower of fields reduces to the first field appearing in A^0_1, namely, it has the factorized form

$$A^{(1|1)}_{\text{phys}} = dx^a A_a \mathbb{Y}^{(0|1)}, \tag{4.23}$$

where $\mathbb{Y}^{(0|1)}$ denotes as usual the PCO. In particular, we observe that (4.23) is A^0_1 in the table above. Moreover, we have shown in (4.21) that, by using carefully the differential operators $d\theta$ and ι_θ, we can write every element of the basis (4.20) by using the basis elements $e^{0|1}_{n,i}$, $n = 0, 1, 2, 3$ only. We could then be induced to consider the BV action to be built by ghost, gauge field, antifield, and antighost of the following form:

$$c^0_0, A^0_1, A^{*0}_2, c^{*0}_3. \tag{4.24}$$

This would be analogous to consider the elements of the diagonal of (4.21). However, this is not the case; indeed, we know that the gauge field is made of four pieces,

$$A^{1|1} = A^1_0 + A^0_1 + A^{-1}_2 + A^{-2}_3, \tag{4.25}$$

and this is required by covariance on a generic supermanifold. An analogous argument holds for the antifield and the antighost, then we need the whole set of fields in (4.22). Let us consider now the structure of the products of the action (4.14); a simple power counting shows that we have to take into account not only the fields of (4.22), but also those of the form (4.18). Indeed, recall that

$$\mathfrak{M}_2 : \Omega^{(p|1)} \otimes \Omega^{(q|1)} \to \Omega^{(p+q|1)}. \tag{4.26}$$

This means we generate all other possible sets, even if we started from the fields in (4.22). For example, we generate forms of $\Omega^{(6|1)}$ from the two-products

$$\mathfrak{M}_2(c^{*3}_0, c^{*3}_0) \in \Omega^{(6|1)}. \tag{4.27}$$

In particular, this term would appear in the action as

$$B^{-6}_3 \wedge \mathfrak{M}_2(c^{*3}_0, c^{*3}_0), \tag{4.28}$$

where $B^{-6}_3 \in \Omega^{(-3|1)}$. That shows we have to include also the fields with negative form degree, which, in turn, will enter in the multiproducts themselves.

These arguments show that the definitions of the multiproducts require fields with any form degree. Their structure is given by the geometric data of the supermanifold considered. Hence, the A_∞ structure of the multiproducts and the L_∞ structure of the gauge transformations are the building blocks of an action on a supermanifold with fields with nonzero and nonmaximum picture number.

Now, we have to show that there is no nontrivial field redefinition which allows us to restrict the space of fields to those of (4.22). For that, we start from the BV symplectic form and we rewrite it as follows:

$$\omega_{\mathrm{BV}} = \langle \mathcal{A}, \mathcal{A} \rangle = \int_{\mathcal{SM}} \mathcal{A}^{(\bullet|1)} \wedge \mathcal{A}^{(\bullet|1)}, \tag{4.29}$$

and the integral selects the forms with total form degree equal to three. We claim that with the construction described in the previous paragraphs we prove that there is no unconstrained field redefinition establishing the equivalence between the BV theory written using all possible components of $\mathcal{A}$ with any form number and the BV theory written using the fields of the box (4.22) only. Let us consider a term of (4.29) made as

$$\begin{aligned}
\langle A^{(p|1)}, A^{(3-p|1)} \rangle &= \langle A^i_{n,q} e^{q|1}_{n,i}, A^j_{3-n,-q} e^{-q|1}_{3-n,i} \rangle = \int_{\mathcal{SM}} \mathrm{Tr}\left\{ \sum_{p=0}^{\infty} (-1)^{|A^{j(p)}_{3-n,-q}|} A^{i,(p)}_{n,q} A^{j,(p)}_{3-n,-q} (d\theta_j)^{p+q} (d\theta_i)^p \iota_i^p \delta(d\theta_i) \iota_j^{p+q} \delta(d\theta_j) \right\} \\
&= \int_{\mathcal{SM}} \mathrm{Tr}\left\{ \sum_{p=0}^{\infty} (-1)^{|A^{j(p)}_{3-n,-q}|} A^{i,(p)}_{n,q} A^{j,(p)}_{3-n,-q} (-1)^q [\iota_j^q (d\theta_j)^{p+q}] (d\theta_i)^p \iota_i^p \delta(d\theta_i) \iota_j^p \delta(d\theta_j) \right\} \\
&= \int_{\mathcal{SM}} \mathrm{Tr}\left\{ \sum_{p=0}^{\infty} (-1)^{|A^{j(p)}_{3-n,-q}|} A^{i,(p)}_{n,q} A^{j,(p)}_{3-n,-q} \frac{(q+p)!}{p!} (d\theta_i)^p (d\theta_j)^p \iota_i^p \delta(d\theta_i) \iota_j^p \delta(d\theta_j) \right\} \\
&= \int_{\mathcal{SM}} \mathrm{Tr}\left\{ \sum_{p=0}^{\infty} (-1)^{|A^{j(p)}_{3-n,-q}|} \tilde{A}^{i,(p)}_{n,q} \tilde{A}^{j,(p)}_{3-n,-q} (d\theta_i)^p (d\theta_j)^p \iota_i^p \delta(d\theta_i) \iota_j^p \delta(d\theta_j) \right\} \\
&= \langle \tilde{A}^i_{n,q} e^{0|1}_{n,i}, \tilde{A}^j_{3-n,-q} e^{0|1}_{3-n,i} \rangle = \langle \tilde{A}^i_{n,q} \tilde{A}^j_{3-n,-q} e^{0|1}_{n,i}, e^{0|1}_{3-n,i} \rangle,
\end{aligned} \tag{4.30}$$

where $\sim$ is used to denote the redefinition of the fields in order to absorb the combinatorial coefficients.

To establish the explicit equivalence of the two BV symplectic forms, we end up with constraints of the fields. The BV from (4.29) with only the fields and antifields $c^{(0|1)}$, $A^{(1|1)}$, $A^{*(2|1)}$, $c^{*(3|1)}$ reads

$$\begin{aligned}
\omega_{\mathrm{BV},A} &= \langle c^{(0|1)} + A^{(1|1)} + A^{*(2|1)} + c^{*(3|1)}, c^{(0|1)} + A^{(1|1)} + A^{*(2|1)} + c^{*(3|1)} \rangle \\
&= 2\langle c^{(0|1)}, c^{*(3|1)} \rangle + 2\langle A^{(1|1)}, A^{*(2|1)} \rangle = 2\sum_{n=0}^{3} [\langle c_n^{(-n|1)}, c_{3-n}^{*(n|1)} \rangle + \langle A_n^{(1-n|1)}, A_{3-n}^{*(n-1|1)} \rangle] \\
&= 2\sum_{n=0}^{3} [\langle c^i_{n,-n} e^{(-n|1)}_{n,i}, c^{*j}_{3-n,n} e^{(n|1)}_{3-n,j} \rangle + \langle A^i_{n,1-n} e^{(1-n|1)}_{n,i}, A^{*j}_{3-n,n-1} e^{(n-1|1)}_{3-n,j} \rangle] \\
&= 2\sum_{n=0}^{3} [\langle \tilde{c}^i_{n,-n} e^{(0|1)}_{n,i}, \tilde{c}^{*j}_{3-n,n} e^{(0|1)}_{3-n,j} \rangle + \langle \tilde{A}^i_{n,1-n} e^{(0|1)}_{n,i}, \tilde{A}^{*j}_{3-n,n-1} e^{(0|1)}_{3-n,j} \rangle] \\
&= 2\sum_{n=0}^{3} [\langle (\tilde{c}^i_{n,-n} \tilde{c}^{*j}_{3-n,n} + \tilde{A}^i_{n,1-n} \tilde{A}^{*j}_{3-n,n-1}) e^{(0|1)}_{n,i}, e^{(0|1)}_{3-n,j} \rangle] = \sum_{n=0}^{3} [\langle B^{ij}_n e^{(0|1)}_{n,i}, e^{(0|1)}_{3-n,j} \rangle],
\end{aligned} \tag{4.31}$$

where in the last equation we have written the field in brackets in terms of a single field to make the notation easier. Now, we can repeat the same calculations for the BV symplectic form built with the whole set of fields,

$$\begin{aligned}
\omega_{\mathrm{BV},\mathcal{A}} &= \sum_{p=-\infty}^{\infty} \langle A^{(p|1)}, A^{(3-p|1)} \rangle = \sum_{n=0}^{3} \sum_{p=-\infty}^{\infty} \langle A^i_{n,p-n} e^{p-n|1}_{n,i}, A^j_{3-n,n-p} e^{n-p|1}_{3-n,j} \rangle \\
&= \sum_{n=0}^{3} \sum_{p=-\infty}^{\infty} \langle \tilde{A}^i_{n,p-n} e^{0|1}_{n,i}, \tilde{A}^j_{3-n,n-p} e^{0|1}_{3-n,j} \rangle = \sum_{n=0}^{3} \sum_{p=-\infty}^{\infty} \langle \tilde{A}^i_{n,p-n} \tilde{A}^j_{3-n,n-p} e^{0|1}_{n,i}, e^{0|1}_{3-n,j} \rangle = \sum_{n=0}^{3} \langle \mathcal{B}^{ij}_n e^{0|1}_{n,i}, e^{0|1}_{3-n,j} \rangle.
\end{aligned} \tag{4.32}$$

It is now clear that the two BV symplectic forms could be equivalent after the identification

$$\sum_{n=0}^{3} B_n^{ij} \equiv \sum_{n=0}^{3} \mathcal{B}_n^{ij}, \quad \text{i.e.,} \sum_{n=0}^{3} 2(\tilde{c}^i_{n,-n}\tilde{c}^{*j}_{3-n,n} + \tilde{A}^i_{n,1-n}\tilde{A}^{*j}_{3-n,n-1}) \equiv \sum_{n=0}^{3}\sum_{p=-\infty}^{\infty} \tilde{A}^i_{n,p-n}\tilde{A}^j_{3-n,n-p}. \tag{4.33}$$

However, this equivalence would imply nontrivial constraints on the fields. In particular, it implies

$$\sum_{\substack{p=-\infty \\ p\neq 0,1,2,3}}^{\infty} \tilde{A}^i_{n,p-n}\tilde{A}^j_{3-n,n-p} = 0, \tag{4.34}$$

which is actually a nontrivial condition. Therefore, only under special conditions, the field redefinition can be achieved and in general, this is not possible.

Finally, the complete BV action for SCS at picture number 1 is given by the CS action with the complete sequence of fields,

$$S_{\mathrm{SCS-BV}} = \int_{\mathcal{SM}^{(3|2)}} \mathrm{Tr}\left[\frac{1}{2}\mathcal{A}\wedge d\mathcal{A} + \mathcal{A}\sum_{i=2}^{\infty}\frac{1}{i+1}\mathfrak{M}_i(\mathcal{A},\mathcal{A},\ldots)\right], \qquad \mathcal{A} = \bigoplus_{p=-\infty}^{\infty} A^{(p|1)}. \tag{4.35}$$

We may ask whether it is possible to select and construct a simplified action in terms of a subsector of fields which identifies a consistent, gauge-invariant action. There are many examples in the literature (see, e.g., [24] and references therein) where it is shown that it is possible to build gauge-invariant actions when working with A_n structures or L_n structures, i.e., algebraic structures including multiproducts up to n entries (see [20] for rigorous definitions). However, the multiproducts we are using do not come from an external structure (e.g., promoting the gauge Lie algebra to a nonassociative version [27]) that we can freely modify, but they are defined in terms of the supermanifold geometry itself; therefore, we are not allowed to truncate the multiproducts with more than n entries. Alternatively, we can investigate whether a subsector of our fields might lead to a gauge-invariant action. There is one consistent choice, namely, restricting ourselves to the fields highlighted in (4.24) only. They are proportional to dx and the terms in the action involving higher products vanish; for example, we have

$$A_1^0\wedge\mathfrak{M}_3(A_1^0,A_1^0,A_1^0) = (dx)^4 A_0^0\wedge\mathfrak{M}_3(A_0^0,A_0^0,A_0^0) = 0. \tag{4.36}$$

That leads to a standard BV Chern-Simons. In that case, the condition (4.34) is satisfied by setting all coefficient to zero. Nonetheless, that solution cannot be reached by a nontrivial field redefinition as shown above.

Notice that from the previous arguments it follows that the restriction to the fields in (4.24) is the only (evident) consistent possible restriction, since otherwise we would immediately need all the tower of possible fields as discussed above. Finally, we conclude that the most general BV action for super-Chern-Simons theory is the one described in (4.35).

It is well known in mathematics that any homotpy algebra allows for a quasi-isomorphic strict model (see again [24] for a detailed description of *quasi-isomorphisms* and their importance in physics), namely, a physically equivalent (i.e., with the same physical states) associative model. As briefly recalled in Sec. III A, we have explicitly verified this for the free SCS (see [10] for a full treatment). The theorem treating the quasi-isomorphism of an A_∞ algebra A and its cohomology $H(A,\mathfrak{M}_1)$ is due to Kadeishvili [28] (it can also be found, e.g., in [23]).

C. The supersymmetric term

As largely discussed in [10,29] and as recalled in Sec. II, when considering super Chern-Simons theory with picture number 0, we have to add to the action the term $(W^{(0|0)})^2V^3\wedge\mathbb{Y}^{(0|2)}$. This is necessary in order to have a closed Lagrangian. Recall that the requirement of a closed Lagrangian is related to the possibility of changing the PCO by d-exact terms, without modifying the action,

$$\int_{\mathcal{SM}^{(3|2)}} \mathcal{L}^{(3|0)}\wedge\delta_V\mathbb{Y}^{(0|2)} = \int_{\mathcal{SM}^{(3|2)}} \mathcal{L}^{(3|0)}\wedge d\Lambda^{(-1|2)} = \int_{\mathcal{SM}^{(3|2)}} d\mathcal{L}^{(3|0)}\wedge\Lambda^{(-1|2)} = 0. \tag{4.37}$$

When considering the picture-1 case, the gauge field is not factorized any more; therefore, we do not have explicitly the chance to choose the embedding, i.e., the PCO $\mathbb{Y}$. Moreover, the Lagrangian is closed by definition, being a top integral form. Then the questions for the W^2 term raise: is this term necessary in the present case? How can we establish the connection with the usual picture-0 case? Again, the answer comes from the nontrivial structure of the picture-1 action. Let us focus on the kinetic term. In particular, let us consider $A_2^{-1}\wedge(dA)_1^1$. We can choose the fields such that

$$A_2^{-1} = V^a \wedge V^b \epsilon_{abc} \iota \gamma^c W^{(0|1)}, \qquad (dA)_1^1 = V^a \psi \gamma_a W^{(0|1)}, \tag{4.38}$$

where $(dA)_1^1$ denotes the contribution to $F_1^{(1|1)} = dA_2^{-1}$ when d acts on V^a. Other terms are needed for covariance. These two terms together reproduce exactly the gaugino term of the super Chern-Simons action, with W lifted to a form with picture number 1. This consideration highlights once again the generality and the nontrivial structure of the action written using the whole tower of forms.

V. CONCLUSIONS

We have studied BV super-Chern-Simons theory on supermanifolds at different picture numbers. The results are the following:

(1) We built BV formalism for super-Chern-Simons starting from picture-0 action. We pointed out that the natural space for antifields is given by the integral forms. There is a natural map between the components antifields and superspace antifields. That was never noticed before, and this gives additional strength to integral form formalism for any supersymmetric model or for any model built on supermanifolds.

(2) We have shown that the richness of the multiproducts and nonassociative algebra is not avoidable. That has been proved by studying the BV symplectic form $\omega_{\rm BV}$ and showing that there is no field redefinition which allows us to restrict the set of fields and consequently the powers of multiproducts. In comparison with [27], apparently here the A_∞ algebra is intrinsically nested in the supermanifold geometry and cannot be redefined. That resembles how Haag-Łopuszańsky-Sonhius theorem overcomes the no-go theorem of Coleman-Mandula, since, as has been noticed in [27], the nontrivial structures may emerge if they have a spacetime interpretation (see also [17]).

(3) We build the BV formalism for picture-1 gauge fields finding once more the beautiful geometric action of Chern-Simons for the entire set of fields and antifields as in string field theory. Furthermore, we show how the gaugino mass term is recovered from that action confirming the consistency of our results. The case of extended supersymmetry will be published elsewhere.

(4) Finally, it would be certainly very interesting to consider this theory a general supermanifold such as nonsplit/nonprojected supermanifold [30–32].

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APPENDIX: NOTATIONS

We consider the case of a real supermanifold $\mathcal{SM}^{(3|2)}$; in terms of the coordinates, we define the following differential operators:

$$\partial_a = \frac{\partial}{\partial x^a}, \qquad D_\alpha = \frac{\partial}{\partial \theta^\alpha} - (\gamma^a \theta)_\alpha \partial_a,$$
$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\gamma^a \theta)_\alpha \partial_a, \tag{A1}$$

where the second and the third are known as superderivative and supersymmetry generator, respectively. They satisfy the superalgebra relations

$$[\partial_a, \partial_b] = 0, \ \{D_\alpha, D_\beta\} = -2\gamma^a_{\alpha\beta}\partial_a, \ \{Q_\alpha, Q_\beta\} = 2\gamma^a_{\alpha\beta}\partial_a,$$
$$\{D_\alpha, Q_\beta\} = 0, \quad \{\partial_a, D_\alpha\} = 0, \quad \{\partial_a, Q_\alpha\} = 0. \tag{A2}$$

In three dimensional, for the local subspace, we use the Lorentzian metric $\eta_{ab} = (-,+,+)$, and the real and symmetric Dirac matrices $\gamma^a_{\alpha\beta}$ given by

$$\gamma^0_{\alpha\beta} = (C\Gamma^0) = -\mathbf{1}, \qquad \gamma^1_{\alpha\beta} = (C\Gamma^1) = \sigma^3,$$
$$\gamma^2_{\alpha\beta} = (C\Gamma^2) = -\sigma^1, \qquad C_{\alpha\beta} = i\sigma^2 = \epsilon_{\alpha\beta}. \tag{A3}$$

Numerically, we have $\hat{\gamma}_a^{\alpha\beta} = \gamma^a_{\alpha\beta}$ and $\hat{\gamma}_a^{\alpha\beta} = \eta_{ab}(C\gamma^b C)^{\alpha\beta} = C^{\alpha\gamma}\gamma_{a,\gamma\delta}C^{\delta\beta}$. The conjugation matrix is $\epsilon^{\alpha\beta}$ and a bispinor is decomposed as follows: $R_{\alpha\beta} = R\epsilon_{\alpha\beta} + R_a \gamma^a_{\alpha\beta}$, where $R = -\frac{1}{2}\epsilon^{\alpha\beta}R_{\alpha\beta}$ and $R_a = \mathrm{Tr}(\gamma_a R)$ are a scalar and a vector, respectively. In addition, it is easy to show that $\gamma^{ab}_{\alpha\beta} \equiv \frac{1}{2}[\gamma^a, \gamma^b]_{\alpha\beta} = \epsilon^{abc}\gamma_{c\alpha\beta}$.

The differential of a generic function ϕ is expanded on a basis of forms as follows:

$$d\phi = dx^a \partial_a \phi + d\theta^\alpha \partial_\alpha \phi$$
$$= (dx^a + \theta\gamma^a d\theta)\partial_a \phi + d\theta^\alpha D_\alpha \phi \equiv V^a \partial_a \phi + \psi^\alpha D_\alpha \phi, \tag{A4}$$

where $V^a = dx^a + \theta\gamma^a d\theta$ and $\psi^\alpha = d\theta^\alpha$ which satisfy the Maurer-Cartan equations

$$dV^a = \psi\gamma^a\psi, \qquad d\psi^\alpha = 0. \tag{A5}$$

Given now a generic form Φ, we can compute the supersymmetry variation and translation as a Lie derivative $\mathcal{L}_\epsilon$ with $\epsilon = \epsilon^\alpha Q_\alpha + \epsilon^a \partial_a$ (ϵ^a are the infinitesimal parameters of the translations and ϵ^α are the supersymmetry parameters) and by means of the Cartan formula, we have

$$\delta_\epsilon \Phi = \mathcal{L}_\epsilon \Phi = \iota_\epsilon d\Phi + d\iota_\epsilon \Phi = \iota_\epsilon (dx^a \partial_a \Phi + d\theta^\alpha \partial_\alpha \Phi) + d\iota_\epsilon \Phi$$
$$= (\epsilon^a + \epsilon \gamma^a \theta)\partial_a \Phi + \epsilon^\alpha \partial_\alpha \Phi + d\iota_\epsilon \Phi = \epsilon^a \partial_a \Phi + \epsilon^\alpha Q_\alpha \Phi + d\iota_\epsilon \Phi, \tag{A6}$$

where the term $d\iota_\epsilon \Phi$ is simply a gauge transformation. It follows easily that $\delta_\epsilon V^a = \delta_\epsilon \psi^\alpha = 0$ and $\delta_\epsilon d\Phi = d\delta_\epsilon \Phi$.

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