# Dual equivalence between self-dual and topologically massive $B \wedge F$ models coupled to matter in 3+1 dimensions

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In this work, we revisit the duality between a self-dual nongauge invariant theory and a topological massive theory in 3 + 1 dimensions. The self-dual Lagrangian is composed by a vector field and an antisymmetric field tensor whereas the topological massive Lagrangian is built using a  $B \wedge F$  term. Though the Lagrangians are quite different, they yield to equations of motion that are connected by a simple dual mapping among the fields. We discuss this duality by analyzing the degrees of freedom in both theories and comparing their propagating modes at the classical level. Moreover, we employ the master action method to obtain a fundamental Lagrangian that interpolates between these two theories and makes evident the role of the topological  $B \wedge F$  term in the duality relation. By coupling these theories with matter fields, we show that the duality holds provided a Thirring-like term is included. In addition, we use the master action in order to probe the duality upon the quantized fields. We carried out a functional integration of the fields and compared the resulting effective Lagrangians.

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# I. INTRODUCTION

Dualities are a main theme in present-day physics. By connecting different theories or opposite regimes of a same model, dualities are powerful tools to seek and understand new effects. Notably, string theories are connected by T and S dualities [1,2] and the AdS/CFT correspondence links low-energy gravitational theory in AdS spacetime with a strong coupling regime of a conformal field theory at the boundary [3]. Among the duality processes, the so-called bosonization is of special importance and widely used to investigate nonperturbative properties in quantum field theory and condensed matter systems in low dimensions [4]. In 1+1 dimension, it is possible to establish a fermion-boson correspondence based on the properties of the Fermi surfaces [5]. This duality can be further generalized for non-Abelian fields [6] and even for higher dimensions [7,8]. Recently, the bosonization lead to new 2+1 relations called a web of dualities [9,10].

Another example of duality involves topologically massive gauge theories. A well-known duality occurs between the self-dual (SD) [11] and the Maxwell-Chern-Simons (MCS) [12] models. These two theories describe a single massive particle of spin-1 in 2 + 1 dimensional Minkowski spacetime. Nevertheless, only the MCS model is gauge invariant. The equivalence between the SD and MCS models was initially proved by Deser and Jackiw [12], and over the years, several studies of this equivalence have been carried out in the literature [13–20]. Particularly, by considering couplings with fermionic fields, it was shown in [18] that the models are equivalent provided that a Thirring-like interaction is included. In addition, supersymmetric [21–23] and noncommutative [24,25] extensions to the duality involving the SD and MCS models have been studied in different contexts.

At the heart of this duality, the Chern-Simons term plays a key role. An alternative topological term in 3 + 1dimensions can be formed from a U(1) vector gauge field  $A_{\mu}$  and a rank-2 antisymmetric tensor field  $B_{\mu\nu}$ , also known as the Kalb-Ramond field [26,27]. Such a massive topological term is commonly called the  $B \wedge F$  term [28–33]. Therefore, a natural generalization of the MCS model in four dimensions consists of the Maxwell and Kalb-Ramon fields coupled by a  $B \wedge F$  term [34]. This topologically massive gauge-invariant  $B \wedge F$  theory  $(TM_{B \wedge F})$  is unitary and renormalizable when minimally coupled to fermions, and represents a massive particle of spin-1 [28]. Models

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involving the Kalb-Ramond field have been extensively studied in the literature, specially in connection with string theories [35], quantum field theory [36,37], supersymmetry [38], Lorentz symmetry violation [39–42], black hole solutions [43], cosmology [44], and brane world scenarios [45,46].

A self-dual version of the  $TM_{B \wedge F}$  model was studied in Ref. [47]. It involves the  $B \wedge F$  term in a nongaugeinvariant, first-order model  $(SD_{B \wedge F})$ . Such work showed the classic equivalence between the models, i.e., at the level of the equations of motion, through the gauge embedding procedure [20]. In addition, when interactions with fermionic fields are considered, the duality mapping only is preserved if Thirring-like terms are taken into account, analogously to the SD/MCS case in 2 + 1 dimensions. Yet, the issues regarding the generalization for arbitrary nonconserved matter currents and the proof of quantum duality have not yet been fully elucidated.

The main goal of this work is to provide an alternative method, via master action [48], to prove the duality between the  $SD_{B \wedge F}$  and  $TM_{B \wedge F}$  theories, when the fields of the SD sector couple linearly with nonconserved currents, composed by arbitrary dynamic fields of matter. The master action approach has the advantage of providing a fundamental theory that interpolates between the two models and allows a more direct demonstration of duality at the quantum level. Besides, the master action method is a natural trail for the supersymmetric generalization of the duality studied here [23].

The present work is organized as follows. In Sec. II, we present the  $SD_{B \wedge F}$  and  $TM_{B \wedge F}$  theories in the free case, review their main physical characteristics, and check the classic duality by comparing their equations of motion. Moreover, we built a master Lagrangian density from the  $TM_{B \wedge F}$  model, introducing auxiliary fields in order to obtain a first-order derivative theory. In Sec. III, we include matter couplings in the SD sector and verify whether the equivalence is still compatible. We apply our results to the case of minimal coupling with fermionic matter and compare it with those found in the literature. In Sec. IV, we investigate the equivalence at the quantum level within the path-integral framework. Finally in Sec. V we provide our conclusions and perspectives concerning further investigations.

## II. THE DUALITY AT THE CLASSICAL LEVEL.

In a 2+1 flat spacetime Townsend, Pilch, and Nieuwenhuizen proposed a first-order derivative theory self-dual to the topological Chern-Simons theory [11]. In four dimensions, this kind of duality can be built through a topological  $B \wedge F$  term. In fact, consider a gauge noninvariant  $SD_{B \wedge F}$  model composed by a vector field  $A_{\mu}$  and an antisymmetric 2-tensor field  $B_{\mu\nu}$  governed by the Lagrangian density [28,47]

$$\mathcal{L}_{SD} = \frac{m^2}{2} A_{\mu} A^{\mu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\chi \theta}{4} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} F^{\alpha\beta}, \quad (1)$$

where *m* is a parameter with dimension of mass,  $\theta$  is a dimensionless coupling constant, and  $\chi = \pm 1$  defines either the self-duality (+) or the anti-self-duality (-) to the theory. The field strengths associated with the vector and tensor fields are defined respectively by  $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$  and  $H_{\mu\nu\alpha} = \partial_{\mu}B_{\nu\alpha} + \partial_{\nu}B_{\alpha\mu} + \partial_{\alpha}B_{\mu\nu}$ . The equations of motion for the  $A_{\mu}$  and  $B_{\mu\nu}$  fields are, respectively,

$$m^2 A_{\beta} - \frac{\chi \theta}{2} \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} B^{\mu\nu} = 0, \qquad (2)$$

$$B_{\mu\nu} - \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} A^{\beta} = 0, \qquad (3)$$

and satisfy the constraint relations

$$\partial_{\mu}A^{\mu} = 0, \qquad (4)$$

$$\partial^{\mu}B_{\mu\nu} = 0. \tag{5}$$

Equations (2) and (3) form a set of coupled first-order differential equations that can be rewritten, with the help of relations (4) and (5), in the form of a wave equation given by

$$\left[\Box + \frac{m^2}{\theta^2}\right]\varphi = 0, \tag{6}$$

where  $\varphi$  denotes  $A_{\mu}$  or  $B_{\mu\nu}$  fields. This implies that the firstorder Lagrangian density  $\mathcal{L}_{SD}$  describes the dynamics of a massive vector field. In fact, the field  $B_{\mu\nu}$  is auxiliary and can be removed from the action leading to [39]

$$\mathcal{L}_{SD} = \frac{m^2}{2} A_{\mu} A^{\mu} - \frac{\theta^2}{4} F_{\mu\nu} F^{\mu\nu}, \qquad (7)$$

which is the Lagrangian density for a massive vector field with three propagating degrees of freedom.

In the context of the present work, we are interested in investigating the equivalence between the self-dual model (1) and a second-order gauge-invariant theory. For this purpose, let us consider a topologically massive  $B \wedge F$  model defined as [29,47]

$$\mathcal{L}_{TM} = \frac{\theta^2}{12m^2} H_{\mu\nu\alpha} H^{\mu\nu\alpha} - \frac{\theta^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\chi\theta}{4} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} F^{\alpha\beta}.$$
(8)

Note that the first two terms of  $\mathcal{L}_{TM}$  are invariant under the gauge transformations  $A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\lambda$  and  $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_{\mu}\beta_{\nu} - \partial_{\nu}\beta_{\mu}$ , whereas the variation of the last term yields to a total divergence. The gauge parameter  $\beta_{\mu}$  still has a subsidiary gauge transformation  $\beta_{\mu} \rightarrow \beta_{\mu} + \partial_{\mu}\alpha$  that leaves

 $B_{\mu\nu}$  unchanged. The equations of motions derived from this Lagrangian density are

$$\frac{\theta^2}{2m^2}\partial^{\mu}H_{\mu\nu\lambda} + \frac{\chi\theta}{4}\epsilon_{\nu\lambda\alpha\beta}F^{\alpha\beta} = 0, \qquad (9)$$

$$\theta^2 \partial^{\mu} F_{\mu\lambda} + \frac{\chi \theta}{6} \epsilon_{\mu\nu\alpha\lambda} H^{\alpha\mu\nu} = 0.$$
 (10)

In general, the two fields  $A_{\mu}$  and  $B_{\mu\nu}$  have four and six independent degrees of freedom, respectively. However, due to the gauge symmetry in the theory described by  $\mathcal{L}_{TM}$ , some of them can be eliminated. In order to identify which ones propagate as massive physical modes or which are spurious (gauge-dependent) modes, it is instructive to perform a decomposition in time-space on the equations of motions (9) and (10). For this purpose, let us split  $B_{\mu\nu}$ into the independent components  $B_{0i}$  and  $B_{ij}$  and to introduce spatial vectors  $\vec{X}$  and  $\vec{Y}$  defined by

$$\mathcal{X}^i \equiv -B_{0i}, \qquad \mathcal{Y}^i \equiv \frac{1}{2} \epsilon^{ijk} B_{jk}, \qquad (11)$$

where  $e^{0ijk} = e^{ijk}$ . With these definitions, we obtain a set of coupled second-order differential equations in the form

$$\nabla^2 A^0 + \partial^0 \partial_i A^i + \frac{\chi}{\theta} \partial_i \mathcal{Y}^i = 0, \qquad (12)$$

$$\Box A^{i} - \partial^{i} (\partial_{0} A^{0} + \partial_{j} A^{j}) + \frac{\chi}{\theta} (\epsilon^{ijk} \partial_{k} \chi_{j} + \partial_{0} \mathcal{Y}^{i}) = 0, \quad (13)$$

$$-\nabla^{2} \mathcal{X}_{i} - \partial_{i} \partial^{j} \mathcal{X}_{j} + \epsilon_{ijk} \left( \partial_{0} \partial^{j} \mathcal{Y}^{k} - \frac{\chi m^{2}}{\theta} \partial^{j} A^{k} \right) = 0, \quad (14)$$

$$\partial_0^2 \mathcal{Y}^k + \partial_i \partial^k \mathcal{Y}^i + \epsilon^{ijk} \partial_0 \partial_j \mathcal{X}_i + \frac{\chi m^2}{\theta} (\partial^k A^0 - \partial^0 A^k) = 0.$$
(15)

After some manipulation of these equations, we can formally solve the temporal component  $A^0$  and the 3-vector  $\vec{\mathcal{X}}$  in terms of the other components according to

$$A^{0} = -\frac{1}{\nabla^{2}} \left( \partial^{0} \partial_{i} A^{i}_{(L)} + \frac{\chi}{\theta} \partial_{i} \mathcal{Y}^{i}_{(L)} \right), \tag{16}$$

$$\mathcal{X}_{i}^{(T)} = \frac{1}{\nabla^{2}} \epsilon_{ijk} \left( \partial_{0} \partial^{j} \mathcal{Y}_{(T)}^{k} - \frac{\chi m^{2}}{\theta} \partial^{j} A_{(T)}^{k} \right), \quad (17)$$

where  $v_{(T)}^i \equiv \theta_j^i v^j$  and  $v_{(L)}^i \equiv \omega_j^i v^j$  are the transversal (T) and longitudinal (L) components of a 3-vector  $\vec{v}$ , respectively, with the projectors  $\theta_j^i$  and  $\omega_j^i$  defined by

$$\theta_j^i \equiv \delta_j^i - \omega_j^i, \qquad \omega_j^i \equiv -\frac{\partial_j \partial^i}{\nabla^2}.$$
(18)

Similar procedures can be applied to the components of the  $\vec{A}$  and  $\vec{y}$ , such that

$$\left[\Box + \frac{m^2}{\theta^2}\right] A^i_{(T)} = 0, \qquad (19)$$

$$\left[\Box + \frac{m^2}{\theta^2}\right] \mathcal{Y}^i_{(L)} = 0.$$
 (20)

The form of these solutions reveals that the only physical components are  $A_{(T)}^i$  and  $\mathcal{Y}_{(L)}^i$ , while the others are auxiliary or gauge modes. Furthermore, as the longitudinal part of  $\vec{\mathcal{Y}}$  is curl-free, it propagates as a massive scalar field, i.e.,  $\vec{\mathcal{Y}} = \nabla \phi$ , whose mass depends on the coupling constant  $\theta$ . Thus, the results above show that the  $TM_{B \wedge F}$  theory defined in (8), like the  $SD_{B \wedge F}$  model, contains three massive propagating modes.

To make explicit the hidden duality between the models described above, it is convenient to introduce the dual fields associated with the field strength tensors  $H^{\mu\nu\alpha}$  and  $F^{\mu\nu}$ , respectively, by

$$\tilde{H}_{\mu} \equiv -\frac{\chi\theta}{6m^2} \epsilon_{\mu\nu\alpha\beta} H^{\nu\alpha\beta}, \qquad (21)$$

$$\tilde{F}_{\mu\nu} \equiv \frac{\chi\theta}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}.$$
(22)

In terms of  $\tilde{H}_{\mu}$  and  $\tilde{F}_{\mu\nu}$ , the equations of motion (9) and (10) become

$$m^{2}\tilde{H}_{\beta} - \frac{\theta}{2\chi}\epsilon_{\mu\nu\alpha\beta}\partial^{\mu}\tilde{F}^{\nu\alpha} = 0, \qquad (23)$$

$$\tilde{F}_{\mu\nu} - \frac{\theta}{\chi} \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} \tilde{H}^{\beta} = 0.$$
(24)

A direct comparison between the pairs of equations (2), (3) and (23), (24) shows that the dual fields  $\tilde{H}_{\beta}$  and  $\tilde{F}_{\mu\nu}$  satisfy exactly the same equations obtained for the  $SD_{B\wedge F}$  model when we identify  $A_{\mu} \rightarrow \tilde{H}_{\mu}$  and  $B_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}$ . Therefore, the basic fields of the  $SD_{B\wedge F}$  model correspond to the dual fields of the  $TM_{B\wedge F}$  model. This proves the classical equivalence via equations of motion in the free field case.

However, despite having established the dual connection, the mapping  $A_{\mu} \rightarrow \tilde{H}_{\mu}$  and  $B_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}$  leads to

$$\mathcal{L}_{TM}(\tilde{H},\tilde{F}) = -\frac{m^2}{2}\tilde{H}_{\mu}\tilde{H}^{\mu} + \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{2}B_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (25)$$

wherein the identities  $F_{\mu\nu}F^{\mu\nu} = -1/\theta^2 \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$  and  $H_{\mu\nu\alpha}H^{\mu\nu\alpha} = -6m^4/\theta^2 \tilde{H}_{\mu}\tilde{H}^{\mu}$  were used. Note that (25) does not recover (1) and the equivalence between the two models is not evident. The common origin of these

the sequel.

Lagrangian densities can be better addressed by means of the *master Lagrangian* method, which we will formulate in

#### A. Classic duality via master Lagrangian

The study of dual equivalence among four-dimensional models containing a topological  $B \wedge F$  term was carried out for the first time in Ref. [47], whereby the authors used the dynamical gauge embedding formalism to show the classic duality between (1) and (8). Here, we employ the master Lagrangian method [12,18] that extends and interpolates those two studied models. Moreover, this method allows us to study the duality at the quantum level more directly.

Let us start from Lagrangian density  $\mathcal{L}_{TM}$  in the form (25) written explicitly in terms of the fundamental fields  $A_{\mu}$  and  $B_{\mu\nu}$ . Following [12], we will introduce auxiliary fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  in order to obtain a first-order derivative theory such that

$$\mathcal{L}_{M} = a\Pi_{\mu}\epsilon^{\mu\rho\sigma\delta}\partial_{\rho}B_{\sigma\delta} + b\Pi_{\mu}\Pi^{\mu} + c\Lambda_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}\partial_{\rho}A_{\sigma} + d\Lambda_{\mu\nu}\Lambda^{\mu\nu} - \frac{\chi\theta}{2}\epsilon_{\mu\nu\alpha\beta}B^{\mu\nu}\partial^{\alpha}A^{\beta}, \qquad (26)$$

where *a*, *b*, *c*, and *d* are constant coefficients to be determined. Note that the presence of mass terms for  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  ensures the auxiliary character of these fields.

The functional variation of  $\mathcal{L}_M$  with respect to the auxiliary fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  allows us to write

$$\Pi_{\mu} = -\frac{a}{2b} \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} B^{\alpha\beta}, \qquad (27)$$

$$\Lambda_{\mu\nu} = -\frac{c}{2d} \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} A^{\beta}.$$
 (28)

Substituting (27) and (28) in (26) and imposing  $\mathcal{L}_M = \mathcal{L}_{TM}$ , we obtain the relations

$$\frac{a^2}{b} = \frac{\theta^2}{2m^2},\tag{29}$$

$$\frac{c^2}{d} = -\theta^2. \tag{30}$$

The same procedure can be performed for the fields  $A_{\mu}$  and  $B_{\mu\nu}$ , and we can immediately solve their equations of motion, obtaining the following solutions:

$$A_{\mu} = \frac{2a}{\chi\theta} \Pi_{\mu} + \partial_{\mu}\phi, \qquad (31)$$

$$B_{\mu\nu} = \frac{2c}{\chi\theta}\Lambda_{\mu\nu} + \partial_{\mu}\Sigma_{\nu} - \partial_{\nu}\Sigma_{\mu}, \qquad (32)$$

being  $\phi$  and  $\Sigma_{\mu}$  arbitrary fields. Now, replacing (31) and (32) in (26) and imposing  $\mathcal{L}_{M} = \mathcal{L}_{SD}$ , we obtain

$$b = \frac{m^2}{2},\tag{33}$$

$$d = -\frac{1}{4},\tag{34}$$

such that we can immediately fix  $a = c = \chi \theta / 2$  so that our master Lagrangian takes the final form

$$\mathcal{L}_{M} = \frac{\chi \theta}{2} \Pi_{\mu} \epsilon^{\mu \rho \sigma \delta} \partial_{\rho} B_{\sigma \delta} + \frac{m^{2}}{2} \Pi_{\mu} \Pi^{\mu} + \frac{\chi \theta}{2} \Lambda_{\mu \nu} \epsilon^{\mu \nu \rho \sigma} \partial_{\rho} A_{\sigma} - \frac{1}{4} \Lambda_{\mu \nu} \Lambda^{\mu \nu} - \frac{\chi \theta}{2} \epsilon_{\mu \nu \alpha \beta} B^{\mu \nu} \partial^{\alpha} A^{\beta}.$$
(35)

Accordingly, the Lagrangian density (35) describes both (1) and (8). This mechanism transforms models without gauge invariance into models with this symmetry by adding terms which do not appear on shell. Note that the gauge invariance of  $\mathcal{L}_M$  under  $\delta A_\mu = \partial_\mu \lambda$  and  $\delta B_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu$  with  $\delta \Pi_\mu = \delta \Lambda_{\mu\nu} = 0$  is now evident, while it was a hidden symmetry in the self-dual formulation. With the master method, we were able to establish the relation of equivalence when the coupling to other dynamical fields is considered and we have a simple formalism which accounts for the investigation of the theory at the quantum level.

## III. DUALITY MAPPING WITH A LINEAR MATTER COUPLING

The discussion on the duality developed in the previous section deals only with free theories. However, it is fundamental to ensure that this dual equivalence is also valid in the presence of external sources coupled to the fields in  $\mathcal{L}_M$ . Here and throughout the paper, we will assume only linear couplings with external fields, whose associated currents are composed only of matter fields, represented generically by  $\psi$ . The cases involving nonlinear couplings or when the currents depend explicitly on the gauge or self-dual fields are beyond our present scope.

Let us consider the master Lagrangian (35) added by dynamical matter fields  $\psi$  linearly coupled to the self-dual sector:

$$\mathcal{L}_{M}^{(1)} = \frac{\chi\theta}{2} \Pi_{\mu} \epsilon^{\mu\rho\sigma\delta} \partial_{\rho} B_{\sigma\delta} + \frac{m^{2}}{2} \Pi_{\mu} \Pi^{\mu} + \frac{\chi\theta}{2} \Lambda_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} \partial_{\rho} A_{\sigma} - \frac{1}{4} \Lambda_{\mu\nu} \Lambda^{\mu\nu} - \frac{\chi\theta}{2} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} \partial^{\alpha} A^{\beta} + \Pi_{\mu} J^{\mu} + \Lambda_{\mu\nu} \mathcal{J}^{\mu\nu} + \mathcal{L}(\psi),$$
(36)

where  $\mathcal{L}(\psi)$  represents a generic Lagrangian density responsible for the dynamics of the matter fields, with the corresponding currents being denoted by  $J_{\mu}$  and  $\mathcal{J}_{\mu\nu}$ . Note that due to the lack of gauge symmetry in the self-dual sector, the matter currents  $J_{\mu}$  and  $\mathcal{J}_{\mu\nu}$  are generally not conserved. Also, to make our analysis as general as possible, we will not assume any specific form to the matter sector for now.

First, we will remove the dependency on the gauge fields in Eq. (36). Varying the action  $\int d^4x \mathcal{L}_M^{(1)}$  with respect to the fields  $A_{\mu}$  and  $B_{\mu\nu}$ , we obtain their corresponding equations of motion whose solutions are given by

$$A_{\mu} = \Pi_{\mu} + \partial_{\mu}\phi, \qquad (37)$$

$$B_{\mu\nu} = \Lambda_{\mu\nu} + \partial_{\mu}\Sigma_{\nu} - \partial_{\nu}\Sigma_{\mu}, \qquad (38)$$

and substituting these solutions into Eq. (36) we find  $\mathcal{L}_{M}^{(1)} = \mathcal{L}_{SD}^{(1)}$ , with

$$\mathcal{L}_{SD}^{(1)} = \frac{m^2}{2} \Pi_{\mu} \Pi^{\mu} - \frac{1}{4} \Lambda_{\mu\nu} \Lambda^{\mu\nu} + \frac{\chi \theta}{2} \Pi_{\mu} \epsilon^{\mu\nu\alpha\beta} \partial_{\nu} \Lambda_{\alpha\beta} + \Pi_{\mu} J^{\mu} + \Lambda_{\mu\nu} \mathcal{J}^{\mu\nu} + \mathcal{L}(\psi).$$
(39)

Then,  $\mathcal{L}_{SD}^{(1)}$  is equivalent to the self-dual theory (1) linearly coupled to the matter, as expected.

Next, we will eliminate the fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  from the master Lagrangian  $\mathcal{L}_{M}^{(1)}$ . The equations of motion for these fields are

$$\Pi_{\mu} = -\frac{\chi\theta}{2m^2} \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} B^{\alpha\beta} - \frac{1}{m^2} J_{\mu}, \qquad (40)$$

$$\Lambda_{\mu\nu} = \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} A^{\beta} + 2 \mathcal{J}_{\mu\nu}. \tag{41}$$

Replacing Eqs. (40) and (41) into the master Lagrangian then implies  $\mathcal{L}_{M}^{(1)} = \mathcal{L}_{TM}^{(1)}$ , with

$$\mathcal{L}_{TM}^{(1)} = \frac{\theta^2}{12m^2} H_{\mu\nu\alpha} H^{\mu\nu\alpha} - \frac{\theta^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\chi\theta}{2} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} \partial^{\alpha} A^{\beta} - \frac{\chi\theta}{2m^2} B_{\mu\nu} \epsilon^{\mu\nu\alpha\beta} \partial_{\alpha} J_{\beta} - \frac{1}{2m^2} J_{\mu} J^{\mu} + \chi\theta A_{\mu} \epsilon^{\mu\nu\alpha\beta} \partial_{\nu} \mathcal{J}_{\alpha\beta} + \mathcal{J}_{\mu\nu} \mathcal{J}^{\mu\nu} + \mathcal{L}(\psi).$$
(42)

From the above result, it is clear that the Lagrangian density  $\mathcal{L}_{TM}^{(1)}$  represents the  $TM_{B\wedge F}$  theory (8) interacting with the matter through "magnetic" currents plus Thirringlike terms involving only the matter fields. A similar Lagrangian density to the  $\mathcal{L}_{TM}^{(1)}$  has appeared before in [47]. However, the approach used in [47] was based on the gauge embedding method, different from the one developed here. Also, one may verify that the equations of motion for the fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  in the  $SD_{B\wedge F}$  model (39) and for the gauge fields  $A_{\mu}$  and  $B_{\mu\nu}$  in the  $TM_{B\wedge F}$  model (42) can be cast in the same form by means of the identification

$$\Pi_{\mu} \to \tilde{H}_{\mu} - \frac{1}{m^2} J_{\mu}, \qquad (43)$$

$$\Lambda_{\mu\nu} \to \tilde{F}_{\mu\nu} + 2\mathcal{J}_{\mu\nu}. \tag{44}$$

It is worth noting that the duality symmetry between  $SD_{B\wedge F}/TM_{B\wedge F}$  theories exchanges linear couplings  $\Pi_{\mu}J^{\mu}$ and  $\Lambda_{\mu\nu}\mathcal{J}^{\mu\nu}$ , involving currents not necessarily conserved in the self-dual sector into derivative dual couplings  $A_{\mu}\epsilon^{\mu\nu\alpha\beta}\partial_{\nu}\mathcal{J}_{\alpha\beta}$  and  $B_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}J_{\beta}$  in the gauge sector, whose associated currents are automatically conserved. Moreover, self-interaction matter terms are naturally generated, which will play a decisive role in ensuring the duality in the matter sector, as we shall see in what follows.

#### A. The matter sector

Classically, the duality mapping established in Eqs. (43) and (44) ensures that the Lagrangian densities (39) and (42) are equivalent since the  $SD_{B\wedge F}$  and  $TM_{B\wedge F}$  fields obey the same equations of motion in the presence of external sources. However, for this equivalence between the models to be complete, it is also necessary to verify what happens in the matter sector, when these sources are dynamics.

To this end, we now consider the equation of motion for the matter field  $\psi$ . First, let us focus our attention on the  $SD_{B \wedge F}$  model described by (39), so

$$\frac{\delta}{\delta\psi} \int d^4x \mathcal{L}_{SD}^{(1)} = 0 \Rightarrow \frac{\delta \mathcal{L}(\psi)}{\delta\psi} = -\Pi_{\mu} \frac{\delta J^{\mu}}{\delta\psi} - \Lambda_{\mu\nu} \frac{\delta \mathcal{J}^{\mu\nu}}{\delta\psi},$$
(45)

where  $\frac{\delta \mathcal{L}(\psi)}{\delta \psi}$  is the Lagrangian derivative.

On the other hand, the equations of motion for the fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  are

$$m^{2}\Pi^{\mu} + \frac{\chi\theta}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{\nu}\Lambda_{\alpha\beta} = -J^{\mu}, \qquad (46)$$

$$\frac{1}{2}\Lambda^{\mu\nu} - \frac{\chi\theta}{2}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}\Pi_{\beta} = \mathcal{J}^{\mu\nu}, \qquad (47)$$

and obey the constraints

$$m^2 \partial_\mu \Pi^\mu = -\partial_\mu J^\mu, \tag{48}$$

$$\partial_{\mu}\Lambda^{\mu\nu} = 2\partial_{\mu}\mathcal{J}^{\mu\nu}.$$
 (49)

Inserting (47) into (46), we can eliminate  $\Lambda_{\mu\nu}$  in favor of  $\Pi_{\mu}$  and obtain a second-order differential equation as

$$(\theta^2 \Box + m^2)\Pi^{\mu} = -J^{\mu} - \frac{\theta^2}{m^2} \partial^{\mu} \partial_{\nu} J^{\nu} - \chi \theta \epsilon^{\mu\nu\alpha\beta} \partial_{\nu} \mathcal{J}_{\alpha\beta},$$
(50)

where we used the constraint  $m^2 \partial_{\mu} \Pi^{\mu} = -\partial_{\mu} J^{\mu}$ . Defining the wave operator as  $\hat{R}^{-1} = \Box + \frac{m^2}{\theta^2}$ , we can write

$$\Pi_{\mu} = -\frac{\hat{R}}{\theta^2} \left( J_{\mu} + \frac{\theta^2}{m^2} \partial_{\mu} \partial^{\nu} J_{\nu} + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} \mathcal{J}^{\alpha\beta} \right).$$
(51)

$$\Lambda_{\mu\nu} = -\frac{R}{\theta^2} \left[ -2m^2 \mathcal{J}_{\mu\nu} + 2\theta^2 \partial^\alpha (\partial_\mu \mathcal{J}_{\nu\alpha} - \partial_\nu \mathcal{J}_{\mu\alpha}) \right. \\ \left. + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^\alpha J^\beta \right].$$
(52)

Replacing the solutions (51) and (52) back in the matter equation (45), we come to the result

$$\frac{\delta \mathcal{L}(\psi)}{\delta \psi} = \frac{\hat{R}}{\theta^2} \left[ J_{\mu} + \frac{\theta^2}{m^2} \partial_{\mu} (\partial^{\nu} J_{\nu}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} \mathcal{J}^{\alpha\beta} \right] \frac{\delta J^{\mu}}{\delta \psi} + \frac{\hat{R}}{\theta^2} [-2m^2 \mathcal{J}_{\mu\nu} + 2\theta^2 \partial^{\alpha} (\partial_{\mu} \mathcal{J}_{\nu\alpha} - \partial_{\nu} \mathcal{J}_{\mu\alpha}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} J^{\beta} ] \frac{\delta \mathcal{J}^{\mu\nu}}{\delta \psi}.$$
(53)

This is a nonlocal differential equation, expressed only in

terms of the matter fields. Now, if we start from  $\mathcal{L}_{TM}^{(1)}$ , the equation of motion for the matter field takes the form

$$\frac{\delta}{\delta\psi} \int d^4x \mathcal{L}_{TM}^{(1)} = 0 \Rightarrow \frac{\delta\mathcal{L}(\psi)}{\delta\psi} = \left(\frac{1}{m^2}J_{\mu} - \hat{H}_{\mu}\right) \frac{\delta J^{\mu}}{\delta\psi} + \left(-2\mathcal{J}_{\mu\nu} - \hat{F}_{\mu\nu}\right) \frac{\delta\mathcal{J}^{\mu\nu}}{\delta\psi}, \qquad (54)$$

where we have used the definitions (21) and (22) for the dual fields.

To eliminate the dual fields in (54), we write the equations of motion for  $A_{\mu}$  and  $B_{\mu\nu}$ , obtained from  $\mathcal{L}_{TM}^{(1)}$ , as

$$m^{2}\tilde{H}_{\mu} + \frac{\theta}{2\chi}\epsilon_{\mu\nu\alpha\beta}\partial^{\nu}\tilde{F}^{\alpha\beta} = -\chi\theta\epsilon_{\mu\nu\alpha\beta}\partial^{\nu}\mathcal{J}^{\alpha\beta}, \quad (55)$$

$$-\tilde{F}_{\mu\nu} + \frac{\theta}{\chi} \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} \tilde{H}^{\beta} = \frac{\chi\theta}{m^2} \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} J^{\beta}.$$
 (56)

These equations can be decoupled, and after some algebraic manipulations we get the following results:

$$\tilde{H}_{\mu} = \frac{\hat{R}}{\theta^2} \left[ \frac{\theta^2}{m^2} (\Box J_{\mu} - \partial_{\mu} \partial^{\nu} J_{\nu}) - \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} \mathcal{J}^{\alpha\beta} \right], \quad (57)$$

$$\tilde{F}_{\mu\nu} = -2R\Box \mathcal{J}_{\mu\nu} - \frac{\hat{R}}{\theta^2} [2\theta^2 \partial^\alpha (\partial_\mu \mathcal{J}_{\nu\alpha} - \partial_\nu \mathcal{J}_{\mu\alpha}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^\alpha J^\beta].$$
(58)

Substituting these solutions in Eq. (54) we obtain

$$\frac{\delta \mathcal{L}(\psi)}{\delta \psi} = \left[ \frac{1}{m^2} (1 - \hat{R} \Box) J_{\mu} + \frac{\hat{R}}{\theta^2} \left( \frac{\theta^2}{m^2} \partial_{\mu} \partial^{\nu} J_{\nu} + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} \mathcal{J}^{\alpha\beta} \right) \right] \frac{\delta J^{\mu}}{\delta \psi} \\
+ \left[ 2(\hat{R} \Box - 1) \mathcal{J}_{\mu\nu} + \frac{\hat{R}}{\theta^2} (2\theta^2 \partial^{\alpha} (\partial_{\mu} \mathcal{J}_{\nu\alpha} - \partial_{\nu} \mathcal{J}_{\mu\alpha}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} J^{\beta}) \right] \frac{\delta \mathcal{J}^{\mu\nu}}{\delta \psi}.$$
(59)

Using the definition  $\hat{R}^{-1} = \Box + \frac{m^2}{\theta^2}$ , we can write  $\Box = R^{-1} - \frac{m^2}{\theta^2}$  which implies

$$\frac{\delta \mathcal{L}(\psi)}{\delta \psi} = \frac{\hat{R}}{\theta^2} \left[ J_{\mu} + \frac{\theta^2}{m^2} \partial_{\mu} (\partial^{\nu} J_{\nu}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\nu} \mathcal{J}^{\alpha\beta} \right] \frac{\delta J^{\mu}}{\delta \psi} + \frac{\hat{R}}{\theta^2} \left[ -2m^2 \mathcal{J}_{\mu\nu} + 2\theta^2 \partial^{\alpha} (\partial_{\mu} \mathcal{J}_{\nu\alpha} - \partial_{\nu} \mathcal{J}_{\mu\alpha}) + \chi \theta \epsilon_{\mu\nu\alpha\beta} \partial^{\alpha} J^{\beta} \right] \frac{\delta \mathcal{J}^{\mu\nu}}{\delta \psi}.$$
(60)

By comparing Eqs. (53) and (60), we conclude that the matter sectors of the two models give rise to the same equations of motion. Thus, we have shown that the Lagrangians  $\mathcal{L}_{SD}^{(1)}$  and  $\mathcal{L}_{TM}^{(1)}$  are equivalent and have established the classical duality between the  $SD_{B\wedge F}$  and  $TM_{B\wedge F}$ theories when couplings with dynamical matter fields are considered.

In order to liken our results with the literature, let us consider, as a particular case, a fermionic matter field minimally coupled to the self-dual field  $\Pi_{\mu}$ . Assuming the following identifications:

$$\mathcal{L}(\psi) \to \mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - M)\psi,$$
 (61)

where M is the Dirac field mass, and the fermionic currents are

$$J_{\mu} \to -eJ_{\mu} = -e\bar{\psi}\gamma_{\mu}\psi, \qquad (62)$$

$$\mathcal{J}_{\mu\nu} \to 0,$$
 (63)

with e being a dimensionless coupling constant. The equation of motion for  $\psi$  (60) takes the simple form

$$(i\gamma^{\mu}\partial_{\mu} - M)\psi = \frac{e^2}{\theta^2}\hat{R}J_{\mu}\gamma^{\mu}\psi, \qquad (64)$$

which agrees with the result obtained in [47].

# IV. THE DUALITY AT THE QUANTUM LEVEL

Once we proved the duality between  $SD_{B \wedge F}$  and  $TM_{B \wedge F}$  models at the level of equations of motion, we now check whether this duality is preserved at the quantum level. For this purpose, we adopt the path-integral framework and define the master generating functional as

$$Z(\psi) = \mathcal{N} \int \mathcal{D}A^{\mu} \mathcal{D}B^{\mu\nu} \mathcal{D}\Pi^{\mu} \mathcal{D}\Lambda^{\mu\nu} \exp \left\{ i \int d^4x [\mathcal{L}_M + J_{\mu}\Pi^{\mu} + \mathcal{J}_{\mu\nu}\Lambda^{\mu\nu} + \mathcal{L}(\psi)] \right\},$$
(65)

where  $\mathcal{N}$  is an overall normalization constant. Our aim is to evaluate the effective Lagrangian resulting from the integration over the fields. First, let us integrate out the contribution of the  $SD_{B \wedge F}$  fields.

After the shifts,  $\Pi_{\mu} \rightarrow \Pi_{\mu} + \tilde{H}_{\mu} - \frac{1}{m^2} J_{\mu}$  and  $\Lambda_{\mu\nu} \rightarrow \Lambda_{\mu\nu} + \tilde{F}_{\mu\nu} + 2\mathcal{J}_{\mu\nu}$ , we perform the functional integration in Eq. (65) over the fields  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$ , thereby producing

$$Z(\psi) = \mathcal{N} \int \mathcal{D}A^{\mu} \mathcal{D}B^{\mu\nu} \exp\left[i \int d^4x \mathcal{L}_{\text{eff}}^{(1)}(A, B, \psi)\right], \quad (66)$$

where

$$\mathcal{L}_{\text{eff}}^{(1)}(A, B, \psi) = \frac{\theta^2}{12m^2} H_{\mu\nu\alpha} H^{\mu\nu\alpha} - \frac{\theta^2}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\chi\theta}{2} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} \partial^\alpha A^\beta - \frac{\chi\theta}{2m^2} B_{\mu\nu} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha J_\beta - \frac{1}{2m^2} J_\mu J^\mu + \chi\theta A_\mu \epsilon^{\mu\nu\alpha\beta} \partial_\nu \mathcal{J}_{\alpha\beta} + \mathcal{J}_{\mu\nu} \mathcal{J}^{\mu\nu} + \mathcal{L}(\psi)$$
(67)

is the same Lagrangian density found in Eq. (42).

To integrate over the fields configurations  $A_{\mu}$  and  $B_{\mu\nu}$ , let us first note that the master Lagrangian  $\mathcal{L}_M$  can be rewritten, up to surface terms, as

$$\mathcal{L}_{M} = \frac{\chi \theta}{2} \epsilon^{\mu\nu\alpha\beta} (\Lambda_{\mu\nu} - B_{\mu\nu}) \partial_{\alpha} (A_{\beta} - \Pi_{\beta}) + \mathcal{L}_{SD}.$$
 (68)

In this way, we can make a shift in the gauge fields through  $B_{\mu\nu} \rightarrow B_{\mu\nu} + \Lambda_{\mu\nu}$  and  $A_{\beta} \rightarrow A_{\beta} + \Pi_{\beta}$ , which allows us to rewrite the generating function (65) as

$$Z(\psi) = \mathcal{N} \int \mathcal{D}A^{\mu} \mathcal{D}B^{\mu\nu} \mathcal{D}\Pi^{\mu} \mathcal{D}\Lambda^{\mu\nu} \exp\left\{i \int d^{4}x \left[-\frac{\chi\theta}{2}\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}\partial_{\alpha}A_{\beta} + \mathcal{L}_{SD} + J_{\mu}\Pi^{\mu} + \mathcal{J}_{\mu\nu}\Lambda^{\mu\nu} + \mathcal{L}(\psi)\right]\right\}, \quad (69)$$

such that the  $A_{\mu}$  and  $B_{\mu\nu}$  fields decouple. Then, performing the function integration yields to the following generating functional

$$Z(\psi) = \mathcal{N} \int \mathcal{D}\Pi^{\mu} \mathcal{D}\Lambda^{\mu\nu} \exp\left[i \int d^4x \mathcal{L}_{\text{eff}}^{(2)}(\Pi, \Lambda, \psi)\right],\tag{70}$$

with

$$\mathcal{L}_{\text{eff}}^{(2)}(\Pi,\Lambda,\psi) = \frac{m^2}{2}\Pi_{\mu}\Pi^{\mu} - \frac{1}{4}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \frac{\chi\theta}{2}\Pi_{\mu}\epsilon^{\mu\nu\alpha\beta}\partial_{\nu}\Lambda_{\alpha\beta} + \Pi_{\mu}J^{\mu} + \Lambda_{\mu\nu}\mathcal{J}^{\mu\nu} + \mathcal{L}(\psi), \tag{71}$$

corresponding to the same Lagrangian density (39) previously obtained. It is worth highlighting the physical implications contained in (68). We clearly see that the master Lagrangian  $\mathcal{L}_M$  obtained in (35) is equivalent to self-dual Lagrangian  $\mathcal{L}_{SD}$  added by a purely topological  $B \wedge F$  term, which makes evident the role of the master Lagrangian on the duality symmetry.

The implications of the above results at the quantum level can be explored by considering the functional derivatives of (66) and (70) with respect to the sources. Setting  $J_{\mu} = \mathcal{J}_{\mu\nu} = 0$ , we can establish the following identities to the correlation functions:

$$\left\langle \Pi_{\mu_1}(x_1)\cdots\Pi_{\mu_N}(x_N)\right\rangle_{SD} = \left\langle \tilde{H}_{\mu_1}[B(x_1)]\cdots\tilde{H}_{\mu_N}[B(x_N)]\right\rangle_{TM} + \text{contact terms},\tag{72}$$

$$\left\langle \Lambda_{\mu_1\nu_1}(x_1)\cdots\Lambda_{\mu_N\nu_N}(x_N)\right\rangle_{SD} = \left\langle \tilde{F}_{\mu_1\nu_1}[A(x_1)]\cdots\tilde{F}_{\mu_N\nu_N}[A(x_N)]\right\rangle_{TM} + \text{contact terms.}$$
(73)

These relations show that the classical dual map (43) and (44) is satisfied by all quantum correlation functions of those fields, up to contact terms.

Finally, we now complete the proof of quantum duality between the  $SD_{B \wedge F}/TM_{B \wedge F}$  models by performing the path integration over  $A_{\mu}$  and  $B_{\mu\nu}$  gauge fields in Eq. (66), and over  $\Pi_{\mu}$  and  $\Lambda_{\mu\nu}$  self-dual fields in Eq. (70). For this goal, it is convenient to organize the effective Lagrangians (67) and (71) in a matrix form according to the

$$\mathcal{L} = \frac{1}{2} X^T \hat{\mathcal{O}} X + X^T J, \tag{74}$$

where the wave operator,  $\hat{O}$ , forms a 2 × 2 matrix, X, and J represent vector-tensor duplet of type

$$\boldsymbol{X} = \begin{pmatrix} A_{\mu} \\ B_{\mu\nu} \end{pmatrix}. \tag{75}$$

To accomplish the functional integration, we use the Gaussian path integral formula over a bosonic field X,

$$\int \mathcal{D}X \exp\left[i \int d^4x \left(\frac{1}{2} X^T \hat{\mathcal{O}}X + X^T J\right)\right] = \left[\operatorname{Det}(-i\hat{\mathcal{O}})\right]^{-\frac{1}{2}} \times \exp\left[-i \int d^4x \frac{1}{2} J^T \hat{\mathcal{O}}^{-1} J\right].$$
(76)

In our case, the determinant  $\text{Det}(-i\hat{O})$  is field independent and can be absorbed by the normalization constant. The calculation of propagators  $\hat{O}^{-1}$  is rather lengthy, and the details are in the Appendix. Here we just write the results

$$(\hat{\mathcal{O}}_{SD}^{-1})^{\mu,\alpha\beta;\nu,\lambda\sigma} = \begin{pmatrix} \frac{1}{\theta^2 \Box + m^2} \Theta^{\mu\nu} + \frac{1}{m^2} \omega^{\mu\nu} & \frac{2}{\theta^2 \Box + m^2} S^{\mu\lambda\sigma} \\ -\frac{2}{\theta^2 \Box + m^2} S^{\alpha\beta\nu} & -\frac{2m^2}{\theta^2 \Box + m^2} (P^{(1)})^{\alpha\beta,\lambda\sigma} - 2(P^{(2)})^{\alpha\beta,\lambda\sigma} \end{pmatrix},$$
(77)

and

$$(\hat{\mathcal{O}}_{TM}^{-1})^{\mu,\alpha\beta;\nu,\lambda\sigma} = \begin{pmatrix} \frac{1}{\theta^2 \Box + m^2} \Theta^{\mu\nu} + \frac{\lambda}{\Box} \omega^{\mu\nu} & -\frac{2m^2}{\theta^2 \Box (\theta^2 \Box + m^2)} S^{\mu\lambda\sigma} \\ \frac{2m^2}{\theta^2 \Box (\theta^2 \Box + m^2)} S^{\alpha\beta\nu} & -\frac{2m^2}{\theta^2 \Box + m^2} (P^{(1)})^{\alpha\beta,\lambda\sigma} - \frac{2\xi}{\Box} (P^{(2)})^{\alpha\beta,\lambda\sigma} \end{pmatrix},$$
(78)

where  $\Theta_{\mu\nu}$ ,  $\omega_{\mu\nu}$ ,  $S_{\mu\nu\alpha}$ ,  $P^{(1)}_{\mu\nu,\alpha\beta}$ , and  $P^{(2)}_{\mu\nu,\alpha\beta}$  are projection operators whose definitions and closed algebras are shown in the Appendix. Also,  $\lambda$  and  $\xi$  are convenient gauge-fixing parameters. Note that the physical poles of the two propagators are equal, i.e.,  $\theta^2 \Box + m^2 = 0$ , and confirm that the particle spectrum of both theories are equivalent, so that we may consider the self-dual theory equivalent to  $TM_{B\wedge F}$  theory with the fixed gauge.

The above propagators, together with formula (76), enable us to perform the functional integration in (66) and (70). After completing all tensorial contractions, we obtain the same effective Lagrangian for the matter field

$$\mathcal{L}_{\rm eff}^{(3)}(\psi) = \mathcal{L}(\psi) + \frac{1}{2} \begin{pmatrix} J_{\mu} & \mathcal{J}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} -\frac{1}{\theta^2 \Box + m^2} \eta^{\mu\nu} - \frac{\theta^2}{m^2} \frac{1}{\theta^2 \Box + m^2} \partial^{\mu} \partial^{\nu} & -\frac{\chi\theta}{\theta^2 \Box + m^2} \epsilon^{\mu\lambda\sigma\delta} \partial_{\delta} \\ \frac{\chi\theta}{\theta^2 \Box + m^2} \epsilon^{\alpha\beta\nu\delta} \partial_{\delta} & \frac{2}{\theta^2 \Box + m^2} (\theta^2 \Box P^{(2)} + m^2 \mathcal{I})^{\alpha\beta,\lambda\sigma} \end{pmatrix} \begin{pmatrix} J_{\nu} \\ \mathcal{J}_{\lambda\sigma} \end{pmatrix}.$$
(79)

It is easy to verify that the equation of motion for the matter field obtained from  $\mathcal{L}_{eff}^{(3)}$  (79) is precisely that found in the previous section [see Eqs. (53) or (60)]. Thus, we prove the quantum equivalence between the matter sector of the  $SD_{B\wedge F}/TM_{B\wedge F}$  models. It is worth mentioning that the dynamics of the matter fields is preserved in the functional integration in (66) only if the Thirring-like interactions are added to the diagonal elements of the  $\hat{\mathcal{O}}_{TM}^{-1}$  matrix. Besides, the gauge-dependent parts involving the gauge-fixing parameters are canceled, as it should be.

## V. CONCLUSION

In this work, we revisited the duality between the selfdual and topologically massive models involving the  $B \wedge F$ term in 3 + 1 spacetime dimensions. The study of this duality when couplings with fermionic matter are included was first carried out in [47], through the gauge embedding formalism. Here, we considered another approach, namely the master action method, whereby we obtained a fundamental Lagrangian density that interpolates between the two models and provides direct proof of dual equivalence at both the classical and quantum levels. The master action enabled us to relate the equations of motion of these models via a dual map among fields and currents of both theories, which ensures that they are equivalent at the classical level. In addition, we demonstrated the duality at the quantum level through the path-integral framework. We defined a master generating functional wherein the integration over the different fields provided effective Lagrangians that are the same as those obtained classically. Moreover, after a last functional integration over the bosonic fields, we obtained an effective nonlocal Lagrangian for the matter fields, which proves the equivalence between the matter sectors of the analyzed models.

We assumed that the external currents are linearly coupled with the self-dual fields and are constituted exclusively of the matter fields. We show that these interactions induce "magnetic" couplings involving the gauge fields, in addition to current-current Thirring-like interactions. These types of couplings are, in general, nonrenormalizable by direct power counting [18,22]. However, as in the 2 + 1 dimensional case involving the Maxwell-Chern-Simons model, we may expect that this weakness can be overcome by a 1/N perturbative expansion when the matter field is an *N*-component fermionic field, such that the theory becomes renormalizable. An explicit verification of this issue, as well as a possible extension of our results to the supersymmetric case [23,38], are themes for forthcoming works.

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# APPENDIX: FEYNMAN PROPAGATOR FOR THE $TM_{B \wedge F}$ THEORY

Consider the topologically massive  $B \wedge F$  model defined as

$$S_{TM} = \int d^4x \left[ -\frac{\theta^2}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\theta^2}{12m^2} H^{\mu\nu\alpha} H_{\mu\nu\alpha} - \frac{\chi\theta}{4} \epsilon_{\mu\nu\alpha\beta} B^{\mu\nu} F^{\alpha\beta} \right], \tag{A1}$$

where the first two terms represent a gauge-invariant Maxwell-Kalb-Ramond theory, while the last is a topological  $B \wedge F$  term. The calculation of the Feynman propagator for the theory (A1) can be performed as follows.

First, let us rewrite the integrand in Eq. (A1) on the matrix form

$$\mathcal{L}_{TM} = \frac{1}{2} X^T \hat{\mathcal{O}}_{TM} X, \qquad (A2)$$

with the wave operator,  $\hat{\mathcal{O}}_{TM}$ , being a 2 × 2 matrix, and *X* represents a column vector-tensor as

$$\boldsymbol{X} = \begin{pmatrix} A_{\mu} \\ B_{\mu\nu} \end{pmatrix}.$$
 (A3)

Adding convenient gauge-fixing terms in (A2), namely,  $-\frac{1}{2\lambda}(\partial_{\mu}A^{\mu})^2$  and  $\frac{1}{2\xi}(\partial_{\mu}B^{\mu\nu})^2$ , we can explicitly write the operator  $\hat{\mathcal{O}}_{TM+gf}$ , in the form

TABLE I. Algebra of the spin-projection operators.

	$\Theta^{lpha}_{ u}$	$\omega^{\nu}{}_{\nu}$
$\overline{\Theta_{\mulpha}}$	$\Theta_{\mu u}$	0
$\omega_{\mu\alpha}$	0	$\omega_{\mu u}$
	$(P^{(1)})^{ ho\sigma}{}_{lphaeta}$	$(P^{(2)})^{ ho\sigma}{}_{lphaeta}$
$P^{(1)}_{\mu u ho\sigma}$	$P_{\mu\nu\alpha\beta}^{(1)}$	0
$P^{(2)}_{\mu u ho\sigma}$	0	$P^{(2)}_{\mu ulphaeta}$

$$\hat{\mathcal{O}}_{TM+gf}^{\mu,\alpha\beta;\nu,\lambda\sigma} = \begin{pmatrix} \theta^2 \Box \Theta^{\mu\nu} + \frac{\Box}{\lambda} \omega^{\mu\nu} & -S^{\mu\lambda\sigma} \\ S^{\alpha\beta\nu} & -\frac{\theta^2 \Box}{2m^2} (P^{(1)})^{\alpha\beta,\lambda\sigma} - \frac{\Box}{2\xi} (P^{(2)})^{\alpha\beta,\lambda\sigma} \end{pmatrix},$$
(A4)

where we have introduced the set of spin-projection operators as

$$\Theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \qquad \omega_{\mu\nu} = \frac{\partial_{\mu}\partial_{\nu}}{\Box}, \qquad (A5)$$

$$S_{\mu\nu\alpha} = \frac{\chi\theta}{2} \epsilon_{\mu\nu\alpha\beta} \partial^{\beta}, \qquad (A6)$$

$$P^{(1)}_{\mu\nu,\alpha\beta} = \frac{1}{2} (\Theta_{\mu\alpha}\Theta_{\nu\beta} - \Theta_{\mu\beta}\Theta_{\nu\alpha}), \qquad (A7)$$

$$P^{(2)}_{\mu\nu,\alpha\beta} = \frac{1}{2} (\Theta_{\mu\alpha}\omega_{\nu\beta} - \Theta_{\mu\beta}\omega_{\nu\alpha} + \Theta_{\nu\beta}\omega_{\mu\alpha} - \Theta_{\nu\alpha}\omega_{\mu\beta}), \quad (A8)$$

with  $\Box \equiv \partial_{\mu}\partial^{\mu}$ , and  $\eta_{\mu\nu}$  is the Minkowski metric with signature (+, -, -, -). Note that  $P^{(1)}$  and  $P^{(2)}$  satisfy the tensorial completeness relation:

$$(P^{(1)} + P^{(1)})_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}) \equiv \mathcal{I}_{\mu\nu,\alpha\beta}.$$
 (A9)

The products between the operators defined above satisfy a closed algebra and are summarized in Tables I and II.

The Feynman propagator is defined as  $\hat{\mathcal{O}}_{TM+gf}^{-1}$ . In order to invert the wave operator, we will write it and its inverse generically by

TABLE II. Algebra of the spin-projection operators.

	$S_{lphaeta u}$	$\Theta_{eta\sigma}$	$\omega_{eta\sigma}$
$S^{\mulphaeta}$	$-\frac{\theta^2}{2}\Box\Theta^{\mu}{}_{\nu}$	$S^{\mulpha}{}_{\sigma}$	0
	$S^{\lambda}{}_{lphaeta}$	$(P^{(1)})^{ u\lambda}_{ ho\sigma}$	$(P^{(2)})^{ u\lambda}{}_{ ho\sigma}$
$S_{\mu u\lambda}$	$-rac{ heta^2}{2}\Box P^{(1)}_{\mu ulphaeta}$	$S_{\mu ho\sigma}$	0

$$\mathbf{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \mathbf{O}^{-1} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix}, \quad (A10)$$

which fulfills the relation  $OO^{-1} = I$ , where the general identity matrix I is defined by

$$\mathbf{I} = \begin{pmatrix} I & 0\\ 0 & \mathcal{I} \end{pmatrix},\tag{A11}$$

where *I* and  $\mathcal{I}$  are the identities to the projectors ( $\theta^{\mu\nu}$ ,  $\omega^{\mu\nu}$ ) and ( $P^{(1)}$ ,  $P^{(2)}$ ), respectively. From these preliminary

$$(\hat{\mathcal{O}}_{TM}^{-1})^{\mu,\alpha\beta;\nu,\lambda\sigma} = \begin{pmatrix} \frac{1}{\theta^2 \Box + m^2} \Theta^{\mu\nu} + \frac{\lambda}{\Box} \omega^{\mu\nu} \\ \frac{2m^2}{\theta^2 \Box (\theta^2 \Box + m^2)} S^{\alpha\beta\nu} \end{pmatrix}$$

definitions, we obtain a system of four equations, whose solutions can be written as we get

$$\begin{cases}
A\mathbb{A} + B\mathbb{C} = I \\
A\mathbb{B} + B\mathbb{D} = 0 \\
C\mathbb{A} + D\mathbb{C} = 0 \\
C\mathbb{B} + D\mathbb{D} = \mathcal{I}
\end{cases} \Rightarrow \begin{cases}
\mathbb{A} = (A - BD^{-1}C)^{-1} \\
\mathbb{B} = -A^{-1}B\mathbb{D} \\
\mathbb{C} = -D^{-1}C\mathbb{A} \\
\mathbb{D} = (D - CA^{-1}B)^{-1}
\end{cases}$$
(A12)

After some algebraic manipulations with the set of the operators presented above, the  $TM_{B \wedge F}$  gauge propagator is properly written as

$$-\frac{2m^{2}}{\theta^{2}\square(\theta^{2}\square+m^{2})}S^{\mu\lambda\sigma} \\ -\frac{2m^{2}}{\theta^{2}\square+m^{2}}(P^{(1)})^{\alpha\beta,\lambda\sigma} - \frac{2\xi}{\square}(P^{(2)})^{\alpha\beta,\lambda\sigma} \end{pmatrix}.$$
(A13)

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