Thermodynamics and kinetics of Hawking-Page phase transition

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We study the thermodynamics and the kinetics of the Hawking-Page phase transition in Einstein gravity and in massive gravity based on the underlying free energy landscape. For Einstein gravity, Schwarzschild–anti-de Sitter (AdS) black holes as well as thermal AdS space can be considered as macroscopic emergent states or phases. The stability and phase transition of these states can be determined by free energy landscape topography quantified by the barrier height between the state basins. Due to the thermal fluctuations, a black hole or AdS space has the chance to escape from one phase to another phase. The first passage process describes a system that undergoes such a kinetic process for the first time, and the mean first passage time can typically be used to quantify the kinetic speed. The probabilistic evolution of such a stochastic process can be described by the corresponding Fokker-Planck equation. We derive analytical integral expressions for the mean first passage time and its fluctuations. The results show that the mean first passage time and its fluctuations are closely related to free energy landscape topography through barrier heights and the temperature. The conclusions for the Hawking-Page phase transition in massive gravity are qualitatively similar to those in Einstein gravity. This study provides a systematic way of studying black hole thermodynamics and kinetics of the black holes from free energy landscape topography.

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I. INTRODUCTION

In general relativity, black holes, which are intriguing solutions to Einstein field equations, can be fully described by their spacetime geometries. When taking the effect of the quantum field into account, Hawking demonstrated that the collapsing black holes can emit radiation from just outside the event horizon in the form of a blackbody spectrum [1]. This intriguing discovery revealed the thermal nature of black holes and established a profound relationship between gravity, thermodynamics, and statistical physics. Thus, thermodynamics and statistical physics may provide a complementary description of black hole physics and could even supply some insights into the quantum nature of gravity.

It is well known that macroscopic emergent phases and phase transitions are extremely vital subjects in thermodynamics and statistical physics [2]. If black holes are identified as thermodynamic systems with physical temperature and entropy as proposed by Bekenstein [3], it is very natural to question whether thermodynamical phases can emerge and whether phase transitions can take place in black holes, and, furthermore, what type of kinetics is in the phase transition process. The phase emergence and phase transition of black holes in asymptotically flat spacetime were first investigated by Hut [4] and Davies [5]. Since then, studying the phase emergence and phase transition in black holes has become an intriguing topic and attracted much attention. In this regard, the remarkable analogy between the van der Waals liquid-gas system and charged black holes in AdS space have been extensively investigated [6–12], where the extended phase space thermodynamics for charged AdS black holes was formulated by treating the cosmological constant as thermodynamic pressure.

The Hawking-Page phase transition [13] is another type of phase transition that takes place in asymptotically AdS space. By treating the black hole as a state in the thermodynamic ensemble, it is shown that two stable thermodynamic phases emerge: the thermal AdS space phase and the large Schwarzschild-AdS black hole phase. It has been found that there is a first order phase transition between the thermal AdS space and the large Schwarzschild-AdS black hole at a certain critical temperature. In the context of anti-de Sitter/conformal field theory (AdS/CFT) correspondence [14–16], the Hawking-Page transition can be properly explained as the confinement/deconfinement transition in quantum chromodynamics (QCD) [17]. The seminal work of Hawking and Page has been generalized to the case of modified gravity

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theories [18–25]. Such a formation of a black hole in the phase transition of thermal spacetime may play an important role in the density fluctuation of the very early Universe.

In an ordinary thermodynamic system, the liquid, gas, or solid form of a material is made up of micromolecules. The liquid, gas, or solid form of matter is the macroscopic emergence state of the degrees of freedom from microscopic molecules. A system under constant pressure and temperature conditions can be best described by Gibbs free energy. Various forms of materials may have different Gibbs free energy characterized by different physical properties such as density, viscosity, or molecular structure. The phase transition can be easily analyzed by using the so-called free energy landscape, which is a familiar and widely used concept in physics, chemistry, and biology [26–29]. In the formalism of the free energy landscape, the Gibbs free energy is defined as the continuous function of the order parameter or reaction coordinate of the system. The order parameter or reaction coordinate may be considered as the coarse-grained description which captures the essential characteristics and the microscopic degrees of freedom of the system. In general, in a system with a first order phase transition, the free energy landscape topography, i.e., the shape of Gibbs free energy plotted as a function of the order parameter or reaction coordinate, can have the shape of double basins, with each representing one emergent phase. The system in equilibrium will always stay in the state of lowest Gibbs free energy. If there are two degenerate basins in the free energy landscape topography, the two phases can coexist. However, temperature can change the shape of the Gibbs free energy landscape in order parameter space. By adjusting the temperature, one can raise or lower the free energy of the basins continuously. The state or phase represented by the local basin (minimum) of the free energy landscape has a good chance of switching to the state or phase represented by the global basin (minimum) of the free energy landscape under the thermal fluctuation and vice versa, as long as the barrier between these basins (minima) is small or comparable to the thermal energy at that temperature. In this picture, any state has a lifetime due to the chance of transition to other states under thermal fluctuations.

We are interested not only in the thermodynamic phase emergence and phase transitions of the black holes but also in the dynamical scenario and kinetics of the black hole phase transition. In this paper, we focus on the Hawking-Page phase transitions in Einstein gravity and in massive gravity. Our discussion is based on the free energy landscape formalism for the black hole system. Our main assumptions and proposal are described in detail as follows.

First, we introduce the concept of the order parameter of the AdS black hole. In fact, if we take the black hole seriously as a thermodynamic entity, it is natural to propose that the black hole is a macroscopic emergent state from the microscopic degrees of freedom. From the macroscopic perspective, there could be an order parameter that measures the essential characteristics and counts the microscopic degrees of freedom of the black hole. It should be noted that, by using the concept of the black hole molecule [30–32], the number density of the black hole molecule is introduced as an order parameter of the black hole, which is closely related to the size or radius of the black hole. In the present work, we propose the radius of the AdS black hole as being the appropriate order parameter to formulate the free energy landscape.

Second, we propose that, at the specific temperature, there exists a series of black hole spacetimes, with the horizon radius (the order parameter) ranging from zero to infinity. For the Hawking-Page phase transition in Einstein gravity, these spacetime states are the thermal AdS space, the small and the large Schwarzschild-AdS black holes, and the intermediate transient states during the phase transition, which compose the canonical ensemble we are considering. The value of the order parameter of the thermal AdS space is zero because it does not have a horizon. The values of the order parameters of the small and the large black holes are determined by solving the Hawking temperature formula [Eq. (4) in Sec. II] for the Schwarzschild-AdS black hole by the replacement of the Hawking temperature with the ensemble temperature. There is no direct relationship between the order parameter of the intermediate transient state and the ensemble temperature. In this sense, the intermediate transient states are not solutions to the Einstein field equation, and their order parameters can take arbitrary values.

Third, we quantify the free energy landscape for the Hawking-Page phase transition by specifying every spacetime state in the ensemble a Gibbs free energy. As a solution to the Einstein field equation, the Gibbs free energy of a Schwarzschild-AdS black hole, which can be given by the thermodynamic relation or calculated directly from the Euclidean action, is on shell. The Gibbs free energies of the intermediates are off shell because they are not the solutions to the Einstein field equation. Inspired by the thermodynamics of a Schwarzschild-AdS black hole, the off-shell Gibbs free energy is defined as the on-shell Gibbs free energy by the replacement of the Hawking temperature with the ensemble temperature. The resulting Gibbs free energy can be expressed as the function of the order parameter and ensemble temperature. Based on the free energy landscape, we can uncover the emergence of the phases and the associated phase transition. More precisely, the extrema of the Gibbs free energy topography are considered as the emerged equilibrium states which represent the large or small Schwarzschild-AdS black hole or the thermal AdS space, respectively. By exploring the range of the ensemble temperatures, we can also analyze the phase diagram and explore the thermodynamic stability.

Fourth, inspired by the barrier crossing picture in the free energy landscape topography, we propose that it is possible to study the stochastic dynamics of the Hawking-Page phase transition under thermal fluctuations in terms of the associated probabilistic Fokker-Planck equation on the free energy landscape [33–35]. Notice that the phase transition or the black hole instability here is caused by the intrinsic thermodynamic fluctuations. In this regard, we can write down the corresponding Fokker-Planck equation by treating the Gibbs free energy as the potential function. We can further analyze the final equilibrium stationary distribution of the spacetime states in the ensemble from the Fokker-Planck equation, which is shown to be closely related to the free energy landscape through the Boltzmann law.

Finally, we are interested in getting the information on the speed of the state switching by quantifying the first passage time and its fluctuation in the Hawking-Page phase transition. The potential barrier crossing picture implies that any state in the ensemble can have the chance of switching to other states under thermal fluctuations; i.e., any state can have a lifetime. Since we are considering the stochastic process, the first passage time will be a random variable. The timescale of the phase transition process from one macroscopic state to another can then be characterized by the mean first passage time, which is defined as the average timescale for a stochastic event to first occur. In this regard, we derive analytical integral expressions for not only the mean first passage time but also its fluctuations, and we present the corresponding numerical results. The results show that the mean first passage time and the fluctuations are closely related to the free energy landscape topography through the barrier heights and the temperature.

This paper is organized as follows. In Sec. II, we study the thermodynamics of the Schwarzschild-AdS black hole and Hawking-Page phase transition from the free energy landscape perspective, giving the phase diagram and stability analysis. In Sec. III, the stochastic probabilistic Fokker-Planck equation on the free energy landscape is presented, and the final stationary distribution is also discussed. In Sec. IV, we study the kinetics as a first passage time problem in the Fokker-Planck equation. The analytical expressions and numerical results of the mean first passage time and its fluctuation of kinetics are discussed. In Sec. V, we study the thermodynamics and kinetics of the Hawking-Page phase transition in massive gravity. The conclusion and a discussion are presented in the last section.

II. THERMODYNAMICS OF PHASE EMERGENCE AND HAWKING-PAGE PHASE TRANSITION BASED ON THE FREE ENERGY LANDSCAPE

We are interested in the phase emergence and the corresponding Hawking-Page phase transition of four-dimensional Schwarzschild-AdS black holes. The metric is given by ($G_4 = 1$ units) [13]

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{r^2}{L^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{r^2}{L^2}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{1}$$

where M is the black hole mass and $L=\sqrt{\frac{-3}{\Lambda}}$ is the AdS curvature radius, with Λ being the cosmological constant.

The black hole horizon r_+ is determined by the root of the equation,

$$1-\frac{2M}{r}+\frac{r^2}{L^2}=0. \tag{2}$$

In the case of a Schwarzschild-AdS black hole, the above equation shows that for any positive value of black hole M, there exists only one black hole horizon r_+. In turn, the mass of the black hole can be expressed by using the black hole radius

$$M=\frac{r_+}{2}\left(1+\frac{r_+^2}{L^2}\right). \tag{3}$$

The Hawking temperature is given by

$$T_H=\frac{1}{4\pi r_+}\left(1+\frac{3r_+^2}{L^2}\right). \tag{4}$$

This means that the Hawking temperature of a Schwarzschild-AdS black hole has a minimal value

$$T_{\min}=\frac{\sqrt{3}}{2\pi L}. \tag{5}$$

The Bekenstein-Hawking entropy is given by the area of the event horizon,

$$S=\pi r_+^2. \tag{6}$$

As discussed in the Introduction, we consider the canonical ensemble at the specific temperature T composed of a series of black hole spacetimes with an arbitrary horizon radius. In order to construct the free energy landscape, we need to specify every spacetime state a Gibbs free energy. The on-shell Gibbs free energy, which can be given by the thermodynamic relationship $G=M-T_H S$ or calculated directly from the Euclidean action [13], is generalized to the off-shell Gibbs free energy by replacing the Hawking temperature T_H with the ensemble temperature T, which is explicitly given as follows:

$$G=M-TS=\frac{r_+}{2}\left(1+\frac{r_+^2}{L^2}\right)-\pi T r_+^2. \tag{7}$$

Note that the black hole radius is considered as the order parameter describing the microscopic degree of freedom of

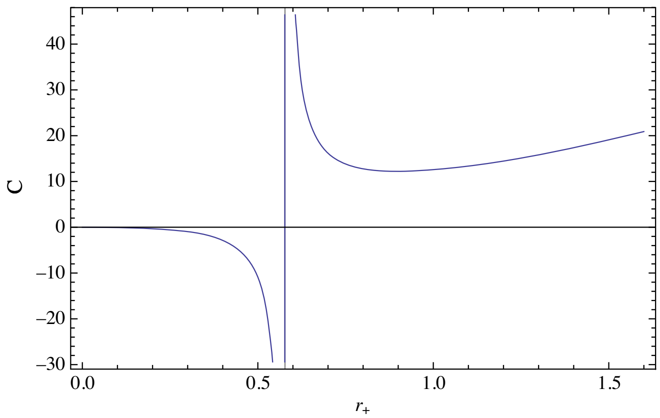

FIG. 1. Heat capacity of a Schwarzschild-AdS black hole as a function of the black hole radius. The vertical line represents the location $r_+ = L/\sqrt{3}$ where heat capacity is divergent.

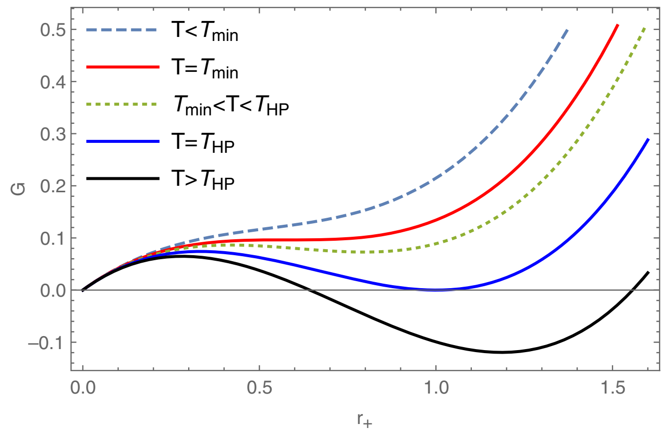

FIG. 2. Gibbs free energy landscapes as a function of black hole radius r_+ at different temperatures.

the system. The off-shell free energy is expressed as a function of the order parameter and ensemble temperature.

For $T > T_{\rm min}$, by replacing the Hawking temperature T_H with the ensemble temperature T in Eq. (4), one can obtain the radii (the order parameters) of the small and the large black holes in the ensemble [36],

$$r_{l,s} = \frac{T}{2\pi T_{\rm min}^2}\left(1 \pm \sqrt{1 - \frac{T_{\rm min}^2}{T^2}}\right), \tag{8}$$

where l/s represents the large/small Schwarzschild-AdS black hole phase.

The small Schwarzschild-AdS black hole is unstable, which can be explicitly shown by computing the heat capacity. The heat capacity is given by

$$C = \frac{\partial M}{\partial T_H} = \frac{2\pi r_+^2(3r_+^2 + L^2)}{(3r_+^2 - L^2)}. \tag{9}$$

We have plotted the heat capacity as a function of the black hole radius in Fig. 1. The heat capacity is divergent at the radius where the large and the small black holes have the same temperature $T_{\rm min}$. The heat capacity of the small black hole branch ($r_+ < L/\sqrt{3}$) is always negative, while it is positive on the large black hole branch ($r_+ > L/\sqrt{3}$). This implies that the small black hole is thermodynamically unstable and the large black hole is stable.

The Gibbs free energy landscape as a function of black hole radius r_+ [36,37] at different temperatures can be plotted, as explicitly shown in Fig. 2. Without loss of generality, we set the AdS curvature radius as $L = 1$ in the following. The Gibbs free energy landscape of the present system is only modulated by the temperature. From these plots, we can easily read off the phase diagram of the system as discussed in the following.

When $T < T_{\rm min}$, there is only one global minimum of the Gibbs free energy landscape at the origin, and the system is in a pure radiation phase or thermal AdS space. At $T = T_{\rm min}$, the Gibbs free energy landscape exhibits an inflection point at $r = L/\sqrt{3}$. Above this temperature, two black hole phases emerge (large and small black holes) with radii given by Eq. (8). When $T_{\rm min} < T < T_{HP}$, the small black hole phase corresponds to a local maximum of the Gibbs free energy, while the large black hole phase is locally stable, being a local minimum on the free energy landscape. The globally stable state phase is still the thermal AdS state. At $T = T_{HP}$, the Gibbs free energy of a black hole phase with radius $r_+ = L$ is degenerate with the Gibbs free energy of a thermal AdS phase at the same temperature. Here, $T_{HP} = \frac{1}{\pi L}$ is known as the Hawking-Page critical temperature. Since the order parameter of the thermal AdS phase is discontinuous with the order parameter of the large black hole phase in the free energy landscape topography, the associated derivative of the free energy will diverge at the Hawking-Page critical temperature, which is a signature of the first order phase transition. Finally, for $T > T_{HP}$ the large black hole phase becomes the absolute minimum of the Gibbs free energy landscape and is a globally stable state. In summary, below the critical temperature T_{HP}, the thermal AdS phase is thermodynamically stable, and above the critical temperature, the large black hole phase is stable. At the critical temperature, both the thermal AdS space phase and the large black hole phase are stable with equal free energy basin depths.

III. PROBABILISTIC FOKKER-PLANCK EQUATION FOR DESCRIBING THE STOCHASTIC DYNAMICS ON THE FREE ENERGY LANDSCAPE

In this section, we study the kinetics of the Hawking-Page phase transition by treating the black hole phase as well as the thermal AdS phase as states in a thermodynamic ensemble. As shown in the last section, we know that the Gibbs free energy landscape as a function of the black hole radius exhibits a double basin shape when the temperature exceeds $T_{\min}$. One naturally regards the black hole radius as the reaction coordinate or order parameter on the free energy landscape.

We consider that there are a large number of states in a thermodynamic ensemble in which one or a group of them can represent a Schwarzschild black hole phase or a thermal AdS phase or any intermediate transient states during the Hawking-Page phase transition. The probability distribution of these states evolving in time should be a function of the order parameter r_+ (black hole radius) and time t. From now on, we use the symbol r to denote the black hole radius r_+ for the sake of simplicity. Thus, the probability distribution is denoted by $\rho(r,t)$. The stochastic kinetics of states under the thermal fluctuation can be described by the probabilistic Fokker-Planck equation, which on the free energy landscape is explicitly given by

$$\frac{\partial \rho(r,t)}{\partial t} = D\frac{\partial}{\partial r}\left\{e^{-\beta G(r)}\frac{\partial}{\partial r}[e^{\beta G(r)}\rho(r,t)]\right\}. \quad (10)$$

In the above equation, the diffusion coefficient D is given by $D = kT/\zeta$, with k being the Boltzmann constant and ζ being the dissipation coefficient. Without loss of generality, we will take $k = \zeta = 1$ in the following. Note that $\beta = 1/kT$ is the inverse temperature of the system, and $G(r)$ is the off-shell Gibbs free energy as a function of the black hole radius r modulated by the temperature.

The final stationary distribution of the probability is determined by $\rho_{st}(r) \propto e^{-\beta G(r)}$, giving the equilibrium Boltzmann distribution, which can be solved directly from the Fokker-Planck equation by setting $\frac{\partial \rho(r,t)}{\partial t} = 0$. The thermodynamic stable state is then determined by the global maximum of the final stationary equilibrium distribution. As shown in Fig. 2, if the initial probability is mainly distributed at the local minimum, most of probability will move into the region of the global minimum. However, there is also a certain probability that the state in the global minimum can escape to the thermodynamically less stable state in the local minimum due to the thermodynamic fluctuations. In this regard, any state can have a lifetime due to the chance of switching to other states under the thermal fluctuations. The mean first passage time can be used to describe the timescale in these kinetic switching processes.

IV. MEAN AND STATISTICAL FLUCTUATIONS OF THE KINETICS FOR THE BLACK HOLE STATE SWITCHING AND HAWKING-PAGE PHASE TRANSITION

A. Mean kinetics for the black hole state switching and Hawking-Page phase transition

In this subsection, we study the mean kinetics through the mean first passage time for the black hole state switching and Hawking-Page phase transition. Our task is to find out the time that it takes for a state starting from one local (global) stable phase to reach another global (local) stable phase. Problems of this sort can be solved by the first passage time. In general, the first passage time is defined as the time required for a state from the local (global) stable phase [described by the local (global) minimum of Gibbs free energy] to reach the transition state (an intermediate of a small black hole phase determined by the maximum of the Gibbs free energy or barrier in the present case) for the first time. Since we are considering a stochastic process caused by thermal fluctuation, the first passage time will be a random variable. Thus, we are particularly interested in the mean first passage time. The mean first passage time quantifies an average timescale for a stochastic event of switching to first occur.

First, let us consider the first passage time of a state starting from the thermal AdS phase and ending at the large black hole phase. Define $\Sigma(t)$ to be the probability that the state has not made a first passage by time t. Suppose there is a perfect absorber placed at the site r_s where the Gibbs free energy attains the local maximum. If the state makes the first passage under the thermal fluctuation, this state leaves the system. In this setup, we have made the assumption that the time taken from the small black hole phase to the large black hole phase is much smaller than the first passage time. Because of the existence of an absorber, the normalization of the probability distribution will not be preserved in this setup. According to the definition, $\Sigma(t)$ is also the probability of a state being in the system at time t. Thus, we have

$$\Sigma(t) = \int_0^{r_s} \rho(r,t)dr. \quad (11)$$

At very late times, the total probability of a state still in the system becomes zero, i.e., $\Sigma(r,t)|_{t\to+\infty} = 0$.

As claimed, the first passage time is a random variable. We denote the distribution of first passage times by $F_p(t)$. Then, the distributions $F_p(t)$ can be given by

$$F_p(t) = -\frac{d\Sigma(t)}{dt}. \quad (12)$$

It is obvious that $F_p(t)dt$ is the probability that a state passes through the intermediate small black hole phase for the first time in the time interval $(t, t+dt)$.

With this time distribution, we can calculate the mean first passage time and its fluctuation. The mean first passage time is defined by

$$\langle t\rangle = \int_0^{+\infty} tF_p(t)dt. \quad (13)$$

In general, the distribution of the first passage times can be numerically simulated from the Fokker-Planck equation by imposing the appropriate boundary conditions. Then, the mean first passage time and its fluctuations can be calculated. Here, we present the analytical expressions for the mean first passage time and its fluctuations [38]. By substituting the relations (11) and (12) between $F_p(t)$ and $\Sigma(t)$ into the definition of the mean first passage time (13) and integrating by parts, one obtains the following expression:

$$\langle t\rangle = \int_0^{+\infty}\Sigma(t)dt = \int_0^{r_s}\rho(r,s=0)dr, \tag{14}$$

where $\rho(r,s)$ is the Laplace transformation of probability distribution $\rho(r,s)$,

$$\rho(r,s) = \int_0^{+\infty}\rho(r,t)e^{-st}dt. \tag{15}$$

Using the Laplace transformation, we can rewrite the Fokker-Planck equation (10) in the form

$$s\rho(r,s) - \delta(r-r_i) = D\frac{\partial}{\partial r}\left\{e^{-\beta G(r)}\frac{\partial}{\partial r}\left[e^{\beta G(r)}\rho(r,s)\right]\right\}, \tag{16}$$

where r_i represents the order parameter of the initial phase. Because we are considering the first passage process of a state starting from the phase of the thermal AdS phase to the large black hole phase, the initial condition of the order parameter should be taken as the radius of the thermal AdS phase, i.e., $r_i = 0^+$. To proceed, we can rewrite the above equation as

$$s\rho(r,s) - \delta(r-r_i) = -\frac{\partial}{\partial r}j(r,s), \tag{17}$$

where

$$j(r,s) = -De^{-\beta G(r)}\frac{\partial}{\partial r}[e^{\beta G(r)}\rho(r,s)]. \tag{18}$$

The boundary condition at $r=0$ is set as the reflecting boundary condition, $j(0,s)=0$, while the boundary condition at $r=r_s$ is set as the absorbing boundary condition, $\rho(r_s,s)=0$. Integrating Eq. (17) from 0 to r_s and using the reflecting boundary condition gives

$$j(r,s) = -\int_0^{r_s}dr'[s\rho(r',s) - -\delta(r'-r_i)]. \tag{19}$$

Combining this expression with Eq. (18) and integrating once more yields the expression for $\rho(r,s)$ as follows:

$$\rho(r,s) = -\frac{1}{D}\int_r^{r_s}dr'\int_0^{r'}dr''[s\rho(r'',s) - \delta(r''-r_i)] \times e^{\beta(G(r'')-G(r'))}. \tag{20}$$

Taking the limit of the above equation as s goes to zero yields

$$\rho(r,s=0) = \frac{1}{D}\int_r^{r_s}dr'\int_0^{r'}dr''\delta(r''-r_i)e^{\beta(G(r'')-G(r'))}. \tag{21}$$

Substituting this equation into Eq. (14) and exchanging the order of integration, one can finally obtain the analytical integration expression for the mean first passage time,

$$\langle t\rangle = \frac{1}{D}\int_0^{r_s}dr\int_0^{r}dr'e^{\beta(G(r)-G(r'))}. \tag{22}$$

The analytical integral expressions for the mean first passage time from the large black hole phase to the thermal AdS phase can also be derived similarly as follows:

$$\langle t\rangle = \frac{1}{D}\int_{r_s}^{r_l}dr\int_r^{+\infty}dr'e^{\beta(G(r)-G(r'))}. \tag{23}$$

With these expressions, we can compute the mean first passage time via numerical integration directly without concern for the time distribution. The numerical results will be presented in Sec. IV C.

B. Statistical fluctuations of the kinetics for the black hole state switching and Hawking-Page phase transition

In principle, we can calculate the fluctuations in kinetics by the nth order moment of the time distribution function of the first passage time by the relationship

$$\langle t^n\rangle = \int_0^{+\infty}t^nF_P(t)dt. \tag{24}$$

In the present paper, we will only concentrate on the $n=2$ cases and calculate the second order fluctuation of the first passage time. By definition (24), one obtains

$$\langle t^2\rangle = 2\int_0^{r_s}dr\int_0^{+\infty}dtt\rho(r,t), \tag{25}$$

where the boundary condition $\Sigma(r,t)|_{t\to+\infty}=0$ is used in the integration by parts. According to the Laplace transformation, we have

$$\frac{\partial}{\partial s}\rho(r,s)\Big|_{s=0} = -\int_0^{+\infty}dtt\rho(r,t). \tag{26}$$

Thus, we get

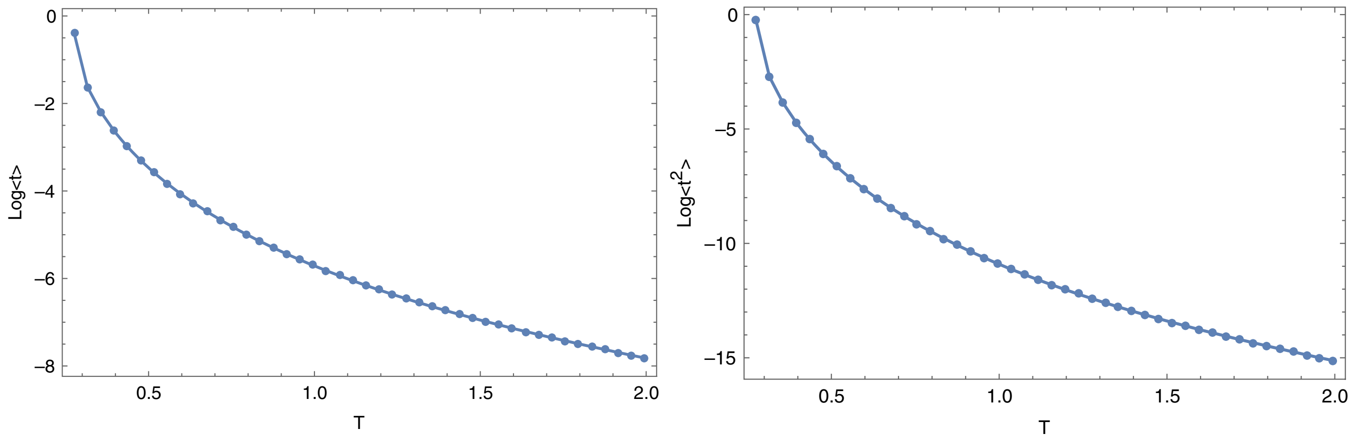

FIG. 3. Mean first passage time $\langle t\rangle$ and $\langle t^2\rangle$ from AdS space phase to the small black hole phase as a function of temperature T. The temperature range is from $T_{\min}$ to a temperature greater than the Hawking-Page critical temperature T_{HP}.

$$
\begin{aligned}
\langle t^2\rangle &= -2\int_0^{r_s} dr \frac{\partial}{\partial s}\rho(r,s)\bigg|_{s=0}\\
&= \frac{2}{D}\int_0^{r_s} dr\int_r^{r_s} dr'\int_0^{r'} dr''\rho(r,s=0)e^{\beta(G(r')-G(r))}\\
&= \frac{2}{D^2}\int_0^{r_s} dr\int_r^{r_s} dr'\int_0^{r'} dr''\int_{r''}^{r_s} dr'''\\
&\quad\times\int_0^{r'''} dr''''\delta(r''''-r_i)e^{\beta(G(r''')-G(r'')+G(r')-G(r))}.
\end{aligned}
\tag{27}
$$

Then, by exchanging the integration order, we can reach the following analytical integration expression, which can be used to compute $\langle t^2\rangle$,

$$
\begin{aligned}
\langle t^2\rangle &= \frac{2}{D^2}\int_0^{r_s} dr\int_0^{r} dr'\int_{r'}^{r_s} dr''\\
&\quad\times\int_0^{r''} dr''' e^{\beta(G(r)-G(r')+G(r'')-G(r'''))}.
\end{aligned}
\tag{28}
$$

Finally, similar analytical integral expressions for the first passage process from the large black hole phase to the thermal AdS phase can be derived, given by

$$
\begin{aligned}
\langle t^2\rangle &= \frac{2}{D^2}\int_{r_s}^{r_l} dr\int_r^{+\infty} dr'\int_{r_s}^{r'} dr''\\
&\quad\times\int_{r''}^{+\infty} dr''' e^{\beta(G(r)-G(r')+G(r'')-G(r'''))}.
\end{aligned}
\tag{29}
$$

C. Illustrations of the mean and fluctuations in kinetics for the black hole state switching and Hawking-Page phase transition

We numerically calculated the mean first passage time $\langle t\rangle$ and its second moment $\langle t^2\rangle$ of the first passage process from the thermal AdS phase to the small black hole phase by utilizing the integral expressions derived in the last two subsections. We mainly consider the temperature dependence of the kinetic switching process through the first passage times. The results are plotted in Fig. 3. Note that the vertical axis in Fig. 3 is in the logarithmic scale. It is shown that the mean first passage time $\langle t\rangle$, as well as its second moment $\langle t^2\rangle$, is a monotonically decreasing function of the temperature.

It should be noted that, although the thermodynamic stability is determined by the Gibbs free energy landscape, the kinetics is determined by the barrier height between the thermal AdS phase and the small black hole phase. The barrier height between the thermal AdS phase and the small black hole phase as a function of temperature is plotted in the left panel of Fig. 4. It is obvious that the barrier height between the thermal AdS phase and the small black hole phase is also a monotonically decreasing function of temperature.

From Fig. 3, we can see that the thermal AdS phase takes less time to switch to the large black hole phase at higher temperature. This is mainly due to the smaller barrier height from the thermal AdS phase and the small black hole phase at higher temperatures. The kinetic behavior on temperature can also be attributed to the more effective thermal diffusion process at higher temperatures. Since the mean first passage time mainly depends on the barrier height and the barrier height is continuous across the Hawking-Page critical temperature, it is shown that the kinetics is smooth across the critical temperature.

In the right panel of Fig. 4, we also plotted the correlation between the kinetics of state switching quantified by the mean first passage time and the barrier height. We can see a strong quantitative correlation between the kinetic time and the barrier height. The higher the barrier in between, the longer it takes for the thermal AdS phase to switch to the large black hole phase. Therefore, we can see that while the free energy value determines the thermal weight or the stability of the states, the free energy barrier determines the kinetics of the state switching of the black holes.

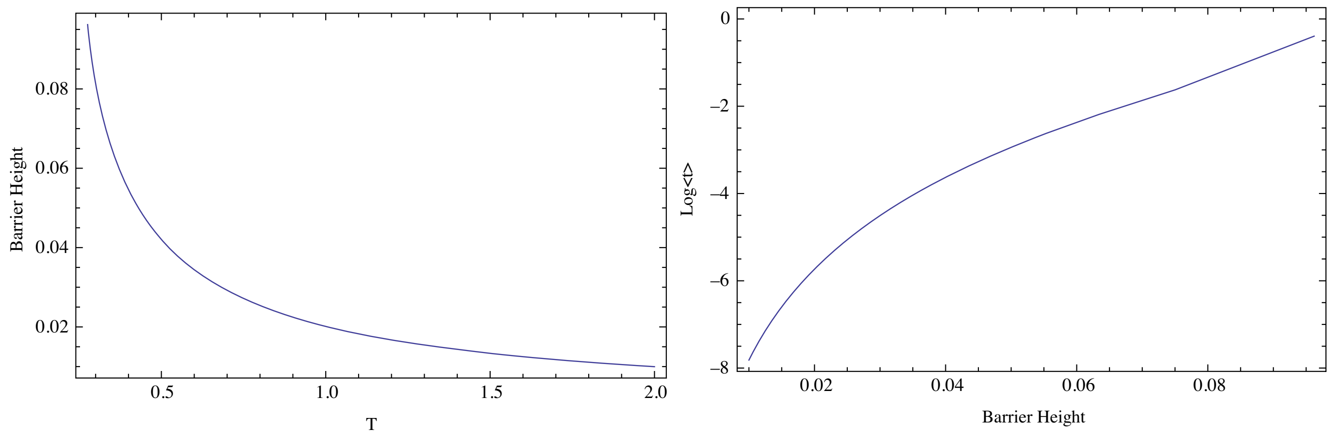

FIG. 4. Left panel: Barrier height from the thermal AdS phase to the large black hole phase as a function of temperature T. Right panel: Correlation between the mean kinetic time and the barrier height.

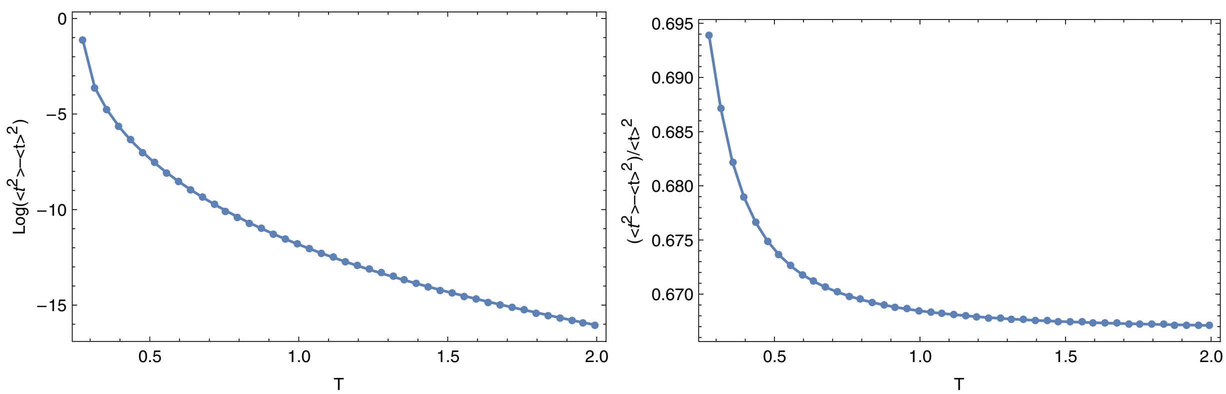

FIG. 5. Fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of first passage time from the thermal AdS phase to the large black hole phase as a function of temperature T.

The second order fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and the relative second order fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time are depicted in Fig. 5. Note that the vertical axis in the left panel of Fig. 5 is in the logarithmic scale. The fluctuation is more significant near the minimal temperature $T_{\min}$ where the unstable small black hole phase appears, and it decreases rapidly when the temperature increases. At very high temperature, the fluctuation approaches zero. The relative fluctuation is also a monotonically decreasing function of temperature. However, the changes of the relative fluctuations are not very significant in the whole temperature range. This may be due to the small changes of the corresponding barrier height. The relative fluctuation reaches a nonzero limit value at very high temperatures. We find at high temperature, $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2 \approx 2/3$, which implies $\langle t^2\rangle \approx \frac{5}{3}\langle t\rangle^2$.

Both the behaviors of the mean and fluctuations in kinetics of switching from the thermal AdS phase to the large black hole phase are determined by the free energy landscape topography through the barrier height. However, the magnitudes of the relative fluctuations in kinetics, as well as the changes in the relative fluctuations in kinetics with respect to the temperatures, are not very large. This indicates that the influence of the thermal fluctuations on the kinetics of switching from the thermal AdS phase to the large black hole phase is limited.

The behavior of the mean first passage time and its second moment from the large black hole phase to the thermal AdS phase as a function of temperature is shown in Fig. 6. In this case, the mean first passage time and its second moment become larger and larger at higher temperature. These trends for the mean first passage time and its second moment imply that it is more difficult for the large black hole phase to escape to the thermal AdS phase at higher temperatures. This is due to the fact that the barrier height from the large black hole phase to the thermal AdS phase becomes higher at higher temperatures, as shown in the left panel of Fig. 7. In the same figure, the right panel shows that the higher the barrier in between, the longer it takes for the large black hole phase to switch to the thermal

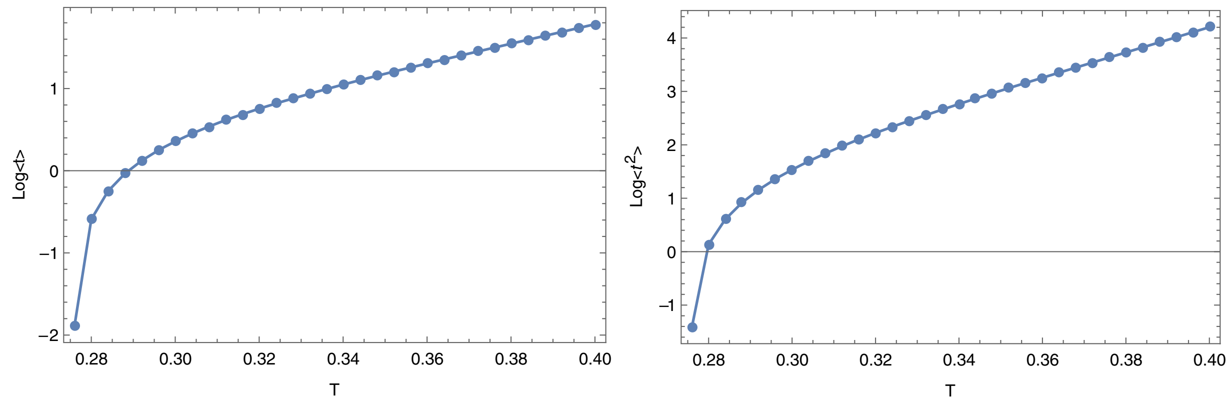

FIG. 6. Mean first passage time $\langle t\rangle$ and $\langle t^2\rangle$ from the large black hole phase to the thermal AdS phase as a function of temperature T.

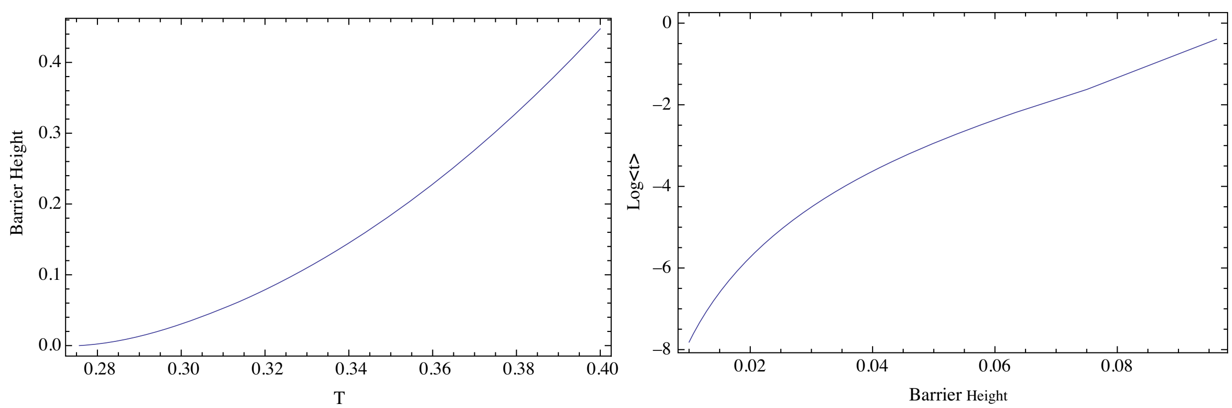

FIG. 7. Left pane: Barrier height from the large black hole phase to the thermal AdS phase as a function of temperature T. Right panel: Correlation between the mean kinetic time and the barrier height.

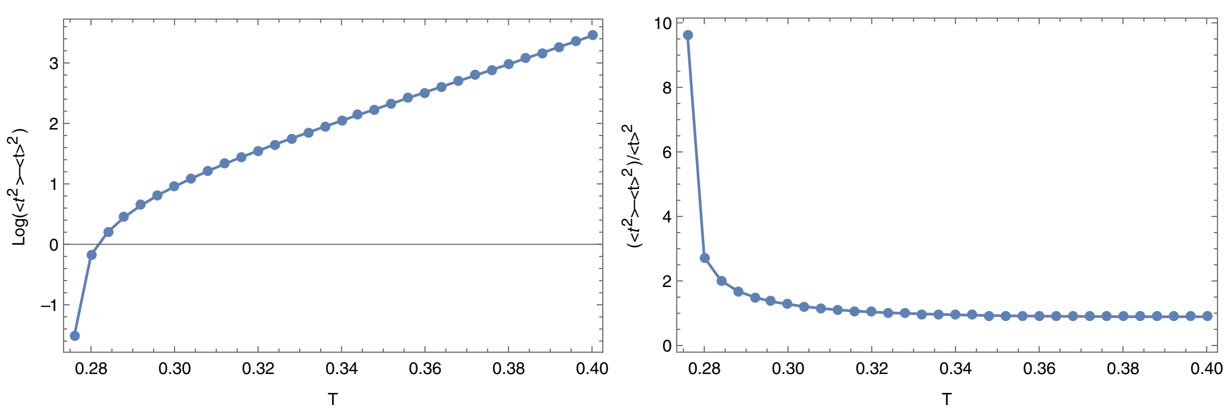

FIG. 8. Fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time from the large black hole phase to the small black hole phase as a function of temperature T.

AdS phase. Therefore, the large black hole becomes more and more stable when increasing the temperature, which is consistent with the results from the free energy landscape topography.

The fluctuation and relative fluctuation of the first passage process from the large black hole to the thermal AdS phase are depicted in Fig. 8. The fluctuation and the relative fluctuation behave differently. The fluctuation is shown to be a monotonic decreasing function of temperature. Similar to the behavior of the relative fluctuation shown in Fig. 6, it will reach a nonzero limit value at high temperature. We find, at high temperature, $\langle t^2\rangle \approx 2\langle t\rangle^2$.

This relation is dramatically different from the relation of the first passage process from the AdS space phase to the large black hole phase, which implies that the time distributions of these two processes at high temperature are different.

The relative fluctuations at low temperatures become large. This indicates the large fluctuations in kinetics. Both the behaviors of the mean and fluctuations in kinetics of switching from the large black hole phase to the thermal AdS phase are determined by the free energy landscape topography through the barrier height. The relative fluctuations in kinetics of switching from the thermal AdS phase to the large black hole phase and from the large black hole phase to the thermal AdS phase depend on how the thermal energy is compared to the barrier height. In Fig. 9, we show the barrier height divided by the temperature as a function of temperature for the AdS space to the large black hole process and that for the inverse process. The red line shows the minor changes of the barrier height divided by the temperature from the AdS space to the large black hole transition process. As a result, the relative fluctuation is not significant in the whole temperature range. The blue line shows that, at low temperatures, the barrier height divided by the temperature is almost close to zero for the process from the large black hole to the AdS space. At low temperatures, we can see that the free energy barrier from the large black hole phase to the thermal AdS phase becomes much smaller compared to the thermal energy, leading to an almost downhill kinetic process. Therefore, the thermal fluctuations will be more significant. The kinetics and associated fluctuations will be more influenced by the thermal fluctuations rather than the barrier height. Therefore, larger relative thermal fluctuations can give rise to larger kinetic fluctuations.

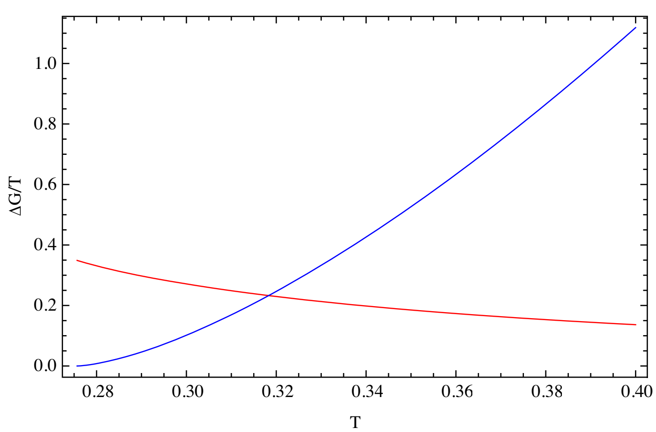

FIG. 9. Barrier height over temperature as a function of temperature for the process from the AdS space phase to the large black hole phase (red line) and the inverse process (blue line).

V. THERMODYNAMICS AND KINETICS OF HAWKING-PAGE PHASE TRANSITION IN MASSIVE GRAVITY

A linear massive gravity theory was developed by Fierz and Pauli [39] by adding interaction terms in the linearized level of general relativity, where the interaction terms were interpreted as the mass terms for the gravitons. Unfortunately, this linear massive gravity theory suffered from the van Dam-Veltman-Zakharov discontinuity problem at the linear level [40,41] and the Boulware-Deser ghost problem at the nonlinear level [42,43]. Recently, a nonlinear massive gravity theory was constructed in [44,45], in which the Boulware-Deser ghost field was eliminated by introducing higher order interaction terms into the Einstein-Hilbert action. It was found that the massive gravity theory has a nontrivial black hole solution with a negative cosmological constant [46]. This solution has been applied to investigate the holographic superconductor model in massive gravity [47,48]. A class of generalized black hole solutions in massive gravity was also constructed, and the thermodynamical properties and phase structure were studied [24] (see [23] for a discussion of the Hawking-Page phase transition in holographic massive gravity). In this section, we will discuss the Hawking-Page phase transition in massive gravity based on the free energy landscape and further study the corresponding kinetics of state switching.

A. Hawking-Page phase transition based on the free energy landscape

We start with a brief review of the recently proposed massive gravity. The action of the four-dimensional ghost-free massive gravity is given by [44–46]

$$I_G = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R + \frac{6}{L^2} + m_g^2\sum_{i=1}^{4} c_i\mathcal{U}_i(g,f)\right], \quad (30)$$

where c_i are constants and m_g is the mass of the graviton. The terms $\mathcal{U}_i(g,f)$ are defined in terms of the symmetric polynomials of the eigenvalues of the matrix $\mathcal{K}^\mu_\nu = \sqrt{g^{\mu\lambda}f_{\lambda\nu}}$, with the reference metric $f_{\mu\nu}$ being a fixed rank-2 symmetric tensor. The terms $\mathcal{U}_i(g,f)$ are explicitly given by

$$\begin{aligned}
\mathcal{U}_1 &= [\mathcal{K}],\\
\mathcal{U}_2 &= [\mathcal{K}]^2 - [\mathcal{K}^2],\\
\mathcal{U}_3 &= [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],\\
\mathcal{U}_4 &= [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}] + 3[\mathcal{K}^2]^2 - 6[\mathcal{K}^4],
\end{aligned}$$

where $\mathcal{K}^2_{\mu\nu} = \mathcal{K}_{\mu\alpha}\mathcal{K}^\alpha_\nu$, and the rectangular brackets denote traces.

The reference metric can be chosen as $f_{\mu\nu} = \mathrm{diag}(0, 0, 1, \sin^2\theta)$ in order to preserve homogeneity and isotropy on the spatial sphere (with the angular coordinates θ and φ parametrizing the spatial sphere), as well as general $r-t$ diffeomorphism invariance. The spherical symmetric black hole solution (with $c_3 = c_4 = 0$) is given by [24]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \tag{31}$$

where the metric function $f(r)$ is given by

$$f(r) = 1 + \frac{r^2}{L^2} - \frac{2M}{r} + \frac{c_1}{2}\mu^2 r + c_2\mu^2. \tag{32}$$

The vacuum solution can be obtained by setting $M = 0$, and then the corresponding metric function $f(r)$ is given by

$$f_0(r) = 1 + \frac{r^2}{L^2} + \frac{c_1}{2}\mu^2 r + c_2\mu^2. \tag{33}$$

Note that the vacuum solution is not an AdS space unless $\mu^2 = 0$.

The black hole horizon which is a null hypersurface is determined by the equation $f(r)|_{r=r_+} = 0$. Then the mass of the black hole can be expressed as

$$M = \frac{r_+}{2}\left(1 + \frac{r_+^2}{L^2} + \frac{c_1}{2}\mu^2 r_+ + c_2\mu^2\right). \tag{34}$$

The Hawking temperature of the black hole can be easily obtained from the surface gravity formula $\kappa = \frac{1}{2}f'(r)|_{r=r_+}$ as

$$T_H = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{L^2} + c_1\mu^2 r_+ + c_2\mu^2\right). \tag{35}$$

The Bekenstein-Hawking entropy is still given by the area of the horizon. Following the strategy of generalizing the on-shell free energy, the off-shell Gibbs free energy in massive gravity is then given by

$$G = M - TS = \frac{r_+}{2}\left(1 + \frac{r_+^2}{L^2} + \frac{c_1}{2}\mu^2 r_+ + c_2\mu^2\right) - \pi T r_+^2. \tag{36}$$

For simplicity, we set all the parameters c_1, c_2, and μ^2 to be positive. In this case, the vacuum solution is a regular solution without a horizon. The vacuum phase is the analog of the thermal AdS space phase in the Einstein gravity case. There is a Hawking-Page phase transition between the vacuum phase and the black hole phase [23,24]. The phase transition can be observed from the plots of the free energy landscape versus the black hole radius r_+ as depicted in Fig. 10.

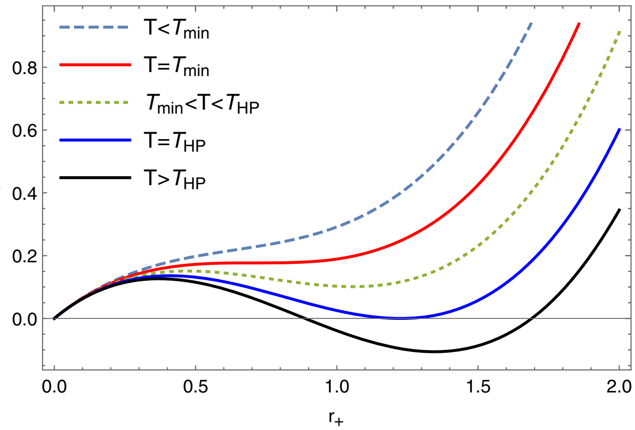

FIG. 10. Gibbs free energy landscapes for the black hole solutions in massive gravity. We take the parameters $c_1 = c_2 = 1$ and $\mu^2 = 0.5$.

From the expression of the temperature, Eq. (35), one can easily obtain the minimal temperature of the black hole, which is given by

$$T_{\min} = \frac{\sqrt{3(1 + c_2\mu^2)}}{2\pi L} + \frac{c_1\mu^2}{4\pi}. \tag{37}$$

For $T > T_{\min}$ a pair of black holes (large/small) or black hole phases can emerge, with radii given by

$$r_{l,s} = \frac{(1 + c_2\mu^2)}{2\pi}\frac{(T - \frac{c_1\mu^2}{4\pi})}{(T_{\min} - \frac{c_1\mu^2}{4\pi})^2} \times \left(1 \pm \sqrt{1 - \left(\frac{4\pi T_{\min} - c_1\mu^2}{4\pi T - c_1\mu^2}\right)^2}\right), \tag{38}$$

where l/s represents the large/small black hole phase. We can also see the small and large black hole phases from the free energy landscape with the small black hole phase corresponding to the local maximum (unstable) and the large black hole phase corresponding to the local or global minimum depending (stable) on the temperature.

The analysis of the Hawking-Page phase transition in massive gravity based on the free energy landscape is qualitatively similar to that of the phase transition in Einstein gravity. When $T < T_{\min}$, there is only one stable phase, i.e., the vacuum phase, corresponding to the global minimum of the Gibbs free energy landscape at the origin. At $T = T_{\min}$, the Gibbs free energy landscape exhibits an inflection point at $r_+ = L\sqrt{1 + c_2\mu^2}/\sqrt{3}$. When $T > T_{\min}$, there are three phases, i.e., the vacuum phase, the small black hole phase, and the large black hole phase. The stable phase is then determined by the free energy landscape. At the Hawking-Page phase transition temperature, the Gibbs free energy of the large black hole phase is degenerate with the Gibbs free energy of the vacuum phase.

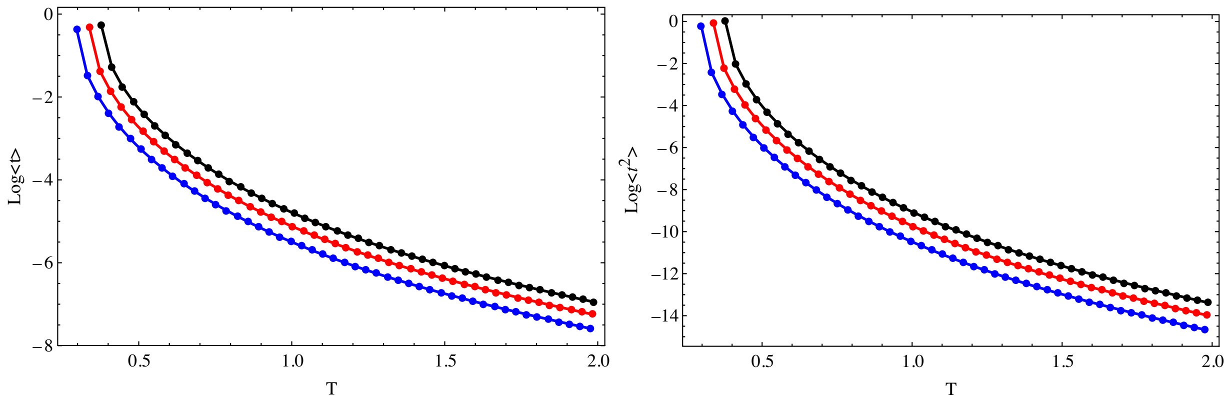

FIG. 11. Mean first passage times $\langle t\rangle$ and $\langle t^2\rangle$ from the thermal vacuum phase to the large black hole phase as functions of temperature T. The blue, red, and black lines represent the cases of $\mu^2 = 0.1$, 0.3, and 0.5, respectively.

The Hawking-Page temperature in massive gravity is then given by

$$T_{HP} = \frac{\sqrt{1 + c_2\mu^2}}{\pi L} + \frac{c_1\mu^2}{4\pi}. \tag{39}$$

When $T_{\min} < T < T_{HP}$, the globally stable phase is still the vacuum phase. For $T > T_{HP}$, the large black hole phase becomes the absolute minimum of the Gibbs free energy landscape and is the globally stable phase. At the critical temperature, both the vacuum phase and the large black hole phase are stable with equal free energy basin depths. Therefore, by analyzing the free energy landscape topography, we can conclude that there is a phase transition from the vacuum phase to the large black hole phase at the Hawking-Page critical temperature.

B. Mean and statistical fluctuations of the kinetics for state switching and Hawking-Page phase transition in massive gravity

Now we discuss the kinetics of the Hawking-Page phase transition in massive gravity. From the viewpoint of the free energy landscape, any state can have the chance to switch to other states under the thermal fluctuations. The switching process can be described by the Fokker-Planck equation on the free energy landscape, and the kinetics of the phase transition is then related to the first passage problem. The analytical expressions of the mean and statistical fluctuations of the kinetics, which are explicitly given by Eqs. (22), (23), (28), and (29), are still applicable to the present case. In the following, we will briefly present the numerical results of the mean and statistical fluctuations of the kinetics in the Hawking-Page phase transition in massive gravity.

Without loss of generality, we fix the parameters as $c_1 = c_2 = 1$. Then we have only one free parameter μ that can vary. In Fig. 11, the mean first passage times $\langle t\rangle$ and $\langle t^2\rangle$ from the thermal vacuum phase to the large black hole phase as functions of the temperature T are presented. It can be observed that, for different parameters μ, the mean first passage time $\langle t\rangle$ and its second moment $\langle t^2\rangle$ are monotonically decreasing functions of temperature. The reason behind this is that the barrier height between the vacuum phase and the small black hole phase is a monotonically decreasing function of temperature. In Fig. 10, it is shown that the Gibbs free energy of the small black hole decreases with increasing temperature, while the Gibbs free energy of the thermal vacuum solution is fixed at zero. Therefore, the effective barrier (i.e., the Gibbs free energy difference between the small black hole phase and the thermal vacuum phase) is lowered at high temperatures, which shortens the timescale of the state switching.

In Fig. 12, we show the numerical results of the fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and the relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time from the thermal vacuum phase to the large black hole phase. The fluctuation and the relative fluctuation are monotonically decreasing functions of temperature. However, the fluctuations change significantly in the whole temperature range, while the relative fluctuation does not change very significantly. At very high temperatures, although the parameter μ is different, the fluctuation approaches zero while the relative fluctuation reaches a nonzero limit value. We find that the relation between $\langle t\rangle^2$ and $\langle t^2\rangle$ at very high temperatures is still given by $\langle t^2\rangle \approx \frac{5}{3}\langle t\rangle^2$.

In Fig. 13, we present the numerical results of the mean first passage times $\langle t\rangle$ and $\langle t^2\rangle$ from the large black hole phase to the thermal vacuum phase. It is shown that the mean first passage times $\langle t\rangle$ and $\langle t^2\rangle$ are monotonically increasing functions of temperature. This is due to the fact that the barrier height between the small black hole phase and the large back hole phase increases with the temperature. As shown in Fig. 10, the Gibbs free energy of the large black hole is significantly affected by the temperature. Meanwhile, the Gibbs free energy of the small black hole is less sensitive to the temperature. Therefore, the barrier height between the small black hole phase and the large

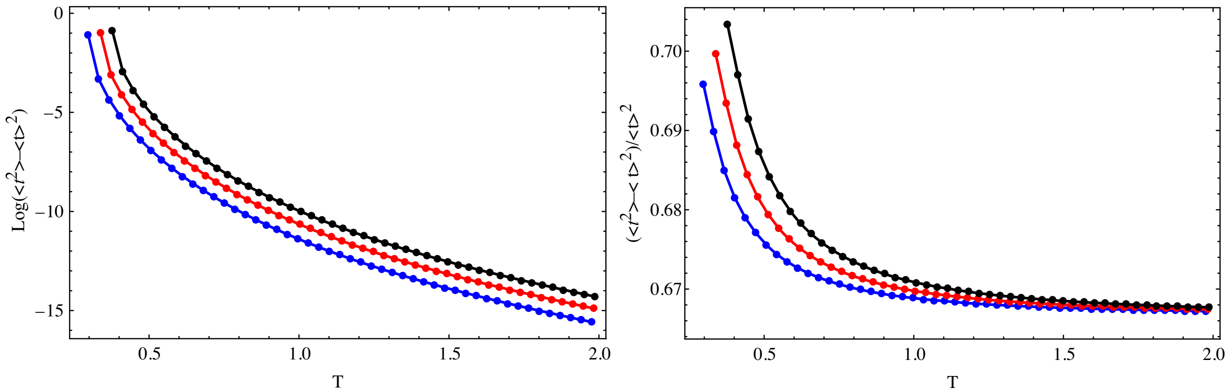

FIG. 12. Fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time from the thermal vacuum phase to the large black hole phase as functions of temperature T.

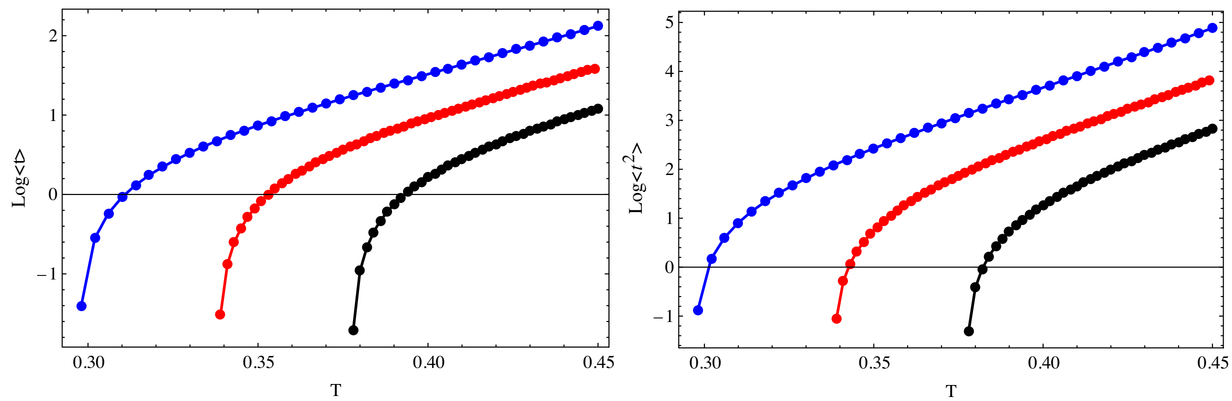

FIG. 13. Mean first passage times $\langle t\rangle$ and $\langle t^2\rangle$ from the large black hole phase to the thermal vacuum phase as functions of temperature T. The blue, red, and black lines represent the cases of $\mu^2 = 0.1$, 0.3, and 0.5, respectively.

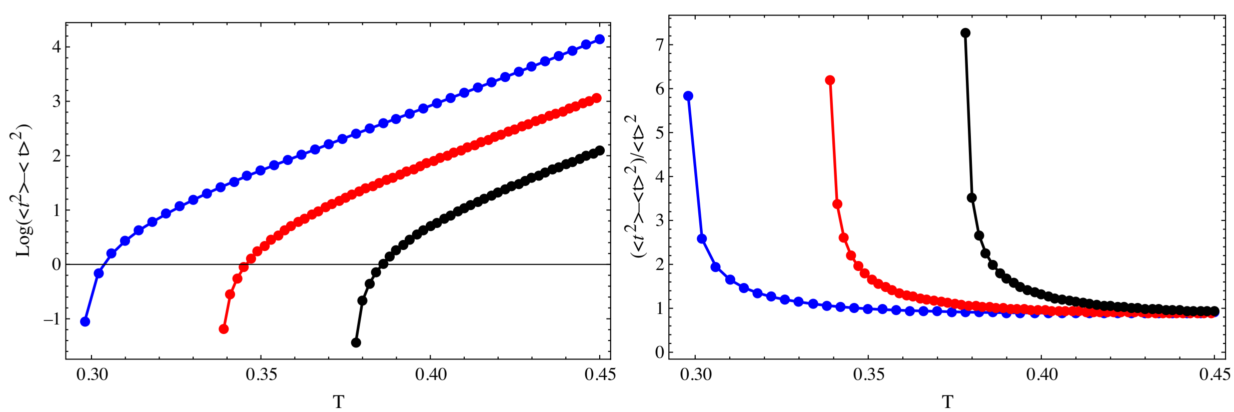

FIG. 14. Fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time from the large black hole phase to the thermal vacuum phase as functions of temperature T.

black hole phase increases with temperature, which leads to a longer timescale of the state switching.

The fluctuation $\langle t^2\rangle - \langle t\rangle^2$ and the relative fluctuation $(\langle t^2\rangle - \langle t\rangle^2)/\langle t\rangle^2$ of the first passage time from the large black hole phase to the thermal vacuum phase are shown in Fig. 14. The fluctuation is the monotonically increasing function of temperature, while the relative fluctuation is the decreasing function of temperature. Once again, it is

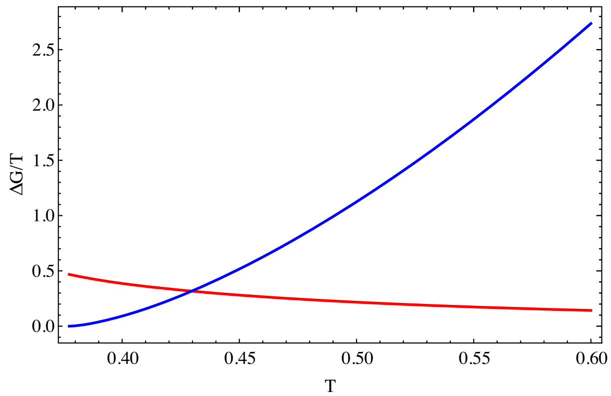

FIG. 15. Barrier height over temperature as a function of temperature for the process from the vacuum phase to the large black hole phase (red line) and the inverse process (blue line). The parameter μ^2 is set to 0.5.

observed that the relative fluctuation will reach a nonzero limit value at high temperature. The relation between $\langle t\rangle^2$ and $\langle t^2\rangle$ at very high temperatures is given by $\langle t^2\rangle \approx 2\langle t\rangle^2$. As discussed in the last section, the relative fluctuation is determined by the competition between the thermal energy and the barrier height. We illustrate the barrier height over temperature $\Delta G/T$ as a function of temperature for the process from the vacuum phase to the large black hole phase and the inverse process in Fig. 15. The red line shows that $\Delta G/T$ is not sensitive to temperature, which implies that the relative fluctuation does not change significantly in the whole temperature range for the process from the thermal vacuum phase to the large black hole phase. The blue line indicates that at low temperature the inverse process from the large black hole phase to the thermal vacuum phase is more influenced by the thermal fluctuation rather than the barrier height.

In summary, the tendencies of the first mean time, its second moment, the fluctuation, and the relative fluctuation for different parameters μ and for different first passage processes in massive gravity are all qualitatively similar to those of the Hawking-Page phase transition in Einstein gravity. This conclusion indicates that the formalism of the free energy landscape in analyzing the Hawking-Page phase transition is universal.

VI. CONCLUSION

In this paper, we have studied both the thermodynamic through the free energy landscape and the kinetic switching process of black hole states and the Hawking-Page phase transition. For Einstein gravity, we consider the Schwarzschild-AdS black holes as well as the thermal AdS state as macroscopic emergent phases. The emergence and Hawking-Page phase transitions are determined by the underlying free energy landscape. Under thermodynamic fluctuations, there is always a chance for black hole or thermal AdS phase switching from one phase to another on the free energy landscape. Any state can have a lifetime due to the chance of switching to other states. The mean first passage time, which is defined as the average timescale for a stochastic event to first occur, is related to the lifetime of the black hole phase or the thermal AdS phase. In order to calculate the mean first passage time, we quantify the free energy landscape for the Hawking-Page phase transition. The kinetics of the black hole state switching can be described by the probabilistic Fokker-Planck equation if treating this process as a stochastic event. Starting with the Fokker-Planck equation, we derive the analytical formulas for the mean first passage time and its second moment. The numerical results show that the timescales and the associated fluctuations are closely related with the topography of the Gibbs free energy landscape through barrier heights. As an additional example, we also consider the thermodynamics and the kinetics of the Hawking-Page phase transition in massive gravity. The conclusions are qualitatively similar to those obtained in Einstein gravity. This study provides a systematic way of studying the black hole thermodynamics and kinetics from the perspective of the free energy landscape.

There are still certain unsolved issues. In the present work, the effect of Hawking radiation is not taken into account. It is natural to consider the effect of Hawking radiation on the stochastic thermal dynamic phase transition between the Schwarzschild-AdS black hole and the thermal AdS phase. The second issue is how to relate the first passage time or switching time to the real lifetime or the stability of the black hole. This type of relationship can help us to establish a concrete foundation for studying the black hole phase transition from the stochastic dynamics perspective. These issues deserve future investigation.

ACKNOWLEDGMENTS

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