New anisotropic sudden singularities and dimensional reduction

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We demonstrate the existence of sudden finite-time singularities, with constant scale factor, expansion rate, and density, in expanding Bianchi type-IX universes with free anisotropic pressures. A new type of nonsimultaneous anisotropic sudden singularity arises because of the divergences of the pressures, which may be of barrel or pancake type. The effect of one or more directions of expansion hitting a sudden singularity is tantamount to dimensional reductions as the nonsingular directions continue expanding and can see the sudden singularity in their past.

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I. INTRODUCTION

Sudden cosmological singularities, first introduced in Ref. [1] and developed systematically in Refs. [2–8] and reviewed in Ref. [9], have attracted widespread interest. They appear in a wide range of gravity theories and solutions. They typically occur when the pressure and scale factor acceleration diverge at a finite time t_s, while the scale factor, density, and Hubble expansion rate remain finite. Thus, all terms in the Friedman equation, or its equivalent in other theories of gravity, remain finite while finite-time singularities occur in the acceleration and conservation equations.

Sudden singularities and their generalized counterparts [3] are weak singularities in the senses of Tipler [10] and Krolak [11], and their conformal diagrams have been constructed in Ref. [12]. Geodesics are unscathed by sudden singularities [13], and the general behavior of the Einstein and geodesic equations in their neighborhood was found in Refs. [5,6,14]. This behavior appears robust in the presence of quantum particle production [7]. The first examples were existence proofs that required unmotivated pressure-density relations so that the density could remain finite while the pressure diverged. However, more recently, generalized singularities of this sort have been found by Barrow and Graham [15] and appear in simple isotropic Friedman universes with a scalar field having power-law self-interaction potentials for a scalar field ϕ of the form $V(\phi) = V_0\phi^n$, $0 < n < 1$. They always develop a finite-time singularity where the Hubble rate and its first derivative are finite, but its second derivative diverges. For noninteger $n > 1$, there is a class of models with even weaker singularities. Infinities first occur at a finite time in the $(k+2)^{th}$ time derivative of the Hubble expansion rate, where $k < n < k+1$ and k is a positive integer [15]. These models inflate, but inflation ends in a singular fashion.

In this paper, we study a new effect in anisotropic cosmological models experiencing nonsimultaneous sudden singularities in all, or some, of their directional scale factors. This can create a form of dimensional reduction in which some directional scale factors experience sudden singularities, while others do not. Those that do not experience the singularities continue expanding as if in a lower-dimensional universe. We use the Bianchi type-IX "mixmaster" universe expanding away from the initial strong curvature at $t = 0$ to illustrate this point and derive the general forms of the evolution of the three expansion scale factors.

In what follows, we set $c = 1 = 8\pi G$. Planck's constant does not appear, and our study is entirely classical. Quantum features can be studied using our paper Ref. [7].

II. THE MIXMASTER MODEL EQUATIONS

The spatially homogeneous diagonal Bianchi IX metric is [16]

$$ds^2 = dt^2 - \gamma_{ab}(t)e^a_\mu e^b_\nu dx^\mu dx^\nu, \tag{1}$$

where

$$\gamma_{ab}(t) = \mathrm{diag}[a^2(t), b^2(t), c^2(t)], \tag{2}$$

and

$$e^a_\mu = \begin{pmatrix} \cos z & \sin z \sin x & 0 \\ -\sin z & \cos z \sin x & 0 \\ 0 & \cos x & 1 \end{pmatrix}. \tag{3}$$

The general relativistic field equations in vacuum Bianchi type IX with scale factors $a(t)$, $b(t)$, $c(t)$, and matter with density ρ and anisotropic pressures p_1, p_2, and

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p_3 [16–18] are most simply expressed by introducing the τ time defined in terms of the comoving proper time t by

$$d\tau = \frac{dt}{abc}, \tag{4}$$

and $'$ denotes $d/d\tau$. The field equations are

$$(\ln a^2)'' + a^4 - (b^2 - c^2)^2 = a^2b^2c^2(\rho - p_a), \tag{5}$$

and their two cyclic permutations obtained under the transform $a \to b \to c \to a$, together with $p_a \to p_b \to p_c$. A first integral, the mixmaster Friedman-like equation, exists and is

$$4\left(\frac{a'b'}{ab} + \frac{b'c'}{bc} + \frac{a'c'}{ac}\right) = a^4 + b^4 + c^4 - 2a^2b^2 - 2c^2(b^2 + a^2) + 4a^2b^2c^2\rho. \tag{6}$$

We see that when $a = b = c$, it reduces to

$$12\frac{a'^2}{a^2} = 4a^6\rho - 3a^4.$$

Restoring the cosmic time derivative $a^3 da/dt = da/d\tau$, we have (overdot denotes d/dt) the standard closed isotropic universe's Friedman equation after the coordinate transform $a \to \frac{a}{2}$:

$$3\frac{\dot{a}^2}{a^2} = \frac{\rho}{3} - \frac{1}{a^2}.$$

When all the quartic terms are dropped in Eq. (5), then they are just like the Bianchi I equations with a curvature term that will dominate the matter at late times so long as $p_i > -\rho/3$. We looked at the flat Bianchi I and VII_0 models in Ref. [4].

III. THE SUDDEN SINGULARITY SCALE FACTOR EVOLUTIONS

In order to establish the existence of anisotropic sudden singularities in the mixmaster metric at late time, we look for the following forms for the scale factors on approach to a finite-time singularity s $t \to t_s$ from below. The scale factors, their first time derivatives, and the density ρ will be assumed to be finite at t_s, but second derivatives of the scale factors, first derivatives of the density, and the principal pressures will be allowed to diverge. Thus, we assume that the asymptotic forms of the scale factors as $t \to t_s$ have the form that we know is part of the general solution of the Einstein equations [5,6]. So, all terms in Eq. (6) will be finite, and Eq. (5) reduce asymptotically to

$$(\ln a^2)'' = -a^2b^2c^2p_a = -C_a p_a, \tag{7}$$

and cyclic. In this $t \to t_s$ limit, we also have

$$a'' \to \ddot{a}a^2b^2c^2, \tag{8}$$

and cyclic, and so we have the simple system of asymptotic equations:

$$(\ln a^2)^{\cdot\cdot} \to -p_a. \tag{9}$$

Therefore, explicitly in the limit $t \to t_s$, we have

$$\left(\frac{\ddot{a}}{a}, \frac{\ddot{b}}{b}, \frac{\ddot{c}}{c}\right) \to -\frac{1}{2}(p_a, p_b, p_c). \tag{10}$$

However, we want to allow the sudden singularity to arise at different times for the motions in the directions of the different scale factors, so we introduce three sudden singularity times t_{sa}, t_{sb}, and $t_{sc} > 0$, thus,[1]

$$a(t) = \left(\frac{t}{t_{sa}}\right)^{q_a}(a_{sa} - 1) + 1 - \left(1 - \frac{t}{t_{sa}}\right)^{n_a}, \tag{11}$$

$$b(t) = \left(\frac{t}{t_{sb}}\right)^{q_b}(a_{sb} - 1) + 1 - \left(1 - \frac{t}{t_{sb}}\right)^{n_b}, \tag{12}$$

$$c(t) = \left(\frac{t}{t_{sc}}\right)^{q_c}(a_{sc} - 1) + 1 - \left(1 - \frac{t}{t_{sc}}\right)^{n_c}. \tag{13}$$

Here, the constants $0 < q_a, q_b, q_c < 1$ and $1 < n_a, n_b, n_c < 2$. Therefore,

$$\ddot{a} = \frac{q_a(q_a - 1)}{t_{sa}^2}\left(\frac{t}{t_{sa}}\right)^{q_a - 2} - n_a(n_a - 1)(t_{sa} - t)^{n_a - 2}, \tag{14}$$

and the forms for $\ddot{b}(t)$ and $\ddot{c}(t)$ are given by in the same form after substitution of the $q's$ and $n's$. The values of $\ddot{a}, \ddot{b}(t)$, and $\ddot{c}(t)$ can each diverge if the values of n_a, n_b, n_c

[1]Technically, we can create a slightly more general but rather cumbersome form by including powers of logarithms. Thus, for $a(t)$, we would have

$$a(t) = \left(\frac{t}{t_{sa}}\right)^{q_a}(a_{sa} - 1) + 1 - \left(1 - \frac{t}{t_{sa}}\right)^{n_a}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{N_j} a_{jk}(t_s - t)^{j/Q}(\log^k[t_s - t])\right\},$$

where $N_j \le j$ is a positive integer and Q is a positive rational. For the corresponding expressions for $b(t)$ and $c(t)$, we replace a_{jk},N_j, and Q by different independent constants satisfying the same inequalities.

are less than 2 and greater than 1, as assumed. This will create complementary divergences in the values of the pressures because of Eq. (10). When $n_a = n_b = n_c$, this is similar to the behavior in the Friedman model first described in [3]. However, it is possible to create new types of anisotropic sudden singularity in acceleration and associated principal pressure by making $n_a \neq n_b \neq n_c$, or allow some directions to avoid a sudden singularity while others experience it. The possibilities are as follows:

(1) $1 < n_a, n_b, n_c < 2$: Sudden singularities in all directions and their associated principal pressures at different times if t_{sa}, t_{sb}, and t_{sc} are unequal, simultaneously if they are equal.
(2) $1 < n_a, n_b < 2$, and $n_c > 2$ and similar for the other two permutations : sudden singularities in two directions and their associated principal pressures (but not in the third).
(3) $1 < n_a < 2$ and $n_b, n_c > 2$ and similar for the other two permutations: sudden singularity in one direction and its principal pressure (but not in the third).

For example, in the second case (2), we have a nonsimultaneous sudden singularity in the a and b directions, with

$$a(t) = \left(\frac{t}{t_{sa}}\right)^{q_a}(a_{sa} - 1) + 1 \to a_{sa}, \tag{15}$$

$$b(t) = \left(\frac{t}{t_{sa}}\right)^{qb}(a_{sb} - 1) + 1 \to a_{sb}, \tag{16}$$

while the expansion parallel to the c direction continues with $n_c > 2$, and there is no singularity in $c(t)$ as $t \to t_{sc}$, and so it continues past the singularities affecting particles moving parallel to the a and b directions for $t > t_{sc}$,

$$\begin{aligned} c(t) &= \left(\frac{t}{t_{sc}}\right)^{q_c}(a_{sc} - 1) + 1 - \left(1 - \frac{t}{t_{sc}}\right)^{n_c} \\ &\to \left(\frac{t}{t_{sc}}\right)^{q_c}(a_{sc} - 1) + 1 - \left(1 - \frac{t}{t_{sc}}\right)^{n_c}. \end{aligned} \tag{17}$$

The relative values of q_c and n_c determine this evolution. Typically, we expect $n_c > q_c$, and so

$$\begin{aligned} c(t) &\to \left(\frac{t}{t_{sc}}\right)^{q_c}(a_{sc} - 1) + 1 - \left(1 - \frac{t}{t_{sc}}\right)^{n_c} \\ &\to 1 - \left(\frac{t}{t_{sc}}\right)^{n_c}(-1)^{n_c}. \end{aligned} \tag{18}$$

For example, for odd n_c, we have for $t > t_{sc}$,

$$c(t) \to \left(\frac{t}{t_{sc}}\right)^{n_c}. \tag{19}$$

For even n_c, we have for $t > t_{sc}$, from (17),

$$c(t) \to \left(\frac{t}{t_{sc}}\right)^{q_c}(a_{sc} - 1) - 1 + \left(\frac{t}{t_{sc}}\right)^{n_c}, \tag{20}$$

and the possibility of a switch between the two asymptotic time dependences.

IV. DIMENSIONAL REDUCTION AND ITS PHYSICAL INTERPRETATION

The possibility of sudden singularities occurring anisotropically at different times is a new feature of this phenomenon. There are no strong curvature singularities associated with any of the sudden singularities, and we expect geodesics to be unscathed by the experience unless the underlying expansion anisotropy contributed strong tidal forces [13]. The appearance of finite-time singularities for the motion of only some of the expansion scale factors is created by the anisotropic pressures. It means that particles moving in the singular directions will hit the pancake or barrel-like sudden singularity, leaving those moving in the directions orthogonal to them unscathed. This has an interesting consequence. Suppose that we repeated our calculations for anisotropic cosmological models with many space dimensions N, then we might have S of those dimensions experiencing sudden singularities (not necessarily all at the same time), eventually leaving $N - S$ to continue expanding. In effect, this is a cosmological dimensional reduction process. If $N - S = 3$, then we would be left with a three-dimensional expanding space. Observers in that space could in principle look back down their past light cones and see consequences of the sudden singularities in the other nonevolving dimensions. This will have consequences for the constants of nature. If the true constants are defined in the N-dimensional space, then in all subspaces of lesser dimension, the apparent constants in their space will be seen to evolve in time on the same timescale that the extra dimensions change on. Observers in an $(N - S)$-dimensional expanding subspace will at first see small variations in quantities like their local fine structure "constant," or the Newtonian gravitation constant, following the overall volume expansion. But when the extra dimensions hit their finite-time singularities, there could be dramatic evolution of the local three-dimensional constants [19–22]. However, the sudden singularities are characterized by the scale factors tending to constant values at the singularity. Since the evolution of the local constants is determined by inverse powers of the mean scale of the extra dimensions, there will no dramatic evolution of the values of local constants toward zero or infinity [23,24]. The global structure of singular cosmological models of this type may prove to be interesting and quite different from that accompanying strong curvature singularities. A case for comparison is of parallel propagated curvature singularities which can result in singularities that are directional—but of a different sort from those discussed here because they possess curvature singularities. [25,26]. We do not expect quantum particle production effects or their classical analog of bulk viscosity because the local expansion rates on which these processes depend are assumed constant on approach to the sudden singularity [7]. Higher-order versions of sudden singularities [3,15] will also be possible in these cosmological models.

V. CONCLUSIONS

We have studied the presence of sudden singularities, with finite scale factors, expansion rates, and matter densities in the most general closed spatially homogeneous universes of Bianchi type IX. They permit divergences in scale factor accelerations and pressures at finite-time singularities, where no curvature invariants diverge. In the presence of anisotropic pressures, we have found a new variety of nonsimultaneous directional sudden singularity which can occur in all or any of the expanding directions. This allows expansion in some directions to end at a sudden singularity while those in other nonsingular directions do not. This creates a new form of dimensional reduction driven by the anisotropic pressures, some of which may diverge at finite time while others remain finite. The expansion continues unaffected in the nonsingular directions and the sudden singularities, and their consequences could be observed in the past of the nonsingular directions.

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