Thermodynamics of two aligned Kerr black holes

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We investigate the first law of thermodynamics in the stationary axisymmetric configurations composed of two Kerr black holes separated by a massless strut. Our analysis employs the recent results obtained for the extended double-Kerr solution and for thermodynamics of the static single and binary black holes. We show that, similar to the electrostatic case, in the stationary binary systems the thermodynamic length ℓ is defined by the formula $\ell = L\exp(\gamma_0)$, where L is the coordinate length of the strut, and γ_0 is the value of the metric function γ on the strut.

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I. INTRODUCTION

In the paper [1], an important notion of *thermodynamic length* was introduced which permits an elegant analytic description of thermodynamics in different *single* black-hole spacetimes. The usefulness of this notion has recently been demonstrated in [2] in application to the *binary* configurations of generic charged static black holes [3,4], for which the first law of thermodynamics has been derived in a concise form. Curiously, even in the absence of charges, when the latter binary configurations are described by the double-Schwarzschild solution [5], the work [2] gives for this special vacuum case a representation of the first law different from the one considered in the well-known paper of Costa and Perry [6]. A natural question arises then, whether the approach developed in the papers [1,2] can be further extended to the binary systems of *rotating* black holes? In the present paper we give a positive answer to this query.

Since the simplest rotating black hole is described by the Kerr vacuum solution [7], in order to accomplish our objective we can restrict ourselves to configurations of two Kerr black-hole constituents kept apart in stationary equilibrium by a massless strut [8]. Such configurations are obtainable in principle from the double-Kerr solution of Kramer and Neugebauer [9] by imposing in it the axis and asymptotic flatness conditions; these, however, were not solved analytically in the original parametrization of the paper [9] without the additional condition of the balance of sources [10] (absence of the strut), the fulfilment of which makes the equilibrium of two Kerr black holes impossible [11]. Sibgatullin's integral method [12] of constructing the exact solutions changed that unpleasant situation drastically, and thanks to it we now have at our disposal various analytical solutions for a pair of interacting Kerr black holes separated by a massless strut which are suitable for the study of the thermodynamic length in the stationary binary systems. Thus, in our paper we are going to consider the configurations of two equal counterrotating Kerr black holes [13,14], of two identical corotating Kerr black holes [15,16], and also the binary system composed of generic Kerr black holes [15,17,18]. We have decided to analyze the configurations of equal counter- and corotating black holes separately from the general case because the thermodynamical properties of these particular two-body systems were already discussed earlier in the literature [19,20], however, using exclusively the general formulas of the usual double-Kerr solution [9] restricted to the subextreme case only and not elaborating the explicit form of the particular cases; besides, the thermodynamic analysis of the corotating case in [20] is essentially based on numerical calculations. Moreover, it is likely to reexamine the case of corotating black holes because the recent paper [21] presents an erroneous study of such system due to employing some quantities characteristic of exclusively the counterrotating configuration and misinterpreting the form of the angular momentum given in [15]. The simple representations of the metrics describing the binary systems and involving the physical parameters will allow us to obtain the concise analytic expressions for the thermodynamic length in all the cases under consideration.

The plan of our paper is as follows. In Sec. II we derive the first law of thermodynamics for a pair of two equal counterrotating black holes. Two possible ways of the derivation of thermodynamic length are discussed. The binary system of identical corotating Kerr black holes is considered in Sec. III, and the binary configurations of unequal Kerr black holes are analyzed in Sec. IV. Concluding remarks are given in Sec. V.

II. TWO EQUAL COUNTERROTATING KERR BLACK HOLES

The solution describing a binary system of equal counterrotating Kerr sources is the vacuum specialization of the Bretón-Manko electrovac solution [13] constructed with the aid of Sibgatullin's method, and its physical parametrization was elaborated in the paper [14], the entire metric being defined by the line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{1}$$

with the metric coefficients f, γ and ω of the form [14]

$$\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{16R^4\sigma^4 R_+ R_- r_+ r_-}, \qquad \omega = -\frac{2\mathrm{Im}[G(\bar{A}+\bar{B})]}{A\bar{A} - B\bar{B}},\\
A &= M^2[4\sigma^2(R_+R_- + r_+r_-) + R^2(R_+r_+ + R_-r_-)] + \{(R-2M)[R(\sigma^2 - a^2) + 2M^3]\\
&\quad + 4M^2a^2\mu\}(R_+r_- + R_-r_+) - 2ia\sigma R(R-2M)(R_+r_- - R_-r_+),\\
B &= 2M\sigma R\{\sigma R(R_+ + R_- + r_+ + r_-) - [2M^2 + ia(R-2M)](R_+ - R_- - r_+ + r_-)\},\\
G &= -zB + M\sigma R\{2M[2\sigma(r_+r_- - R_+R_-) + R(R_-r_- - R_+r_+)]\\
&\quad + (R+2\sigma)[R\sigma - 2M^2 - ia(R-2M)](R_+ - r_-)\\
&\quad + (R-2\sigma)[R\sigma + 2M^2 + ia(R-2M)](R_- - r_+)\},\\
R_\pm &= \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma\right)^2}, \qquad r_\pm = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2},
\end{aligned} \tag{2}$$

where M is the mass of each black hole, a is the angular momentum per unit mass of the lower black hole ($-a$ for the upper black hole, see Fig. 1), R is the coordinate distance between the centers of black holes, while the constant quantity σ representing the half length of each black hole's horizon is given by the formula

$$\sigma = \sqrt{M^2 - a^2\mu}, \qquad \mu \equiv \frac{R-2M}{R+2M}. \tag{3}$$

We would like to emphasize that the metric (1)–(3) describes the configurations of two counterrotating black holes or hyperextreme sources. However, since our interest lies in the black-hole sector of the above solution, the parameters M, a, and R must preserve the reality of σ, which implies $\sigma^2 > 0$.

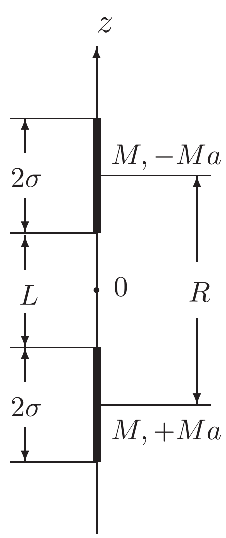

FIG. 1. Location of two equal conterrotating Kerr black holes on the symmetry axis.

It was shown in [14] that each black hole in the solution (1)–(3) satisfies the well-known Smarr mass formula [22]

$$M = \frac{1}{4\pi}\kappa\mathcal{A} + 2\Omega J, \tag{4}$$

where κ is the surface gravity, $\mathcal{A}$ the area of the horizon, Ω the lower black hole horizon's angular velocity, and J the angular momentum of the lower black hole ($-\Omega$ and $-J$ in the case of the upper black hole). Then $J = Ma$, while for $\mathcal{A}$, κ, and Ω the paper [14] gives the expressions[1]

$$\begin{aligned}
\mathcal{A} &= 8\pi M(M+\sigma)\left(1 + \frac{2M}{R}\right),\\
\kappa &= \frac{R\sigma}{2M(M+\sigma)(R+2M)}, \qquad \Omega = \frac{a\mu}{2M(M+\sigma)}.
\end{aligned} \tag{5}$$

Formulas (5) together with the expression of the interaction force [23]

$$\mathcal{F} = \frac{M^2}{R^2 - 4M^2} \tag{6}$$

permit us to elaborate the first law of thermodynamics for the binary system under consideration by following the procedure similar to the one employed in the paper [2]. Passing from the area $\mathcal{A}$ to the entropy S via $S = \frac{1}{4}\mathcal{A}$

[1]Note that in [14] the letter S was used for denoting horizon's area, but in our paper S stands for the entropy.

[24,25], we must take differentials of the quantities S, Ω and $\mathcal{F}$ by considering these as functions of the parameters (M, a, R) or parameters (M, J, R). The second option seems more simple, and to use it, one only has to change a to J/M in the expressions of Ω and σ. After having obtained the form of dS, $d\Omega$, and $d\mathcal{F}$ in terms of dM, dJ, dR, one has to solve the resulting algebraic system for dM, $d\Omega$, and dR, thus yielding the desired expression for dM in terms of dS, dJ, and $d\mathcal{F}$:

$$dM = \frac{R\sigma}{4\pi M(R+2M)(M+\sigma)}dS + \frac{J\mu}{2M^2(M+\sigma)}dJ - \frac{(R-2\sigma)(R^2-4M^2)}{2R^2}d\mathcal{F}, \tag{7}$$

so that we can introduce the temperature T by

$$T = \left.\frac{\partial M}{\partial S}\right|_{J,\mathcal{F}} = \frac{R\sigma}{4\pi M(R+2M)(M+\sigma)}, \tag{8}$$

and from (5) it follows that the above T coincides with the Hawking temperature $T = \kappa/(2\pi)$ [25]. Therefore, taking into account the equality of black holes, we finally arrive at the first law of thermodynamics for the entire system in the form

$$dM_T = 2TdS + 2\Omega dJ - \ell d\mathcal{F},$$
$$M_T = 2M, \qquad \ell = (R-2\sigma)(R^2-4M^2)/R^2. \tag{9}$$

The last term on the right-hand side in (9) determines the contribution of the conical singularity into the first law, and it represents elementary work performed by the strut. It can be easily seen that the thermodynamic length ℓ reduces in the static limit to the respective ℓ for two equal Schwarzschild black holes (see formula (4.5) of [2] in the case $m = M$).

By observing that $R-2\sigma$ is the coordinate length of the strut, and $(R^2-4M^2)/R^2$ is equal to $\exp(\gamma_0)$, γ_0 being the value of the metric function γ on the strut (the part $-\frac{1}{2}R + \sigma \le z \le \frac{1}{2}R - \sigma$ of the symmetry axis), we arrive at the remarkable conclusion that ℓ is defined by the same formula as obtained in [2] for the static case:

$$\ell = Le^{\gamma_0}, \tag{10}$$

where $L = R - 2\sigma$. Of course, this could be a mere coincidence which might be attributed to the same form of the interaction force (6) as in the case of two equal Schwarzschild black holes. So, further evidence is still needed to make sure that (10) holds generically for other stationary binary systems of black holes too.

To conclude this section, we would like to remark that the derivation of the first law (9) can be also performed directly using the parameter set (M, a, R) which enters the formulas (2)–(3). This possibility, as will be seen later, is highly important when the rotational parameter a is related in a complicated way to the angular momentum J. Since in our case J is simply Ma and hence $dJ = Mda + adM$, one will be able to arrive at the correct result for dM in (7) by substituting the differentials da by $(dJ - adM)/M$ throughout the calculations.

III. TWO EQUAL COROTATING KERR BLACK HOLES

The case of two identical corotating Kerr black holes is described by the exact solution worked out in two different representations in the papers [15,16]. The representation involving physical parameters, which is of interest to us for our purposes, is defined by the formulas

$$f = \frac{A\bar{A} - B\bar{B}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{K_0^2 R_+ R_- r_+ r_-},$$
$$\omega = 4a - \frac{2\mathrm{Im}[G(\bar{A}+\bar{B})]}{A\bar{A} - B\bar{B}},$$
$$A = R^2(R_+ - R_-)(r_+ - r_-) - 4\sigma^2(R_+ - r_+)(R_- - r_-),$$
$$B = 2R\sigma[(R+2\sigma)(R_- - r_+) - (R-2\sigma)(R_+ - r_-)],$$
$$G = -zB + R\sigma[2R(R_- r_- - R_+ r_+) + 4\sigma(R_+ R_- - r_+ r_-) - (R^2 - 4\sigma^2)(R_+ - R_- - r_+ + r_-)],$$
$$R_\pm = \frac{-M(\pm 2\sigma + R) + id}{2M^2 + (R+2ia)(\pm\sigma + ia)}\sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma\right)^2},$$
$$r_\pm = \frac{-M(\pm 2\sigma - R) + id}{2M^2 - (R-2ia)(\pm\sigma + ia)}\sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma\right)^2},$$
$$K_0 = \frac{4\sigma^2[(R^2 + 2MR + 4a^2)^2 - 16M^2a^2]}{M^2[(R+2M)^2 + 4a^2]}, \tag{11}$$

where the constant quantities σ and d have the form

$$\sigma = \sqrt{M^2 - a^2 + d^2(R^2 - 4M^2 + 4a^2)^{-1}},$$
$$d = \frac{2Ma(R^2 - 4M^2 + 4a^2)}{R^2 + 2MR + 4a^2}. \tag{12}$$

Like in the previous case of counterrotating black holes, the parameters M and R denote, respectively, the mass of each black hole and the coordinate distance between the centers of black holes (see Fig. 2); however, now the rotational parameter a is not equal exactly to the angular momentum J of the black hole per unit mass M, but its relation to J is determined by the following cubic equation:

$$J = \frac{Ma[(R+2M)^2 + 4a^2]}{R^2 + 2MR + 4a^2}. \tag{13}$$

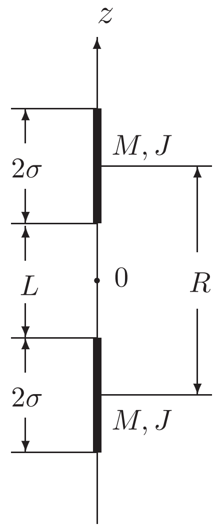

FIG. 2. Location of two identical corotating Kerr black holes on the symmetry axis.

The black-hole sector of the metric (11)–(12) corresponds to the real-valued σ, while the hyperextreme Kerr sources, which are of no interest to us in this paper, are described by the pure imaginary σ.

Each black hole in the binary system verifies identically the Smarr formula (4), and the known thermodynamical characteristics which we will need for the derivation of the first law of thermodynamics are written down below:

$$S = \frac{2\pi M[(R+2M)^2+4a^2][(R+2M)(M+\sigma)-2a^2]}{(R+2\sigma)(R^2+2MR+4a^2)},$$
$$T = \frac{\sigma(R+2\sigma)(R^2+2MR+4a^2)}{4\pi M[(R+2M)^2+4a^2][(R+2M)(M+\sigma)-2a^2]},$$
$$\Omega = \frac{(M-\sigma)(R^2+2MR+4a^2)}{2Ma[(R+2M)^2+4a^2]},$$
$$\mathcal{F} = \frac{M^2[(R+2M)^2-4a^2]}{(R^2-4M^2+4a^2)[(R+2M)^2+4a^2]}, \tag{14}$$

where we have given the expression of the entropy S instead of the horizon area $\mathcal{A}$,[2] and the temperature T instead of the surface gravity κ.

The parameter set that we must employ during the calculations is (M, a, R) which does not include explicitly the angular momentum J. Therefore, we have to follow the procedure outlined at the end of the previous section, i.e., we should treat the quantities S, Ω, and $\mathcal{F}$ as functions of M, a, and R, and after taking differentials dS, $d\Omega$, and $d\mathcal{F}$ we must change the differential da to the combination of the differentials dJ, dM, and dR via the formula obtainable from (13), namely,

$$\begin{aligned} da = {} & \frac{1}{M[R(R+2M)^3+8a^2(R^2+MR-2M^2+2a^2)]} \\ & \times \{(R^2+2MR+4a^2)^2 dJ \\ & - a[(R+2M)^2(R^2+4MR+8a^2) \\ & + 16a^2(m^2+a^2)]dM \\ & + 2M^2 a[(R+2M^2)^2-4a^2]dR\}. \end{aligned} \tag{15}$$

Then it only remains to solve the system of three algebraic equations for dM, $d\Omega$, and dR, and the expression for dM multiplied by 2 (due to equality of black holes) finally provides us with the first law of thermodynamics for the binary configuration of two identical corotating Kerr black holes:

$$\begin{aligned} dM_T &= 2TdS + 2\Omega dJ - \ell d\mathcal{F}, \\ M_T &= 2M, \\ \ell &= \frac{(R-2\sigma)(R^2-4M^2+4a^2)[(R+2M)^2+4a^2]}{(R^2+2MR+4a^2)^2-16M^2a^2}, \end{aligned} \tag{16}$$

with the coefficients T and Ω defined by (14). One can see that the first law (16) has the same structure as in (9). Although the thermodynamic length ℓ in (16) has a more complicated form than the respective ℓ in (9), it is still not difficult to verify that the new ℓ obeys formula (10) too: the coordinate length L of the strut is the same as in the previous "counterrotating" case ($L = R - 2\sigma$) and the value of $\exp(\gamma_0)$ calculated with the aid of formulas (11)–(12) coincides with ℓ/L in (16).

We now turn to the general case of rotating Kerr black holes.

IV. TWO GENERIC KERR BLACK HOLES

The general solution describing a system of two aligned Kerr black holes separated by a massless strut is defined by the formulas [15,17,18]

[2]There is a misprint in the formula (26) of [15] for horizon's area: the last term in the numerator must read $-2a^2$.

$$
\begin{aligned}
f &= \frac{A\bar{A} - B\bar{B}}{(A+B)(\bar{A}+\bar{B})}, \qquad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{16|\sigma_1|^2|\sigma_2|^2 K_0^2 \tilde{R}_+\tilde{R}_-\tilde{r}_+\tilde{r}_-}, \qquad \omega = 2a - \frac{2\mathrm{Im}[G(\bar{A}+\bar{B})]}{A\bar{A} - B\bar{B}},\\
A &= [R^2 - (\sigma_1+\sigma_2)^2](R_+ - R_-)(r_+ - r_-) - 4\sigma_1\sigma_2(R_+ - r_-)(R_- - r_+),\\
B &= 2\sigma_1(R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+) + 2\sigma_2(R^2 + \sigma_1^2 - \sigma_2^2)(r_- - r_+)\\
&\quad + 4R\sigma_1\sigma_2(R_+ + R_- - r_+ - r_-),\\
G &= -zB + \sigma_1(R^2 - \sigma_1^2 + \sigma_2^2)(R_- - R_+)(r_+ + r_- + R)\\
&\quad + \sigma_2(R^2 + \sigma_1^2 - \sigma_2^2)(r_- - r_+)(R_+ + R_- - R)\\
&\quad - 2\sigma_1\sigma_2\{2R[r_+r_- - R_+R_- - \sigma_1(r_- - r_+) + \sigma_2(R_- - R_+)]\\
&\quad + (\sigma_1^2 - \sigma_2^2)(r_+ + r_- - R_+ - R_-)\},\\
r_\pm &= \mu_0^{-1}\frac{(\pm\sigma_1 - m_1 - ia_1)[(R+M)^2 + a^2] + 2a_1[m_1 a + iM(R+M)]}{(\pm\sigma_1 - m_1 + ia_1)[(R+M)^2 + a^2] + 2a_1[m_1 a - iM(R+M)]}\tilde{r}_\pm,\\
R_\pm &= -\mu_0\frac{(\pm\sigma_2 + m_2 - ia_2)[(R+M)^2 + a^2] - 2a_2[m_2 a - iM(R+M)]}{(\pm\sigma_2 + m_2 + ia_2)[(R+M)^2 + a^2] - 2a_2[m_2 a + iM(R+M)]}\tilde{R}_\pm,\\
\tilde{r}_\pm &= \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm \sigma_1\right)^2}, \qquad \tilde{R}_\pm = \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma_2\right)^2},
\end{aligned}
\tag{17}
$$

where the constants K_0 and μ_0 have the form[3]

$$
\begin{aligned}
K_0 &= \frac{[(R+M)^2 + a^2][R^2 - (m_1 - m_2)^2 + a^2] - 8m_1m_2a^2}{m_1m_2[(R+M)^2 + a^2]},\\
\mu_0 &= \frac{R+M-ia}{R+M+ia},
\end{aligned}
\tag{18}
$$

and the quantities σ_1 and σ_2 representing the half lengths of the horizons of black holes are given by the expressions

$$
\begin{aligned}
\sigma_1 &= \sqrt{m_1^2 - a_1^2 + 4m_2a_1d_1}\\
\sigma_2 &= \sqrt{m_2^2 - a_2^2 + 4m_1a_2d_2},\\
d_1 &= \frac{[m_1(a_1 - a_2 + a) + Ra_1][(R+M)^2 + a^2] + m_2a_1a^2}{[(R+M)^2 + a^2]^2},\\
d_2 &= \frac{[m_2(a_2 - a_1 + a) + Ra_2][(R+M)^2 + a^2] + m_1a_2a^2}{[(R+M)^2 + a^2]^2}.
\end{aligned}
\tag{19}
$$

The arbitrary real parameters of the metric (17)–(19) are m_1, m_2, a_1, a_2, and R, five in total, and the upper black hole has mass m_1 and angular momentum per unit mass a_1, while the lower black hole is endowed with mass m_2 and angular momentum per unit mass a_2, so that $a_1 = j_1/m_1$, $a_2 = j_2/m_2$, j_1 and j_2 being angular momenta of the upper and lower black hole, respectively (see Fig. 3); we note that these masses and angular momenta are Komar quantities [26]. As usual, the parameter R denotes the coordinate distance between the centers of black holes. The total mass M and total angular momentum J of the binary system have the form

$$
M = m_1 + m_2, \qquad J = m_1a_1 + m_2a_2, \tag{20}
$$

and a is related to the aforementioned five parameters by the cubic equation

$$
(R^2 - M^2 + a^2)(a_1 + a_2 - a) + 2(R+M)(J - Ma) = 0, \tag{21}
$$

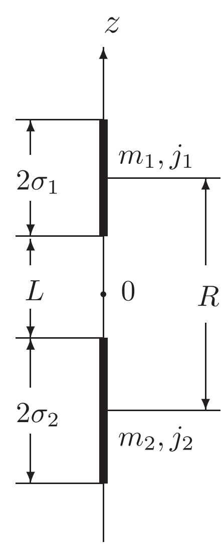

FIG. 3. Location of two unequal Kerr black holes on the symmetry axis. The coordinate length of the strut L is equal to $R - \sigma_1 - \sigma_2$.

[3]We have rectified misprints in the formula (13) of [17] for K_0 and formulas (5) of [17] for d_1 and d_2.

so that its role is similar to that of a from the previous section. Note that the particular case of equal counterrotating black holes follows from the general formulas by setting $m_1 = m_2 = M$, $a_2 = -a_1 = \alpha$, $a = 0$, while the corotating case of equal black holes corresponds to the parameter choice $m_1 = m_2 = M$, $a_1 = a_2 = \alpha$, with a formal redefinition $a \to 2a$ and changing α to a by means of (21).

For the entropies (S_1, S_2), temperatures (T_1, T_2), horizon's angular velocities (Ω_1, Ω_2) of each black hole, and the interaction force $\mathcal{F}$ we have the expressions

$$\begin{aligned}
\frac{S_1}{\pi} &= \frac{\sigma_1}{2\pi T_1} = \frac{\{(m_1+\sigma_1)[(R+M)^2+a^2] - 2m_1a_1a\}^2 + a_1^2(R^2-M^2+a^2)^2}{[(R+M)^2+a^2][(R+\sigma_1)^2-\sigma_2^2]},\\
\frac{S_2}{\pi} &= \frac{\sigma_2}{2\pi T_2} = \frac{\{(m_2+\sigma_2)[(R+M)^2+a^2] - 2m_2a_2a\}^2 + a_2^2(R^2-M^2+a^2)^2}{[(R+M)^2+a^2][(R+\sigma_2)^2-\sigma_1^2]},\\
\Omega_1 &= \frac{m_1-\sigma_1}{2m_1a_1}, \qquad \Omega_2 = \frac{m_2-\sigma_2}{2m_2a_2},\\
\mathcal{F} &= \frac{m_1m_2[(R+M)^2-a^2]}{(R^2-M^2+a^2)[(R+M)^2+a^2]},
\end{aligned} \tag{22}$$

that must now be cleverly used for the derivation of the first law of thermodynamics. To avoid the resolution of the cubic equation (21), it appears that the best strategy to tackle the derivation problem is to work with the parameter set $\{m_1, m_2, j_1, R, a\}$, for which purpose it is necessary first to change a_1 and a_2 to j_1/m_1 and j_2/m_2 in the formulas (22), as well as in the expressions for σ_1 and σ_2. Then we have to solve equation (21) for j_2, yielding

$$j_2 = \frac{m_2\{m_1a[(R+M)^2+a^2] - j_1[(R+m_1)^2-m_2^2+a^2]\}}{m_1[(R+m_2)^2-m_1^2+a^2]}, \tag{23}$$

and make another substitution in the formulas involved, this time changing j_2 by means of (23). As a result, we have the necessary formulas rewritten in the desired parameter set and can proceed in a standard way. We must take differentials of the quantities S_1, S_2, Ω_1, Ω_2 and $\mathcal{F}$ by considering these as functions of m_1, m_2, j_1, R, a and changing the differentials da to dj_2 by means of (23). The resulting system of five algebraic equations must be solved for dm_1, dm_2, $d\Omega_1$, $d\Omega_2$, and dR, thus giving us the first law of thermodynamics as the sum of dm_1 and dm_2:

$$\begin{aligned}
dM &= T_1dS_1 + T_2dS_2 + \Omega_1dj_1 + \Omega_2dj_2 - \ell d\mathcal{F},\\
M &= m_1 + m_2,\\
\ell &= \frac{(R-\sigma_1-\sigma_2)(R^2-M^2+a^2)[(R+M)^2+a^2]}{(R^2+MR+a^2)^2-(m_1-m_2)^2(R+M)^2-4m_1m_2a^2}.
\end{aligned} \tag{24}$$

Therefore, as it follows from (24), the structure of the thermodynamic length ℓ in the general case remains the same as in two previous particular cases—it is determined by the formula (10) because the coordinate length of the strut for the unequal black holes is $L = R - \sigma_1 - \sigma_2$, and the part ℓ/L in (24) coincides exactly with the corresponding value of $\exp(\gamma_0)$, as can be easily verified. It is quite surprising that the very cumbersome intermediate calculations have eventually led us to an elegant final result for ℓ proving the universal character of the formula (10).

V. CONCLUDING REMARKS

In the present paper we have succeeded in extending the notion of thermodynamic length ℓ further to binary stationary systems of black holes and found the explicit concise form of ℓ in the case of three different binary configurations. The thermodynamic length permits one to derive analytically the first law of thermodynamics in a consistent way, and the physical parametrization of the solutions describing the systems of black holes simplifies considerably the derivation procedure. It is remarkable that ℓ in the stationary case turns out to be determined by the same formula (10) as in the static vacuum and electrostatic cases, and it admits the same geometrical interpretation as given in [2]—the area of the worldsheet of the strut per unit time. This suggests in particular that most probably the notion of thermodynamic length is also applicable to the stationary electrovac configurations of black holes as well. At least this is certainly true in the case of the Bretón-Manko solution [13] for two equal counterrotating Kerr-Newman black holes [27]. Using the physical parametrization of this solution obtained in [28] it can be actually shown that the corresponding first law of thermodynamics and the thermodynamic length have the form

$$\begin{aligned}
dM_T &= 2TdS + 2\Omega dJ + 2\Phi dQ - \ell d\mathcal{F},\\
M_T &= 2M \qquad \ell = (R-2\sigma)(R^2-4M^2+4Q^2)/R^2,
\end{aligned} \tag{25}$$

(Q is the charge and Φ the electric potential) and ℓ verifies formula (10). In the absence of rotation, one recovers the result obtained for ℓ in [2]. We are going to consider the thermodynamics of rotating charged binary black holes in a separate publication.

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