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Sixth post-Newtonian local-in-time dynamics of binary systems

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Using a recently introduced method [D. Bini, T. Damour, and A. Geralico, Phys. Rev. Lett. **123**, 231104 (2019)], which splits the conservative dynamics of gravitationally interacting binary systems into a nonlocal-in-time part and a local-in-time one, we compute the local part of the dynamics at the sixth post-Newtonian (6PN) accuracy. Our strategy combines several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, effective-field-theory, gravitational self-force, effective one-body, and Delaunay averaging. The full functional structure of the local 6PN Hamiltonian (which involves 151 numerical coefficients) is derived, but contains four undetermined numerical coefficients. Our 6PN-accurate results are complete at orders G^3 and G^4, and the derived $O(G^3)$ scattering angle agrees, within our 6PN accuracy, with the computation of [Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Phys. Rev. Lett. **122**, 201603 (2019)]. All our results are expressed in several different gauge-invariant ways. We highlight, and make a crucial use of, several aspects of the hidden simplicity of the mass-ratio dependence of the two-body dynamics.

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I. INTRODUCTION

A new method for analytically computing the conservative dynamics of gravitationally interacting binary systems has been recently introduced [1]. This method draws its efficiency from combining in a specific way results coming from several different theoretical formalisms: post-Newtonian (PN), post-Minkowskian (PM), multipolar-post-Minkowskian (MPM), effective-field-theory (EFT), gravitational self-force (SF), effective one-body (EOB), and Delaunay averaging. We have recently applied this method to the derivation of the fifth post-Newtonian (5PN), and fifth-and-a-half post-Newtonian (5.5PN) dynamics [2]. Here, we extend the application of this method to the sixth post-Newtonian (6PN) level.

Let us recall the main idea and the various complementary steps of our strategy. As the main purpose of the present paper is to present our new, 6PN-level results, we will be as brief as possible in guiding the reader through the results presented below. For more details and references, see Refs. [1,2].

The main idea of our strategy is to decompose, from the start, the total reduced[1] two-body conservative action ($S_{\rm tot}$) in two separate pieces: a nonlocal-in-time part ($S_{\rm nonloc,f}$) and a local-in-time part ($S_{\rm loc,f}$). This decomposition is done at some given PN accuracy, say nPN, and yields (when $n \geq 4$) an action of the form

$$S^{\leq n\rm PN}_{\rm tot}[x_1(s_1), x_2(s_2)] = S^{\leq n\rm PN}_{\rm loc,f}[x_1(s_1), x_2(s_2)] + S^{\leq n\rm PN}_{\rm nonloc,f}[x_1(s_1), x_2(s_2)]. \quad (1.1)$$

Here each action piece is a time-symmetric functional of the worldlines of the two bodies, say $x_1(s_1)$ and $x_2(s_2)$. The meaning of the additional subscript f (which stands for "flexibility factor") will be discussed below.

The fact that the PN-approximated dynamics of a gravitationally interacting system must include, starting at the 4PN level, a nonlocal-in-time part was discovered in Ref. [6] by using the PN-matched [6–10] multipolar-post-Minkowskian (MPM) formalism [11]. The description of the 4PN-level nonlocal-in-time (henceforth abbreviated as "nonlocal") dynamics by an action was initiated in Ref. [12] (later refined in Ref. [13]) within the EFT approach to the dynamics [14] of binary systems and their coupling to radiation [15,16]. However, the nonlocal action considered in Refs. [12,13] is a Schwinger-Keldysh-type, in-in, action, with doubled fields, that is not appropriate to the Tetrode-Fokker-type approach we are using. The corresponding appropriate time-symmetric 4PN-level nonlocal action was first written down in Ref. [17]. See Refs. [18–21] for later discussions of this 4PN nonlocal action.

[1]The reduced two-body action is defined as the two-worldline action obtained by integrating out the mediating field from the original particle-plus-field action. It was introduced in electromagnetism by Schwarzschild, Tetrode and Fokker (see Ref. [3] for references and further developments). Its generalization to the gravitational two-body interaction was introduced in the PN context in Ref. [4], and in the PM context in Ref. [5].

The extension of the nonlocal action to the 5PN level was obtained in Refs. [22,23], with extension to the 5.5PN level in the latter reference. The derivation of these nonlocal actions in Refs. [17,23] was obtained by combining information from the MPM formalism, with special properties of the 1PN-accurate interaction of a gravitationally system with an external tidal field [24,25]. Here, we need the extension of the nonlocal part of the action to the 6PN level. As emphasized in Refs. [26,27], the EFT approach [14–16,28] is useful in this respect and gives a guide for writing the nonlocal part of the action beyond the leading order. We are, however, confused by the meaning of some of the equations presented in Refs. [26,27] because they seem to refer to non conservative systems that should be treated by a doubled-field Schwinger-Keldysh-type, while we are interested in the Tetrode-Fokker-type time-symmetric action for conservative systems. There is also a lack of explicit proof (beyond the 5PN level, which was explicitly treated in Ref. [23], see also the Appendix A of Ref. [27]) that the multipole moments to be used in the tail-transported nonlocal action are the same as the "canonical" (or "algorithmic") moments, M_L, S_L, parametrizing the fully nonlinear multipolar structure of gravitationally radiating systems in the MPM formalism [11]. In addition, a consistent 6PN-level evaluation of the nonlocal action requires (as will be made clear below) that the multipole moments, M_L, S_L, parametrizing the exterior MPM gravitational field be expressed as functionals of the source variables. The MPM formalism succeeded in doing this task, and its appropriate results will be used below. Recent work [27] provides some partial checks of this circle of ideas at the level of the logarithmic terms associated with nonlocal correlations,[2] in the restricted case of circular motions. Our work here will provide further checks concerning elliptic motions.

II. NONLOCAL ACTION AT THE 6PN ORDER

The starting point for our method is to have in hand an explicit expression for the nonlocal part of the action, $S_{\rm nonloc,f}$. At the 6PN accuracy, the nonlocal action can be linearly decomposed into its $4+5+6$PN piece, and its 5.5PN piece

$$S^{\leq 6\rm PN}_{\rm nonloc,f} = S^{4+5+6\rm PN}_{\rm nonloc,f} + S^{5.5\rm PN}_{\rm nonloc}. \tag{2.1}$$

The 5.5PN piece (which is independent of the flexibility factor f) has already been treated in Ref. [2] and will not be further discussed here. In view of the work recalled above, the $4+5+6$PN piece reads

$$S^{4+5+6\rm PN}_{\rm nonloc,f}[x_1(s_1), x_2(s_2)] = -\int dt H^{4+5+6\rm PN}_{\rm nonloc,f}(t), \tag{2.2}$$

with

$$H^{4+5+6\rm PN}_{\rm nonloc,f}(t) = \frac{G\mathcal{M}}{c^3}\mathrm{Pf}_{2r^f_{12}(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t'). \tag{2.3}$$

Here, $\mathcal{M}$ denotes the total ADM conserved mass-energy of the binary system,

$$r^f_{12}(t) = f(t) r^h_{12}(t), \tag{2.4}$$

is a flexed version of the radial distance between the two bodies ($r^h_{12}(t)$ denoting the harmonic-coordinate distance and $f(t)$ being a function of the instantaneous state of the system), while $\mathcal{F}^{\rm split}_{\rm 2PN}(t,t')$ is the time-split version of the fractionally 2PN-accurate gravitational-wave energy flux (absorbed and) emitted by the (conservative) system. It can be decomposed as

$$\begin{aligned}\mathcal{F}^{\rm split}_{\rm 2PN}(t,t') = \frac{G}{c^5}[&F^{\rm split}_{I_2}(t,t') + \eta^2 F^{\rm split}_{I_3,J_2}(t,t') \\ &+ \eta^4 F^{\rm split}_{I_4,J_3}(t,t')],\end{aligned} \tag{2.5}$$

with

$$\begin{aligned}F^{\rm split}_{I_2}(t,t') &= \frac{1}{5} I^{(3)}_{ab}(t) I^{(3)}_{ab}(t'), \\ F^{\rm split}_{I_3,J_2}(t,t') &= \frac{1}{189} I^{(4)}_{abc}(t) I^{(4)}_{abc}(t') + \frac{16}{45} J^{(3)}_{ab}(t) J^{(3)}_{ab}(t'), \\ F^{\rm split}_{I_4,J_3}(t,t') &= \frac{1}{9072} I^{(5)}_{abcd}(t) I^{(5)}_{abcd}(t') \\ &\quad + \frac{1}{84} J^{(4)}_{abc}(t) J^{(4)}_{abc}(t'),\end{aligned} \tag{2.6}$$

where $\eta \equiv 1/c$ and the superscript in parenthesis denotes repeated time-derivatives. The multipole moments I_L, J_L denote here the values of the canonical moments M_L, S_L parametrizing (in a minimal, gauge-fixed way) the exterior field (and therefore the relevant coupling between the system and a long-wavelength external radiation field) when they are reexpressed as explicit functionals of the instantaneous state of the binary system. We employ here the notation used[3] in the early works on the PN-matched MPM formalism [7,8,32] in which the source-related values of the algorithmic multipole moments, $M_L = I_L[\text{source}]$, $S_L = J_L[\text{source}]$, were obtained with 1PN fractional accuracy. The latter accuracy suffices for the contribution involving $F^{\rm split}_{I_3,J_2}(t,t')$ [and *a fortiori* $F^{\rm split}_{I_4,J_3}(t,t')$]. However, for the first contribution involving $F^{\rm split}_{I_2}(t,t')$ we need the 2PN-accurate value of the quadrupole moment expressed in terms of the material source [33,34]. We need also to use the explicit form of the 2PN-accurate dynamics

[2]The fact that nonlocal interactions generate logarithmic terms was pointed out in Refs. [29,30].

[3]In more recent developments [31] the notation I_L, J_L refers to slightly different source-related moments, with a difference starting at order $\frac{1}{c^5}$ which is, anyway, not relevant to the present work.

of a binary system in harmonic coordinates [35,36], and its relation [37] to the 2PN-accurate Hamiltonian in Arnowitt-Deser-Misner coordinates [38].

The nonlocal Hamiltonian can be further decomposed into

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,f}}(t) = H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}} + \Delta^{\mathrm{f-h}}H(t), \tag{2.7}$$

where, replacing $\mathcal{M} = \frac{H}{c^2}$ where H is the Hamiltonian,[4] and introducing an intermediate length scale s,

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) = -\frac{GH}{c^5}\mathrm{Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\mathrm{split}}_{\mathrm{2PN}}(t,t+\tau) + 2\frac{GH}{c^5}\mathcal{F}^{\mathrm{split}}_{\mathrm{2PN}}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right), \tag{2.8}$$

and

$$\Delta^{\mathrm{f-h}}H(t) = +2\frac{GH}{c^5}\mathcal{F}^{\mathrm{split}}_{\mathrm{2PN}}(t,t)\ln\left(f(t)\right). \tag{2.9}$$

The corresponding local Hamiltonians are defined so that

$$H_{\mathrm{tot}} = H_{\mathrm{loc,h}} + H_{\mathrm{nonloc,h}} = H_{\mathrm{loc,f}} + H_{\mathrm{nonloc,f}}. \tag{2.10}$$

In view of Eq. (2.7), we have

$$H_{\mathrm{loc,h}} = H_{\mathrm{loc,f}} + \Delta^{\mathrm{f-h}}H(t), \tag{2.11}$$

where it should be noted that $\Delta^{\mathrm{f-h}}H(t)$ is (like $f(t)$) a local function of the dynamical variables.

Depending on the various sections of this paper, we shall work either with the "h-route" nonlocal Hamiltonian $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}$, or the flexed "f-route" local Hamiltonian $H^{4+5+6\mathrm{PN}}_{\mathrm{loc,f}}$. As discussed in [2], the use of a suitable flexibility factor $f(t)$ within our strategy allows one to cleanly separate the determination of the local Hamiltonian $H_{\mathrm{loc,f}}$ from the nonlocal physics. The present paper will focus on the explicit computation of the f-route local Hamiltonian $H_{\mathrm{loc,f}}$ [under the sole assumption that $f(t) = O(\nu)$]. We leave to a separate work a full study of the complementary nonlocal Hamiltonian $H_{\mathrm{nonloc,f}}$, and the determination of the flexibility factor $f(t)$.

III. COMPUTING THE DELAUNAY AVERAGE OF THE NONLOCAL-IN-TIME H-ROUTE HAMILTONIAN

The first stage of our strategy consists of computing the Delaunay average of the nonlocal h-route Hamiltonian $H_{\mathrm{nonloc,h}}$, Eq. (2.8). This computation is conveniently separated into several successive steps: (1) computing the 2PN-accurate multipole moments entering $\mathcal{F}^{\mathrm{split}}_{\mathrm{2PN}}(t,t')$; (2) using a generic 2PN quasi-Keplerian parametrization of the motion; (3) computing the quasi-Keplerian parameters in harmonic coordinates; (4) computing the quasi-Keplerian parameters in EOB coordinates; (5) evaluating the multipole moments along the orbit; and finally, (6) computing the Delaunay-average of the h-route nonlocal Hamiltonian in harmonic coordinates.

In this section, we shall use as (rescaled) energy and angular momentum variables

$$\bar{E} \equiv \frac{H - Mc^2}{\mu}, \qquad j \equiv \frac{J}{GM\mu}. \tag{3.1}$$

Beware that we shall also use other rescaled energy variables in other sections.

A. The 2PN-accurate multipole moments in harmonic coordinates

In this subsection, x^i and $v^i \equiv \frac{dx^i}{dt}$ denote the harmonic-coordinate *relative* center-of-mass position and velocity of a two-body system. One also uses the shorthand notation $L_i \equiv \epsilon_{ijk}x^j v^k$. Using the standard notation for the symmetric and tracefree part of a tensor T, $T_{\langle ijk\ldots\rangle}$, and for the tensor product of two or more vectors $x_i x_j x_k \ldots = x_{ijk\ldots}$, the following results hold

$$\begin{aligned}
x_{\langle ijk\rangle} &= x_{ijk} - \frac{3}{5}x^2 x_{(i}\delta_{jk)},\\
x_{\langle ij}v_{k\rangle} &= x_{(ij}v_{k)} - \frac{1}{5}x^2\delta_{(ij}v_{k)} - \frac{2}{5}(\mathbf{v}\cdot\mathbf{x})\delta_{(ij}x_{k)},\\
x_{\langle i}v_{jk\rangle} &= v_{(ij}x_{k)} - \frac{1}{5}v^2\delta_{(ij}x_{k)} - \frac{2}{5}(\mathbf{v}\cdot\mathbf{x})\delta_{(ij}v_{k)},\\
x_{\langle ijkl\rangle} &= x_{ijkl} - \frac{6}{7}x^2\delta_{(ij}x_{kl)} + \frac{3}{35}x^4\delta_{(ij}\delta_{kl)},\\
L_{\langle i}x_{jk\rangle} &= L_{(i}x_{jk)} - \frac{1}{5}x^2\delta_{(ij}L_{k)},\\
L_{\langle i}x_{jkl\rangle} &= L_{(i}x_{jkl)} - \frac{3}{7}x^2\delta_{(ij}L_k x_{l)},
\end{aligned} \tag{3.2}$$

where $x^2 = \mathbf{x}\cdot\mathbf{x} = x^i x^i$, $\mathbf{v}\cdot\mathbf{x} = v^i x^i$, etc., and where parentheses denote symmetrization (with weight one).

The mass quadrupole moment, I_{ij} at the 2PN accuracy [33,34], the mass octupole moment I_{ijk}, and mass hexadecapole moment, I_{ijkl} at the 1PN accuracy [7], the spin quadrupole moment, J_{ij}, and the spin octupole moment, J_{ijk}, at the 1PN accuracy [8,32], have the following expressions [39]

$$\begin{aligned}
I_{ij} &= \mu[C_1 x_{\langle ij\rangle} + C_2 v_{\langle ij\rangle} + C_3 x_{\langle i}v_{j\rangle}],\\
I_{ijk} &= \mu[B_1 x_{\langle ijk\rangle} + B_2 x_{\langle ij}v_{k\rangle} + B_3 x_{\langle i}v_{jk\rangle}],\\
I_{ijkl} &= \mu(1-3\nu)x_{\langle ijkl\rangle},\\
J_{ij} &= \mu[D_1 L_{\langle i}x_{j\rangle} + D_2 L_{\langle i}v_{j\rangle}],\\
J_{ijk} &= \mu(1-3\nu)L_{\langle i}x_{jk\rangle},
\end{aligned} \tag{3.3}$$

where the various parameters $C_1, C_2, \ldots$ etc., are listed in Table I.

[4] At the present level, we can use the 2PN-accurate Hamiltonian.

TABLE I. Parameters entering the multipolar moments used in the 2PN flux.

C_1	$1+\eta^2[\frac{29}{42}(1-3\nu)v^2-\frac{1}{7}(5-8\nu)\frac{GM}{r}]$
	$+\eta^4[\frac{GM}{r}v^2(\frac{2021}{756}-\frac{5947}{756}\nu-\frac{4833}{756}\nu^2)+\frac{G^2M^2}{r^2}(\frac{355}{252}-\frac{953}{126}\nu+\frac{337}{252}\nu^2)$
	$+v^2(\frac{253}{504}-\frac{1835}{504}\nu+\frac{3545}{504}\nu^2)+\frac{GM}{r}\dot{r}^2(-\frac{131}{756}+\frac{907}{756}\nu-\frac{1273}{756}\nu^2)]$
C_2	$\eta^2r^2\{\frac{11}{21}(1-3\nu)+\eta^2[\frac{GM}{r}(\frac{106}{27}-\frac{335}{189}\nu-\frac{985}{189}\nu^2)+v^2(\frac{41}{126}-\frac{337}{126}\nu+\frac{733}{126}\nu^2)+\dot{r}^2(\frac{5}{63}-\frac{25}{63}\nu+\frac{25}{63}\nu^2)]\}$
C_3	$2\eta^2r\dot{r}\{-\frac{2}{7}+\frac{6}{7}\nu+\eta^2[v^2(-\frac{13}{63}+\frac{101}{63}\nu-\frac{209}{63}\nu^2)+\frac{GM}{r}(-\frac{155}{108}+\frac{4057}{756}\nu+\frac{209}{108}\nu^2)]\}$
B_1	$\sqrt{1-4\nu}\{-1+\eta^2[\frac{GM}{r}(\frac{5}{6}-\frac{13}{6}\nu)+v^2(-\frac{5}{6}+\frac{19}{6}\nu)]\}$
B_2	$\sqrt{1-4\nu}(1-2\nu)\eta^2r\dot{r}$
B_3	$-\sqrt{1-4\nu}(1-2\nu)\eta^2r^2$
D_1	$\sqrt{1-4\nu}\{-1+\eta^2[\frac{GM}{r}(-\frac{27}{14}-\frac{15}{7}\nu)+v^2(-\frac{13}{28}+\frac{17}{7}\nu)]\}$
D_2	$\sqrt{1-4\nu}r\dot{r}(-\frac{5}{28}-\frac{5}{14}\nu)\eta^2$

B. Generic 2PN quasi-Keplerian parametrization of elliptic motion (valid in all coordinates)

The 2PN quasi-Keplerian parametrization [40–42] of elliptic motion, in polar coordinates, (r,ϕ), is the following

$$\begin{aligned} r &= a_r(1-e_r\cos u),\\ \ell &= n(t-t_P) = u-e_t\sin u+f_t\sin V+g_t(V-u),\\ \bar{\phi} &= \frac{\phi-\phi_P}{K} = V+f_\phi\sin 2V+g_\phi\sin 3V, \end{aligned} \tag{3.4}$$

where

$$V(u) = 2\arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right]. \tag{3.5}$$

Here a_r is the semimajor axis of the orbit, e_t, e_r, e_ϕ are three kinds of eccentricities, K is the periastron advance and $n=\frac{2\pi}{T}$ is the circular frequency of the radial motion. This representation is valid (at 2PN) in any (usual) coordinate systems: harmonic, ADM, or EOB. The gauge-invariant quantities K and n are numerically the same in all coordinates, while the quasi-Keplerian elements a_r, e_t, e_r, e_ϕ depend on the coordinate system. We will distinguish them by decorating them with an extra label; for example e_t^h for the harmonic coordinate expression, e_t^e for the EOB coordinate expression, etc. To ease the notation, we will omit the extra label specification when it is clear from the context what are the coordinates used. Most of the time we will (as in our previous works) use rescaled versions of many physical quantities. Notably, we use a dimensionless radial distance $r=r^{\rm phys}/(GM\eta^2)$ and a dimensionless radial period $T=T^{\rm phys}/(GM\eta^3)$.

We recall that

$$\begin{aligned} V'(u) &= \frac{\sqrt{1-e_\phi^2}}{1-e_\phi\cos u}, \qquad \sin V = \frac{\sqrt{1-e_\phi^2}\sin u}{1-e_\phi\cos u},\\ \cos V &= \frac{\cos u-e_\phi}{1-e_\phi\cos u}. \end{aligned} \tag{3.6}$$

These relations imply, for example, the following explicit expression for $\ell(u)$

$$\begin{aligned} \ell &= u-e_t\sin u+f_t\frac{\sqrt{1-e_\phi^2}\sin u}{1-e_\phi\cos u}+g_t(V-u)\\ &= (1-g_t)u-e_t\sin u+f_t\sqrt{1-e_\phi^2}\frac{\sin u}{1-e_\phi\cos u}\\ &\quad+2g_t\arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right], \end{aligned} \tag{3.7}$$

or, equivalently, replacing u in terms of V, the explicit expression for $\ell(V)$

$$\begin{aligned} \ell &= 2(1-g_t)\arctan\left[\sqrt{\frac{1-e_\phi}{1+e_\phi}}\tan\frac{V}{2}\right]\\ &\quad-e_t\sqrt{1-e_\phi^2}\frac{\sin V}{1+e_\phi\cos V}+f_t\sin V+g_tV. \end{aligned} \tag{3.8}$$

C. 2PN expressions of the orbital parameters in harmonic coordinates

To get gauge-invariant expressions for the Keplerian elements one needs to relate them to the conserved 2PN-accurate energy and angular momentum [43]. We made use of explicit (3PN-accurate) results in the literature [39,42]. [Note that, at the 3PN level, one needs to transform away some harmonic-gauge-related logarithms.]

We list in Table II the 2PN-accurate expressions of the *harmonic-coordinate* orbital parameters, as functions of the conserved energy and angular momentum of the system, as defined in Eq. (3.1). We use the shorthand notation

$$e_{\rm N}^2(\bar{E},j)\equiv 1+2\bar{E}j^2. \tag{3.9}$$

It is also useful to have the inverse expressions, *i.e.*, $\bar{E}$ and j expressed in terms of a_r and e_t:

TABLE II. 2PN expressions of the harmonic-coordinates orbital parameters, as functions of the conserved energy and angular momentum of the system, Eq. (3.1).

n	$(-2\bar{E})^{3/2}[1+\eta^2\frac{-15+\nu}{8}(-2\bar{E})+\eta^4\frac{(-2\bar{E})^2}{128}(555+30\nu+11\nu^2+\frac{192}{\sqrt{-2\bar{E}}j}(-5+2\nu))]$
a_r	$\frac{1}{(-2\bar{E})}\{1+\eta^2\frac{-7+\nu}{4}(-2\bar{E})+\eta^4\frac{(-2\bar{E})^2}{16}[1+\nu^2+\frac{16}{(-2\bar{E})j^2}(7\nu-4)]\}$
e_t^2	$e_{\rm N}^2+\frac{(-2\bar{E})}{4}\eta^2[-8+8\nu-(-2\bar{E})j^2(-17+7\nu)]+\frac{(-2\bar{E})^2}{8}[4(3+18\nu+5\nu^2)-(-2\bar{E})j^2(112-47\nu+16\nu^2)$ $-\frac{16}{(-2\bar{E})j^2}(-4+7\nu)-24\sqrt{-2\bar{E}}j(-5+2\nu)+\frac{24}{\sqrt{-2\bar{E}}j}(-5+2\nu)]\eta^4$
e_r^2	$e_{\rm N}^2+2\bar{E}j^2[\frac{5}{4}(3-\nu)(-2\bar{E})+\frac{-6+\nu}{j^2}]\eta^2$ $+\frac{(-2\bar{E})^2}{8}[2(30+74\nu+\nu^2)-(80-45\nu+4\nu^2)(-2\bar{E})j^2-\frac{32}{(-2\bar{E})j^2}(-4+7\nu)]\eta^4$
e_ϕ^2	$e_{\rm N}^2+2\bar{E}j^2[(\frac{15}{4}-\frac{1}{4}\nu)(-2\bar{E})-\frac{6}{j^2}]\eta^2$ $-\frac{(-2\bar{E})^2}{16(-2\bar{E})j^2}[-416+91\nu+15\nu^2-2(-2\bar{E})j^2(-20+17\nu+9\nu^2)+(-2\bar{E})^2j^4(160-31\nu+3\nu^2)]\eta^4$
f_t	$-\frac{(-2\bar{E})^{3/2}e_{\rm N}}{8j}\nu(-15+\nu)\eta^4$
g_t	$-\frac{3}{2}\frac{(-2\bar{E})^{3/2}}{j}(-5+2\nu)\eta^4$
f_ϕ	$\eta^4\frac{e_{\rm N}^2}{8j^4}(1+19\nu-3\nu^2)$
g_ϕ	$-\eta^4\frac{1}{32}\frac{e_{\rm N}^3}{j^4}\nu(-1+3\nu)$
K	$1+\frac{3}{j^2}\eta^2+\frac{\eta^4}{4}[\frac{3(-2\bar{E})}{j^2}(-5+2\nu)+\frac{15}{j^4}(7-2\nu)]$

$$
\begin{aligned}
\bar{E} &= -\frac{1}{2a_r}-\frac{(-7+\nu)}{8a_r^2}\eta^2+\left[-\nu^2+7\nu-25-\frac{(-32+56\nu)}{(1-e_t^2)}\right]\frac{\eta^4}{16a_r^3},\\
j &= \sqrt{a_r(1-e_t^2)}+\left[-(-3+\nu)(1-e_t^2)^{1/2}-\frac{(1-\nu)}{(1-e_t^2)^{1/2}}\right]\frac{\eta^2}{\sqrt{a_r}}\\
&\quad+\left[\frac{1}{2}(-5-3\nu)(1-e_t^2)^{1/2}+\frac{15}{2}-3\nu+\frac{1}{2}\frac{(6\nu+\nu^2+3)}{(1-e_t^2)^{1/2}}+\frac{1}{2}\frac{(-15+6\nu)}{(1-e_t^2)}-\frac{1}{2}\frac{(-7+12\nu+\nu^2)}{(1-e_t^2)^{3/2}}\right]\frac{\eta^4}{a_r^{3/2}},
\end{aligned}
\tag{3.10}
$$

from which one gets (we defined the usual periastron advance parameter $k\equiv K-1$)

$$
\begin{aligned}
n &= \frac{1}{a_r^{3/2}}+\frac{(-\frac{9}{2}+\frac{1}{2}\nu)\eta^2}{a_r^{5/2}}+\left[\frac{147}{8}-\frac{25}{8}\nu+\frac{3}{8}\nu^2+\frac{3(-5+2\nu)}{2\sqrt{1-e_t^2}}+\frac{3(-4+7\nu)}{2(1-e_t^2)}\right]\frac{\eta^4}{a_r^{7/2}},\\
k &= \frac{3}{(1-e_t^2)}\frac{\eta^2}{a_r}+\left[\frac{-\frac{87}{4}+\frac{15}{2}\nu}{(1-e_t^2)}+\frac{\frac{129}{4}-\frac{27}{2}\nu}{(1-e_t^2)^2}\right]\frac{\eta^4}{a_r^2},\\
e_r &= e_t\left[1+\left(4-\frac{3}{2}\nu\right)\frac{\eta^2}{a_r}+\left(-\frac{29}{8}\nu+\frac{3}{8}\nu^2+\left(\frac{15}{2}-3\nu\right)\frac{1}{(1-e_t^2)^{1/2}}+\frac{(4-7\nu)}{(1-e_t^2)}\right)\frac{\eta^4}{a_r^2}\right],\\
e_\phi &= e_t\left[1+(4-\nu)\frac{\eta^2}{a_r}+\left(-\frac{(96\nu-240)}{32(1-e_t^2)^{1/2}}+\frac{3\nu(\nu-27)}{32}-\frac{(5\nu+9)(3\nu-32)}{32(1-e_t^2)}\right)\frac{\eta^4}{a_r^2}\right],\\
f_t &= -\frac{e_t\nu(-15+\nu)}{8a_r^2\sqrt{1-e_t^2}}\eta^4,\\
g_t &= -\frac{3(-5+2\nu)}{2a_r^2\sqrt{1-e_t^2}}\eta^4,\\
f_\phi &= -\frac{e_t^2(-1-19\nu+3\nu^2)}{8a_r^2(1-e_t^2)^2}\eta^4,\\
g_\phi &= -\frac{e_t^3\nu(-1+3\nu)}{32a_r^2(1-e_t^2)^2}\eta^4.
\end{aligned}
\tag{3.11}
$$

TABLE III. 2PN expressions of the EOB-coordinates orbital parameters, as functions of the conserved energy and angular momentum of the system, Eq. (3.1).

n	$(-2\bar{E})^{3/2}[1+\eta^2\frac{-15+\nu}{8}(-2\bar{E})+\eta^4\frac{(-2\bar{E})^2}{128}(555+30\nu+11\nu^2+\frac{192}{\sqrt{-2\bar{E}}j}(-5+2\nu))]$
a_r	$\frac{1}{(-2\bar{E})}+\frac{\nu-3}{4}\eta^2-\frac{\eta^4}{8j^2}[8(4-\nu)+\bar{E}j^2(1+\nu^2)]$
e_t	$e_{\rm N}+\frac{\eta^2\bar{E}}{2e_{\rm N}}[4+j^2\bar{E}(17+\nu)]+\frac{\eta^4\bar{E}}{e_{\rm N}^3}[e_{\rm N}^4(2\nu-5)\frac{6\bar{E}}{j\sqrt{-2\bar{E}}}-\frac{\bar{E}^3j^4}{8}(\nu^2-102\nu-607)+\frac{\bar{E}j^2}{2}(19\nu+90)+5\bar{E}(\nu-3)+\frac{2}{j^2}(\nu-4)]$
e_r	$e_{\rm N}+\frac{\eta^2\bar{E}}{2e_{\rm N}}[-8+j^2\bar{E}(\nu-7)]-\frac{\eta^4\bar{E}}{8e_{\rm N}^3j^2}[(\nu^2+42\nu-79)\bar{E}^3j^6-(5\nu-52)\bar{E}^2j^2-80(\nu-5)\bar{E}j^2+32(4-\nu)]$
e_ϕ	$e_{\rm N}+\frac{\eta^2\bar{E}}{2e_{\rm N}}[-12+j^2\bar{E}(\nu-15)]+\frac{\eta^4\bar{E}}{8e_{\rm N}^3j^2}[(\nu^2+90\nu-415)j^6\bar{E}^3-4(107\nu-30)j^4\bar{E}^2-40(13\nu-15)j^2\bar{E}-16(9\nu-13)]$
f_t	0
g_t	$\frac{6(5-2\nu)\bar{E}^2}{j\sqrt{-2\bar{E}}}\eta^4$
f_ϕ	$-\frac{e_{\rm N}^2(6\nu-1)}{8j^4}\eta^4$
g_ϕ	0
K	$1+\frac{3}{j^2}\eta^2+\frac{\eta^4}{4}[\frac{3(-2\bar{E})}{j^2}(-5+2\nu)+\frac{15}{j^4}(7-2\nu)]$

D. 2PN quasi-Keplerian orbital parameters in EOB coordinates

The 2PN-accurate quasi-Keplerian representation (3.4) is also valid in EOB coordinates. As we shall need to transform the harmonic-coordinate 2PN expressions of the orbital parameters into their EOB counterparts, it is very useful to express both as functions of the conserved energy and angular momentum of the system. The relations $a_r(\bar{E},j)$ and $e_r(\bar{E},j)$ in EOB coordinates are easily obtained by evaluating the reduced energy $\bar{E}=\hat{H}\equiv\frac{H-Mc^2}{\mu}$, and angular momentum $j\equiv\frac{J}{GM\mu}$ at the periastron ($r=a_r(1-e_r)$, $u=0$, $p_r=0$) and the apoastron ($r=a_r(1+e_r)$, $u=\pi$, $p_r=0$). The resulting expressions are listed in Table III below. From these relations one finds in particular

$$e_t=e_r\left[1-3\frac{\eta^2}{a_r}-\frac{\eta^4}{2a_r^2}\left(\frac{2(9e_r^2-5-\nu)}{(1-e_r^2)}-\frac{3(2\nu-5)}{\sqrt{1-e_r^2}}\right)\right],$$
$$e_\phi=e_r\left[1+\frac{\eta^2}{a_r}+\eta^4\frac{(6-7\nu-e_r^2)}{a_r^2(1-e_r^2)}\right]. \tag{3.12}$$

E. Evaluating the multipole moments along the orbit

Let us turn to the definition (2.6) of the 2PN split-flux. The various multipole moments are functions of $r(t)$ and $\phi(t)$ and their derivatives up to the fifth order. In order to compute the Delaunay average of the nonlocal Hamiltonian (2.3) it is convenient to work with the "mean anomaly", i.e., the angular variable ℓ, with respect to which all scalar functions are periodic with period 2π. [Note that $d\ell/dt=n=$ cst.] Therefore, we first compute the multipole moments as functions of u and u', and then replace $u=u(\ell,e_t,\nu)$, $u'=u'(\ell',e_t,\nu)$. Finally, we will take the *partie finie* in ℓ' and the average over ℓ.

We need to invert the 2PN-accurate generalized Kepler equation

$$\ell=u-e_t\sin u+f_t\sin V+g_t(V-u), \tag{3.13}$$

where $V=V(u;e_t,a_r)$. At 1PN (i.e., when neglecting $f_t=O(\eta^4)=g_t$) this inversion is well known, because Eq. (3.13) then reduces to the usual Kepler equation. Namely,

$$u=\ell+\sum_{n=1}^{N}c_n^{\rm 1PN}(e_t)\sin(n\ell)+O(\eta^4), \tag{3.14}$$

with the notation

$$c_n^{\rm 1PN}(e_t)\equiv\frac{2}{n}{\rm BesselJ}(n,ne_t). \tag{3.15}$$

Evidently, the exact inversion of Kepler's equation necessitates to take the upper limit $N=\infty$, but all our computations are done with a finite upper limit N, chosen large enough to end up with the required accuracy on the eccentricity expansion of the redshift z_1.

The inversion of Eq. (3.13) at 2PN is obtained first by expressing f_t, g_t and V as functions of e_t and a_r, and then by looking for an $O(\eta^4)$-modified relation of the type

$$u=\ell+\sum_{n=1}^{N}\left(c_n^{\rm 1PN}(e_t)+\frac{\eta^4}{a_r^2}\phi_n(e_t;\nu)\right)\sin(n\ell). \tag{3.16}$$

Substituting Eq. (3.16) into Eq. (3.13), and expanding in series of e_t, one straightforwardly obtains the expressions listed in Table IV, where terms only up to e_t^{10} (included) are shown. The 2PN coefficients $\phi_n(e_t;\nu)$ depend at most

TABLE IV. The expressions of the various $\phi_j(e_t;\nu)$ entering Eq. (3.16).

$\phi_1(e_t;\nu)$	$(-\frac{15}{2}+\frac{9}{8}\nu+\frac{1}{8}\nu^2)e_t+(-\frac{15}{8}+\frac{93}{64}\nu-\frac{3}{64}\nu^2)e_t^3+(-\frac{255}{128}-\frac{1}{512}\nu^2+\frac{423}{512}\nu)e_t^5+(\frac{17995}{24576}\nu-\frac{127}{73728}\nu^2-\frac{5425}{3072})e_t^7$ $+(\frac{1290347}{1966080}\nu-\frac{1741}{1966080}\nu^2-\frac{158029}{98304})e_t^9$
$\phi_2(e_t;\nu)$	$(\frac{15}{16}\nu-\frac{75}{8}+\frac{3}{16}\nu^2)e_t^2+(-\frac{11}{96}\nu^2+\frac{55}{32}\nu)e_t^4+(\frac{139}{256}\nu-\frac{115}{64}+\frac{3}{256}\nu^2)e_t^6+(-\frac{9}{2560}\nu^2+\frac{4913}{7680}\nu-\frac{1127}{768})e_t^8$ $+(-\frac{349}{276480}\nu^2-\frac{12595}{9216}+\frac{3475}{6144}\nu)e_t^{10}$
$\phi_3(e_t;\nu)$	$(\frac{17}{64}\nu^2+\frac{49}{64}\nu-\frac{95}{8})e_t^3+(\frac{2273}{1024}\nu+\frac{745}{256}-\frac{231}{1024}\nu^2)e_t^5+(\frac{2613}{40960}\nu-\frac{2253}{1024}+\frac{2229}{40960}\nu^2)e_t^7+(\frac{209349}{327680}\nu-\frac{19533}{16384}-\frac{3539}{327680}\nu^2)e_t^9$
$\phi_4(e_t;\nu)$	$(\frac{71}{192}\nu^2-\frac{975}{64}+\frac{35}{64}\nu)e_t^4+(-\frac{129}{320}\nu^2+\frac{955}{128}+\frac{49}{16}\nu)e_t^6+(\frac{387}{2560}\nu^2-\frac{2827}{768}-\frac{6107}{7680}\nu)e_t^8+(-\frac{23495}{32256}+\frac{741}{896}\nu-\frac{8639}{241920}\nu^2)e_t^{10}$
$\phi_5(e_t;\nu)$	$(\frac{1167}{5120}\nu+\frac{523}{1024}\nu^2-\frac{5049}{256})e_t^5+(\frac{542539}{122880}\nu-\frac{50195}{73728}\nu^2+\frac{44521}{3072})e_t^7+(-\frac{2439751}{344064}-\frac{16009117}{6881280}\nu+\frac{473695}{1376256}\nu^2)e_t^9$
$\phi_6(e_t;\nu)$	$(-\frac{1645}{64}+\frac{899}{1280}\nu^2-\frac{65}{256}\nu)e_t^6+(\frac{117137}{17920}\nu+\frac{6451}{256}-\frac{19851}{17920}\nu^2)e_t^8+(-\frac{20475}{4096}\nu-\frac{397035}{28672}+\frac{100713}{143360}\nu^2)e_t^{10}$
$\phi_7(e_t;\nu)$	$(-\frac{844747}{860160}\nu-\frac{723943}{1504}+\frac{355081}{368640}\nu^2)e_t^7+(\frac{56824273}{1376256}-\frac{6902693}{3932160}\nu^2+\frac{270188581}{27525120}\nu)e_t^9$
$\phi_8(e_t;\nu)$	$(-\frac{55697}{26880}\nu+\frac{47259}{35840}\nu^2-\frac{271981}{6144})e_t^8+(-\frac{3959051}{1451520}\nu^2+\frac{273379}{18432}\nu+\frac{50476225}{774144})e_t^{10}$
$\phi_9(e_t;\nu)$	$(\frac{16541017}{9175040}\nu^2-\frac{101441437}{27525120}\nu-\frac{80363041}{1376256})e_t^9$
$\phi_{10}(e_t;\nu)$	$(-\frac{59782109}{774144}-\frac{7804319}{1290240}\nu+\frac{5719087}{2322432}\nu^2)e_t^{10}$

quadratically on ν, and are even (respectively odd) polynomials in e_t, when n is even (respectively odd).

As a preparation for using the Delaunay-averaging technique, it is useful to express the motion in terms of the two independent angles entering the action-angle description of equatorial motion: the angle ℓ measuring the periodicity in the radial motion, and the angle g measuring the mean periastron precession. These two angles are canonically conjugated to two corresponding action variables, traditionally denoted as L and G. [Modulo some rescalings, the link between the Delaunay action variables L, G and the usual action variables is $L=I_3=I_r+I_\phi$ and $G=I_\phi$.] The radial motion is entirely expressed in terms of the sole angle ℓ, while one must separate in the azimuthal motion (given, on shell, by $\phi(\ell)=K\ell+W(\ell)$, where $K\equiv 1+k$) the contributions coming from ℓ and from g (which is equal to $k\ell$ on shell):

$$\phi(\ell,g)=\ell+g+W(\ell). \tag{3.17}$$

Here, $W(\ell)$ is a periodic function of ℓ, say

$$\begin{aligned} W(\ell)&=\sum_{k=1}^{N}\left[P_k^{(0)}(e_t,\nu)+\frac{\eta^2}{a_r}P_k^{(2)}(e_t,\nu)\right.\\ &\quad\left.+\frac{\eta^4}{a_r^2}P_k^{(4)}(e_t,\nu)\right]\sin(k\ell)\\ &\equiv\sum_{k=1}^{N}c_k(e_t,\nu,a_r,\eta)\sin(k\ell). \end{aligned} \tag{3.18}$$

The structure of the angular motion is then of the form

$$\phi(\ell,g)=\ell+g+\sum_{k=1}^{N}c_k(e_t,\nu,a_r,\eta)\sin(k\ell), \tag{3.19}$$

where we recall that one must consider the angle g (mean periastron argument) as an independent angular variable. On-shell we have (remembering the notation $K\equiv 1+k$)

$$\begin{aligned} \ell^{\text{onshell}}&=\ell_0+nt,\\ g^{\text{onshell}}&=k\ell=g_0+knt. \end{aligned} \tag{3.20}$$

One then computes

$$e^{i\phi(\ell,g)}=e^{i\ell}e^{ig}e^{iW(\ell)},\qquad e^{-i\phi(\ell,g)}=e^{-i\ell}e^{-ig}e^{-iW(\ell)}, \tag{3.21}$$

with

$$e^{iW(\ell)}=1+e_t\sum_{k=-N}^{N}d_k^{(1)}e^{ik\ell}+e_t^2\sum_{k=-N}^{N}d_k^{(2)}e^{ik\ell}+\ldots \tag{3.22}$$

expanded in series of e_t. The Cartesian coordinates of the relative (equatorial) motion are then expressed as the following doubly periodic functions of ℓ and g:

$$\begin{aligned} x(\ell,g)&=\frac{1}{2}r_h(\ell)(e^{i\phi(\ell,g)}+e^{-i\phi(\ell,g)}),\\ y(\ell,g)&=\frac{1}{2i}r_h(\ell)(e^{i\phi(\ell,g)}-e^{-i\phi(\ell,g)}). \end{aligned} \tag{3.23}$$

Finally, one computes all the on-shell time derivatives entering the definition of the multipole moments by using, in view of Eqs. (3.20), $d\ell/dt=n$ and $dg/dt=kn=O(\eta^2)+O(\eta^4)$.

F. Computing the Delaunay-average of the nonlocal Hamiltonian in harmonic coordinates

We are now ready to sketch the computation of the Delaunay-average of the h-route nonlocal Hamiltonian, i.e., the average of the action-angle Hamiltonian $H^{4+5+6\text{PN}}_{\text{nonloc,h}}(L,G,\ell,g)$ over the two angles ℓ,g:

$$\langle H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}\rangle \equiv \oint \frac{d\ell}{2\pi}\frac{dg}{2\pi} H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(L,G,\ell,g). \quad (3.24)$$

We recall that

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) = -\frac{GH_{2\mathrm{PN}}}{c^5}\mathrm{Pf}_{2s/c}\int\frac{d\tau}{|\tau|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t+\tau) + 2\frac{GH_{2\mathrm{PN}}}{c^5}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t)\ln\left(\frac{r^h_{12}(t)}{s}\right), \quad (3.25)$$

with

$$\frac{H_{2\mathrm{PN}}}{Mc^2} = 1+\nu\bar{E}\eta^2 = 1-\frac{\nu}{2a^h_r}\eta^2+\frac{\nu}{8(a^h_r)^2}(7-\nu)\eta^4+O(\eta^6). \quad (3.26)$$

One must, in principle, express the nonlocal integrand entering Eq. (3.25) in terms of two quadruplets of Delaunay variables, (L,G,ℓ,g) and (L',G',ℓ',g'), where the first quadruplet refers to the state of the system at time t while the second refers to the state at the shifted time $t'=t+\tau$. [The Delaunay variables (L,G,ℓ,g) are action-angle variables for the main part of the Hamiltonian, to which $H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}$ is added as a first-order perturbation. In practice, it suffices to use the 2PN-accurate Hamiltonian.] This yields an integrand which can be expressed as a multi-Fourier series of the general form

$$\mathcal{F}^{\mathrm{split}}(L,G,\ell,g;L',G',\ell',g') = \sum_{p,m,p',m'} C_{p,m,p',m'}e^{i(p\ell+mg+p'\ell'+m'g')}, \quad (3.27)$$

where the relative integers p,m,p',m' are summed from $-\infty$ to $+\infty$, and where the coefficients $C_{p,m,p',m'}$ are functions of (L,G,L',G').

As shown in Ref. [17], one can first use a nonlocal shift of the phase-space variables to replace the second quadruplet by its on-shell value in terms of (L,G,ℓ,g) and of the time shift τ. In other words, we can insert in Eq. (3.27)

$$(L',G',\ell',g') \mapsto (L,G,\ell+n(L,G)\tau, g+k(L,G)n(L,G)\tau), \quad (3.28)$$

where we used the simple equations of motion of the Delaunay variables (L,G,ℓ,g).

TABLE V. 6PN coefficients of the averaged nonlocal Hamiltonian (with scale $2r^h_{12}/c$) in harmonic coordinates.

Coefficient	Expression
$\mathcal{A}^{6\mathrm{PN}}(e^h_t)$	$-8\nu^2+\frac{1238}{7}+\frac{6578}{105}\nu+(\frac{32}{5}\nu^2+\frac{1173532}{2835}+\frac{38266}{63}\nu)\gamma$ $+(\frac{3125212}{2835}-\frac{35362}{945}\nu+\frac{622648}{945}\nu^2)\ln(2)+(-\frac{2673}{7}\nu^2-243+\frac{29889}{28}\nu)\ln(3)$ $+[\frac{252377}{315}+\frac{34409}{90}\nu-\frac{75104}{135}\nu^2+(\frac{577921}{105}\nu-\frac{260872}{945}+\frac{2815}{3}\nu^2)\gamma+(-\frac{8467217}{315}\nu^2+\frac{39588209}{945}\nu-\frac{8908616}{945})\ln(2)$ $+(-\frac{837621}{140}\nu+\frac{2071089}{560}+\frac{3052323}{560}\nu^2)\ln(3)+(\frac{9765625}{9072}-\frac{9765625}{1512}\nu+\frac{9765625}{1008}\nu^2)\ln(5)](e^h_t)^2$ $+[\frac{13092029}{1680}\nu-\frac{1114139}{648}-\frac{52844503}{7560}\nu^2+(\frac{35023}{4}\nu^2+\frac{1693451}{140}\nu-\frac{1047607}{315})\gamma$ $+(-\frac{235370381}{1260}\nu+\frac{1392676751}{3780}\nu^2+\frac{55700171}{945})\ln(2)+(-\frac{929042703}{8960}\nu+\frac{18495459}{160}\nu^2+\frac{989253}{1120})\ln(3)$ $+(\frac{904296875}{5376}\nu-\frac{25390625}{864}-\frac{166015625}{756}\nu^2)\ln(5)](e^h_t)^4$ $+[\frac{598387141}{10080}\nu-\frac{1164531499}{30240}\nu^2-\frac{411052325}{13608}+(\frac{4585927}{1890}+\frac{154063}{4}\nu^2-\frac{33879}{20}\nu)\gamma$ $+(-\frac{1792419163}{5670}+\frac{34283848589}{34020}\nu-\frac{101367784649}{34020}\nu^2)\ln(2)+(\frac{6204343797}{4480}\nu-\frac{7349467203}{35840}-\frac{80018658513}{35840}\nu^2)\ln(3)$ $+(\frac{99453653125}{746496}-\frac{198307890625}{435456}\nu+\frac{2578115234375}{1741824}\nu^2)\ln(5)$ $+(-\frac{96889010407}{124416}\nu+\frac{96889010407}{82944}\nu^2+\frac{96889010407}{746496})\ln(7)](e^h_t)^6$ $+[\frac{40216186627}{161280}\nu-\frac{107831692469}{725760}-\frac{739210729}{5376}\nu^2+(\frac{3745525}{32}\nu^2+\frac{477783529}{10080}-\frac{81947429}{960}\nu)\gamma$ $+(\frac{2521461747193}{816480}+\frac{496432848889}{12960}\nu^2-\frac{9704902576499}{544320}\nu)\ln(2)+(\frac{501042990087}{573440}+\frac{1148629836951}{81920}\nu^2-\frac{9521506906299}{2293760}\nu)\ln(3)$ $+(-\frac{19630345703125}{3981312}\nu^2-\frac{677663178125}{3096576}-\frac{81800483984375}{111476736}\nu)\ln(5)$ $+(-\frac{15648846319351}{11943936}-\frac{68694308378563}{3981312}\nu^2+\frac{145243406006069}{15925248}\nu)\ln(7)](e^h_t)^8$ $+[-\frac{97074060217}{201600}+\frac{405149913757}{537600}\nu-\frac{609781084333}{1612800}\nu^2+(\frac{182458353}{640}\nu^2-\frac{617936773}{1920}\nu+\frac{52105469}{280})\gamma$ $+(-\frac{9975543985969843}{27216000}\nu^2+\frac{300764760469259}{1814400}\nu-\frac{40023535388969}{1701000})\ln(2)$ $+(-\frac{1205456030071641}{22937600}\nu+\frac{910396452443931}{114688000}+\frac{240491992467807}{7168000}\nu^2)\ln(3)$ $+(\frac{972942783453125}{222953472}\nu-\frac{114839474809375}{111476736}+\frac{2227079123046875}{222953472}\nu^2)\ln(5)$ $+(\frac{8183105251126477}{1592524800}-\frac{67003553703806461}{1990656000}\nu+\frac{33107071745082307}{318504960}\nu^2)\ln(7)](e^h_t)^{10}$
$\mathcal{B}^{6\mathrm{PN}}(e^h_t)$	$-\frac{16}{5}\nu^2-\frac{19133}{63}\nu-\frac{586766}{2835}+(\frac{130436}{945}-\frac{2815}{6}\nu^2-\frac{577921}{210}\nu)(e^h_t)^2+(-\frac{35023}{8}\nu^2-\frac{1693451}{280}\nu+\frac{1047607}{630})(e^h_t)^4$ $+(-\frac{154063}{8}\nu^2+\frac{33879}{40}\nu-\frac{4585927}{3780})(e^h_t)^6+(\frac{81947429}{1920}\nu-\frac{477783529}{20160}-\frac{3745525}{64}\nu^2)(e^h_t)^8$ $+(\frac{617936773}{3840}\nu-\frac{182458353}{1280}\nu^2-\frac{52105469}{560})(e^h_t)^{10}$

After this replacement the crucial nonlocal τ integral in Eq. (3.25) can be explicitly evaluated by using the basic formula (where γ denotes Euler's constant)

$$\mathrm{Pf}_{2s/c}\int_{-\infty}^{+\infty}\frac{d\tau}{|\tau|}e^{i(p'+m'k)n\tau} = -2\ln\left(|(p'+m'k)n|\frac{2se^{\gamma}}{c}\right). \tag{3.29}$$

Using the latter formula for evaluating the τ integral in Eq. (3.25) yields a result which is a function of (L, G, ℓ, g). Adding the local term $2\frac{G^2H_{2\mathrm{PN}}}{c^5}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t)\ln(\frac{r^h_{12}(t)}{s})$, we can finally evaluate the double average over the two angular variables ℓ and g.

For conceptual clarity, we have assumed here that we were using the (2PN-accurate) Delaunay action variables L, G as arguments in the double-Fourier expansion of the nonlocal integrand (3.27). However, in practice, it suffices to use the Keplerian elements a^h_r and e^h_t entering the 2PN-accurate quasi-Keplerian representation (3.4) of the elliptic motion. At the end of the day, the method presented above leads to an explicit expression for the Delaunay-averaged h-route nonlocal Hamiltonian of the form

$$\langle H^h_{\mathrm{nonloc,h}}\rangle \equiv \langle H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}\rangle = F^h(a^h_r, e^h_t), \tag{3.30}$$

with

$$\begin{aligned}F^h(a^h_r,e^h_t) &= \frac{\nu^2}{(a^h_r)^5}[\mathcal{A}^{4\mathrm{PN}}(e^h_t)+\mathcal{B}^{4\mathrm{PN}}(e^h_t)\ln a^h_r]\\ &+\frac{\nu^2}{(a^h_r)^6}[\mathcal{A}^{5\mathrm{PN}}(e^h_t)+\mathcal{B}^{5\mathrm{PN}}(e^h_t)\ln a^h_r]\\ &+\frac{\nu^2}{(a^h_r)^7}[\mathcal{A}^{6\mathrm{PN}}(e^h_t)+\mathcal{B}^{6\mathrm{PN}}(e^h_t)\ln a^h_r].\end{aligned} \tag{3.31}$$

The coefficients entering this decomposition are independent of the intermediate scale s, and are obtained as expansions in powers of e^h_t that we have computed up to the order $O((e^h_t)^{10})$ included. The values of the 4PN and 5PN coefficients have been given in Ref. [2]. We list the 6PN coefficients $\mathcal{A}^{6\mathrm{PN}}(e^h_t)$, $\mathcal{B}^{6\mathrm{PN}}(e^h_t)$, in Table V. [We use here $G=1=c$, and we recall that a^h_r has been adimensionalized by GM.]

IV. DERIVING THE H-ROUTE NONLOCAL EOB HAMILTONIAN

An important ingredient of our method is to translate the h-route nonlocal averaged Hamiltonian computed in the previous section into a canonically equivalent EOB Hamiltonian. This is done by parametrizing the corresponding h-route nonlocal EOB Hamiltonian by means of the usual EOB potentials, in some fixed EOB gauge. At this stage of our computation, it is most convenient to use the p_r gauge (introduced in Ref. [44]).

Explicitly, we look for a rescaled squared effective EOB Hamiltonian of the general form (where $u=1/r=GM/r^{\mathrm{phys}}$, $p_r=p_r^{\mathrm{phys}}/\mu$, $p_\phi=p_\phi^{\mathrm{phys}}/(GM\mu)\equiv j$)

$$\begin{aligned}\hat{H}^2_{\mathrm{eff}} &= A(u;\nu)(1+p^2_\phi u^2+A(u;\nu)\bar{D}(u;\nu)p^2_r\\ &+\hat{Q}(u,p_r;\nu)),\end{aligned} \tag{4.1}$$

with potentials $A(u;\nu)$, $\bar{D}(u;\nu)$ and

$$\begin{aligned}\hat{Q}(u,p_r;\nu) &= p^4_r q_4(u;\nu)+p^6_r q_6(u;\nu)\\ &+p^8_r q_8(u;\nu)+p^{10}_r q_{10}(u;\nu)+\ldots.\end{aligned} \tag{4.2}$$

All the potentials $A(u;\nu)$, $\bar{D}(u;\nu)$, $\hat{Q}(u,p_r;\nu)$ reduce to their Schwarzschild values when $\nu\to 0$: $A(u;0)=1-2u$, $\bar{D}(u;0)=1$, $\hat{Q}(u,p_r;0)=0$, and can be expanded in powers of ν away from the test-mass limit:

$$\begin{aligned}A(u;\nu) &= 1-2u+\nu a^{\nu^1}(u)+\nu^2 a^{\nu^2}(u)+\nu^3 a^{\nu^3}(u)+\ldots\\ \bar{D}(u;\nu) &= 1+\nu\bar{d}^{\nu^1}(u)+\nu^2\bar{d}^{\nu^2}(u)+\nu^3\bar{d}^{\nu^2}(u)+\ldots\\ q_4(u;\nu) &= \nu q_4^{\nu^1}(u)+\nu^2 q_4^{\nu^2}(u)+\nu^3 q_4^{\nu^3}(u)+\ldots\\ q_6(u;\nu) &= \nu q_6^{\nu^1}(u)+\nu^2 q_6^{\nu^2}(u)+\nu^3 q_6^{\nu^3}(u)+\ldots\\ q_8(u;\nu) &= \nu q_8^{\nu^1}(u)+\nu^2 q_8^{\nu^2}(u)+\nu^3 q_8^{\nu^3}(u)+\ldots.\end{aligned} \tag{4.3}$$

Each EOB potential can be decomposed in a local part and a nonlocal one:

$$\begin{aligned}A &= A^{\mathrm{loc,h}}+A^{\mathrm{nonloc,h}} = A^{\mathrm{loc,f}}+A^{\mathrm{nonloc,f}},\\ \bar{D} &= \bar{D}^{\mathrm{loc,h}}+\bar{D}^{\mathrm{nonloc,h}} = \bar{D}^{\mathrm{loc,f}}+\bar{D}^{\mathrm{nonloc,f}},\\ \hat{Q} &= \hat{Q}^{\mathrm{loc,h}}+\hat{Q}^{\mathrm{nonloc,h}} = \hat{Q}^{\mathrm{loc,f}}+\hat{Q}^{\mathrm{nonloc,f}}.\end{aligned} \tag{4.4}$$

The nonlocal parts start at 4PN.They can be treated as first-order perturbations of the local parts, which start at 2PN (and also at 2PM). We, indeed, recall that the EOB formalism has the remarkable feature to describe both the 1PN-accurate dynamics and the 1PM one, by a Schwarzschild effective metric. This means that all the local contributions to $A-(1-2u)$, $\bar{D}-1$ and Q start at order $u^2=O(G^2)$ or more (and contain a factor ν). [The main A potential actually starts to deviate from $1-2u$ by a term $2\nu u^3$.] For clarity, we have indicated that the precise values of both the local and nonlocal EOB potentials will depend on the choice of the flexibility factor $f(t)$ used in defining the Pf scale $r^f_{12}=f(t)r^h_{12}$ entering the nonlocal action (2.3). The h-route is defined by choosing the default value $f=1$, while the (tuned) f-route is defined by choosing a value $f=1+\nu O(\eta^2)$ determined in the way explained at 5PN in Ref. [2]. As a consequence the difference between the h and f values of any quantity

starts at 5PN and at the second-self-force (2SF) order, *i.e.*, the order $O(\nu^2)$ in $\hat{H}^2_{\rm eff}$. This corresponds to the order $O(\nu^3)$ in the usual Hamiltonian. Indeed, the universal EOB energy map says that the usual center-of-mass Hamiltonian of the system, $H = Mc^2 + \cdots$ is given by

$$H = Mc^2\sqrt{1 + 2\nu(\hat{H}_{\rm eff} - 1)}, \tag{4.5}$$

where one should note the factor ν in front of $\hat{H}_{\rm eff}$.

To simplify the notation, we shall denote the nonlocal part of the *squared* effective EOB Hamiltonian as

$$\delta^{\rm h,f}\hat{H}^2_{\rm eff} \equiv [\hat{H}^2_{\rm eff}]^{4+5+6{\rm PN}}_{\rm nonloc,h,f}. \tag{4.6}$$

It is related to the corresponding (h-route or f-route) nonlocal Hamiltonian via

$$H^{4+5+6{\rm PN}}_{\rm nonloc,h,f} = \frac{\mu M}{2H\hat{H}_{\rm eff}}\delta^{\rm h,f}\hat{H}^2_{\rm eff}. \tag{4.7}$$

The prefactor on the right-hand side of Eq. (4.7) is given (at the 2PN accuracy) by

$$\frac{\mu M}{2H\hat{H}_{\rm eff}} = M\left(\frac{1}{2}\nu + \frac{\nu(\nu+1)}{4a^e_r}\eta^2 + \frac{\nu(-\nu+3\nu^2-1)}{16(a^e_r)^2}\eta^4 + O(\eta^6)\right), \tag{4.8}$$

where a^e_r denotes the EOB-coordinate semi-major axis.

With this notation, the squared effective EOB Hamiltonian reads

$$\hat{H}^2_{\rm eff} = \hat{H}^2_{\rm eff,loc,h,f} + \delta^{\rm h,f}\hat{H}^2_{\rm eff}, \tag{4.9}$$

where

$$\hat{H}^2_{\rm eff,loc,h,f} = A^{\rm loc,h,f}[1 + A^{\rm loc,h,f}\bar{D}^{\rm loc,h,f}p^2_r + p^2_\phi u^2 + \hat{Q}^{\rm loc,h,f}], \tag{4.10}$$

and

$$\delta^{\rm h,f}\hat{H}^2_{\rm eff} = [1 + 2(1-2u)p^2_r + p^2_\phi u^2]\delta^{\rm h,f}A + (1-2u)^2 p^2_r \delta^{\rm h,f}\bar{D} + (1-2u)\delta^{\rm h,f}\hat{Q}. \tag{4.11}$$

At the $4+5+6$PN accuracy, the expressions for the nonlocal EOB potentials read

$$\begin{aligned}\delta A &= a^{\rm nonloc}_5 u^5 + a^{\rm nonloc}_6 u^6 + a^{\rm nonloc}_7 u^7,\\ \delta\bar{D} &= \bar{d}^{\rm nonloc}_4 u^4 + \bar{d}^{\rm nonloc}_5 u^5 + \bar{d}^{\rm nonloc}_6 u^6,\\ \delta\hat{Q} &= p^4_r(q^{\rm nonloc}_{43}u^3 + q^{\rm nonloc}_{44}u^4 + q^{\rm nonloc}_{45}u^5)\\ &\quad + p^6_r(q^{\rm nonloc}_{62}u^2 + q^{\rm nonloc}_{63}u^3 + q^{\rm nonloc}_{64}u^4)\\ &\quad + p^8_r(q^{\rm nonloc}_{81}u + q^{\rm nonloc}_{82}u^2 + q^{\rm nonloc}_{83}u^3)\\ &\quad + p^{10}_r(q^{\rm nonloc}_{10,0} + q^{\rm nonloc}_{10,1}u + q^{\rm nonloc}_{10,2}u^2),\end{aligned} \tag{4.12}$$

etc., where each coefficient will be decomposed in "constant," and "logarithmically running" parts according to the scheme: $a^{\rm nonloc}_5 = a^{\rm nl,c}_5 + a^{\rm nl,log}_5 \ln(u)$, etc. To ease the notation, we have suppressed on each nonlocal quantity the extra label h or f specifying whether this is computed by the h-route or the f-route. A term $\propto q^{\rm nonloc}_{2p,q}u^q$ belongs to the n-PN approximation with $n = p + q - 1$. Note that, contrary to the local EOB potentials that must start at order u^2 at least, the nonlocal ones, being obtained by matching a nonlocal action by means of a nearzone eccentricity (or p_r) expansion, include, at high orders in p_r powers of u that are smaller than 2.

Having clarified the meaning of the nonlocal parts of the EOB potentials, we can now determine the values of the h-route nonlocal EOB potentials, $a^{\rm nonloc,h}_n$, $\bar{d}^{\rm nonloc,h}_n$, $q^{\rm nonloc,h}_{2p,q}$ that are gauge-equivalent to the h-route nonlocal Hamiltonian computed in the previous section. These values are determined by writing the equality between the corresponding Delaunay-averaged perturbed Hamiltonians, namely

$$\langle H^{\rm eob}_{\rm nonloc,h}\rangle = \langle H^h_{\rm nonloc,h}\rangle, \tag{4.13}$$

where the left-hand side is the Delaunay average of the EOB-parametrized Hamiltonian

$$\langle H^{\rm eob}_{\rm nonloc,h}\rangle \equiv \frac{\mu M}{2H\hat{H}_{\rm eff}}\oint\frac{d\ell}{2\pi}\frac{dg}{2\pi}\delta^{\rm h}\hat{H}^2_{\rm eff}, \tag{4.14}$$

and where the right-hand side is the function $F^h(a^h_r, e^h_t)$ computed in the previous section, see Eq. (3.30). The equality Eq. (4.13) expresses the requirement that the EOB nonlocal dynamics is canonically equivalent to the original nonlocal dynamics, described by Eq. (2.3) (see Ref. [23]). The computations needed to evaluate $\langle H^{\rm eob}_{\rm nonloc,h}\rangle$ are similar to the computations described above (with the simplifying feature that one only works with an Hamiltonian given as a function of the instantaneous state of the system). One uses the EOB version of the 2PN-accurate quasi-Keplerian representation of elliptic motions, as described in the previous section.

Finally, the identification Eq. (4.13) uniquely determines, from the knowledge of the function $F^h(a^h_r, e^h_t)$, Eq. (3.31), all the coefficients parametrizing the nonlocal EOB potentials Eq. (4.12). We give the resulting values in Table VI, up to the eight power of p_r. Indeed, we will not need in the following the coefficient $q_{10}(u;\nu)$ of p^{10}_r.

TABLE VI. h-route nonlocal EOB coefficients.

$a_7^{\mathrm{nl,c}}$	$(\frac{206740}{567}\ln(2)+\frac{12664}{105}-\frac{4617}{14}\ln(3)-\frac{5044}{405}\gamma)\nu$ $+(-\frac{1139672}{945}\ln(2)+\frac{10132}{105}+\frac{10449}{7}\ln(3)+\frac{101272}{315}\gamma)\nu^2$ $+(-\frac{112}{5}+32\gamma+\frac{1214624}{945}\ln(2)-\frac{4860}{7}\ln(3))\nu^3$,
$a_7^{\mathrm{nl,log}}$	$-\frac{2522}{405}\nu+\frac{50636}{315}\nu^2+16\nu^3$,
$\bar{d}_6^{\mathrm{nl,c}}$	$(-\frac{6381680}{189}\ln(2)+\frac{2043541}{2835}+\frac{1765881}{140}\ln(3)-\frac{64096}{45}\gamma+\frac{9765625}{2268}\ln(5))\nu$ $+(\frac{28429312}{189}\ln(2)-\frac{3576231}{70}\ln(3)+\frac{167906}{105}+\frac{302752}{105}\gamma-\frac{9765625}{378}\ln(5))\nu^2$ $+(-\frac{9908480}{63}\ln(2)-\frac{744704}{945}+\frac{9765625}{252}\ln(5)+\frac{2944}{3}\gamma+\frac{1275021}{28}\ln(3))\nu^3$,
$\bar{d}_6^{\mathrm{nl,log}}$	$-\frac{32048}{45}\nu+\frac{151376}{105}\nu^2+\frac{1472}{3}\nu^3$
$q_{45}^{\mathrm{nl,c}}$	$(\frac{70925884}{63}\ln(2)+\frac{13212013}{5670}-\frac{3873663}{16}\ln(3)-\frac{8787109375}{27216}\ln(5)-\frac{617716}{315}\gamma)\nu$ $+(\frac{92560887}{280}\ln(3)-\frac{12619052648}{2835}\ln(2)-\frac{1437979}{63}+\frac{632344}{315}\gamma+\frac{7755859375}{4536}\ln(5))\nu^2$ $+(-\frac{177316}{35}+\frac{11263031264}{2835}\ln(2)+\frac{16544}{9}\gamma-\frac{4091796875}{2268}\ln(5)+\frac{2908467}{20}\ln(3))\nu^3$
$q_{45}^{\mathrm{nl,log}}$	$-\frac{308858}{315}\nu+\frac{316172}{315}\nu^2+\frac{8272}{9}\nu^3$
$q_{64}^{\mathrm{nl,c}}$	$(-\frac{211076833264}{14175}\ln(2)-\frac{137711989}{28350}-\frac{9678652821}{5600}\ln(3)+\frac{447248}{1575}\gamma+\frac{153776136875}{23328}\ln(5)$ $+\frac{96889010407}{116640}\ln(7))\nu$ $+(\frac{44592947739}{2800}\ln(3)+\frac{2411178384736}{42525}\ln(2)-\frac{126070663}{4725}-\frac{26848}{175}\gamma-\frac{796015515625}{27216}\ln(5)$ $-\frac{96889010407}{19440}\ln(7))\nu^2$ $+(-\frac{40513708}{4725}-\frac{109566260523}{5600}\ln(3)+\frac{1424826953125}{54432}\ln(5)+\frac{96889010407}{12960}\ln(7)+\frac{2368}{5}\gamma$ $-\frac{431564554688}{8505}\ln(2))\nu^3$
$q_{64}^{\mathrm{nl,log}}$	$\frac{223624}{1575}\nu-\frac{13424}{175}\nu^2+\frac{1184}{5}\nu^3$
$q_{83}^{\mathrm{nl,c}}$	$(\frac{5196312336176}{35721}\ln(2)+\frac{17515638027261}{313600}\ln(3)-\frac{63886617280625}{1016064}\ln(5)-\frac{29247366220639}{933120}\ln(7)$ $-\frac{709195549}{132300})\nu$ $+(-\frac{177055674739808}{297675}\ln(2)-\frac{43719724468071}{156800}\ln(3)+\frac{366449151015625}{1524096}\ln(5)+\frac{26506549233199}{155520}\ln(7)$ $-\frac{1746293}{70})\nu^2$ $+(\frac{57604236136064}{99225}\ln(2)+\frac{10467583300341}{39200}\ln(3)-\frac{73366198046875}{381024}\ln(5)-\frac{7709596970957}{38880}\ln(7)$ $-\frac{154862}{21})\nu^3$
$q_{83}^{\mathrm{nl,log}}$	0

V. COMPUTING THE 1SF TIME-AVERAGED REDSHIFT TO EIGHTH ORDER IN ECCENTRICITY AND DERIVING ITS EOB COUNTERPART

The second pillar of our method is to combine the information extracted from the analytical knowledge of the nonlocal part of the dynamics with a knowledge obtained from self-force calculations, which gives information about the *total*, local plus nonlocal, near-zone dynamics, at the first order in mass ratio $q=\frac{m_1}{m_2}$ beyond the test-mass limit. Indeed, Refs. [45–47] have found a relation between the m_1-dependence of the Hamiltonian of a two-body system, and the (regularized) redshift [48,49] $z_1=ds_1/dt$ of particle 1 in the gravitational field created by the two particles. We have developed efficient tools in previous work [50,51] for tapping information by such self-force computations. The current limitation of this technique (for non-spinning bodies) is not the PN accuracy (which can be pushed to extremely high levels [52,53]) but rather the order of expansion in the eccentricity of the considered elliptic motion of a small mass m_1 around a large mass m_2. Here, we have extended our previous results [1,2,54,55] by computing the first-order-self-force (1SF) correction to the time-averaged redshift $\langle z_1\rangle=\langle ds_1/dt\rangle$ [49] of body 1 to the eighth order in eccentricity and through the 9.5PN accuracy. Obtaining the eight order in eccentricity is, by itself, a major technical endeavour, and is crucial to allow us to inform the terms $\sim q_8(u;\nu)p_r^8$ in the total EOB effective Hamiltonian, and thereby to reach the 6PN approximation.

The gauge-invariant 1SF observable we are using is defined as follows. One initially considers the averaged redshift $\langle z_1\rangle$ as a function of the two adimensionalized frequencies of an elliptic motion: $\hat{\Omega}_r=Gm_2\Omega_r$ and $\hat{\Omega}_\phi=Gm_2\Omega_\phi$ and of the mass ratio $q=m_1/m_2$. The 1SF expansion of the latter function yields:

$$\langle z_1\rangle(\hat{\Omega}_r,\hat{\Omega}_\phi,q)=\langle z_1\rangle(\hat{\Omega}_r,\hat{\Omega}_\phi,0)+q\delta z_1(\hat{\Omega}_r,\hat{\Omega}_\phi)+O(q^2), \tag{5.1}$$

TABLE VII. List of the various coefficients entering the self-force based expression of $\delta z_1^{e^8}(u)$.

C_3	$\frac{15}{64}$
C_4	$\frac{3001}{384}-\frac{287}{4096}\pi^2$
$C_5^{\rm c}$	$\frac{4597}{96}-\frac{162109375}{2304}\ln(5)-\frac{11332791}{1280}\ln(3)+\frac{55}{6}\gamma+\frac{15967961}{90}\ln(2)-\frac{474715}{196608}\pi^2$
$C_5^{\rm ln}$	$\frac{55}{12}$
$C_6^{\rm c}$	$-\frac{9863051}{40320}+\frac{96889010407}{442368}\ln(7)-\frac{64481546637}{114688}\ln(3)-\frac{5977}{240}\gamma-\frac{16605499789}{5040}\ln(2)+\frac{1466047}{196608}\pi^2+\frac{4761539921875}{3096576}\ln(5)$
$C_6^{\rm ln}$	$-\frac{5977}{480}$
$C_{13/2}$	$+\frac{7527343}{145152}\pi$
$C_7^{\rm c}$	$-\frac{18761241007}{870912}-\frac{12511111253459}{1492992}\ln(7)-\frac{75643996671875}{5225472}\ln(5)-\frac{53971661}{45360}\gamma$ $+\frac{13950859695883}{408240}\ln(2)+\frac{32462513613}{2240}\ln(3)-\frac{368710657}{33554432}\pi^4+\frac{205074667027}{113246208}\pi^2$
$C_7^{\rm ln}$	$-\frac{53971661}{90720}$
$C_{15/2}$	$-\frac{107115666451}{162570240}\pi$
$C_8^{\rm c}$	$-\frac{6488211537}{11200}\gamma\ln(3)-\frac{45307496529}{11200}\ln(2)\ln(3)+\frac{111806640625}{12096}\ln(5)\gamma+\frac{111806640625}{12096}\ln(5)\ln(2)-\frac{10769592586}{525}\gamma\ln(2)$ $+\frac{1919773074129997}{10059033600}+\frac{111806640625}{24192}\ln(5)^2+\frac{1922666600157935849}{14014218240}\ln(7)$ $-\frac{6488211537}{22400}\ln(3)^2+\frac{22363}{45}\gamma^2-\frac{555027930119}{14175}\ln(2)^2-\frac{5263490413}{453600}\gamma-\frac{396348077586606421}{1571724000}\ln(2)-\frac{4261220414023638519}{35323904000}\ln(3)$ $-\frac{2926}{3}\zeta(3)+\frac{2572903425668796875}{103004504064}\ln(5)+\frac{472342810483}{805306368}\pi^4-\frac{83426620549601}{4529848320}\pi^2$
$C_8^{\rm ln}$	$-\frac{5263490413}{907200}+\frac{22363}{45}\gamma+\frac{111806640625}{24192}\ln(5)-\frac{5384796293}{525}\ln(2)-\frac{6488211537}{22400}\ln(3)$
$C_8^{\rm ln^2}$	$+\frac{22363}{180}$
$C_{17/2}$	$+\frac{76704522232619}{15088550400}\pi$
$C_9^{\rm c}$	$\frac{9273891051462598777}{3814050240000}-\frac{7041196288536323}{687194767360}\pi^4-\frac{534085235454726901}{2536715059200}\pi^2-\frac{6382001}{350}\gamma^2+\frac{148431462289177}{198450}\ln(2)^2+\frac{76287}{5}\zeta(3)$ $+\frac{445208365512387}{10035200}\ln(3)^2-\frac{3307792499609375}{32514048}\ln(5)^2-\frac{8816899947037}{663552}\ln(7)^2-\frac{29787070952219425357}{273277255680}\ln(7)$ $+\frac{24221957299249248118 1}{143026884000}\ln(2)+\frac{3446985257889680656 25}{360515764224}\ln(5)-\frac{8635972471496543618 01}{1607237632000}\ln(3)-\frac{8816899947037}{331776}\gamma\ln(7)$ $-\frac{8816899947037}{331776}\ln(2)\ln(7)+\frac{40278774263897}{99225}\gamma\ln(2)+\frac{445208365512387}{5017600}\gamma\ln(3)+\frac{1075881868211907}{5017600}\ln(2)\ln(3)$ $-\frac{3307792499609375}{16257024}\ln(5)\gamma+\frac{42671896046383}{174636000}\gamma-\frac{3307792499609375}{16257024}\ln(5)\ln(2)$
$C_9^{\rm ln}$	$+\frac{42653978392783}{349272000}-\frac{8816899947037}{663552}\ln(7)-\frac{6382001}{350}\gamma+\frac{40278774263897}{198450}\ln(2)+\frac{445208365512387}{10035200}\ln(3)-\frac{3307792499609375}{32514048}\ln(5)$
$C_9^{\rm ln^2}$	$-\frac{6382001}{1400}$
$C_{19/2}^{\rm c}$	$\frac{3162423854803}{95256000}\pi\gamma-\frac{3345263881047}{784000}\pi\ln(3)+\frac{1900712890625}{72576}\pi\ln(5)-\frac{5131372911332653}{95256000}\pi\ln(2)-\frac{4411156827190136540051 3}{154661262852096000}\pi$ $-\frac{29555363129}{2721600}\pi^3$
$C_{19/2}^{\rm ln}$	$+\frac{3162423854803}{190512000}\pi$

where the $q=0$ term is the test-mass (Schwarzschild) result. The 1SF redshift is the function $\delta z_1(\hat{\Omega}_r,\hat{\Omega}_\phi)$, which can be alternatively expressed as a function of the unperturbed (Schwarzschild-background) semi-latus rectum $p^{\rm phys}\equiv Gm_2 p$ and eccentricity e. Denoting $u\equiv\frac{1}{p}$, the function $\delta z_1(u,e)$ is obtained as an expansion in powers of e, say

$$\delta z_1(u,e)=\delta z_1^{e^0}(u)+e^2\delta z_1^{e^2}(u)+\cdots+e^8\delta z_1^{e^8}(u)+O(e^{10}), \tag{5.2}$$

where each coefficient $\delta z_1^{e^{2n}}(u)$ is computed as a PN expansion (i.e., an expansion in powers of u) up to some order.

At the 4PN approximation, the functions $\delta z_1^{e^{2n}}(u)$ have been determined up to the order $O(e^{20})$ in Refs. [55,56]. Higher-PN order computations of the functions $\delta z_1^{e^2}(u)$ and $\delta z_1^{e^4}(u)$ were done in Refs. [54,55] through the 9.5PN order (i.e., up to $u^{9.5}$), while the term $\delta z_1^{e^6}(u)$ was computed to the same accuracy in our recent 5PN-level works [1,2]. For the present 6PN-level work, we needed to extend this determination to the function $\delta z_1^{e^8}(u)$. Our result for this function (up to the 9.5PN order) reads:

$$\begin{aligned}\delta z_1^{e^8}(u)=&\,C_3u^3+C_4u^4+(C_5^{\rm c}+C_5^{\rm ln}\ln u)u^5+(C_6^{\rm c}+C_6^{\rm ln}\ln u)u^6+C_{13/2}u^{13/2}\\&+(C_7^{\rm c}+C_7^{\rm ln}\ln u)u^7+C_{15/2}u^{15/2}+(C_8^{\rm c}+C_8^{\rm ln}\ln u+C_8^{\rm ln^2}\ln^2 u)u^8+C_{17/2}u^{17/2}\\&+(C_9^{\rm c}+C_9^{\rm ln}\ln u+C_9^{\rm ln^2}\ln^2 u)u^9+(C_{19/2}^{\rm c}+C_{19/2}^{\rm ln}\ln u)u^{19/2}+O_{\ln(u)}(u^{10}),\end{aligned} \tag{5.3}$$

where the coefficients C_i are listed in Table VII.

TABLE VIII. List of the various coefficients entering the self-force based expression of $q_8(u)$.

B_1	$-\frac{27734375}{126}\ln(5)+\frac{6591861}{350}\ln(3)+\frac{21668992}{45}\ln(2)-\frac{35772}{175}$
B_2	$\frac{13841287201}{17280}\ln(7)-\frac{393786545409}{156800}\ln(3)-\frac{16175693888}{1575}\ln(2)+\frac{875090984375}{169344}\ln(5)+\frac{5790381}{2450}$
$B_{5/2}$	$+\frac{5994461}{12700800}\pi$
B_3	$-\frac{29247366220639}{933120}\ln(7)-\frac{63886617280625}{1016064}\ln(5)+\frac{5196312336176}{35721}\ln(2)+\frac{17515638027261}{313600}\ln(3)-\frac{2843819611}{529200}$
$B_{7/2}$	$+\frac{12986592749}{22759833600}\pi$
$B_4^{\rm c}$	$\frac{25659132742}{606375}\gamma-\frac{2458476234653610278}{1719073125}\ln(2)$
	$+\frac{835937500}{27}\ln(5)\ln(2)+\frac{835937500}{27}\ln(5)\gamma-\frac{21240840924}{1225}\ln(2)\ln(3)$
	$-\frac{4080804948}{875}\gamma\ln(3)-\frac{3558749575168}{55125}\gamma\ln(2)-\frac{806339890542506373}{1379840000}\ln(3)+\frac{581245383137875}{1824768}\ln(5)$
	$+\frac{946254728855647813}{1642291200}\ln(7)+\frac{417968750}{27}\ln(5)^2-\frac{2040402474}{875}\ln(3)^2-\frac{60734915396608}{496125}\ln(2)^2$
	$-\frac{303760055}{11010048}\pi^2-\frac{68634427305713}{185220000}$
$B_4^{\rm ln}$	$+\frac{12829566371}{606375}-\frac{1779374787584}{55125}\ln(2)-\frac{2040402474}{875}\ln(3)+\frac{417968750}{27}\ln(5)$
$B_{9/2}$	$+\frac{341391869291507}{18435465216000}\pi$
$B_5^{\rm c}$	$-\frac{4856666007821}{6306300}\gamma+\frac{624385478329109678 27}{5005940940}\ln(2)+\frac{63988691154965631875}{37970989056}\ln(5)$
	$-\frac{675460646171875}{889056}\ln(5)\ln(2)-\frac{675460646171875}{889056}\ln(5)\gamma+\frac{1155153739426227}{1372000}\ln(2)\ln(3)$
	$+\frac{496094995065267}{1372000}\gamma\ln(3)+\frac{5087539076789248}{3472875}\gamma\ln(2)-\frac{1259557135291}{12960}\ln(2)\ln(7)-\frac{1259557135291}{12960}\gamma\ln(7)$
	$+\frac{400332056150861177697}{1757916160000}\ln(3)-\frac{63730386553687696488 73}{1067489280000}\ln(7)$
	$-\frac{1259557135291}{25920}\ln(7)^2-\frac{675460646171875}{1778112}\ln(5)^2+\frac{496094995065267}{2744000}\ln(3)^2+\frac{9330506645499392}{3472875}\ln(2)^2$
	$-\frac{740234446559}{176160768}\pi^2+\frac{828093816579233391 31}{13984850880000}$
$B_5^{\rm ln}$	$-\frac{4856666007821}{12612600}+\frac{2543769538394624}{3472875}\ln(2)-\frac{1259557135291}{25920}\ln(7)+\frac{496094995065267}{2744000}\ln(3)-\frac{675460646171875}{1778112}\ln(5)$
$B_{11/2}^{\rm c}$	$-\frac{18492868322811}{1097600}\pi\ln(3)+\frac{20550060546875}{222264}\pi\ln(5)-\frac{627504931547331563}{3333960000}\pi\ln(2)$
	$+\frac{1587378124097}{3333960000}\pi\gamma+\frac{37938867020822625604207}{240584186658816000}\pi-\frac{14835309571}{95256000}\pi^3$
$B_{11/2}^{\rm ln}$	$+\frac{1587378124097}{6667920000}\pi$

The gauge-invariant information contained in the 1SF-accurate (first order in mass ratio) function $\delta z_1^{e^8}$ can then be converted (by extending the results of Ref. [47]) into the corresponding $O(\nu)$ contribution to the EOB potential $q_8(u;\nu)$ parametrizing the term $q_8(u;\nu)p_r^8 \in \hat{Q}(u,p_r)$. More precisely, writing as above

$$q_8(u;\nu)=\nu q_8^{\nu^1}(u)+\nu^2 q_8^{\nu^2}(u)+\nu^3 q_8^{\nu^3}(u)+\ldots, \quad (5.4)$$

the 1SF result $\delta z_1^{e^8}(u)$, Eq. (5.3), leads to the determination of the $O(\nu)$ coefficient $q_8^{\nu^1}(u)$ to a reduced (fractional) 5.5PN accuracy. [In view of Eq. (4.12), such an accuracy corresponds to an absolute 8.5PN accuracy of the Hamiltonian, which is more than enough for reaching our aimed 6PN accuracy.] We find the following $u^{5.5}$ accurate value for $q_8^{\nu^1}(u)$:

$$\begin{aligned} q_8^{\nu^1}(u) &= B_1 u + B_2 u^2 + B_{5/2}u^{5/2} + B_3 u^3 + B_{7/2}u^{7/2} \\ &\quad + (B_4^{\rm c} + B_4^{\rm ln}\ln u)u^4 + B_{9/2}u^{9/2} \\ &\quad + (B_5^{\rm c} + B_5^{\rm ln}\ln u)u^5 + (B_{11/2}^{\rm c} + B_{11/2}^{\rm ln}\ln u)u^{11/2} \\ &\quad + O_{\ln(u)}(u^6), \end{aligned} \quad (5.5)$$

where the various coefficients are listed in Table VIII.

VI. DETERMINING THE LOCAL PART OF THE EOB POTENTIALS AT ORDER ν^1

The next step of our strategy is to derive the *local* part of the EOB Hamiltonian by subtracting the nonlocal part of the EOB potentials (obtained in Sec. IV) from their complete local-plus-nonlocal parts (obtained in Sec. V from self-force computations). As the self-force computation is only accurate to linear order in ν, we thereby determine the local part of the EOB potentials only at the first order in ν. The nonlocal part we computed was of the h-type (and was determined exactly in ν). However, recalling that we will always consider flexibility factors of the type $f=1+O(\nu)$, the h-route and f-route versions of both the nonlocal and the local Hamiltonians only differ at the second self-force order, *i.e.*, by terms of order $O(\nu^3)$ in the physical Hamiltonians, $H_{\rm nonloc,h,f}$, $H_{\rm loc,h,f}$, corresponding to terms of order $O(\nu^2)$ in the corresponding squared effective Hamiltonians, $\hat{H}^2_{\rm eff}$.

The values of the local EOB potentials at $4+5+6$PN, obtained from our results so far, can be written as:

$$
\begin{aligned}
a_{4+5+6\mathrm{PN,loc,f}} &= \left[\left(\frac{2275}{512}\pi^2-\frac{4237}{60}\right)\nu+\left(\frac{41}{32}\pi^2-\frac{221}{6}\right)\nu^2\right]u^5+\left[\left(-\frac{1026301}{1575}+\frac{246367}{3072}\pi^2\right)\nu+a^{(\nu)}_{6,f}\right]u^6\\
&\quad+\left[\left(-\frac{2800873}{262144}\pi^4+\frac{608698367}{1769472}\pi^2-\frac{1469618167}{907200}\right)\nu+a^{(\nu)}_{7,f}\right]u^7,\\
\bar{d}_{4+5+6\mathrm{PN,loc,f}} &= \left[\left(\frac{1679}{9}-\frac{23761}{1536}\pi^2\right)\nu+\left(-260+\frac{123}{16}\pi^2\right)\nu^2\right]u^4+\left(\frac{331054}{175}\nu-\frac{63707}{512}\nu\pi^2+\bar{d}^{(\nu)}_{5,f}\right)u^5\\
&\quad+\left[\left(\frac{229504763}{98304}\pi^2+\frac{135909}{262144}\pi^4-\frac{99741733409}{6350400}\right)\nu+\bar{d}^{(\nu)}_{6,f}\right]u^6,\\
q_{4,4+5+6\mathrm{PN,loc,f}} &= (20\nu+q^{(\nu)}_{43})u^3+\left[\left(-\frac{93031}{1536}\pi^2+\frac{1580641}{3150}\right)\nu+q^{(\nu)}_{44,f}\right]u^4\\
&\quad+\left[\left(\frac{81030481}{65536}\pi^2-\frac{3492647551}{423360}\right)\nu+q^{(\nu)}_{45,f}\right]u^5,\\
q_{6,4+5+6\mathrm{PN,loc,f}} &= \left(-\frac{9}{5}\nu+q^{(\nu)}_{62}\right)u^2+\left(\frac{123}{10}\nu+q^{(\nu)}_{63,f}\right)u^3\\
&\quad+\left[\left(-\frac{9733841}{327680}\pi^2-\frac{112218283}{294000}\right)\nu+q^{(\nu)}_{64,f}\right]u^4,\\
q_{8,5+6\mathrm{PN,loc,f}} &= q_{82}(\nu)u^2+\left(-\frac{7447}{560}\nu+q^{(\nu)}_{83}\right)u^3,\\
q_{10,6\mathrm{PN,loc,f}} &= q_{10,2}(\nu)u^2.
\end{aligned}
\tag{6.1}
$$

Here, the first coefficients in each line (except in the last two lines) belong to the 4PN level, and are equivalent to results obtained in Ref. [23]. The explicit values of $q^{(\nu)}_{43}$ and $q^{(\nu)}_{62}$ are

$$
q^{(\nu)}_{43}=-83\nu^2+10\nu^3,\qquad q^{(\nu)}_{62}=-\frac{27}{5}\nu^2+6\nu^3. \tag{6.2}
$$

As exemplified by these coefficients, we introduced here the general notation $C^{(\nu)}$ to denote all the contributions to any ν-dependent coefficient $C(\nu)$ that are nonlinear in ν, i.e.,

$$
C(\nu)=C^{\nu^1}\nu+C^{(\nu)};\quad \text{with}\quad C^{(\nu)}=C^{\nu^2}\nu^2+C^{\nu^3}\nu^3+\ldots \tag{6.3}
$$

The second coefficients in each line (and the first on the penultimate line) belong to the 5PN level, and were determined in our recent work [1], modulo two unknown coefficients at order ν^2. They read

$$
\begin{aligned}
a^{(\nu)}_{6,f} &= a^{\nu^2}_6\nu^2+4\nu^3,\\
\bar{d}^{(\nu)}_{5,f} &= \bar{d}^{\nu^2}_5\nu^2+\left(\frac{1069}{3}-\frac{205}{16}\pi^2\right)\nu^3,\\
q^{(\nu)}_{44,f} &= \left(-\frac{2075}{3}+\frac{31633}{512}\pi^2\right)\nu^2+\left(640-\frac{615}{32}\pi^2\right)\nu^3,\\
q^{(\nu)}_{63,f} &= -\frac{69}{5}\nu^2+116\nu^3-14\nu^4,
\end{aligned}
\tag{6.4}
$$

where $a^{\nu^2}_{6,f}$ and $\bar{d}^{\nu^2}_{5,f}$ are the only two numerical coefficients left undetermined at 5PN by our method.

Finally, the values of the 6PN-level coefficients are determined at the linear-in-ν level by our self-force computation and can be written as

$$
\begin{aligned}
a_{7,\mathrm{loc,f}}(\nu) &= \left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a^{(\nu)}_{7,f},\\
\bar{d}_{6,\mathrm{loc,f}}(\nu) &= \left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}^{(\nu)}_{6,f},\\
q_{45,\mathrm{loc,f}}(\nu) &= \left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q^{(\nu)}_{45,f},\\
q_{64,\mathrm{loc,f}}(\nu) &= \left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + q^{(\nu)}_{64,f},\\
q_{83,\mathrm{loc,f}}(\nu) &= -\frac{7447}{560}\nu + q^{(\nu)}_{83,f}.
\end{aligned}
\tag{6.5}
$$

At this stage, we have no information about the nonlinear-in-ν coefficients $a^{(\nu)}_{7,f}$, $\bar{d}^{(\nu)}_{6,f}$, $q^{(\nu)}_{45,f}$, $q^{(\nu)}_{64,f}$, and $q^{(\nu)}_{83,f}$. Let us, however, anticipate on the results of the following section, where we will show how to determine the four ν-nonlinear coefficients $\bar{d}^{(\nu)}_{6,f}$, $q^{(\nu)}_{45,f}$, $q^{(\nu)}_{64,f}$, $q^{(\nu)}_{83,f}$, in terms of only two free numerical parameters, namely $\bar{d}^{\nu^2}_6$, and $q^{\nu^2}_{45,f}$. In addition, we will find that $a^{(\nu)}_{7,f}$ is at most cubic in ν. Our final results will then read:

$$
\begin{aligned}
a^{(\nu)}_{7,f} &= a^{\nu^2}_7\nu^2 + a^{\nu^3}_7\nu^3,\\
\bar{d}^{(\nu)}_{6,f} &= \bar{d}^{\nu^2}_6\nu^2 + \left(\frac{45089}{72} - \frac{44489}{1536}\pi^2 - \bar{d}^{\nu^2}_5 - 15a^{\nu^2}_6\right)\nu^3 - 48\nu^4,\\
q^{(\nu)}_{45,f} &= q^{\nu^2}_{45}\nu^2 + \left(-\frac{474899}{216} + \frac{36677}{1152}\pi^2 - \frac{14}{3}\bar{d}^{\nu^2}_5\right)\nu^3 + \left(-\frac{7375}{6} + \frac{1435}{32}\pi^2\right)\nu^4,\\
q^{(\nu)}_{64,f} &= \left(-\frac{21996581}{21000} + \frac{156397}{1280}\pi^2\right)\nu^2 + \left(\frac{6977}{6} - \frac{29665}{256}\pi^2\right)\nu^3 + \left(-\frac{3640}{3} + \frac{287}{8}\pi^2\right)\nu^4,\\
q^{(\nu)}_{83,f} &= -\frac{963}{56}\nu^2 - \frac{117}{10}\nu^3 - 147\nu^4 + 18\nu^5.
\end{aligned}
\tag{6.6}
$$

In these results, the two coefficients $a^{\nu^2}_6$ and $\bar{d}^{\nu^2}_5$ come from the 5PN level, while the new undetermined 6PN-level numerical coefficients are $a^{\nu^2}_7$, $a^{\nu^3}_7$, $\bar{d}^{\nu^2}_6$, and $q^{\nu^2}_{45}$. [The origin of these undetermined coefficients will be discussed below.]

The coefficient $q_{10,2}(\nu)$ of $p^{10}_r u^2$ cannot be extracted from our $O(e^8)$ self-force results, but it can be derived from the exact knowledge of the 2PM ($O(G^2)$) EOB Hamiltonian [57], as will be shown below. Its value is

$$
q_{10,2}(\nu) = -\frac{11}{21}\nu - \frac{11}{7}\nu^2 - \frac{20}{7}\nu^3 - \frac{5}{3}\nu^4 + 6\nu^5. \tag{6.7}
$$

Note the remarkable fact that the $4+5+6$PN-accurate local $O(\nu)$ EOB Hamiltonian is *logarithm free*. Not only all the $\ln u$ terms present in the nonlocal EOB potentials have disappeared (as expected because they have been known for a long time to be linked to the time nonlocality), but even the various numerical logarithms $\ln 2, \ln 3, \ldots$, as well as Euler's constant γ have all disappeared. Only rational numbers, and $\pi^2 \sim \zeta(2)$ enter the $O(\nu)$ local Hamiltonian. In addition, the fractional powers of u have also disappeared because they only come from the nonlocal 5.5PN action.

Note finally that, contrary to the nonlocal EOB potentials shown above, there are no contributions to the local EOB potentials featuring powers of u strictly smaller than 2. This follows from the fact that the PM expansion of the exact potential Q starts at order G^2 [57]. Contributions to Q involving powers u^n with $n < 2$ can only enter the nonlocal part of the Hamiltonian, where they come from having expanded the nonlocal Hamiltonian as a formally infinite series of powers of p^2_r [23].

VII. USING THE MASS-RATIO DEPENDENCE OF THE SCATTERING ANGLE TO DETERMINE MOST OF THE $\nu^{n\geq 2}$ STRUCTURE OF THE 6PN f-ROUTE LOCAL HAMILTONIAN

Up to this stage, our method has only determined (besides the full nonlocal part of the Hamiltonian) the linear-in-ν part of the local Hamiltonian. The next stage of our method is to use the special ν-dependence of the scattering angle pointed out in Ref. [58] to determine most of the nonlinear dependence on ν of the local Hamiltonian. (See [59] for a generalization of this approach to the dynamics of spinning bodies.) This is done by going through several steps.

A. Going from the p_r-gauge to the energy-gauge

As a first step, it is convenient to transform the above p_r-gauge form of the local EOB effective Hamiltonian, (4.10), to its (H-type) energy-gauge version, defined by writing

$$
\hat{H}^{2\,\mathrm{EG}}_{\mathrm{eff,loc,f}}(u, p_r, j; \nu) = H^2_S + (1-2u)\hat{Q}^{\mathrm{EG}}_{H\mathrm{loc,f}}(u, H_S; \nu), \tag{7.1}
$$

where H_S denotes the (rescaled) Schwarzschild Hamiltonian, i.e., the square root of

$$H_S^2(u,p_r,j)=(1-2u)[1+(1-2u)p_r^2+j^2u^2], \tag{7.2}$$

and where

$$\begin{aligned}\hat{Q}^{\rm EG}_{H{\rm loc,f}}(u,H_S;\nu)&=u^2q^H_{2\rm EG}(H_S;\nu)+u^3q^H_{3\rm EG}(H_S;\nu)\\&+u^4q^{H{\rm loc,f}}_{4\rm EG}(H_S;\nu)+u^5q^{H{\rm loc,f}}_{5\rm EG}(H_S;\nu)\\&+u^6q^{H{\rm loc,f}}_{6\rm EG}(H_S;\nu)+u^7q^{H{\rm loc,f}}_{7\rm EG}(H_S;\nu).\end{aligned} \tag{7.3}$$

We have added a label "H" on $\hat{Q}^{\rm EG}_{H{\rm loc,f}}$ and its u-expansion coefficients, as a reminder that we use here the H-version of the energy gauge, by contrast to its E-version [58]. This means that $\hat{Q}^H$ is directly written as a function of the phase-space variable q, p, via the argument $H_S(u,p_r,j)$. In the E-version of the energy-gauge $\hat{Q}$ is written as a function of u and the effective energy $\hat{\mathcal{E}}_{\rm eff}$:

$$\begin{aligned}\hat{Q}^{\rm EG}_{E{\rm loc,f}}(u,\hat{\mathcal{E}}_{\rm eff};\nu)&=u^2q^E_{2\,\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu)+u^3q^E_{3\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu)\\&+u^4q^{E{\rm loc,f}}_{4\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu)+u^5q^{E{\rm loc,f}}_{5\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu)\\&+u^6q^{E{\rm loc,f}}_{6\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu)+u^7q^{E{\rm loc,f}}_{7\rm EG}(\hat{\mathcal{E}}_{\rm eff};\nu).\end{aligned} \tag{7.4}$$

The difference between the two sequences of expansion coefficients only start at the $u^4\propto G^4$ level, so that the first two functions[5] coincide with each other: $q^H_{2\rm EG}(\gamma;\nu)=q^E_{2\rm EG}(\gamma;\nu)$, $q^H_{3\rm EG}(\gamma;\nu)=q^E_{3\rm EG}(\gamma;\nu)$. We henceforth denote them simply as $q_{2\,\rm EG}(\gamma;\nu)$ and $q_{3\rm EG}(\gamma;\nu)$. [See below for the link between the higher-order coefficients.] We did not put any extra label "loc, f" on the first two coefficients because the effect of the flexibility coefficient f only starts at the G^4 level.

The energy-dependent coefficient $q^{H{\rm loc,f}}_{n\rm EG}(\gamma;\nu)$ belongs to the n-PM approximation because $u^n=(GM/(r^{\rm phys}c^2))^n$ is proportional to G^n. The 2PM coefficient $q_{2\rm EG}(\gamma;\nu)$ is known exactly. It has been first obtained in Ref. [57], and then confirmed in Refs. [60–62]. The 3PM coefficient $q_{3\rm EG}(\gamma;\nu)$ has so far only be derived (as a closed-form function of γ and ν) in Refs. [61,62]. Its 5PN expansion was confirmed in Ref. [1], and its 6PN expansion was recently confirmed in Refs. [2,63,64]. We will give below the details of our derivation of the 6PN-accurate value of $q_{3\rm EG}(\gamma;\nu)$. The higher PM-order coefficients $q^{H{\rm loc,f}}_{n\rm EG}(\gamma;\nu)$ are currently only known in their PN-expanded versions, say

$$\begin{aligned}q^{H{\rm loc,f}}_{n\rm EG}(\gamma;\nu)&=q^0_{n\rm EG}(\nu)+q^1_{n\rm EG}(\nu)(\gamma^2-1)\\&+q^2_{n\rm EG}(\nu)(\gamma^2-1)^2+\cdots.\end{aligned} \tag{7.5}$$

[5]When γ is used, as here, to denote the argument of $q^{H{\rm loc,f}}_{n\rm EG}$, it is understood as a mathematical argument, to be later replaced by $H_S(u,p_r,j)$.

We recall that the properties of the EOB formalism are such that the full potential $Q(u,\gamma;\nu)$ vanishes in the test-mass limit $\nu\to 0$, so that each PN expansion coefficient $q^p_{n\rm EG}(\nu)$ must be $\sim\nu+\nu^2+\cdots$ when $\nu\to 0$.

The PN expansions of all the energy-gauge coefficients $q^{H,{\rm loc,f}}_{n\rm EG}(\gamma)$ are determined from the corresponding p_r-gauge coefficients entering the Hamiltonian (notably the 6PN-level ones $a_7^{(\nu)}$, $\bar{d}_6^{(\nu)}$, $q_{45}^{(\nu)}$, $q_{64}^{(\nu)}$ and $q_{83}^{(\nu)}$) by computing the canonical transformation connecting the two gauges. The structure of this canonical transformation is

$$g(r,p_r)=(rp_r)[g_{2\rm PN}+g_{3\rm PN}+g_{4\rm PN}+g_{5\rm PN}+g_{6\rm PN}], \tag{7.6}$$

where the factor rp_r would describe an identity transformation, and where the leading-order term is at the 2PN (and 2PM) level, and reads

$$g_{2\rm PN}=\frac{3}{2}\eta^4\frac{\nu}{r^2}. \tag{7.7}$$

The 2PN ($g_{2\rm PN}$) and 3PN ($g_{3\rm PN}$) terms were derived in Ref. [57]; the 4PN one ($g_{4\rm PN}$) was derived in Appendix A of Ref. [65]; and the 5PN one ($g_{5\rm PN}$) was derived in our recent work [2]. We have extended the determination of the canonical transformation $g(r,p_r)$ to the 6PN level. This is done by using the method of undetermined coefficients. The looked-for $g_{6\rm PN}$ is parametrized as

$$\begin{aligned}g_{6\rm PN}=\frac{1}{r^2}\Bigg[&\frac{w_1j^4}{r^5}p_r^2+\frac{w_2j^8}{r^8}+w_3p_r^8+\frac{w_4}{r^4}+\frac{w_5}{r}p_r^6\\&+\frac{w_6j^4}{r^4}p_r^4+\frac{w_7j^2}{r^2}p_r^6+\frac{w_8j^2}{r^4}p_r^2+\frac{w_9j^2}{r^3}p_r^4\\&+\frac{w_{10}j^6}{r^6}p_r^2+\frac{w_{11}}{r^2}p_r^4+\frac{w_{12}}{r^3}p_r^2+\frac{w_{13}j^2}{r^5}\\&+\frac{w_{14}j^4}{r^6}+\frac{w_{15}j^6}{r^7}\Bigg],\end{aligned} \tag{7.8}$$

with unknown coefficients $w_1,\ldots,w_{15}$. The values of these coefficients are then determined by imposing that the two (effective, squared) Hamiltonians (4.1) (with $A=A_{\rm loc,f}$, etc.) and (7.1) are equivalent (at the 6PN accuracy) through this canonical transformation.

The explicit expressions of the 6PN coefficients $w_1\ldots w_{15}$ will be displayed later, in their final form, in Table IX, after we determine, using our strategy, all possible unknowns.

B. Computing the f-route local scattering angle

The next step in the determination of many of the nonlinear-in-ν coefficients in the local EOB Hamiltonian proceeds through the computation of the corresponding scattering angle, $\chi^{\rm loc,f}$. This is most efficiently done in the energy-gauge.

TABLE IX. Final form of the coefficients $w_1 \ldots w_{15}$ parametrizing the 6PN canonical transformation, Eq. (7.8).

w_1	$-\frac{236879}{80640}\nu - \frac{6753}{1792}\nu^2 - \frac{197}{320}\nu^3 - \frac{4965}{256}\nu^4 - \frac{1417}{128}\nu^5$
w_2	$-\frac{33}{512}\nu - \frac{99}{512}\nu^2 - \frac{45}{128}\nu^3 - \frac{105}{512}\nu^4 + \frac{189}{256}\nu^5$
w_3	$-\frac{2123}{10752}\nu - \frac{2123}{3584}\nu^2 - \frac{965}{896}\nu^3 - \frac{965}{1536}\nu^4 + \frac{579}{256}\nu^5$
w_4	$(\frac{1483514111}{9437184}\pi^2 - \frac{228466894127}{190512000} + \frac{45303}{1048576}\pi^4)\nu + (\frac{10486361}{84000} + \frac{1}{12}\bar{d}_6^{\nu^2} - \frac{132333}{40960}\pi^2 + \frac{1}{12}a_6^{\nu^2} - \frac{1}{60}\bar{d}_5^{\nu^2} - \frac{1}{20}q_{45}^{\nu^2})\nu^2$ $+(-\frac{500837}{61440}\pi^2 + \frac{213103}{1440} + \frac{3}{20}\bar{d}_5^{\nu^2} - \frac{5}{4}a_6^{\nu^2})\nu^3 + \frac{3}{32}\nu^4 + \frac{21}{16}\nu^5$
w_5	$-\frac{284849}{80640}\nu - \frac{9951}{1792}\nu^2 - \frac{8829}{2240}\nu^3 - \frac{5595}{256}\nu^4 - \frac{463}{128}\nu^5$
w_6	$-\frac{11}{20}\nu - \frac{33}{20}\nu^2 - 3\nu^3 - \frac{7}{4}\nu^4 + \frac{63}{10}\nu^5$
w_7	$-\frac{869}{1792}\nu - \frac{2607}{1792}\nu^2 - \frac{1185}{448}\nu^3 - \frac{395}{256}\nu^4 + \frac{711}{128}\nu^5$
w_8	$(-\frac{48669205}{12582912}\pi^2 - \frac{781859}{17640})\nu + (-\frac{2898667}{20160} + \frac{781985}{49152}\pi^2)\nu^2 + (\frac{119531}{720} - \frac{741625}{49152}\pi^2)\nu^3 + (-\frac{117029}{1152} + \frac{7175}{1536}\pi^2)\nu^4 + \frac{753}{64}\nu^5$
w_9	$-\frac{138127}{26880}\nu - \frac{14067}{1792}\nu^2 - \frac{1423}{320}\nu^3 - \frac{7199}{256}\nu^4 - \frac{1527}{128}\nu^5$
w_{10}	$-\frac{77}{256}\nu - \frac{231}{256}\nu^2 - \frac{105}{64}\nu^3 - \frac{245}{256}\nu^4 + \frac{441}{128}\nu^5$
w_{11}	$(-\frac{574296619}{125829120}\pi^2 - \frac{200657371}{3528000})\nu + (-\frac{158915363}{1008000} + \frac{9227423}{491520}\pi^2)\nu^2 + (\frac{491849}{2880} - \frac{1750235}{98304}\pi^2)\nu^3 + (-\frac{329935}{2304} + \frac{16933}{3072}\pi^2)\nu^4 + \frac{687}{128}\nu^5$
w_{12}	$(-\frac{4561111909}{5292000} + \frac{15680782981}{110100480}\pi^2)\nu + (-\frac{2689283}{172032}\pi^2 + \frac{134682217}{1764000} + \frac{4}{35}q_{45}^{\nu^2})\nu^2 + (-\frac{1559441}{4320} - \frac{8}{15}\bar{d}_5^{\nu^2} + \frac{3831013}{184320}\pi^2)\nu^3$ $+(\frac{409}{72} - \frac{205}{384}\pi^2)\nu^4 - \frac{33}{8}\nu^5$
w_{13}	$(-\frac{2031118237}{7056000} + \frac{5643368761}{110100480}\pi^2)\nu + (-\frac{416103}{57344}\pi^2 + \frac{4901243}{196000} + \frac{3}{70}q_{45}^{\nu^2})\nu^2 + (-\frac{231013}{1440} - \frac{1}{5}\bar{d}_5^{\nu^2} + \frac{692411}{61440}\pi^2)\nu^3$ $+(\frac{1343}{48} - \frac{205}{128}\pi^2)\nu^4 - \frac{9}{2}\nu^5$
w_{14}	$(-\frac{9733841}{8388608}\pi^2 - \frac{2868989}{235200})\nu + (-\frac{2745397}{67200} + \frac{156397}{32768}\pi^2)\nu^2 + (\frac{48821}{960} - \frac{148325}{32768}\pi^2)\nu^3 + (-\frac{16513}{768} + \frac{1435}{1024}\pi^2)\nu^4 + \frac{747}{128}\nu^5$
w_{15}	$-\frac{44651}{80640}\nu - \frac{717}{1792}\nu^2 + \frac{207}{320}\nu^3 - \frac{1425}{256}\nu^4 - \frac{433}{128}\nu^5$

Several procedures (discussed in Refs. [57,58]) can be used to compute the expansion of $\chi^{\mathrm{loc,f}}(\gamma, j)$ in powers of $\frac{1}{j} \propto G$, at a fixed value of the EOB effective energy $\gamma \equiv \hat{\mathcal{E}}_{\mathrm{eff}}$. One uses the fact that, given any (local) Hamiltonian, the corresponding scattering angle of hyperboliclike motions is given by the integral ($u = 1/r$) [66]

$$\frac{1}{2}(\chi(\gamma, j) + \pi) = -\int_0^{u_{\max}} \frac{\partial}{\partial j} p_r(u; \gamma, j) \frac{du}{u^2}, \tag{7.9}$$

where $u_{\max} = u_{\max}(\gamma, j) = 1/r_{\min}$ corresponds to the distance of closest approach of the two bodies, and where the radial momentum $p_r = p_r(u; \gamma, j)$ is obtained from writing the energy conservation at a given angular momentum. When using the H-version of the energy gauge, Eq. (7.1) directly defines the squared effective Hamiltonian, $\hat{H}^{\mathrm{2EG}}_{\mathrm{eff,loc,f}}(u, p_r, j; \nu)$, as a function of p_r, j and u. To obtain p_r as a function of $\gamma \equiv \hat{\mathcal{E}}_{\mathrm{eff}}$ one should then iteratively solve for p_r (in a PM expanded way, i.e., using the scaling $u \mapsto Gu$ and $j \mapsto G^{-1}j$) the energy conservation law

$$\begin{aligned}\gamma^2 = \hat{\mathcal{E}}^2_{\mathrm{eff}} &= \hat{H}^{\mathrm{2EG}}_{\mathrm{eff,loc,f}}(u, p_r, j; \nu) \\ &= H_S^2 + (1 - 2u)\hat{Q}^{\mathrm{EG}}_{H\mathrm{loc,f}}(u, H_S; \nu),\end{aligned} \tag{7.10}$$

where $H_S(u, p_r, j)$ was defined in Eq. (7.2), and $\hat{Q}^{\mathrm{EG}}_{H\mathrm{loc,f}}(u, H_S; \nu)$ in Eq. (7.3). The computation of the function $p_r(\hat{\mathcal{E}}_{\mathrm{eff}}, j)$ is simpler when using the E-version of the energy gauge, i.e., Eq. (7.4). Indeed, in that case the EOB mass-shell condition reads

$$-\frac{\hat{\mathcal{E}}^2_{\mathrm{eff}}}{1 - 2u} + 1 + (1 - 2u)p_r^2 + j^2u^2 + \hat{Q}^{\mathrm{EG}}_{E\mathrm{loc,f}}(u, \hat{\mathcal{E}}_{\mathrm{eff}}; \nu) = 0, \tag{7.11}$$

which is a *linear equation* in $p_r^2(\hat{\mathcal{E}}_{\mathrm{eff}}, j, u)$ whose exact solution reads (denoting again $\gamma \equiv \hat{\mathcal{E}}_{\mathrm{eff}}$)

$$p_r^2(\gamma, j, u) = \frac{\gamma^2 - (1 - 2u)(1 + j^2u^2 + \hat{Q}^{\mathrm{EG}}_{E\mathrm{loc,f}}(u, \gamma; \nu))}{(1 - 2u)^2}. \tag{7.12}$$

In both cases, one expands $p_r(\gamma, j, u)$, as it appears in Eq. (7.9), in powers of $u \mapsto Gu$, say

$$p_r = p_r^{(0)} + Gp_r^{(1)} + G^2p_r^{(2)} + \ldots, \tag{7.13}$$

whose first two terms read

$$p_r^{(0)} = \sqrt{-1 + \gamma^2 - j^2u^2}, \tag{7.14}$$

and

$$p_r^{(1)} = \frac{(-1 + 2\gamma^2 - j^2u^2)u}{\sqrt{-1 + \gamma^2 - j^2u^2}}. \tag{7.15}$$

All the integrals that appear in the PM expansion of Eq. (7.9) are elementary and are evaluated (following [40]) by using Hadamard's *partie finie*.

The scattering angle is then obtained as a PM expansion of the form

$$\frac{1}{2}\chi^{\mathrm{loc,f}}(\gamma,j;\nu)=\sum_{n\geq 1}\frac{\chi_n^{\mathrm{loc,f}}(\gamma;\nu)}{j^n}. \tag{7.16}$$

Here, each n-PM-order expansion coefficient $\chi_n^{\mathrm{loc,f}}(\gamma;\nu)$ is determined from the value of the corresponding n-PM-order energy-gauge coefficient $q_{n\mathrm{EG}}^{H\mathrm{loc,f}}(\gamma;\nu)$, or $q_{n\mathrm{EG}}^{E\mathrm{loc,f}}(\gamma;\nu)$, together with the values of the lower PM-order coefficients.

Denoting, for brevity,

$$\Delta\chi_n(\gamma)\equiv\chi_n(\gamma)-\chi_n^{\mathrm{Schw}}(\gamma), \tag{7.17}$$

the scattering-angle coefficients obtained from the *E-version* $\hat{Q}^{\mathrm{EG}}_{E\mathrm{loc,f}}(u,\gamma;\nu)$ corrected of the energy gauge [which is simpler to implement in view of the explicit expression (7.12)] read

$$\begin{aligned}
\Delta\chi_2(\gamma)&=-\frac{\pi}{4}q_2(\gamma),\qquad \Delta\chi_3(\gamma)=-\frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}q_2(\gamma)-\sqrt{\gamma^2-1}q_3(\gamma),\\
\Delta\chi_4(\gamma)&=\pi\left[\frac{3}{16}q_2(\gamma)^2-\frac{9}{16}(-1+5\gamma^2)q_2(\gamma)-\frac{3}{8}(-1+3\gamma^2)q_3(\gamma)-\frac{3}{8}q_4^E(\gamma)(\gamma^2-1)\right],\\
\Delta\chi_5(\gamma)&=\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}}q_2(\gamma)^2+\left[2\sqrt{\gamma^2-1}q_3(\gamma)-\frac{2}{3}\frac{(60\gamma^2-5+64\gamma^6-120\gamma^4)}{(\gamma^2-1)^{3/2}}\right]q_2(\gamma)\\
&\quad-2\frac{(8\gamma^4+1-8\gamma^2)}{\sqrt{\gamma^2-1}}q_3(\gamma)-\frac{4(4\gamma^2-1)\sqrt{\gamma^2-1}}{3}q_4^E(\gamma)-\frac{4(\gamma^2-1)^{3/2}}{3}q_5^E(\gamma),\\
\Delta\chi_6(\gamma)&=-\frac{5}{32}q_2(\gamma)^3+\left(-\frac{45}{64}+\frac{225}{64}\gamma^2\right)q_2(\gamma)^2+\left[\left(\frac{45}{16}\gamma^2-\frac{15}{16}\right)q_3(\gamma)+\left(-\frac{15}{16}+\frac{15}{16}\gamma^2\right)q_4^E(\gamma)\right.\\
&\quad\left.-\frac{17325}{256}\gamma^4+\frac{4725}{128}\gamma^2-\frac{525}{256}\right]q_2(\gamma)+\left(\frac{15}{32}\gamma^2-\frac{15}{32}\right)q_3(\gamma)^2+\left(-\frac{1575}{64}\gamma^4+\frac{525}{32}\gamma^2-\frac{75}{64}\right)q_3(\gamma)\\
&\quad+\left(-\frac{525}{64}\gamma^4+\frac{225}{32}\gamma^2-\frac{45}{64}\right)q_4^E(\gamma)+\left(-\frac{75}{32}\gamma^4-\frac{15}{32}+\frac{45}{16}\gamma^2\right)q_5^E(\gamma)+\left(-\frac{15}{32}-\frac{15}{32}\gamma^4+\frac{15}{16}\gamma^2\right)q_6^E(\gamma).
\end{aligned} \tag{7.18}$$

The first three equations above (for χ_2, χ_3, χ_4) agree with the corresponding ones in Ref. [58].

While the E-version of the energy-gauge is more simply connected to the scattering angle, the H-version is more simply connected to the usual p_r-gauge EOB Hamiltonian. This is why we use the H-version in practice, as indicated in Eq. (7.5). Let us therefore complete the above E-type scattering-angle results by the transformation between the E-type coefficients, $q_{n\mathrm{EG}}^{E\mathrm{loc,f}}(\gamma;\nu)$, and the H-type ones, $q_{n\mathrm{EG}}^{H\mathrm{loc,f}}(\gamma;\nu)$. We recall that the first two, q_2 and q_3, are the same. To write the link between the higher-order ones, it is convenient to provisionally use as common argument for these functions $x\equiv\gamma^2$. By writing that Eqs. (7.10) and (7.11) define the same mass-shell constraint one finds (where, for uniformity, we have left the labels E or H on $q_2^E=q_2^H=q_2$ and $q_3^E=q_3^H=q_3$):

$$\begin{aligned}
q_4^H(x)&=q_4^E(x)+q_2^E(x)\frac{dq_2^E(x)}{dx},\\
q_5^H(x)&=(q_3^E(x)-2q_2^E(x))\frac{dq_2^E(x)}{dx}+q_5^E(x)+\frac{dq_3^E(x)}{dx}q_2^E(x),\\
q_6^H(x)&=\frac{1}{2}\frac{d^2q_2^E(x)}{dx^2}q_2^E(x)^2+q_2^E(x)\left(\frac{dq_2^E(x)}{dx}\right)^2+(q_4^E(x)-2q_3^E(x))\frac{dq_2^E(x)}{dx}\\
&\quad+(q_3^E(x)-2q_2^E(x))\frac{dq_3^E(x)}{dx}+q_6^E(x)+q_2^E(x)\frac{dq_4^E(x)}{dx},\\
q_7^H(x)&=(-2q_2^E(x)^2+q_2^E(x)q_3^E(x))\frac{d^2q_2^E(x)}{dx^2}+\frac{1}{2}\frac{d^2q_3^E(x)}{dx^2}q_2^E(x)^2+(q_3^E(x)-4q_2^E(x))\left(\frac{dq_2^E(x)}{dx}\right)^2\\
&\quad+\left[2\frac{dq_3^E(x)}{dx}q_2^E(x)+q_5^E(x)-2q_4^E(x)\right]\frac{dq_2^E(x)}{dx}+(q_4^E(x)-2q_3^E(x))\frac{dq_3^E(x)}{dx}\\
&\quad+(q_3^E(x)-2q_2^E(x))\frac{dq_4^E(x)}{dx}+q_7^E(x)+q_2^E(x)\frac{dq_5^E(x)}{dx},
\end{aligned} \tag{7.19}$$

which can also be written in the reverse direction:

$$
\begin{aligned}
q_4^E(x) &= -\frac{dq_2^H(x)}{dx} q_2^H(x) + q_4^H(x),\\
q_5^E(x) &= \frac{dq_2^H(x)}{dx}[-q_3^H(x) + 2q_2^H(x)] - \frac{dq_3^H(x)}{dx} q_2^H(x) + q_5^H(x),\\
q_6^E(x) &= -\frac{dq_4^H(x)}{dx} q_2^H(x) + \left(\frac{dq_2^H(x)}{dx}\right)^2 q_2(x) - \frac{dq_2^H(x)}{dx}(q_4^H(x) - 2q_3^H(x)) + \frac{1}{2}\frac{d^2q_2^H(x)}{dx^2}[q_2^H(x)]^2\\
&\quad - \frac{dq_3^H(x)}{dx}(q_3^H(x) - 2q_2^H(x)) + q_6^H(x),\\
q_7^E(x) &= -q_2^H(x)(-q_3^H(x) + 2q_2^H(x))\frac{d^2q_2^H(x)}{dx^2} + \frac{1}{2}\frac{d^2q_3^H(x)}{dx^2} q_2^H(x)^2 + (q_3^H(x) - 4q_2^H(x))\left(\frac{dq_2^H(x)}{dx}\right)^2\\
&\quad + \left[2\frac{dq_3^H(x)}{dx} q_2^H(x) - q_5^H(x) + 2q_4^H(x)\right]\frac{dq_2^H(x)}{dx}\\
&\quad + (-q_4^H(x) + 2q_3^H(x))\frac{dq_3^H(x)}{dx} + (-q_3^H(x) + 2q_2^H(x))\frac{dq_4^H(x)}{dx} - \frac{dq_5^H(x)}{dx} q_2^H(x) + q_7^H(x).
\end{aligned}
\tag{7.20}
$$

C. Using the mass-ratio dependence of the f-route local scattering angle

Applying the scattering-angle results derived in the previous subsection to our PN-expanded parametrization of the H-type energy-gauge coefficients, Eq. (7.5), yields explicit, PN-expanded (6PN-accurate) expressions for the scattering angle. [These can also be obtained by directly evaluating the integral (7.9) in a PN-expanded way]. Let us only give here one specific example:

$$
\begin{aligned}
\chi_5^{\rm loc,f} &= \frac{1}{5p_\infty^5} - \frac{2}{p_\infty^3}\eta^2 + \frac{32 - 8\nu}{p_\infty}\eta^4 + \left[320 + \left(-\frac{1168}{3} + \frac{41}{8}\pi^2\right)\nu + 24\nu^2\right] p_\infty \eta^6\\
&\quad + \left[640 + \left(\frac{5069}{144}\pi^2 - \frac{227059}{135}\right)\nu + \left(-\frac{287}{24}\pi^2 + \frac{7342}{9}\right)\nu^2 - 40\nu^3\right] p_\infty^3 \eta^8\\
&\quad + \left[\frac{1792}{5} + \left(-\frac{1460479}{525} + \frac{111049}{960}\pi^2\right)\nu + \left(\frac{41026}{15} - \frac{40817}{640}\pi^2 - \frac{4}{15}\bar{d}_5^{\nu^2}\right)\nu^2\right.\\
&\quad \left. + \left(-\frac{11108}{9} + \frac{451}{24}\pi^2\right)\nu^3 + 56\nu^4\right] p_\infty^5 \eta^{10} + \left[\left(\frac{93031}{2304}\pi^2 - \frac{498343703}{604800}\right)\nu + \left(\frac{2827607}{1152} - \frac{31633}{768}\pi^2\right)\nu^2\right.\\
&\quad \left. + \left(\frac{205}{16}\pi^2 - \frac{253361}{96}\right)\nu^3 + \frac{212879}{384}\nu^4 + \frac{63}{64}\nu^5 - 2q_{\rm 3EG}^4 - 4q_{\rm 4EG}^3 - \frac{4}{3}q_{\rm 5EG}^2\right] p_\infty^7 \eta^{12}.
\end{aligned}
\tag{7.21}
$$

Here, we used as energy variable the EOB asymptotic momentum p_∞, defined as

$$
p_\infty^2 \equiv \gamma^2 - 1. \tag{7.22}
$$

This quantity naturally appears in the PM-expanded mass-shell condition, see Eq. (7.14), and is also a convenient PN-expansion parameter $p_\infty^2 \mapsto \eta^2 p_\infty^2$. We recall that the $q_{n\rm EG}^k$'s appearing in Eq. (7.21) are the coefficients of the expansion in powers of p_∞^2 of the H-type $q_{n\rm EG}^{H\rm loc,f}(\gamma;\nu)$ coefficients, see Eq. (7.5).

Having in hands the expressions of the $\chi_n^{\rm loc,f}(p_\infty;\nu)$'s (which we shall indifferently denote as $\chi_n^{\rm loc,f}(\gamma;\nu)$), let us now consider the following energy-rescaled versions of these coefficients

$$
\tilde{\chi}_n^{\rm loc,f}(\gamma;\nu) \equiv [h(\gamma;\nu)]^{n-1}\chi_n^{\rm loc,f}(\gamma;\nu), \tag{7.23}
$$

where

$$
h(\gamma;\nu) \equiv \sqrt{1 + 2\nu(\gamma - 1)} = \frac{H}{Mc^2}. \tag{7.24}
$$

Reference [58] has shown that the *total* (local plus non-local) scattering angle satisfied the following condition:

$$
C_n^{\rm tot}: \tilde{\chi}_n^{\rm tot}(\gamma;\nu) = P_{d_n}^{\nu}(\nu); \quad \text{with} \quad d_n \equiv \left[\frac{n-1}{2}\right]. \tag{7.25}
$$

TABLE X. List of the f-route EOB potentials in p_r-gauge.

$a_5^{\mathrm{loc,f}}$	$(-\frac{4237}{60}+\frac{2275}{512}\pi^2)\nu+(\frac{41}{32}\pi^2-\frac{221}{6})\nu^2$
$a_6^{\mathrm{loc,f}}$	$(-\frac{1026301}{1575}+\frac{246367}{3072}\pi^2)\nu+a_6^{\nu^2}\nu^2+4\nu^3$
$a_7^{\mathrm{loc,f}}$	$(-\frac{2800873}{262144}\pi^4+\frac{608698367}{1769472}\pi^2-\frac{1469618167}{907200})\nu+a_7^{\nu^2}\nu^2+a_7^{\nu^3}\nu^3$
$\bar{d}_4^{\mathrm{loc,f}}$	$(\frac{1679}{9}-\frac{23761}{1536}\pi^2)\nu+(\frac{123}{16}\pi^2-260)\nu^2$
$\bar{d}_5^{\mathrm{loc,f}}$	$(\frac{331054}{175}-\frac{63707}{512}\pi^2)\nu+\bar{d}_5^{\nu^2}\nu^2+(-\frac{205}{16}\pi^2+\frac{1069}{3})\nu^3$
$\bar{d}_6^{\mathrm{loc,f}}$	$(\frac{229504763}{98304}\pi^2+\frac{135909}{262144}\pi^4-\frac{99741733409}{6350400})\nu+\bar{d}_6^{\nu^2}\nu^2+(\frac{45089}{72}-\frac{44489}{1536}\pi^2-\bar{d}_5^{\nu^2}-15a_6^{\nu^2})\nu^3-48\nu^4$
$q_{43}^{\mathrm{loc,f}}$	$20\nu-83\nu^2+10\nu^3$
$q_{44}^{\mathrm{loc,f}}$	$(\frac{1580641}{3150}-\frac{93031}{1536}\pi^2)\nu+(-\frac{2075}{3}+\frac{31633}{512}\pi^2)\nu^2+(640-\frac{615}{32}\pi^2)\nu^3$
$q_{45}^{\mathrm{loc,f}}$	$(\frac{81030481}{65536}\pi^2-\frac{3492647551}{423360})\nu+q_{45}^{\nu^2}\nu^2+(-\frac{14}{3}\bar{d}_5^{\nu^2}+\frac{36677}{1152}\pi^2-\frac{474899}{216})\nu^3+(\frac{1435}{32}\pi^2-\frac{7375}{6})\nu^4$
$q_{62}^{\mathrm{loc,f}}$	$-\frac{9}{5}\nu-\frac{27}{5}\nu^2+6\nu^3$
$q_{63}^{\mathrm{loc,f}}$	$\frac{123}{10}\nu-\frac{69}{5}\nu^2+116\nu^3-14\nu^4$
$q_{64}^{\mathrm{loc,f}}$	$(-\frac{9733841}{327680}\pi^2-\frac{112218283}{294000})\nu+(\frac{156397}{1280}\pi^2-\frac{21996581}{21000})\nu^2+(\frac{6977}{6}-\frac{29665}{256}\pi^2)\nu^3+(\frac{287}{8}\pi^2-\frac{3640}{3})\nu^4$
$q_{82}^{\mathrm{loc,f}}$	$\frac{6}{7}\nu+\frac{18}{7}\nu^2+\frac{24}{7}\nu^3-6\nu^4$
$q_{83}^{\mathrm{loc,f}}$	$-\frac{7447}{560}\nu-\frac{963}{56}\nu^2-\frac{117}{10}\nu^3-147\nu^4+18\nu^5$
$q_{10,2}^{\mathrm{loc,f}}$	$-\frac{11}{21}\nu-\frac{11}{7}\nu^2-\frac{20}{7}\nu^3-\frac{5}{3}\nu^4+6\nu^5$

Here, and below, the notation $P_k^{\gamma}(\nu)$ denotes a generic polynomial of degree $\leq k$, with γ- (or, equivalently, p_∞-) dependent coefficients.

In Ref. [2] we pointed out the simplification brought in the determination of the local Hamiltonian by choosing a flexibility factor $f(t)$ in the definition of the Pf scale $r_{12}^f = f(t)r_{12}^h$ such that the condition C_n^{tot} *separately* applies to the nonlocal contribution $\chi_n^{\mathrm{nonloc,f}}(\gamma;\nu)$, and to the local one $\chi_n^{\mathrm{loc,f}}(\gamma;\nu)$. [We recall that $\chi_n^{\mathrm{tot}}(\gamma;\nu)=\chi_n^{\mathrm{loc,f}}(\gamma;\nu)+\chi_n^{\mathrm{nonloc,f}}(\gamma;\nu)$, because the nonlocal part can be treated as a first-order perturbation.] We showed there that it was always possible to construct such a flexibility factor $f=1+O(\frac{\nu}{c^2})$ at the 1PN fractional accuracy. We will show in a separate work that this holds also at the 2PN fractional accuracy, of relevance to the present study. This choice of such a tuned f allows us to separate the determination of the f-route local Hamiltonian, from the discussion of the corresponding nonlocal contribution to the scattering angle, $\chi_n^{\mathrm{nonloc,f}}(\gamma;\nu)$.

We shall then enforce the condition (with $d_n \equiv [\frac{n-1}{2}]$)

$$C_n^{\mathrm{loc,f}}: \tilde{\chi}_n^{\mathrm{loc,f}}(\gamma;\nu)=P_{d_n}^{\gamma}(\nu); \tag{7.26}$$

i.e.,

$$C_n^{\mathrm{loc,f}}: \tilde{\chi}_n^{\mathrm{loc,f}}(\gamma;\nu)=c_{n0}(\gamma)+c_{n1}(\gamma)\nu+\cdots+c_{nd_n}(\gamma)\nu^{d_n}. \tag{7.27}$$

This condition yields strong constraints on the ν-dependence of the various coefficients in the local Hamiltonians (in any gauge), and allows one to determine most of the coefficients entering the (usual) local Hamiltonian $H^{\mathrm{loc,f}}(r,p_r,j)$.

Applying the condition $C_n^{\mathrm{loc,f}}$ for $n=2,\ldots,7$, we could determine the nonlinear ν-dependence of the coefficients entering the 6PN-accurate p_r-gauge effective Hamiltonian $\hat{H}^2_{\mathrm{eff\ loc,f}}$, except for the following *four* numerical coefficients

$$a_7^{\nu^2}, a_7^{\nu^3}, \bar{d}_6^{\nu^2}, \quad \text{and} \quad q_{45}^{\nu^2}. \tag{7.28}$$

We recall that, at the 5PN level, we could determine the nonlinear ν-dependence of the EOB potentials except for *two* numerical coefficients: $a_6^{\nu^2}$, and $\bar{d}_5^{\nu^2}$. We list in Table X the knowledge of the coefficients parametrizing the f-route local EOB potentials. We note that among the 52 coefficients entering the $5+6$PN local EOB potentials our method allowed to determine 46. To complete the previous information we also list in Tables XI, XII the parameters entering the H-type and E-type energy-gauge (squared) effective Hamiltonian for $n\geq 3$ and $n\geq 4$, respectively. [We recall that $q_3^E=q_3^H$.]

The situation is even more impressive if one considers the usual Hamiltonian, expressed in terms of the effective one by Eq. (4.5), as a function of u, p_r, and $p^2\equiv p_r^2+j^2u^2$,

$$H^{\mathrm{loc,f,6PN}}=\sum_{0\leq k\leq 7,0\leq l\leq 5,k+l\leq 7} C_{2k}^{(2l)}(\nu)p^{2k}p_r^{2l}u^{7-k-l}. \tag{7.29}$$

Indeed, the ν-dependence of this 6PN-level Hamiltonian (see Table XIII) contains 151 (or 147, if we consider that four coefficients start at $O(\nu^2)$) numerical coefficients, and our method determines $151-4=147$ (or $147-4=143$) of them. The ν-dependent coefficients $C_{2k}^{(2l)}(\nu)=\sum_n C_{2k,n}^{(2l)}\nu^n$ are listed in Table XIII.

TABLE XI. List of the p_∞^2-expansion coefficients of the u-coefficients in the H-type energy-gauge (squared) effective Hamiltonian for $n \geq 3$, see Eqs. (7.3) and (7.5).

$q^0_{3\mathrm{EG}}$	5ν
$q^1_{3\mathrm{EG}}$	$27\nu - \frac{23}{4}\nu^2$
$q^2_{3\mathrm{EG}}$	$\frac{1021}{80}\nu - \frac{445}{16}\nu^2 + \frac{49}{8}\nu^3$
$q^3_{3\mathrm{EG}}$	$-\frac{213}{2240}\nu - \frac{2409}{320}\nu^2 + \frac{437}{16}\nu^3 - \frac{407}{64}\nu^4$
$q^4_{3\mathrm{EG}}$	$-\frac{50557}{80640}\nu + \frac{1149}{1792}\nu^2 + \frac{489}{320}\nu^3 - \frac{6735}{256}\nu^4 + \frac{835}{128}\nu^5$
$q^0_{4\mathrm{EG}}$	$\nu(\frac{175}{3} - \frac{41}{32}\pi^2) - \frac{7}{2}\nu^2$
$q^1_{4\mathrm{EG}}$	$(\frac{5632}{45} - \frac{33601}{6144}\pi^2)\nu + (-\frac{405}{4} + \frac{123}{64}\pi^2)\nu^2 + \frac{13}{2}\nu^3$
$q^2_{4\mathrm{EG}}$	$(-\frac{93031}{12288}\pi^2 + \frac{699761}{7200})\nu + (\frac{31633}{4096}\pi^2 - \frac{77443}{480})\nu^2 + (-\frac{615}{256}\pi^2 + 130)\nu^3 - \frac{293}{32}\nu^4$
$q^3_{4\mathrm{EG}}$	$(-\frac{40415563}{1411200} - \frac{9733841}{4194304}\pi^2)\nu + (-\frac{6742919}{67200} + \frac{156397}{16384}\pi^2)\nu^2 + (\frac{80153}{480} - \frac{148325}{16384}\pi^2)\nu^3 + (-\frac{7223}{48} + \frac{1435}{512}\pi^2)\nu^4 + \frac{185}{16}\nu^5$
$q^0_{5\mathrm{EG}}$	$(-\frac{29917}{6144}\pi^2 + \frac{44357}{360})\nu + (-\frac{2387}{24} + \frac{205}{64}\pi^2)\nu^2 + \frac{9}{4}\nu^3$
$q^1_{5\mathrm{EG}}$	$(\frac{15540691}{25200} - \frac{2590847}{61440}\pi^2)\nu + (\frac{1}{5}\bar{d}_5^{\nu^2} - \frac{15581}{80} + \frac{347673}{20480}\pi^2)\nu^2 + (-\frac{1763}{256}\pi^2 + \frac{5131}{24})\nu^3 - \frac{93}{16}\nu^4$
$q^2_{5\mathrm{EG}}$	$(-\frac{807638471}{1764000} + \frac{30033990443}{440401920}\pi^2)\nu + (\frac{3}{35}q_{45}^{\nu^2} + \frac{6466655}{114688}\pi^2 - \frac{299020817}{588000})\nu^2 + (-\frac{7356287}{245760}\pi^2 + \frac{289783}{1440} - \frac{2}{5}\bar{d}_5^{\nu^2})\nu^3$ $+(-\frac{41833}{128} + \frac{5535}{512}\pi^2)\nu^4 + \frac{657}{64}\nu^5$
$q^0_{6\mathrm{EG}}$	$(\frac{541363}{10240}\pi^2 - \frac{69733}{350})\nu + (\frac{11717}{60} + a_6^{\nu^2} + \frac{1}{5}\bar{d}_5^{\nu^2} + \frac{17857}{5120}\pi^2)\nu^2 + (\frac{326}{3} - \frac{287}{64}\pi^2)\nu^3 - \frac{11}{8}\nu^4$
$q^1_{6\mathrm{EG}}$	$(\frac{45303}{524288}\pi^4 - \frac{195178823647}{47628000} + \frac{100876235443}{165150720}\pi^2)\nu + (\frac{1}{6}\bar{d}_6^{\nu^2} + \frac{11}{70}q_{45}^{\nu^2} + \frac{7}{6}a_6^{\nu^2} + \frac{596127373}{588000} + \frac{593223}{28672}\pi^2 + \frac{1}{6}\bar{d}_5^{\nu^2})\nu^2$ $+(-\frac{431999}{30720}\pi^2 - \frac{78703}{90} - \frac{9}{10}\bar{d}_5^{\nu^2} - \frac{5}{2}a_6^{\nu^2})\nu^3 + (-\frac{28463}{96} + \frac{205}{16}\pi^2)\nu^4 + \frac{73}{16}\nu^5$
$q^0_{7\mathrm{EG}}$	$(-\frac{5556443}{524288}\pi^4 - \frac{37569023551}{4762800} + \frac{1163042866561}{990904320}\pi^2)\nu + (\frac{83453959}{29400} - \frac{675041}{8960}\pi^2 + a_7^{\nu^2} + \frac{25}{6}a_6^{\nu^2} + \frac{1}{14}q_{45}^{\nu^2} + \frac{1}{6}\bar{d}_6^{\nu^2} - \frac{1}{30}\bar{d}_5^{\nu^2})\nu^2$ $+(a_7^{\nu^3} + \frac{373067}{12288}\pi^2 - \frac{1}{2}\bar{d}_5^{\nu^2} - \frac{5}{2}a_6^{\nu^2} - \frac{222749}{144})\nu^3 + (-\frac{3021}{32} + \frac{615}{128}\pi^2)\nu^4 + \frac{13}{16}\nu^5$

TABLE XII. List of the p_∞^2-expansion coefficients [similarly to Eq. (7.5)] of the E-type energy gauge (squared) effective Hamiltonian, Eq. (7.4), for $n \geq 4$.

$q^0_{4E,\mathrm{EG}}$	$(\frac{175}{3} - \frac{41}{32}\pi^2)\nu - \frac{7}{2}\nu^2$
$q^1_{4E,\mathrm{EG}}$	$(\frac{5632}{45} - \frac{33601}{6144}\pi^2)\nu + (-\frac{441}{4} + \frac{123}{64}\pi^2)\nu^2 + \frac{13}{2}\nu^3$
$q^2_{4E,\mathrm{EG}}$	$(\frac{699761}{7200} - \frac{93031}{12288}\pi^2)\nu + (-\frac{90403}{480} + \frac{31633}{4096}\pi^2)\nu^2 + (-\frac{615}{256}\pi^2 + \frac{601}{4})\nu^3 - \frac{293}{32}\nu^4$
$q^3_{4E,\mathrm{EG}}$	$(-\frac{9733841}{4194304}\pi^2 - \frac{40415563}{1411200})\nu + (\frac{156397}{16384}\pi^2 - \frac{7498919}{67200})\nu^2 + (\frac{102833}{480} - \frac{148325}{16384}\pi^2)\nu^3 + (\frac{1435}{512}\pi^2 - \frac{8789}{48})\nu^4 + \frac{185}{16}\nu^5$
$q^0_{5E,\mathrm{EG}}$	$(-\frac{29917}{6144}\pi^2 + \frac{44357}{360})\nu + (\frac{205}{64}\pi^2 - \frac{2747}{24})\nu^2 + \frac{9}{4}\nu^3$
$q^1_{5E,\mathrm{EG}}$	$(\frac{15540691}{25200} - \frac{2590847}{61440}\pi^2)\nu + (-\frac{29501}{80} + \frac{347673}{20480}\pi^2 + \frac{1}{5}\bar{d}_5^{\nu^2})\nu^2 + (-\frac{1763}{256}\pi^2 + \frac{6499}{24})\nu^3 - \frac{93}{16}\nu^4$
$q^2_{5E,\mathrm{EG}}$	$(-\frac{807638471}{1764000} + \frac{30033990443}{440401920}\pi^2)\nu + (\frac{3}{35}q_{45}^{\nu^2} - \frac{472730717}{588000} + \frac{6466655}{114688}\pi^2)\nu^2 + (-\frac{7356287}{245760}\pi^2 - \frac{2}{5}\bar{d}_5^{\nu^2} + \frac{965323}{1440})\nu^3$ $+(\frac{5535}{512}\pi^2 - \frac{57457}{128})\nu^4 + \frac{657}{64}\nu^5$
$q^0_{6E,\mathrm{EG}}$	$(-\frac{69733}{350} + \frac{541363}{10240}\pi^2)\nu + (a_6^{\nu^2} - \frac{5083}{60} + \frac{37537}{5120}\pi^2 + \frac{1}{5}\bar{d}_5^{\nu^2})\nu^2 + (\frac{1775}{12} - \frac{287}{64}\pi^2)\nu^3 - \frac{11}{8}\nu^4$
$q^1_{6E,\mathrm{EG}}$	$(\frac{100876235443}{165150720}\pi^2 - \frac{195178823647}{47628000} + \frac{45303}{524288}\pi^4)\nu + (\frac{1}{6}\bar{d}_5^{\nu^2} - \frac{109708309}{196000} + \frac{11}{70}q_{45}^{\nu^2} + \frac{7}{6}a_6^{\nu^2} + \frac{1754467}{28672}\pi^2 + \frac{1}{6}\bar{d}_6^{\nu^2})\nu^2$ $+(-\frac{963359}{30720}\pi^2 + \frac{186533}{360} - \frac{9}{10}\bar{d}_5^{\nu^2} - \frac{5}{2}a_6^{\nu^2})\nu^3 + (\frac{205}{16}\pi^2 - \frac{42773}{96})\nu^4 + \frac{73}{16}\nu^5$
$q^0_{7E,\mathrm{EG}}$	$(-\frac{5556443}{524288}\pi^4 - \frac{37569023551}{4762800} + \frac{1163042866561}{990904320}\pi^2)\nu + (\frac{1}{6}\bar{d}_6^{\nu^2} + \frac{39167441}{44100} + \frac{25}{6}a_6^{\nu^2} - \frac{1}{30}\bar{d}_5^{\nu^2} + \frac{1}{14}q_{45}^{\nu^2} - \frac{348401}{53760}\pi^2 + a_7^{\nu^2})\nu^2$ $+(a_7^{\nu^3} - \frac{49799}{144} + \frac{46379}{12288}\pi^2 - \frac{1}{2}\bar{d}_5^{\nu^2} - \frac{5}{2}a_6^{\nu^2})\nu^3 + (\frac{615}{128}\pi^2 - \frac{4921}{32})\nu^4 + \frac{13}{16}\nu^5$

TABLE XIII. Coefficients entering the 6PN real EOB Hamiltonian (7.29).

Coefficient	Powers	Value
$C_{14}^{(0)}(\nu)$	$p_r^0 p^{14} u^0$	$\frac{33}{2048}\nu+\frac{33}{2048}\nu^2+\frac{45}{2048}\nu^3+\frac{15}{512}\nu^4+\frac{35}{1024}\nu^5+\frac{63}{2048}\nu^6+\frac{33}{2048}\nu^7$
$C_{12}^{(0)}(\nu)$	$p_r^0 p^{12} u^1$	$\frac{21}{1024}\nu+\frac{21}{1024}\nu^2+\frac{21}{1024}\nu^3-\frac{35}{512}\nu^5-\frac{189}{1024}\nu^6-\frac{231}{1024}\nu^7$
$C_{10}^{(0)}(\nu)$	$p_r^0 p^{10} u^2$	$-\frac{7}{512}\nu-\frac{7}{512}\nu^2-\frac{15}{512}\nu^3-\frac{5}{64}\nu^4-\frac{35}{256}\nu^5+\frac{63}{512}\nu^6+\frac{693}{512}\nu^7$
$C_{8}^{(0)}(\nu)$	$p_r^0 p^{8} u^3$	$\frac{5}{256}\nu-\frac{5}{256}\nu^2-\frac{1}{256}\nu^3+\frac{7}{128}\nu^4+\frac{45}{128}\nu^5+\frac{385}{256}\nu^6-\frac{1155}{256}\nu^7$
$C_{6}^{(0)}(\nu)$	$p_r^0 p^{6} u^4$	$-\frac{5}{128}\nu+(\frac{385}{384}-\frac{41}{1024}\pi^2)\nu^2+(\frac{373}{384}-\frac{41}{1024}\pi^2)\nu^3+\frac{1}{32}\nu^4+(-\frac{925}{192}+\frac{205}{1024}\pi^2)\nu^5-\frac{595}{128}\nu^6+\frac{1155}{128}\nu^7$
$C_{4}^{(0)}(\nu)$	$p_r^0 p^{4} u^5$	$\frac{7}{64}\nu+(\frac{1141}{480}-\frac{1619}{8192}\pi^2)\nu^2+(\frac{1153}{240}-\frac{2275}{8192}\pi^2)\nu^3+(\frac{10105}{8192}\pi^2-\frac{21721}{960})\nu^4+(-\frac{123}{128}\pi^2+\frac{347}{16})\nu^5$ $+\frac{357}{64}\nu^6-\frac{693}{64}\nu^7$
$C_{2}^{(0)}(\nu)$	$p_r^0 p^{2} u^6$	$-\frac{21}{32}\nu+(\frac{254113}{12288}\pi^2-\frac{8478053}{50400})\nu^2+(\frac{7466063}{50400}-\frac{234685}{12288}\pi^2+\frac{1}{4}a_6^{\nu^2})\nu^3+(\frac{10861}{240}-\frac{1}{4}a_6^{\nu^2}-\frac{6169}{2048}\pi^2)\nu^4$ $+(-\frac{507}{16}+\frac{369}{256}\pi^2)\nu^5-\frac{91}{32}\nu^6+\frac{231}{32}\nu^7$
$C_{0}^{(0)}(\nu)$	$p_r^0 p^{0} u^7$	$-\frac{33}{16}\nu+(-\frac{2081602903}{1814400}-\frac{2800873}{524288}\pi^4+\frac{756731519}{3538944}\pi^2)\nu^2+(\frac{1}{2}a_6^{\nu^2}+\frac{133421}{3072}\pi^2+\frac{1}{2}a_7^{\nu^2}-\frac{4823719}{12600})\nu^3$ $+(\frac{1}{2}a_7^{\nu^3}+\frac{1}{2}a_6^{\nu^2}+\frac{6169}{2048}\pi^2-\frac{11251}{240})\nu^4+(\frac{167}{12}-\frac{41}{64}\pi^2)\nu^5+\frac{7}{16}\nu^6-\frac{33}{16}\nu^7$
$C_{10}^{(2)}(\nu)$	$p_r^2 p^{10} u^1$	$\frac{63}{256}\nu+\frac{63}{256}\nu^2+\frac{21}{64}\nu^3+\frac{105}{256}\nu^4+\frac{105}{256}\nu^5+\frac{63}{256}\nu^6$
$C_{8}^{(2)}(\nu)$	$p_r^2 p^{8} u^2$	$\frac{35}{128}\nu+\frac{35}{32}\nu^2+\frac{135}{128}\nu^3+\frac{55}{64}\nu^4-\frac{25}{128}\nu^5-\frac{105}{64}\nu^6$
$C_{6}^{(2)}(\nu)$	$p_r^2 p^{6} u^3$	$-\frac{5}{32}\nu-\frac{185}{32}\nu^2-\frac{81}{16}\nu^3-\frac{239}{32}\nu^4-\frac{269}{32}\nu^5+\frac{135}{32}\nu^6$
$C_{4}^{(2)}(\nu)$	$p_r^2 p^{4} u^4$	$\frac{3}{16}\nu+(\frac{611}{48}-\frac{25729}{8192}\pi^2)\nu^2+(-\frac{13921}{8192}\pi^2-\frac{779}{24})\nu^3+(-\frac{13921}{8192}\pi^2-\frac{1189}{48})\nu^4+(-\frac{69}{16}+\frac{369}{256}\pi^2)\nu^5-\frac{45}{8}\nu^6$
$C_{2}^{(2)}(\nu)$	$p_r^2 p^{2} u^5$	$-\frac{5}{16}\nu+(\frac{36359}{2048}\pi^2-\frac{447313}{1400})\nu^2+(-\frac{4267103}{8400}-\frac{1}{4}\bar{d}_5^{\nu^2}+\frac{47511}{2048}\pi^2)\nu^3+(\frac{45409}{2048}\pi^2-\frac{1}{4}\bar{d}_5^{\nu^2}-\frac{16871}{48})\nu^4$ $+(-\frac{41}{16}\pi^2+\frac{767}{48})\nu^5+\frac{75}{16}\nu^6$
$C_{0}^{(2)}(\nu)$	$p_r^2 p^{0} u^6$	$\frac{7}{8}\nu+(-\frac{137379058337}{12700800}+\frac{271118011}{196608}\pi^2+\frac{135909}{524288}\pi^4)\nu^2+(\frac{8602849}{12600}+\frac{1}{2}a_6^{\nu^2}-\frac{3}{2}\bar{d}_5^{\nu^2}-\frac{238105}{6144}\pi^2+\frac{1}{2}\bar{d}_6^{\nu^2})\nu^3$ $+(-\frac{15}{2}a_6^{\nu^2}-\frac{99253}{6144}\pi^2+\frac{18713}{144})\nu^4+(\frac{517}{24}-\frac{41}{64}\pi^2)\nu^5-\frac{9}{4}\nu^6$
$C_{6}^{(4)}(\nu)$	$p_r^4 p^{6} u^2$	$\frac{35}{32}\nu-\frac{5}{32}\nu^2+\frac{35}{32}\nu^3+\nu^4+\frac{31}{32}\nu^5+\frac{15}{16}\nu^6$
$C_{4}^{(4)}(\nu)$	$p_r^4 p^{4} u^3$	$\frac{15}{16}\nu+\frac{141}{16}\nu^2-6\nu^3-\frac{63}{16}\nu^4-\frac{57}{8}\nu^5-\frac{15}{4}\nu^6$
$C_{2}^{(4)}(\nu)$	$p_r^4 p^{2} u^4$	$-\frac{3}{8}\nu+(\frac{93031}{6144}\pi^2-\frac{924983}{6300})\nu^2+(\frac{23773}{1575}-\frac{467}{1536}\pi^2)\nu^3+(-\frac{469}{12}-\frac{21793}{2048}\pi^2)\nu^4+(\frac{615}{128}\pi^2-\frac{903}{8})\nu^5+\frac{15}{4}\nu^6$
$C_{0}^{(4)}(\nu)$	$p_r^4 p^{0} u^5$	$\frac{1}{4}\nu+(-\frac{18410298107}{4233600}+\frac{83902033}{131072}\pi^2)\nu^2+(\frac{2968391}{6300}+\frac{1}{2}q_{45}^{\nu^2}-\frac{201851}{3072}\pi^2)\nu^3+(-\frac{799151}{432}-\frac{7}{3}\bar{d}_5^{\nu^2}+\frac{555389}{9216}\pi^2)\nu^4$ $+(\frac{205}{16}\pi^2-\frac{4153}{12})\nu^5$
$C_{4}^{(6)}(\nu)$	$p_r^6 p^{4} u^2$	$-\frac{27}{80}\nu^2-\frac{27}{20}\nu^3-\frac{9}{40}\nu^4+\frac{9}{80}\nu^5+\frac{9}{8}\nu^6$
$C_{2}^{(6)}(\nu)$	$p_r^6 p^{2} u^3$	$\frac{5}{4}\nu-\frac{331}{40}\nu^2-\frac{57}{40}\nu^3-\frac{243}{10}\nu^4-\frac{309}{20}\nu^5-\nu^6$
$C_{0}^{(6)}(\nu)$	$p_r^6 p^{0} u^4$	$\frac{1}{2}\nu+(-\frac{109101883}{588000}-\frac{9733841}{655360}\pi^2)\nu^2+(\frac{156397}{2560}\pi^2-\frac{23052881}{42000})\nu^3+(-\frac{29665}{512}\pi^2+\frac{29551}{60})\nu^4$ $+(\frac{287}{16}\pi^2-\frac{32533}{60})\nu^5-\frac{5}{2}\nu^6$
$C_{2}^{(8)}(\nu)$	$p_r^8 p^{2} u^2$	$-\frac{3}{14}\nu^2-\frac{6}{7}\nu^3-\frac{3}{2}\nu^4+\frac{9}{14}\nu^5+\frac{3}{2}\nu^6$
$C_{0}^{(8)}(\nu)$	$p_r^8 p^{0} u^3$	$-\frac{1787}{224}\nu^2-\frac{7311}{560}\nu^3-\frac{837}{140}\nu^4-\frac{921}{14}\nu^5+6\nu^6$
$C_{0}^{(10)}(\nu)$	$p_r^{10} p^{0} u^2$	$-\frac{11}{42}\nu^2-\frac{11}{14}\nu^3-\frac{10}{7}\nu^4-\frac{5}{6}\nu^5+3\nu^6$

VIII. VALUES OF THE 6PN-ACCURATE f- ROUTE LOCAL SCATTERING ANGLE AT PM ORDERS G^3, G^4, G^5, AND G^6

Having determined most of the coefficients parametrizing the f-route local Hamiltonian we can write down the (PN-expanded) values of the corresponding successive n-PM contributions, χ_n, to the scattering angle. The results are more compactly expressed when writing them in terms of the difference between the energy-rescaled angle (7.23) and the corresponding test-mass (Schwarzschild value).

Let us first recall that the *exact* values of the Schwarzschild scattering angle coefficients are

$$\chi_1^{\rm Schw}(p_\infty)=\frac{1}{p_\infty}+2p_\infty,\qquad \chi_2^{\rm Schw}(p_\infty)=\pi\left(\frac{3}{2}+\frac{15}{8}p_\infty^2\right),\qquad \chi_3^{\rm Schw}(p_\infty)=-\frac{1}{3p_\infty^3}+\frac{4}{p_\infty}+24p_\infty+\frac{64}{3}p_\infty^3,$$
$$\chi_4^{\rm Schw}(p_\infty)=\pi\left(\frac{105}{8}+\frac{315}{8}p_\infty^2+\frac{3465}{128}p_\infty^4\right),\qquad \chi_5^{\rm Schw}(p_\infty)=\frac{1}{5p_\infty^5}-\frac{2}{p_\infty^3}+\frac{32}{p_\infty}+320p_\infty+640p_\infty^3+\frac{1792}{5}p_\infty^5,$$
$$\chi_6^{\rm Schw}(p_\infty)=\pi\left(\frac{1155}{8}+\frac{45045}{64}p_\infty^2+\frac{135135}{128}p_\infty^4+\frac{255255}{512}p_\infty^6\right),$$
$$\chi_7^{\rm Schw}(p_\infty)=-\frac{1}{7p_\infty^7}+\frac{8}{5p_\infty^5}-\frac{16}{p_\infty^3}+\frac{320}{p_\infty}+4480p_\infty+14336p_\infty^3+\frac{86016}{5}p_\infty^5+\frac{49152}{7}p_\infty^7. \tag{8.1}$$

We then find that the differences $\tilde{\chi}_n^{\rm loc,f}-\chi_n^{\rm Schw}$ [recalling the definition Eq. (7.23)] read

$$\pi^{-1}(\tilde{\chi}_2^{\rm loc}-\chi_2^{\rm Schw})=0,\quad \nu^{-1}(\tilde{\chi}_3^{\rm loc}-\chi_3^{\rm Schw})=-\frac{1}{3p_\infty}\eta^2-\frac{47}{12}\eta^4p_\infty-\frac{313}{24}\eta^6p_\infty^3-\frac{749}{320}p_\infty^5\eta^8-\frac{7519}{4480}\eta^{10}p_\infty^7+\frac{211469}{161280}\eta^{12}p_\infty^9,$$
$$\begin{aligned}\nu^{-1}\pi^{-1}(\tilde{\chi}_4^{\rm loc,f}-\chi_4^{\rm Schw})=&-\frac{15}{4}\eta^4+\left(-\frac{557}{16}+\frac{123}{256}\pi^2\right)\eta^6p_\infty^2+\left(-\frac{4601}{96}+\frac{33601}{16384}\pi^2\right)\eta^8p_\infty^4\\&+\left(-\frac{3978707}{134400}+\frac{93031}{32768}\pi^2\right)\eta^{10}p_\infty^6+\left(\frac{29201523}{33554432}\pi^2+\frac{5058313}{627200}\right)\eta^{12}p_\infty^8,\end{aligned}$$
$$\begin{aligned}\nu^{-1}(\tilde{\chi}_5^{\rm loc,f}-\chi_5^{\rm Schw})=&\frac{2}{5p_\infty^3}\eta^2+\left(-\frac{121}{10}+\frac{1}{5}\nu\right)\frac{\eta^4}{p_\infty}+\left(-\frac{19457}{60}+\frac{59}{10}\nu+\frac{41}{8}\pi^2\right)\eta^6p_\infty\\&+\left(-\frac{41}{24}\nu\pi^2+\frac{10681}{144}\nu+\frac{5069}{144}\pi^2-\frac{4572503}{4320}\right)\eta^8p_\infty^3\\&+\left(\frac{111049}{960}\pi^2+\frac{23407}{5760}\nu\pi^2-\frac{4}{15}\nu\bar{d}_5^{\nu^2}-\frac{55558621}{33600}-\frac{573577}{4320}\nu\right)\eta^{10}p_\infty^5\\&+\left(-\frac{4}{35}\nu q_{45}^{\nu^2}-\frac{184881}{4480}\pi^2+\frac{1219303}{20160}\nu\pi^2-\frac{16844006729}{21168000}\nu+\frac{15827493497}{42336000}\right)\eta^{12}p_\infty^7,\end{aligned}$$
$$\begin{aligned}\nu^{-1}\pi^{-1}(\tilde{\chi}_6^{\rm loc,f}-\chi_6^{\rm Schw})=&\left(-\frac{625}{4}+\frac{105}{16}\nu+\frac{615}{256}\pi^2\right)\eta^6+\left(-\frac{1845}{512}\nu\pi^2+\frac{257195}{8192}\pi^2+\frac{10065}{64}\nu-\frac{224113}{192}\right)\eta^8p_\infty^2\\&+\left(-\frac{15}{32}\nu a_6^{\nu^2}-\frac{15}{32}\nu\bar{d}_5^{\nu^2}-\frac{61855}{32768}\nu\pi^2+\frac{2321185}{16384}\pi^2+\frac{4625}{192}\nu-\frac{20420849}{6720}\right)\eta^{10}p_\infty^4\\&+\left(-\frac{35}{64}\nu\bar{d}_5^{\nu^2}-\frac{5}{64}\nu\bar{d}_6^{\nu^2}-\frac{4911465305}{25165824}\pi^2-\frac{11437991}{8960}\nu-\frac{35}{64}\nu a_6^{\nu^2}-\frac{15}{64}\nu q_{45}^{\nu^2}\right.\\&\left.+\frac{2363865}{65536}\nu\pi^2-\frac{679545}{16777216}\pi^4+\frac{1343882527}{10160640}\right)\eta^{12}p_\infty^6,\end{aligned}$$
$$\begin{aligned}\nu^{-1}(\tilde{\chi}_7^{\rm loc,f}-\chi_7^{\rm Schw})=&-\frac{3}{7p_\infty^5}\eta^2+\left(\frac{227}{28}-\frac{3}{7}\nu\right)\frac{\eta^4}{p_\infty^3}+\left(-\frac{60377}{168}+\frac{339}{14}\nu+\frac{41}{8}\pi^2-\frac{1}{7}\nu^2\right)\frac{\eta^6}{p_\infty}\\&+\left(\frac{33131}{192}\pi^2-\frac{221}{28}\nu^2+\frac{158129}{112}\nu-\frac{152237341}{20160}-\frac{123}{4}\nu\pi^2\right)\eta^8p_\infty\\&+\left(\frac{378953}{384}\pi^2-\frac{18343}{128}\nu\pi^2+\frac{41}{8}\nu^2\pi^2-\frac{64315}{336}\nu^2-8\nu a_6^{\nu^2}\right.\\&\left.-4\nu\bar{d}_5^{\nu^2}+\frac{2208701}{480}\nu-\frac{6769922309}{201600}\right)\eta^{10}p_\infty^3\\&+\left(\frac{4}{5}\bar{d}_5^{\nu^2}\nu^2-\frac{284141687}{69120}\pi^2-\frac{3474679}{3840}\nu\pi^2-\frac{64}{5}\nu\bar{d}_5^{\nu^2}-\frac{196222844821}{16934400}-\frac{1811850763}{1008000}\nu\right.\\&\left.+\frac{596213}{720}\nu^2-\frac{128}{5}\nu a_6^{\nu^2}-\frac{8}{5}\nu a_7^{\nu^2}-\frac{8}{5}\nu^2a_7^{\nu^3}-\frac{8}{5}\nu\bar{d}_6^{\nu^2}-\frac{12}{5}\nu q_{45}^{\nu^2}-\frac{11471}{384}\nu^2\pi^2+\frac{666241}{40960}\pi^4\right)\eta^{12}p_\infty^5.\end{aligned} \tag{8.2}$$

These results for the scattering angle provide a lot of new information that offers gauge-invariant checks for future independent computations of the dynamics of binary systems.

In particular, using the fact (explicitly proven in Ref. [67]) that the nonlocal dynamics starts contributing to the scattering angle only at $O(G^4)$, so that $\chi_3^{\rm tot}=\chi_3^{\rm loc}$, our result above for $\chi_3^{\rm loc}$ actually describes the total 3PM-level scattering angle. The corresponding explicit 6PN-accurate expression of the unrescaled, and unsubtracted 3PM-level scattering angle (which is equivalent to the simpler rescaled, subtracted result above) reads

$$\chi_3=-\frac{1}{3p_\infty^3}+\frac{4}{p_\infty}+(24-8\nu)p_\infty+\left(\frac{64}{3}-36\nu+8\nu^2\right)p_\infty^3+\left(-\frac{91}{5}\nu+34\nu^2-8\nu^3\right)p_\infty^5 +\left(\frac{69}{70}\nu+\frac{51}{5}\nu^2-32\nu^3+8\nu^4\right)p_\infty^7+\left(\frac{1447}{5040}\nu-\frac{93}{56}\nu^2-\frac{27}{10}\nu^3+30\nu^4-8\nu^5\right)p_\infty^9+O(p_\infty^{11}). \tag{8.3}$$

This result is in agreement with the corresponding 6PN-level term in the PN expansion of the 3PM-level recent result of [61,62]. It has also been recently obtained in Refs. [63,64].

Let us emphasize that our results also provide a complete, 6PN-accurate value for the 4PM-level scattering angle $\chi_4=\chi_4^{\rm loc,f}+\chi_4^{\rm nonloc,f}$. We will discuss separately the 6PN-accurate nonlocal contribution $\chi_4^{\rm nonloc,f}$. Let us, for completeness, exhibit the unrescaled, unsubtracted value of $\chi_4^{\rm loc,f}$. It reads

$$\begin{aligned}\pi^{-1}\chi_4^{\rm loc,f}=&\left(\frac{105}{8}-\frac{15}{4}\nu\right)\\&+\left[\frac{315}{8}+\left(-\frac{109}{2}+\frac{123}{256}\pi^2\right)\nu+\frac{45}{8}\nu^2\right]p_\infty^2\\&+\left[\frac{3465}{128}+\left(\frac{33601}{16384}\pi^2-\frac{19597}{192}\right)\nu+\left(\frac{4827}{64}-\frac{369}{512}\pi^2\right)\nu^2-\frac{225}{32}\nu^3\right]p_\infty^4\\&+\left[\left(-\frac{1945583}{33600}+\frac{93031}{32768}\pi^2\right)\nu+\left(\frac{1937}{16}-\frac{94899}{32768}\pi^2\right)\nu^2+\left(-\frac{2895}{32}+\frac{1845}{2048}\pi^2\right)\nu^3+\frac{525}{64}\nu^4\right]p_\infty^6\\&+\left[\left(\frac{3879719}{313600}+\frac{29201523}{33554432}\pi^2\right)\nu+\left(\frac{4843207}{89600}-\frac{469191}{131072}\pi^2\right)\nu^2+\left(\frac{444975}{131072}\pi^2-\frac{15875}{128}\right)\nu^3\right.\\&\left.+\left(\frac{104755}{1024}-\frac{4305}{4096}\pi^2\right)\nu^4-\frac{4725}{512}\nu^5\right]p_\infty^8+O(p_\infty^{10}).\end{aligned} \tag{8.4}$$

Finally, concerning our results above for the 5PM, 6PM, and 7PM local scattering angles, if we transcribe them in terms of the unrescaled coefficients, $\chi_5^{\rm loc,f},\chi_6^{\rm loc,f},\chi_7^{\rm loc,f}$, they contain (in spite of the presence of undetermined parameters at the $O(\nu^2)$ level) a lot of new information, both for the linear-in-ν contributions, and for many terms involving higher powers of ν.

IX. RADIAL ACTION AND ITS HIDDEN STRUCTURE

In Ref. [2] we pointed out the existence of a hidden simplicity in the mass-ratio-dependence of the (rescaled) radial action

$$I_r(\gamma,j)=\frac{1}{2\pi}\oint p_r dr, \tag{9.1}$$

when it is expressed in terms of the EOB effective energy $\gamma=\hat{\mathcal{E}}_{\rm eff}$ (or equivalently p_∞) and of the rescaled angular momentum $j=J/(GM\mu)$. We work here with dimensionless scaled variables $I_r=I_r^{\rm phys}/(GM\mu)$, $p_r=p_r^{\rm phys}/\mu$, $r=r^{\rm phys}/GM$.

This hidden simplicity consists in noting the remarkably simple ν-dependence of the coefficients $I_n(\gamma;\nu)$ entering the following way of writing the 6PN-accurate expression for I_r:

$$\begin{aligned}I_r^{\rm loc,f}(\gamma,j)=&-j+I_0^S(\gamma)+\frac{I_1^S(\gamma)}{hj}+\frac{I_3(\gamma;\nu)}{(hj)^3}\\&+\frac{I_5(\gamma;\nu)}{(hj)^5}+\frac{I_7(\gamma;\nu)}{(hj)^7}\\&+\frac{I_9(\gamma;\nu)}{(hj)^9}+\frac{I_{11}(\gamma;\nu)}{(hj)^{11}}.\end{aligned} \tag{9.2}$$

First, the second term $I_0(\gamma)$ in this expression is independent of ν and equal to the analytic continuation (in γ) of χ_1 [68]

$$I_0^S(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{1-\gamma^2}}, \tag{9.3}$$

and, second, and most importantly, after having factored out the same power of $\frac{1}{h}$ as the power of $\frac{1}{j}$, the numerator $I_{2n+1}(\gamma;\nu)$ is a *polynomial* in ν of degree n:

$$I_{2n+1}(\gamma;\nu) = I_{2n+1}^S(\gamma) + \sum_{k=1}^{n} I_{2n+1}^{\nu^k}(\gamma)\nu^k. \tag{9.4}$$

The latter polynomial structure was not pointed out in previous discussions [40,68] of the radial action. Several conditions are needed to reveal it: the use of the effective EOB energy $\hat{\mathcal{E}}_{\rm eff}$ as energy variable, and a PN-complete account of each coefficient $I_{2n+1}(\hat{\mathcal{E}}_{\rm eff};\nu)$. We note in this respect that Eq. (3.10) of Ref. [40] used the specific binding energy $(H - Mc^2)/\mu$ as energy variable, and that Eq. (4.29) of Ref. [68] is a *PN-incomplete* 2PM truncation of I_r, which does not satisfy the simple rule (9.4).

As pointed out (and proven) in our previous work [2], the ν^0 terms (corresponding to the $\nu \to 0$ limit) in Eq. (9.4) can be exactly computed (for all values of n) because they correspond [like the term $I_0^S(\gamma)$] to the test-mass dynamics, described by a Schwarzschild metric of mass $M = m_1 + m_2$. The *exact* values of the most 6PN-relevant ν^0, Schwarzschildlike, terms read

$$\begin{aligned}
I_1^S(\gamma) &= -\frac{3}{4} + \frac{15}{4}\gamma^2,\\
I_3^S(\gamma) &= \frac{35}{64} - \frac{315}{32}\gamma^2 + \frac{1155}{64}\gamma^4,\\
I_5^S(\gamma) &= -\frac{231}{256} + \frac{9009}{256}\gamma^2 - \frac{45045}{256}\gamma^4 + \frac{51051}{256}\gamma^6\\
I_7^S(\gamma) &= \frac{32175}{16384} - \frac{546975}{4096}\gamma^2 + \frac{10392525}{8192}\gamma^4\\
&\quad - \frac{14549535}{4096}\gamma^6 + \frac{47805615}{16384}\gamma^8\\
I_9^S(\gamma) &= -\frac{323323}{65536} + \frac{33948915}{65536}\gamma^2 - \frac{260275015}{32768}\gamma^4\\
&\quad + \frac{1301375075}{32768}\gamma^6 - \frac{5019589575}{65536}\gamma^8\\
&\quad + \frac{3234846615}{65536}\gamma^{10}.
\end{aligned} \tag{9.5}$$

Let us only cite the $\gamma \to 1$ value of the last Schwarzschildlike coefficient entering Eq. (9.2) (which suffices at the 6PN accuracy)

$$I_{11}^S(\gamma) = \frac{14196819}{256} + O(\gamma^2 - 1). \tag{9.6}$$

The most useful consequence of the expression (9.2) for the radial action is that it condenses the *irreducible* (post-test-mass) information about the 6PN local dynamics in a rather small number of coefficients, namely the fifteen energy-dependent coefficients $I_{2n+1}^{\nu^k}(\gamma)$, with $1 \le k \le n$ and $1 \le n \le 5$. Our 6PN-accurate computation yields these coefficients in the form of a PN expansion, i.e., an expansion in powers of $p_\infty^2 \equiv \gamma^2 - 1$. [Note that the so-defined quantity p_∞^2 is negative for bound states.] We found, for example,

$$\begin{aligned}
I_3^{\nu^1}(\gamma) &= -\frac{5}{2}\eta^4 + \left(\frac{41}{128}\pi^2 - \frac{557}{24}\right)p_\infty^2\eta^6\\
&\quad + \left(-\frac{4601}{144} + \frac{33601}{24576}\pi^2\right)p_\infty^4\eta^8\\
&\quad + \left(-\frac{3978707}{201600} + \frac{93031}{49152}\pi^2\right)p_\infty^6\eta^{10}\\
&\quad + \left(\frac{9733841}{16777216}\pi^2 + \frac{5058313}{940800}\right)p_\infty^8\eta^{12}.
\end{aligned} \tag{9.7}$$

The other ν-dependent contributions can be read off Table XIV, which lists the PN expansions of the full coefficients $I_{2n+1}(\gamma;\nu) = I_{2n+1}^S(\gamma) + \sum_{k=1}^{n} I_{2n+1}^{\nu^k}(\gamma)\nu^k$.

We recall that the periastron-advance parameter is derived from the radial action as follows [40]:

$$K \equiv 1 + k \equiv \frac{\Phi}{2\pi} = -\partial_j I_r(\gamma, j). \tag{9.8}$$

Inserting the expression (9.2) in the latter formula yields

$$\begin{aligned}
k(\gamma, j) &= \frac{I_1^S(\gamma)}{hj^2} + 3\frac{I_3(\gamma;\nu)}{h^3 j^4}\\
&\quad + 5\frac{I_5(\gamma;\nu)}{h^5 j^6} + 7\frac{I_7(\gamma;\nu)}{h^7 j^8}\\
&\quad + 9\frac{I_9(\gamma;\nu)}{h^9 j^{10}} + 11\frac{I_{11}(\gamma;\nu)}{h^{11} j^{12}},
\end{aligned} \tag{9.9}$$

where the various coefficients $I_n(\gamma,\nu)$ are listed in Table XIV.

Recently, Ref. [68] pointed out that the periastron precession $\Phi(\gamma, j) - 2\pi = 2\pi k(\gamma, j)$ could (under some conditions) be identified with a suitably defined analytic continuation of $\chi(\gamma, j) + \chi(\gamma, -j)$. The ν-structure of the formula (9.9) is then seen to be a consequence of the rule, Eqs. (7.26), (7.27), found in Ref. [58], about the polynomial ν-structure of the energy-rescaled scattering angle $h^{n-1}\chi_n(\gamma,\nu)$. We then tried to replace the imposition of the constraint (7.27) by the imposition of the polynomiality constraint (9.4) directly to the radial action [or, equivalently to the periastron precession $k(\gamma, j)$, Eq. (9.9)]. However, imposing the polynomiality constraints (9.4) or (9.9) is not equivalent, and, in fact, significantly *weaker* than imposing the conditions (7.27). Imposing the conditions (9.4) or (9.9) at the 6PN level leaves undetermined many more coefficients than imposing (7.27). This non equivalence essentially follows from the fact that $I_{2n+1}(\gamma;\nu)$ is proportional to

TABLE XIV. List of the various coefficients $I_n(\gamma,\nu)$ (expressed in terms of $p_\infty^2 \equiv \gamma^2 - 1 < 0$) entering the expression (9.2) of the radial action.

$I_0(p_\infty)$	$\frac{1}{\sqrt{-p_\infty^2}}(1+2\eta^2 p_\infty^2)$
$I_1(p_\infty)$	$3\eta^2 + \frac{15}{4}\eta^4 p_\infty^2$
$I_3(p_\infty;\nu)$	$\frac{35}{4}\eta^4 + \frac{105}{4}p_\infty^2\eta^6 + \frac{1155}{64}p_\infty^4\eta^8$ $+[-\frac{5}{2}\eta^4 + (\frac{41}{128}\pi^2 - \frac{557}{24})p_\infty^2\eta^6 + (-\frac{4601}{144} + \frac{33601}{24576}\pi^2)p_\infty^4\eta^8 + (-\frac{3978707}{201600} + \frac{93031}{49152}\pi^2)p_\infty^6\eta^{10}$ $+(\frac{9733841}{16777216}\pi^2 + \frac{5058313}{940800})p_\infty^8\eta^{12}]\nu$
$I_5(p_\infty;\nu)$	$\frac{231}{4}\eta^6 + \frac{9009}{32}p_\infty^2\eta^8 + \frac{27027}{64}p_\infty^4\eta^{10} + \frac{51051}{256}p_\infty^6\eta^{12}$ $+[(\frac{123}{128}\pi^2 - \frac{125}{2})\eta^6 + (-\frac{224113}{480} + \frac{51439}{4096}\pi^2)p_\infty^2\eta^8 + (\frac{464237}{8192}\pi^2 - \frac{20420849}{16800})p_\infty^4\eta^{10}$ $+(-\frac{135909}{8388608}\pi^4 - \frac{982293061}{12582912}\pi^2 + \frac{1343882527}{25401600})p_\infty^6\eta^{12}]\nu$ $+[\frac{21}{8}\eta^6 + (-\frac{369}{256}\pi^2 + \frac{2013}{32})p_\infty^2\eta^8 + (\frac{925}{96} - \frac{3}{16}a_6^{\nu^2} - \frac{3}{16}\bar{d}_5^{\nu^2} - \frac{12371}{16384}\pi^2)p_\infty^4\eta^{10}$ $+(-\frac{7}{32}\bar{d}_5^{\nu^2} - \frac{7}{32}a_6^{\nu^2} - \frac{3}{32}q_{45}^{\nu^2} - \frac{1}{32}\bar{d}_6^{\nu^2} - \frac{11437991}{22400} + \frac{472773}{32768}\pi^2)p_\infty^6\eta^{12}]\nu^2$
$I_7(p_\infty;\nu)$	$\frac{32175}{64}\eta^8 + \frac{109395}{32}p_\infty^2\eta^{10} + \frac{2078505}{256}p_\infty^4\eta^{12}$ $+[(\frac{425105}{24576}\pi^2 - \frac{248057}{288})\eta^8 + (\frac{2310485}{16384}\pi^2 - \frac{99111883}{13440})p_\infty^2\eta^{10}$ $+(-\frac{109665759605}{75497472}\pi^2 - \frac{28658940509}{3386880} + \frac{81987555}{8388608}\pi^4)p_\infty^4\eta^{12}]\nu$ $+[(-\frac{1025}{256}\pi^2 + \frac{18925}{96})\eta^8 + (-\frac{1290275}{24576}\pi^2 + \frac{1089349}{576} - \frac{15}{8}a_6^{\nu^2} - \frac{5}{8}\bar{d}_5^{\nu^2})p_\infty^2\eta^{10}$ $+(-\frac{103473815}{196608}\pi^2 - \frac{15}{32}q_{45}^{\nu^2} - \frac{405}{32}a_6^{\nu^2} - \frac{15}{16}a_7^{\nu^2} + \frac{832072211}{161280} - \frac{135}{32}\bar{d}_5^{\nu^2} - \frac{15}{32}\bar{d}_6^{\nu^2} + \frac{25215}{65536}\pi^4)p_\infty^4\eta^{12}]\nu^2$ $+[-\frac{45}{16}\eta^8 + (-\frac{7595}{64} + \frac{3075}{1024}\pi^2)p_\infty^2\eta^{10} + (\frac{15}{32}\bar{d}_5^{\nu^2} - \frac{15}{16}a_7^{\nu^3} + \frac{5585}{32} - \frac{477255}{65536}\pi^2 + \frac{15}{32}a_6^{\nu^2})p_\infty^4\eta^{12}]\nu^3$
$I_9(p_\infty;\nu)$	$\frac{323323}{64}\eta^{10} + \frac{11316305}{256}p_\infty^2\eta^{12}$ $+[(\frac{121807}{1024}\pi^2 - \frac{6817563}{640})\eta^{10} + (-\frac{551913398477}{113246208}\pi^2 + \frac{387365405}{8388608}\pi^4 - \frac{23711330921}{345600})p_\infty^2\eta^{12}]\nu$ $+[(-\frac{7}{16}\bar{d}_5^{\nu^2} + \frac{572999}{128} - \frac{35}{16}a_6^{\nu^2} - \frac{1755159}{16384}\pi^2)\eta^{10}$ $+(-\frac{385}{32}\bar{d}_5^{\nu^2} - \frac{21}{32}q_{45}^{\nu^2} - \frac{35}{8}a_7^{\nu^2} + \frac{176505}{32768}\pi^4 - \frac{122488583}{49152}\pi^2 - \frac{1925}{32}a_6^{\nu^2} - \frac{35}{32}\bar{d}_6^{\nu^2} + \frac{6532266163}{115200})p_\infty^2\eta^{12}]\nu^2$ $+[(\frac{10045}{1024}\pi^2 - \frac{42665}{96})\eta^{10} + (\frac{105}{16}a_6^{\nu^2} + \frac{35}{16}\bar{d}_5^{\nu^2} + \frac{13076035}{98304}\pi^2 - \frac{35}{8}a_7^{\nu^3} - \frac{11754113}{2304})p_\infty^2\eta^{12}]\nu^3$ $+[\frac{385}{128}\eta^{10} + (\frac{291655}{1536} - \frac{10045}{2048}\pi^2)p_\infty^2\eta^{12}]\nu^4$
$I_{11}(p_\infty;\nu)$	$\frac{14196819}{256}\eta^{12}$ $+[(-\frac{3236467169}{30240} - \frac{188085303629}{50331648}\pi^2 + \frac{350055909}{8388608}\pi^4)\eta^{12}]\nu$ $+[(-\frac{21}{32}\bar{d}_6^{\nu^2} + \frac{529515}{65536}\pi^4 + \frac{2062272503}{22400} - \frac{1911}{32}a_6^{\nu^2} - \frac{9}{32}q_{45}^{\nu^2} - \frac{63}{16}a_7^{\nu^2} - \frac{179354853}{65536}\pi^2 - \frac{273}{32}\bar{d}_5^{\nu^2})\eta^{12}]\nu^2$ $+[(-\frac{63}{16}a_7^{\nu^3} + \frac{315}{32}a_6^{\nu^2} + \frac{24980025}{65536}\pi^2 - \frac{978061}{64} + \frac{63}{32}\bar{d}_5^{\nu^2})\eta^{12}]\nu^3$ $+[(-\frac{38745}{2048}\pi^2 + \frac{428085}{512})\eta^{12}]\nu^4$ $+[-\frac{819}{256}\eta^{12}]\nu^5$

$\chi_{2n+2}(\gamma,\nu)$ and therefore misses the ν-information contained in the odd scattering-angle coefficients $\chi_{2n+1}(\gamma,\nu)$.

Finally, let us recall the well-known fact that the gauge-invariant relation between energy and angular momentum along circular orbits can be conveniently obtained by setting $I_r = 0$ in Eq. (9.2). The resulting equation,

$$j = I_0^S(\gamma) + \frac{I_1^S(\gamma)}{hj} + \frac{I_3(\gamma;\nu)}{(hj)^3} + \frac{I_5(\gamma;\nu)}{(hj)^5} + \frac{I_7(\gamma;\nu)}{(hj)^7} + \frac{I_9(\gamma;\nu)}{(hj)^9} + \frac{I_{11}(\gamma;\nu)}{(hj)^{11}}, \tag{9.10}$$

can then be easily perturbatively solved to get either $\frac{1}{j^2}$ as an expansion in powers of p_∞^2, or p_∞^2 as an expansion in powers of $\frac{1}{j^2}$, say

$$1-\gamma^2 = -p_\infty^2 = \frac{1}{j^2} + \frac{2}{j^4} + (9-2\nu)\frac{1}{j^6} + \left(54 - \frac{154}{3}\nu + \frac{41}{32}\pi^2\nu\right)\frac{1}{j^8} + \cdots, \tag{9.11}$$

or, equivalently,

$$\gamma \equiv \hat{E}_{\rm eff}^{\rm loc,f,circ} = 1 - \frac{1}{2j^2} - \frac{9}{8j^4} + \left(-\frac{81}{16} + \nu\right)\frac{1}{j^6} + \left(-\frac{3861}{128} + \frac{157}{6}\nu - \frac{41}{64}\pi^2\nu\right)\frac{1}{j^8} + \cdots. \tag{9.12}$$

The local contribution to the circular energy then straightforwardly follows:

$$E^{\rm loc,f,circ}(j) = M\sqrt{1+2\nu(\hat{E}_{\rm eff}^{\rm loc,f,circ}-1)}. \tag{9.13}$$

Here, we have simply indicated the 3PN-accurate beginning of these expansions. It is easy to use our results to derive the 6PN-accurate local circular energy. We leave to future work the completion of these results to the full 6PN level, obtained by adding the $4+5+6$PN nonlocal contribution.

X. POST-MINKOWSKIAN VIEW OF THE DETERMINATION OF THE LOCAL DYNAMICS

At any given PN accuracy, our new method is able to determine most of the structure of the two-body dynamics except for a relatively small number of numerical coefficients. When working at the 5PN accuracy, only two numerical coefficients are left undetermined in the local dynamics, namely $a_6^{\nu^2}$ and $\bar{d}_5^{\nu^2}$. When working at the 6PN accuracy, four more numerical coefficients are left undetermined, namely $a_7^{\nu^2}, a_7^{\nu^3}, \bar{d}_6^{\nu^2}$, and $q_{45}^{\nu^2}$. Let us clarify the basic reason underlying this incompleteness, in a way which will allow us to anticipate the number and structure of the higher-order analogs of these undetermined parameters. This is easily done by working within a PM-expanded scheme, and by using some of the structural results of PM gravity discussed in Ref. [58]. It was found there that the general structure of the PM coefficients $q_n^E(\gamma,\nu)$ of the EOB Q potential in the E-type energy gauge was

$$\begin{aligned} q_n^E(\gamma,\nu) &= q_{n,0}(\gamma) + \frac{q_{n,1}(\gamma)}{h(\gamma,\nu)} + \cdots + \frac{q_{n,n-1}(\gamma)}{h^{n-1}(\gamma,\nu)} \\ &= \sum_{k=0}^{n-1} \frac{q_{n,k}(\gamma)}{h^k(\gamma,\nu)}, \end{aligned} \tag{10.1}$$

with the constraint

$$\sum_{k=0}^{n-1} q_{n,k}(\gamma) = 0. \tag{10.2}$$

The important structural information in the expression (10.1) is the fact that the ν-dependence of $q_n^E(\gamma,\nu)$ is entirely described through the powers of $h(\gamma,\nu)$ entering the denominators. All the corresponding numerators $q_{n,k}(\gamma)$ depend only on the EOB effective energy $\gamma = \hat{\mathcal{E}}_{\rm eff}$. The constraint (10.2) expresses the fact that

$$\lim_{\nu\to 0} q_n^E(\gamma,\nu) = 0, \tag{10.3}$$

i.e., the basic feature of the EOB formalism that the $\nu \to 0$ limit of the EOB mass-shell condition reduces to a geodesic in a Schwarzshild metric of mass M. Let us also note that the behavior of the coefficients $q_n^E(\gamma,\nu)$ in the $\gamma^2 \to 1$ limit,

$$\begin{aligned} q_{n\rm EG}^{E\rm loc,f}(\gamma;\nu) &= q_{n\rm E,EG}^0(\nu) + q_{n\rm E,EG}^1(\nu)(\gamma^2-1) \\ &\quad + q_{n\rm E,EG}^2(\nu)(\gamma^2-1)^2 + \ldots . \end{aligned} \tag{10.4}$$

is smooth, i.e., the expansion coefficients, and notably the first one, $q_{n\rm E,EG}^0(\nu)$, are all finite [and $O(\nu)$].

As explicitly discussed in the 3PM case, $n=3$, in Ref. [58], there are more constraints on the n energy-dependent coefficients $q_{n,k}(\gamma)$ which determine some of them in terms of the lower PM orders. Indeed, let us first insert the decomposition (10.1) in the expressions (7.18) relating the PM-expansion coefficients $q_n^E(\gamma,\nu)$ of the EOB potential to the PM-expansion coefficients $\chi_n(\gamma,\nu)$ of the scattering angle. This yields explicit expressions for the $\chi_n(\gamma,\nu)$'s in terms of the $q_n^E(\gamma,\nu)$'s. For instance, at the lowest PM order ($n=2$) we have

$$\begin{aligned} \chi_2(\gamma,\nu) &= \chi_2^{\rm Schw}(\gamma) - \frac{\pi}{4} q_2(\gamma,\nu) \\ &= \chi_2^{\rm Schw}(\gamma) - \frac{\pi}{4}\left(q_{2,0}(\gamma) + \frac{q_{2,1}(\gamma)}{h(\gamma,\nu)}\right), \end{aligned} \tag{10.5}$$

so that

$$\tilde{\chi}_2(\gamma,\nu) \equiv h\chi_2 = h\left(\chi_2^{\rm Schw}(\gamma) - \frac{\pi}{4} q_{2,0}(\gamma)\right) - \frac{\pi}{4} q_{2,1}(\gamma). \tag{10.6}$$

Imposing the condition that $\tilde{\chi}_2$ is independent of ν reduces to imposing that the coefficient of $h(\gamma,\nu)$ on the right-hand side vanishes. This yields the constraint

$$\chi_2^{\rm Schw}(\gamma) - \frac{\pi}{4} q_{2,0}(\gamma) = 0, \tag{10.7}$$

which determines $q_{2,0}(\gamma)$ in terms of $\chi_2^{\rm Schw}(\gamma)$. The summed constraint (10.2) then determines $q_{2,1}(\gamma) = -q_{2,0}(\gamma)$. One then recovers the result [66]

$$q_2(\gamma,\nu) = \frac{4}{\pi}\chi_2^{\rm Schw}(\gamma)\left(1-\frac{1}{h}\right) = \frac{3}{2}(5\gamma^2-1)\left(1-\frac{1}{h}\right). \tag{10.8}$$

In other words, the 2PM dynamics is entirely determined by the test-mass scattering angle.

At the 3PM level the three coefficients entering

$$q_3(\gamma,\nu) = q_{3,0}(\gamma) + \frac{q_{3,1}(\gamma)}{h(\gamma,\nu)} + \frac{q_{3,2}(\gamma)}{h^2(\gamma,\nu)}, \tag{10.9}$$

satisfy two constraints. First, the sum constraint (10.2), i.e.,

$$q_{3,0}(\gamma) + q_{3,1}(\gamma) + q_{3,2}(\gamma) = 0, \tag{10.10}$$

and then the condition that $\tilde{\chi}_3(\gamma,\nu) \equiv h^2\chi_3(\gamma,\nu)$ be linear in ν. The second Eq. (7.18) allows one to express $\chi_3(\gamma,\nu)$ in

terms of the $q_{3,k}(\gamma)$'s. It is easily seen that inserting the expression (10.8) of $q_2(\gamma,\nu)$ in the second Eq. (7.18) yields $\tilde{\chi}_3(\gamma,\nu)$ in the form of a polynomial in h, namely,

$$\begin{aligned}\tilde{\chi}_3(\gamma,\nu) &= h^2\chi_3^{\rm Schw}(\gamma) - \frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}h^2 q_2(\gamma,\nu)\\ &\quad - \sqrt{\gamma^2-1}h^2 q_3(\gamma,\nu)\\ &= h^2\chi_3^{\rm Schw} - \frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}\frac{3}{2}(5\gamma^2-1)(h^2-h)\\ &\quad - \sqrt{\gamma^2-1}(h^2 q_{3,0}(\gamma) + h q_{3,1}(\gamma) + q_{3,2}(\gamma)).\end{aligned} \tag{10.11}$$

As $h^2 = 1+2\nu(\gamma-1)$, the condition to be linear in ν (at a fixed value of γ) is equivalent (for a polynomial in h with γ-dependent coefficients) to having the structure $c_0 + c_2 h^2$. This gives one constraint, namely the vanishing of the coefficient of h^1. This constraint determines the coefficient $q_{3,1}(\gamma)$ to have the value [58]

$$q_{3,1}(\gamma) = \frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}. \tag{10.12}$$

The two remaining coefficients $q_{3,0}(\gamma)$, $q_{3,2}(\gamma)$ then satisfy the single sum constraint (10.10). The conclusion is that the general solution of the 3PM constraints is a Q potential of the form

$$\begin{aligned}q_3(\gamma,\nu) &= q_{3,1}(\gamma)\left(\frac{1}{h(\gamma,\nu)}-1\right)\\ &\quad + q_{3,2}(\gamma)\left(\frac{1}{h^2(\gamma,\nu)}-1\right),\end{aligned} \tag{10.13}$$

where $q_{3,1}(\gamma)$ is determined from (10.12), and where $q_{3,2}(\gamma)$ is, at this stage, left undetermined by the general PM-EOB constraints of Ref. [58]. On the other hand, let us assume that one has somehow determined (maybe to some limited PN accuracy) the value of the gauge-invariant 3PM scattering angle, which must have the structure

$$\chi_3(\gamma,\nu) = \chi_{3,0}(\gamma) + \frac{\chi_{3,2}(\gamma)}{h^2(\gamma,\nu)}. \tag{10.14}$$

Let us now insert in the second Eq. (7.18) the expressions of $\chi_3(\gamma,\nu)$ and $q_3(\gamma,\nu)$ as polynomials in $\frac{1}{h}$ (with coefficients depending only on γ), i.e., Eqs. (10.9) and (10.14). As both sides are polynomials in $\frac{1}{h}$, we can identify the coefficients of $\frac{1}{h^2}$ on both sides. Indeed, we are dealing here with expressions depending on ν only through the energy parameter $h(\gamma,\nu)$. Therefore, two functions of γ and ν, which can written as polynomials in $\frac{1}{h}$, can be equal only if all the γ-dependent (but crucially ν-independent) coefficients of the various powers of $\frac{1}{h}$ agree with each other. This yields the simple link:

$$\chi_{3,2}(\gamma) = -\sqrt{\gamma^2-1}\,q_{3,2}(\gamma). \tag{10.15}$$

In addition, using the fact that $\chi_3^{\rm Schw}(\gamma) = \chi_3(\gamma,0) + \chi_{3,2}(\gamma)$, we can rewrite Eq. (10.14) as

$$\chi_3(\gamma,\nu) = \chi_3^{\rm Schw}(\gamma) - 2\nu\frac{(\gamma-1)\chi_{3,2}(\gamma)}{h^2(\gamma,\nu)}. \tag{10.16}$$

This formula shows that the function $\chi_{3,2}(\gamma)$ parametrizes the deviation of $\chi_3(\gamma,\nu)$ away from its test-mass limit $\lim_{\nu\to 0}\chi_3(\gamma,\nu) = \chi_3^{\rm Schw}(\gamma)$. Let us again emphasize (following Ref. [58]) that even if one knows only the linear-in-ν (1SF) expansion of the 3PM scattering angle, Eq. (10.16) shows that this suffices to fully determine the function $\chi_{3,2}(\gamma)$. Then having extracted the function $\chi_{3,2}(\gamma)$ from the 1SF expansion of $\chi_3(\gamma,\nu)$, we can compute $q_{3,2}(\gamma)$ from Eq. (10.15), and thereby obtain the full 3PM dynamics by using Eqs. (10.12), (10.13).

As our method, when applied at any PN approximation, determines (in particular) the 1SF expansion of the local dynamics, we see that it will determine the function $q_{3,2}(\gamma)$ with the PN accuracy with which we work. This is why we could determine the 6PN expansion of $q_{3,2}(\gamma)$, i.e., of the local 3PM dynamics.

In view of the several independent 6PN-accurate confirmations ([63,64], and the present work) of the value of $\chi_3(\gamma,\nu)$ derived in Refs. [61,62], we shall assume in the following that $q_{3,2}(\gamma)$ is exactly known, namely (using the notation[6] of [58])

$$q_{3,2}(\gamma) = -\frac{\bar{C}^B(\gamma)}{\gamma-1}, \tag{10.17}$$

with

$$\begin{aligned}\bar{C}^B(\gamma) &= \frac{2}{3}\gamma(14\gamma^2+25)\\ &\quad + 4\frac{4\gamma^4-12\gamma^2-3}{\sqrt{\gamma^2-1}}\operatorname{arcsinh}\left(\sqrt{\frac{\gamma-1}{2}}\right).\end{aligned} \tag{10.18}$$

This assumption will allow us to simplify the discussion of the determination of the higher PM-order coefficients.

Let us indeed indicate how the above 2PM and 3PM results extend at higher PM levels. This will allow us to clarify the effectiveness (associated with a partial

[6]The coefficients $q_{3,0}(\gamma), q_{3,1}(\gamma), q_{3,2}(\gamma)$ are respectively denoted $A(\gamma)$, $B(\gamma)$, $C(\gamma)$ there, with $\bar{C}(\gamma) \equiv -(\gamma-1)C(\gamma)$.

ineffectiveness) of our method in determining (or leaving undetermined) the parameters entering the local dynamics.

The structure of the 4PM-level EOB Q potential is

$$q_4^E(\gamma,\nu) = q_{4,0}(\gamma) + \frac{q_{4,1}(\gamma)}{h(\gamma,\nu)} + \frac{q_{4,2}(\gamma)}{h^2(\gamma,\nu)} + \frac{q_{4,3}(\gamma)}{h^3(\gamma,\nu)}, \tag{10.19}$$

with the usual constraint

$$q_{4,0}(\gamma) + q_{4,1}(\gamma) + q_{4,2}(\gamma) + q_{4,3}(\gamma) = 0. \tag{10.20}$$

The third Eq. (7.18) leads to an expression for $\chi_4(\gamma,\nu)$ of the form

$$\chi_4(\gamma,\nu) = \chi_4^{\rm Schw}(\gamma) - \frac{3\pi}{8}(\gamma^2-1)q_4^E(\gamma,\nu) + K[q_2(\gamma,\nu), q_3(\gamma,\nu)], \tag{10.21}$$

where $K[q_2,q_3]$ denotes some *known* terms, namely

$$K[q_2,q_3] = \pi\left[\frac{3}{16}q_2^2(\gamma,\nu) - \frac{9}{16}(5\gamma^2-1)q_2(\gamma,\nu) - \frac{3}{8}(3\gamma^2-1)q_3(\gamma,\nu)\right]. \tag{10.22}$$

Inserting the $\frac{1}{h}$ parametrization (10.19) of $q_4^E(\gamma,\nu)$, together with the above explicit expressions of $q_2(\gamma,\nu)$, and $q_3(\gamma,\nu)$ (as polynomials in $\frac{1}{h}$), then leads to an expression for $\chi_4(\gamma,\nu)$ having also the structure of a polynomial in $\frac{1}{h}$, say

$$\chi_4(\gamma,\nu) = \chi_{4,0}^q(\gamma) + \frac{\chi_{4,1}^q(\gamma)}{h(\gamma,\nu)} + \frac{\chi_{4,2}^q(\gamma)}{h^2(\gamma,\nu)} + \frac{\chi_{4,3}^q(\gamma)}{h^3(\gamma,\nu)}, \tag{10.23}$$

where the superscript q means that all the coefficients $\chi_{4,k}^q(\gamma)$ are explicit expressions in the $q_{n,k}$'s.

The rule found in [58] restricts $h^3\chi_4(\gamma,\nu)$ to be linear in ν. This is equivalent to the following *restricted polynomial structure* for $\chi_4(\gamma,\nu)$:

$$\chi_4(\gamma,\nu) = \frac{\chi_{4,1}(\gamma)}{h(\gamma,\nu)} + \frac{\chi_{4,3}(\gamma)}{h^3(\gamma,\nu)}, \tag{10.24}$$

with the constraint

$$\chi_{4,1}(\gamma) + \chi_{4,3}(\gamma) = \chi_4^{\rm Schw}(\gamma). \tag{10.25}$$

For the general reason already explained above, the equality (for all values of ν) between two functions of γ and ν, which can both be written as polynomials in $\frac{1}{h}$ with γ-dependent (but crucially ν-independent) coefficients, implies the equality of the γ-dependent coefficients of all the various powers of $\frac{1}{h}$. We therefore conclude that the coefficients $q_{4,k}$ must satisfy the two equations

$$\chi_{4,0}^q(\gamma) = 0; \qquad \chi_{4,2}^q(\gamma) = 0. \tag{10.26}$$

In view of Eq. (10.21), the latter two equations are respectively linear in $q_{4,0}(\gamma)$ and $q_{4,2}(\gamma)$, and contain "source terms" provided both by $\chi_4^{\rm Schw}(\gamma)$ and by $K[q_2(\gamma,\nu), q_3(\gamma,\nu)]$. We can then solve the system of the two equations (10.26) for $q_{4,0}(\gamma)$, and $q_{4,2}(\gamma)$. This yields the (unique) solution

$$\begin{aligned} q_{4,2}(\gamma) &= a_2(\gamma) + b_2(\gamma)q_{3,2}(\gamma), \\ q_{4,0}(\gamma) &= -a_1(\gamma) - a_2(\gamma) - b_2(\gamma)q_{3,2}(\gamma), \end{aligned} \tag{10.27}$$

where we denoted

$$\begin{aligned} a_2(\gamma) &= \frac{9}{8}\frac{(5\gamma^2-1)^2}{\gamma^2-1}, \\ b_2(\gamma) &= -\frac{3\gamma^2-1}{\gamma^2-1}, \\ a_1(\gamma) &= -\frac{1875\gamma^6 - 2529\gamma^4 + 905\gamma^2 - 59}{16(\gamma^2-1)^2}. \end{aligned} \tag{10.28}$$

Let us now consider the sum constraint, Eq. (10.20). The latter constraint, together with the solution (10.27), yields the following expression for $q_{4,1}(\gamma)$:

$$q_{4,1}(\gamma) = a_1(\gamma) - q_{4,3}(\gamma). \tag{10.29}$$

In other words, the exact structure of the 4PM Q coefficient is

$$\begin{aligned} q_4^E(\gamma,\nu) = {} & a_1(\gamma)\left(\frac{1}{h(\gamma,\nu)} - 1\right) \\ & + (a_2(\gamma) + b_2(\gamma)q_{3,2}(\gamma))\left(\frac{1}{h^2(\gamma,\nu)} - 1\right) \\ & + q_{4,3}(\gamma)\left(\frac{1}{h^3(\gamma,\nu)} - \frac{1}{h(\gamma,\nu)}\right). \end{aligned} \tag{10.30}$$

In this expression, $q_{3,2}(\gamma)$ can (as far as we know) be replaced by (10.17), so that the only undetermined function of γ is the last coefficient $q_{4,3}(\gamma)$. The latter can be determined by the knowledge of $\chi_4(\gamma,\nu)$. Indeed, the link (10.21) between q_4 and χ_4 implies that the coefficient $\chi_{4,3}$ of $\frac{1}{h^3}$ in the $\frac{1}{h}$-polynomial expression (10.24) of χ_4 is directly linked to $q_{4,3}(\gamma)$ via

$$\chi_{4,3}(\gamma) = -\frac{3\pi}{8}(\gamma^2-1)q_{4,3}(\gamma). \tag{10.31}$$

In addition, we note that Eq. (10.24) can be rewritten as

$$\chi_4(\gamma,\nu) = \frac{\chi_4^{\rm Schw}(\gamma)}{h(\gamma,\nu)} + \chi_{4,3}(\gamma)\left(\frac{1}{h^3(\gamma,\nu)} - \frac{1}{h(\gamma,\nu)}\right) = \frac{\chi_4^{\rm Schw}(\gamma)}{h(\gamma,\nu)} - 2\nu(\gamma-1)\frac{\chi_{4,3}(\gamma)}{h^3(\gamma,\nu)}. \tag{10.32}$$

The latter expression clearly shows that the knowledge of the $O(G^4)$ scattering angle $\chi_4(\gamma,\nu)$ at the *linear* order in ν (1SF order) suffices to determine the exact function $\chi_{4,3}(\gamma)$, and thereby to have the full ν dependence of the scattering angle, as defined by the expression (10.32).

We have derived above the 4PM scattering angle $\chi_4(\gamma,\nu)$ with 6PN accuracy. Using the representation (10.32) we can transcribe our results into the following corresponding 6PN knowledge of the more primitive function $\chi_{4,3}(\gamma)$:

$$\chi_{4,3}(\gamma) = \pi\left[\frac{15}{4p_\infty^2} + \frac{391}{8} - \frac{123}{256}\pi^2 + \left(\frac{4597}{48} - \frac{35569}{16384}\pi^2\right)p_\infty^2 + \left(\frac{372943}{5600} - \frac{217695}{65536}\pi^2\right)p_\infty^4 + \left(-\frac{4976527}{1881600} - \frac{49220339}{33554432}\pi^2\right)p_\infty^6 + O(p_\infty^8)\right], \tag{10.33}$$

where $p_\infty^2 = \gamma^2 - 1$. The latter result can then be transcribed in a corresponding 6PN-accurate knowledge of the function $q_{4,3}(\gamma)$, and thereby of the full 4PM Q potential $q_4^E(\gamma,\nu)$, using Eq. (10.30). We note in passing that the result (10.33) implies for $q_{4,3}(\gamma)$ a behavior in the small p_∞ limit of the form

$$q_{4,3}(\gamma) = -\frac{10}{p_\infty^4} + \left(-\frac{391}{3} + \frac{41}{32}\pi^2\right)\frac{1}{p_\infty^2} + \left(-\frac{4597}{18} + \frac{35569}{6144}\pi^2\right) + \left(-\frac{372943}{2100} + \frac{72565}{8192}\pi^2\right)p_\infty^2 + \left(\frac{4976527}{705600} + \frac{49220339}{12582912}\pi^2\right)p_\infty^4 + O(p_\infty^6), \tag{10.34}$$

so that the corresponding contribution to $q_4^E(\gamma,\nu)$, Eq. (10.30), reads

$$\begin{aligned} &q_{4,3}(\gamma)\left(\frac{1}{h^3(\gamma,\nu)} - \frac{1}{h(\gamma,\nu)}\right) \\ &= \frac{10}{p_\infty^2}\nu + \left[-15\nu^2 + \left(\frac{767}{6} - \frac{41}{32}\pi^2\right)\nu\right] \\ &\quad + \left[\frac{75}{4}\nu^3 + \left(-188 + \frac{123}{64}\pi^2\right)\nu^2 + \left(\frac{4033}{18} - \frac{33601}{6144}\pi^2\right)\nu\right]p_\infty^2 \\ &\quad + \left[-\frac{175}{8}\nu^4 + \left(\frac{3685}{16} - \frac{615}{256}\pi^2\right)\nu^3 + \left(\frac{31633}{4096}\pi^2 - \frac{13921}{48}\right)\nu^2 + \left(-\frac{93031}{12288}\pi^2 + \frac{6514457}{50400}\right)\nu\right]p_\infty^4 \\ &\quad + \left[\frac{1575}{64}\nu^5 + \left(\frac{1435}{512}\pi^2 - \frac{12635}{48}\right)\nu^4 + \left(\frac{58775}{192} - \frac{148325}{16384}\pi^2\right)\nu^3 + \left(-\frac{1114333}{8400} + \frac{156397}{16384}\pi^2\right)\nu^2 \right. \\ &\quad \left. + \left(-\frac{6859063}{235200} - \frac{9733841}{4194304}\pi^2\right)\nu\right]p_\infty^6 + O(p_\infty^8). \end{aligned} \tag{10.35}$$

This contribution is singular as $p_\infty \to 0$. However, it is easily checked that the other contributions to $q_4^E(\gamma,\nu)$ in Eq. (10.30) cancel this low-velocity singularity and leave a finite result,

$$q_4^E(\gamma,\nu) = \left(\frac{175}{3} - \frac{41}{32}\pi^2\right)\nu - \frac{7}{2}\nu^2 + O(p_\infty^2), \tag{10.36}$$

in agreement with the result listed in Table XII.

Let us sketch the extension of these results to the ≥ 5 PM orders. [See Appendix for more technical details.] Again the basic trick is to express all dynamical functions as polynomials in $\frac{1}{h}$, with γ-dependent coefficients. This trick is efficient because the PM-EOB results Eqs. (7.18) involve no explicit ν dependence. In turn, this property follows from the basic fact that the 1PM-accurate EOB dynamics is ν-independent when expressed in terms of the EOB effective energy $\gamma = \hat{\mathcal{E}}_{\rm eff}$ [66].

The structure of the 5PM Q potential reads

$$q_5^E(\gamma,\nu) = q_{5,0}(\gamma) + \frac{q_{5,1}(\gamma)}{h(\gamma,\nu)} + \frac{q_{5,2}(\gamma)}{h^2(\gamma,\nu)} + \frac{q_{5,3}(\gamma)}{h^3(\gamma,\nu)} + \frac{q_{5,4}(\gamma)}{h^4(\gamma,\nu)}, \tag{10.37}$$

with the usual constraint

$$q_{5,0}(\gamma) + q_{5,1}(\gamma) + q_{5,2}(\gamma) + q_{5,3}(\gamma) + q_{5,4}(\gamma) = 0. \tag{10.38}$$

The fourth Eq. (7.18) leads to a corresponding expression for $\chi_5(\gamma,\nu)$ of the form

$$\chi_5(\gamma,\nu) = \chi_5^{\rm Schw}(\gamma) - \frac{4(\gamma^2-1)^{3/2}}{3} q_5^E(\gamma) + K[q_2,q_3,q_4], \tag{10.39}$$

where the "known" contribution, $K[q_2,q_3,q_4]$, which involves previous PM orders, q_2, q_3, and q_4, will be found in Appendix.

The rule restricting the ν structure of $\chi_5(\gamma,\nu)$ [58] is equivalent to imposing:

$$\chi_5(\gamma,\nu) = \chi_{5,0}(\gamma) + \frac{\chi_{5,2}(\gamma)}{h^2(\gamma,\nu)} + \frac{\chi_{5,4}(\gamma)}{h^4(\gamma,\nu)}, \tag{10.40}$$

with the constraint

$$\chi_{5,0}(\gamma) + \chi_{5,2}(\gamma) + \chi_{5,4}(\gamma) = \chi_5^{\rm Schw}(\gamma). \tag{10.41}$$

Imposing this structure on the expression following from Eq. (10.39) then yields two constraints expressing the vanishing of the terms $\propto \frac{1}{h}$ and $\propto \frac{1}{h^3}$. This yields two equations of the type

$$\begin{aligned} q_{5,1}(\gamma) &= \text{known}, \\ q_{5,3}(\gamma) &= \text{known}, \end{aligned} \tag{10.42}$$

whose explicit form will be found in Appendix.

In addition, we have the third equation (10.38). The latter equation yields an expression for $q_{5,0}(\gamma)$ of the form

$$q_{5,0}(\gamma) = -q_{5,2}(\gamma) - q_{5,4}(\gamma) + \text{known}. \tag{10.43}$$

At the end of the day, we have a general expression for $q_5(\gamma,\nu)$ of the form

$$q_5^E(\gamma,\nu) = \text{known} + q_{5,2}(\gamma)\left(\frac{1}{h^2(\gamma,\nu)} - 1\right) + q_{5,4}(\gamma)\left(\frac{1}{h^4(\gamma,\nu)} - 1\right), \tag{10.44}$$

where "known" means here

$$\text{known} = q_{5,1}(\gamma)\left(\frac{1}{h} - 1\right) + q_{5,3}(\gamma)\left(\frac{1}{h^3} - 1\right), \tag{10.45}$$

with $q_{5,1}(\gamma)$ and $q_{5,3}(\gamma)$ given in Eqs. (A3).

The expression (10.44) involves only two undetermined (at this stage) parameters $q_{5,2}(\gamma)$ and $q_{5,4}(\gamma)$. As before (*mutatis mutandis*), the two remaining parameters $q_{5,2}(\gamma)$ and $q_{5,4}(\gamma)$ would be determined by the knowledge of the two corresponding coefficients in $\chi_5(\gamma,\nu)$, Eq. (10.40), namely $\chi_{5,2}(\gamma)$ and $\chi_{5,4}(\gamma)$. Indeed, we have the two equations

$$\begin{aligned} \chi_{5,2}(\gamma) &= \frac{45\gamma^4 - 34\gamma^2 + 7}{3(\gamma^2-1)^{1/2}} q_{3,2}(\gamma) - \frac{4}{3}(\gamma^2-1)^{3/2} q_{5,2}(\gamma) - \frac{3(14\gamma^2-5)(-1+5\gamma^2)^2}{4(\gamma^2-1)^{1/2}}, \\ \chi_{5,4}(\gamma) &= -\frac{4}{3}(\gamma^2-1)^{3/2} q_{5,4}(\gamma), \end{aligned} \tag{10.46}$$

where we recall that the 3PM-level function $q_{3,2}(\gamma)$ is known.

However, there is now a difference with what happened at lower PM orders. Indeed, we can rewrite Eq. (10.40) in the form

$$\chi_5(\gamma,\nu) = \chi_5^{\rm Schw}(\gamma) + \chi_{5,2}(\gamma)\left(\frac{1}{h^2(\gamma,\nu)} - 1\right) + \chi_{5,4}(\gamma)\left(\frac{1}{h^4(\gamma,\nu)} - 1\right), \tag{10.47}$$

or, equivalently,

$$\chi_5(\gamma,\nu) = \chi_5^{\rm Schw}(\gamma) - \frac{4}{h^4}\left[\nu(\gamma-1)\left(\chi_{5,4}(\gamma) + \frac{1}{2}\chi_{5,2}(\gamma)\right) + \nu^2(\gamma-1)^2(\chi_{5,4}(\gamma) + \chi_{5,2}(\gamma))\right], \tag{10.48}$$

In other words, after factoring $1/h^4$ the difference $\chi_5(\gamma,\nu) - \chi_5^{\rm Schw}(\gamma)$ has a ν structure of the type $\sim\nu+\nu^2$. By contrast, we previously had a difference $\chi_4(\gamma,\nu) - \frac{\chi_4^{\rm Schw}(\gamma)}{h(\gamma,\nu)}$ of the type $\sim\nu$. This change of ν dependence (from

$\sim\nu$ to $\sim\nu+\nu^2$) implies that, at the 5PM level, there appear two independent functions of γ parametrizing the scattering function, after having taken into account the general structural information about its ν dependence, while there appeared only one function of γ at the 3PM and 4PM levels. As a consequence, our method (which completes a linear-in-ν self-force information by a general ν-dependence information) is able to get complete PN-expanded results at the 3PM and 4PM levels (up to the PN accuracy it uses). However, at the 5PM (and also 6PM) levels, it can only determine one combination of the two independent functions of γ appearing at these levels (namely the function parametrizing the coefficient of ν among the total $\sim\nu+\nu^2$ dependence just mentioned). More specifically, at the 5PM level, one sees from Eq. (10.48) that, when working at some given PN accuracy, our method will be able to determine, within this PN accuracy, the PN expansion of the function $\chi_{5,4}(\gamma)+\frac{1}{2}\chi_{5,2}(\gamma)$, but will leave (partially) undetermined that of the complementary combination $\chi_{5,4}(\gamma)+\chi_{5,2}(\gamma)$. Using our results, we find

$$\begin{aligned}\chi_{5,4}(\gamma)+\frac{1}{2}\chi_{5,2}(\gamma)=&\frac{4}{p_\infty^3}+\left(-\frac{41}{16}\pi^2+\frac{587}{3}\right)\frac{1}{p_\infty}\\&+\left(-\frac{10507}{576}\pi^2+\frac{480263}{540}\right)p_\infty\\&+\left(-\frac{715139}{11520}\pi^2+\frac{30034567}{18900}\right)p_\infty^3\\&+\left(\frac{1160329}{161280}\pi^2+\frac{411639569}{1176000}\right)p_\infty^5\\&+O(p_\infty^7),\end{aligned}\tag{10.49}$$

which is, indeed, fully determined to our 6PN accuracy, while

$$\begin{aligned}&\chi_{5,4}(\gamma)+\chi_{5,2}(\gamma)\\&=-\frac{8}{p_\infty^3}+\left(-\frac{406}{9}+\frac{41}{24}\pi^2\right)\frac{1}{p_\infty}\\&\quad+\left(\frac{112333}{270}+\frac{4}{15}\bar{d}_5^{\nu^2}-\frac{18487}{5760}\pi^2\right)p_\infty\\&\quad+\left(\frac{4}{35}q_{45}^{\nu^2}+\frac{2}{15}\bar{d}_5^{\nu^2}+\frac{1993193869}{1323000}-\frac{5049671}{80640}\pi^2\right)p_\infty^3\\&\quad+O(p_\infty^5),\end{aligned}\tag{10.50}$$

involves the undetermined parameters $\bar{d}_5^{\nu^2}$ and $q_{45}^{\nu^2}$.

When translating this knowledge in terms of the EOB Q potential (in E-type energy gauge[7]), this means that our method is able to determine the function $q_{5,4}(\gamma)+\frac{1}{2}q_{5,2}(\gamma)$, but leaves partly undetermined the complementary function $q_{5,4}(\gamma)+q_{5,2}(\gamma)$. Concerning the other coefficient functions $q_{5,k}(\gamma)$, with $k=0$, 1, 3, parametrizing $q_5(\gamma,\nu)$, the generalization of the reasoning explained above for $q_4(\gamma,\nu)$ shows that they are fully determined in terms of the lower PM information.

One can check that a similar situation occurs at the 6PM level, where the structure of the scattering angle reads

$$\begin{aligned}\chi_6(\gamma,\nu)=&\frac{\chi_6^{\rm Schw}(\gamma)}{h(\gamma,\nu)}+\chi_{6,3}(\gamma)\left(\frac{1}{h^3(\gamma,\nu)}-\frac{1}{h(\gamma,\nu)}\right)\\&+\chi_{6,5}(\gamma)\left(\frac{1}{h^5(\gamma,\nu)}-\frac{1}{h(\gamma,\nu)}\right).\end{aligned}\tag{10.51}$$

The two independent functions $\chi_{6,3}(\gamma),\chi_{6,5}(\gamma)$ parametrize a structure $\sim h^{-5}(\nu+\nu^2)$. Similarly to Eq. (10.48), this can be made manifest by introducing the following two combinations (with γ-dependent coefficients) of $\chi_{6,3}(\gamma)$ and $\chi_{6,5}(\gamma)$, say

$$\begin{aligned}\hat{\chi}_{6,\nu}(\gamma)&\equiv-4(\gamma-1)\left(\frac{1}{2}\chi_{6,3}(\gamma)+\chi_{6,5}(\gamma)\right),\\\hat{\chi}_{6,\nu^2}(\gamma)&\equiv-4(\gamma-1)^2(\chi_{6,3}(\gamma)+\chi_{6,5}(\gamma)),\end{aligned}\tag{10.52}$$

such that Eq. (10.51) reads

$$\chi_6(\gamma,\nu)=\frac{\chi_6^{\rm Schw}(\gamma)}{h(\gamma,\nu)}+\frac{\nu\hat{\chi}_{6,\nu}(\gamma)+\nu^2\hat{\chi}_{6,\nu^2}(\gamma)}{h^5(\gamma,\nu)}.\tag{10.53}$$

Again, our method can only determine one combination [namely $\hat{\chi}_{6,\nu}(\gamma)$] of the two functions $\chi_{6,3}(\gamma),\chi_{6,5}(\gamma)$. When translating this knowledge in terms of the EOB Q potential (in energy gauge), this means that our method will be able to determine only one combination of the two functions $q_{6,3}(\gamma)$,and $q_{6,5}(\gamma)$, via the link of Eqs. (A4). On the other hand, the other coefficient functions $q_{6,k}(\gamma)$, with $k=0$, 1, 2, 4, parametrizing $q_6(\gamma,\nu)$ are fully determined in terms of lower PM information.

At 7PM, one finds that there are *three* independent functions of γ, namely $\chi_{7,2}(\gamma),\chi_{7,4}(\gamma)$, and $\chi_{7,6}(\gamma)$. They are linked to their EOB counterparts $q_{7,2}(\gamma)$, $q_{7,4}(\gamma)$ and $q_{7,6}(\gamma)$ (and to lower PM functions) via the relations Eqs. (A5). The three functions $\chi_{7,2}(\gamma),\chi_{7,4}(\gamma)$ and $\chi_{7,6}(\gamma)$ parametrize a ν dependence of the type $\sim(\nu+\nu^2+\nu^3)/h^6$. More precisely, there are three combinations of $\chi_{7,2}(\gamma)$, $\chi_{7,4}(\gamma)$, and $\chi_{7,6}(\gamma)$, say

$$\begin{aligned}\hat{\chi}_{7,\nu}(\gamma)&\equiv-2(\gamma-1)(\chi_{7,2}(\gamma)+2\chi_{7,4}(\gamma)+3\chi_{7,6}(\gamma)),\\\hat{\chi}_{7,\nu^2}(\gamma)&\equiv-4(\gamma-1)^2(2\chi_{7,2}(\gamma)+3\chi_{7,4}(\gamma)+3\chi_{7,6}(\gamma)),\\\hat{\chi}_{7,\nu^3}(\gamma)&\equiv-8(\gamma-1)^3(\chi_{7,2}(\gamma)+\chi_{7,4}(\gamma)+\chi_{7,6}(\gamma)),\end{aligned}\tag{10.54}$$

such that

[7]The relations we gave above then allow one to translate the q_n^E's into their H-type correspondents q_n^H.

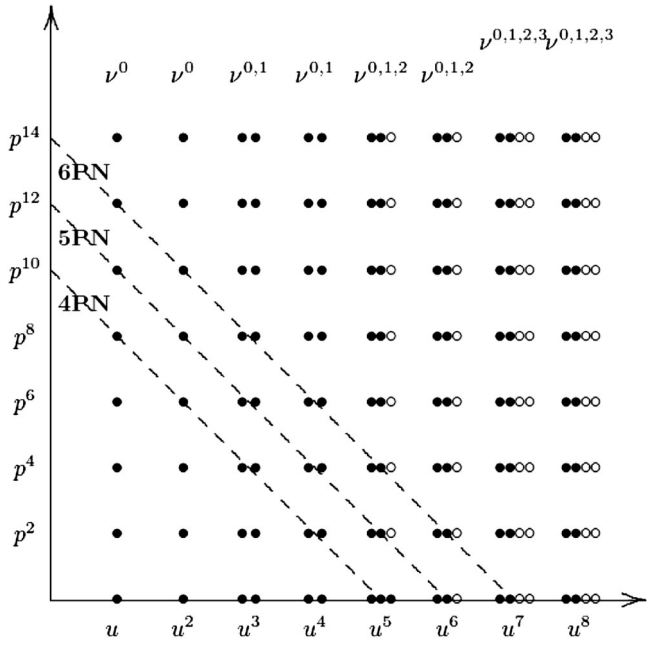

FIG. 1. Schematic representation of the irreducible information contained, at each post-Minkowskian level (keyed by a power of $u = GM/r$), in the local dynamics. Each vertical column of dots describes the post-Newtonian expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle. The various columns at a given post-Minkowskian level correspond to increasing powers of the symmetric mass-ratio ν. See text for details.

$$\chi_7(\gamma,\nu) = \chi_7^{\rm Schw}(\gamma) + \frac{\nu\hat{\chi}_{7,\nu}(\gamma) + \nu^2\hat{\chi}_{7,\nu^2}(\gamma) + \nu^3\hat{\chi}_{7,\nu^3}(\gamma)}{h^6(\gamma,\nu)}. \tag{10.55}$$

The situation is similar at the 8PM level, with three independent functions of $\gamma\chi_{8,3}(\gamma)$, $\chi_{8,5}(\gamma)$, and $\chi_{8,7}(\gamma)$, related to their corresponding EOB functions $q_{8,3}(\gamma)$, $q_{8,5}(\gamma)$, and $q_{8,7}(\gamma)$ via the relations Eqs. (A6). The three functions $\chi_{8,3}(\gamma)$, $\chi_{8,5}(\gamma)$, and $\chi_{8,7}(\gamma)$ parametrize a $\sim(\nu+\nu^2+\nu^3)/h^7$ structure for the difference $\chi_8(\gamma,\nu) - \frac{1}{h}\chi_8^{\rm Schw}(\gamma)$. And again our method can only determine one combination of these three functions.

We summarize in a pictorial manner the irreducible information contained, at each PM level, in the local dynamics in Fig. 1. The horizontal axis indicates the successive PM orders, while the vertical axis indicates successive PN orders, keyed by powers of p^2 (representing $p_\infty^2 \equiv \gamma^2 - 1$ when working in the energy-gauge). This figure displays the information contained either in the PM-expansion coefficients $\chi_n(\gamma,\nu)$ of χ, or in the PM-expansion coefficients $q_n^E(\gamma,\nu)$ of $\hat{Q}^E(u,\gamma;\nu)$. [We have explained above the (recursive) one-to-one map between these two sequences of coefficients.] By *irreducible* information we mean the building blocks that depend only on γ and that parametrize the ν-dependence of the coefficients $\chi_n(\gamma,\nu)$ or $q_n^E(\gamma,\nu)$. For instance, at the PM level $n=3$ (or u^3 in $\hat{Q}^E(u,\gamma;\nu)$), the 3PM local dynamics is fully described by Eq. (10.14), which we write again for conceptual clarity,

$$\chi_3(\gamma,\nu) = \chi_{3,0}(\gamma) + \frac{\chi_{3,2}(\gamma)}{h^2(\gamma,\nu)}, \tag{10.56}$$

i.e., by *two* independent functions of γ: $\chi_{3,0}(\gamma)$ and $\chi_{3,2}(\gamma)$. One half of this information comes from the test-mass limit, $\nu\to 0$ (namely $\chi_{3,0}(\gamma)+\chi_{3,2}(\gamma) = \chi_3^{\rm Schw}(\gamma)$), while the other half is encoded in the 1SF (linear in ν) expansion of $\chi_3(\gamma,\nu)$. This is clear if one rewrites Eq. (10.56) in the form of Eq. (10.16), i.e.,

$$\chi_3(\gamma,\nu) = \chi_3^{\rm Schw}(\gamma) - 2\nu\frac{(\gamma-1)\chi_{3,2}(\gamma)}{h^2(\gamma,\nu)}. \tag{10.57}$$

Here we are talking about the PM expansion. When working within a PN approximation scheme, some of the functions of γ entering as irreducible building blocks are only known in their PN-expanded forms, i.e., only a limited number of terms in their expansion in powers of $p_\infty^2 \equiv \gamma^2 - 1$ is known. For instance, we derived here, by working at the 6PN approximation, the first five terms of the function $\chi_{3,2}(\gamma)$, in the form of the related function

$$\bar{C}(\gamma) = \frac{\gamma-1}{p_\infty}\chi_{3,2}(\gamma) = -(\gamma-1)q_{3,2}(\gamma), \tag{10.58}$$

namely

$$\bar{C}^{\rm 6PN}(\gamma) = 4 + 18p_\infty^2 + \frac{91}{10}p_\infty^4 - \frac{69}{140}p_\infty^6 - \frac{1447}{10080}p_\infty^8 + O(p_\infty^{10}). \tag{10.59}$$

See Eq. (10.33) for the analogous result at the 4PM level.

Having in mind this PN-expansion of the γ-dependent irreducible PM building blocks $\chi_{n,k}(\gamma)$, we represent in Fig. 1 each such building block $\chi_{n,k}(\gamma)$ by a vertical line of filled circles. At the 1PM and 2PM levels there is only one irreducible building block, and therefore only one vertical line of dots. Moreover, these building blocks can be entirely deduced from the test-mass limit, i.e., they are encoded in the $\nu\to 0$ limit (or Schwarzschild limit) of the scattering angle. At the 3PM level, there are two independent irreducible functions of γ, represented as two vertical sequences of filled circles in the figure. One can think of the left column of dots as being of order ν^0 in the SF expansion, and therefore as being entirely deducible from the Schwarzschild limit. By contrast, the right column of

dots represents (modulo some h-dependent prefactor) a 1SF-level information, i.e., it is encoded in the $O(\nu^1)$ term in the expansion of $\chi_3(\gamma,\nu)$ in powers of ν. At the 4PM level we have again only two vertical sequences of dots, say one encoded in the $\nu \to 0$ limit, and the other one representing a fresh 1SF information encoded (modulo some h-dependent factor) in the $O(\nu^1)$ term in the ν-expansion of $\chi_4(\gamma,\nu)$. [Note in passing that the ν-dependence of the 4PM EOB potential $q_4(\gamma,\nu)$ deduced from $\chi_4(\gamma,\nu)$ is more involved than the one of $\chi_4(\gamma,\nu)$. In particular, the $O(\nu^1)$ term in $q_4(\gamma,\nu)$ is partly determined by the $O(\nu^1)$ information present at the 3PM level, and by fresh $O(\nu^1)$ information contained in $\chi_4(\gamma,\nu)$.]

At the 5PM and 6PM levels, we have three independent building blocks [see Eqs. (10.48) and (10.53)], represented as three vertical columns of dots. Again the left column can be thought of as being $O(\nu^0)$ (and Schwarzschildlike), the middle column as being $O(\nu^1)$ and 1SF-determined, while the third column is now $O(\nu^2)$, i.e., encoded at the 2SF level. The only knowledge we currently have of this 5PM third column is its lowest PN approximation, i.e., the filled circle located at u^5 on the horizontal axis. Indeed, this term $\sim\nu^2 u^5 p^0$ was determined by the computation of the 4PN dynamics. In the p_r gauge, it is described by the contribution $+(\frac{41}{32}\pi^2 - \frac{221}{6})\nu^2$ to the coefficient $a_5^{\rm loc,f}$ of the u^5 term in the EOB radial $A(u,\nu)$ potential. The 5PN approximation consists of collecting the terms along the second slanted line represented in Fig. 1. We see that the slanted 5PN line passes through two of the $O(\nu^2)$ third vertical columns. In the current implementation of our method, the third (and higher) vertical columns, corresponding to $O(\nu^{\geq 2})$ (2SF and higher) contributions are left undetermined. We highlight this fact by using empty circles to represent these columns. This visually explains the origin of the two coefficients left undetermined by our method at 5PN. The empty circle in the u^5 column corresponds to $\bar{d}_5^{\nu^2}$, while the empty circle at the u^6 location on the horizontal axis corresponds to $a_6^{\nu^2}$.

At the 7PM and 8PM levels, we have four independent building blocks parametrizing a $\sim\nu^0+\nu^1+\nu^2+\nu^3$ structure [see Eq. (10.55)]. When considering the 6PN, upper slanted line, we now understand clearly why there were four extra coefficients left undetermined by our method at 6PN. Namely: one in the u^5 $O(\nu^2)$ third vertical column ($q_{4,5}^{\nu^2} p_r^4 u^5$); one in the u^6 $O(\nu^2)$ third vertical column ($\bar{d}_6^{\nu^2} p_r^2 u^6$); and two on the u^7 location on the horizontal axis, linked to the third and fourth columns ($\nu^2 a_7^{\nu^2} u^7$ and $\nu^3 a_7^{\nu^3} u^7$).

Looking at Fig. 1, we can also see what information could give a 7PN-level extension of our method (completed by a 6.5PN-level purely nonlocal dynamics). It would: (i) provide a 7PN-level test of the 3PM dynamics of Refs. [61,62]; (ii) improve the knowledge of the 4PM dynamics at the 7PN level; (iii) improve the knowledge of the $O(\nu^1)$-encoded local dynamics at the 5PM, 6PM, 7PM, and 8PM levels; but (iv) leave undetermined six numerical coefficients encoding effects of the type

$$\nu^2(u^5p^6+u^6p^4+u^7p^2+u^8)+\nu^3(u^7p^2+u^8). \quad (10.60)$$

[In the p_r-gauge all the powers of p have to be interpreted as being powers of p_r.] Note that the current lack of determination of coefficients entering ν^2 and ν^3 effects is not a conceptual limitation of our method. It is rather a technical limitation of the current development of SF theory which cannot yet compute any genuine $O(\nu^2)$ effects. [See, however, [69] for significant progress toward that goal.] The combination of our method with a 2SF-level technology would allow one to cover, in principle, many more dots in the plane of Fig. 1.

XI. CONCLUDING REMARKS

We have extended the application of a new approach to binary dynamics [1] to the 6PN level. Our approach has allowed us to derive an almost complete expression for the 6PN-level action, given by the sum of a 4PN + 5PN + 5.5PN + 6PN nonlocal action, Eqs. (2.1), (2.2), (2.3), and of a local one $\int p dq - H_{\rm loc,f}^{4+5+6{\rm PN}} dt$. We succeeded in determining the full functional structure of $H_{\rm loc,f}^{\leq 6{\rm PN}}$ (which contains 151 numerical coefficients), except for four coefficients: three ν^3-level coefficients, and one ν^4-level one (when counting powers of ν in the unrescaled Hamiltonian $H = Mc^2 + \cdots$). One of the crucial tools in our derivation of $H_{\rm loc,f}^{\leq 6{\rm PN}}$ has been the computation of the Detweiler-Barack-Sago redshift invariant along eccentric orbits in a Schwarzschild spacetime, up to the eight power of the eccentricity and the 9.5th power of the inverse semi-latus rectum. This computation alone has been the most time-consuming element of our work, and has extended the frontier of analytical gravitational self-force theory.

We have expressed our final results in five different gauge-invariant ways: (i) in terms of the PM-expanded scattering angle (see Eqs. (7.16), (7.18) and discussion in the text); (ii) in terms of the PN-expanded radial action (see Eqs. (9.2), (9.5), with results summarized in Table XIV); (iii) in terms of the p_r-gauge effective EOB Hamiltonian (see Eqs. (6.1)–(6.7) as well as the summary in Table X); (iv) in terms of the H-type energy-gauge effective EOB Hamiltonian (see the defining relation in Eq. (7.4) and results listed in Table XI); and also, (v) in terms of the irreducible building blocks parametrizing a general PM dynamics [see Sec. X, and notably Eqs. (10.33) and (10.59)].

Among our new results, let us emphasize: (1) the obtention of the 6PN-accurate $O(G^3)$ scattering angle χ_3 [see notably Eq. (8.3)], in agreement with the PM computation of Refs. [61,62] (and with the PN computations of Refs. [63,64]); (2) the obtention (without any undetermined parameters) of the 6PN-accurate, 4PM ($O(G^4)$) *local* scattering angle $\chi_4^{\rm loc,f}$ [see notably Eq. (8.4)]; (3) the obtention of the linear-in-ν contributions to the 6PN-accurate 5PM, 6PM and 7PM *local* scattering angles $\chi_5^{\rm loc,f}$, $\chi_6^{\rm loc,f}$, $\chi_7^{\rm loc,f}$ [see Eqs. (8.2)]; (4) the derivation of the explicit link between the PM-expanded scattering angle and the PM-expanded EOB Q potential (in energy gauge) at the 5PM, 6PM and 7PM levels. We leave to future work the derivation of the *nonlocal* contributions to the scattering angle, and the associated explicit determination of the tuned flexibility factor $f(t)$ used here to define the local part of the dynamics.

Finally, in Sec. X we have discussed the synergistic interplay between four approaches to binary dynamics: post-Minkowskian, effective-one-body, gravitational self-force, and post-Newtonian (see Fig. 1). Effective-one-body theory offers an efficient framework for combining gauge-invariant information coming from post-Newtonian, post-Minkowskian, and gravitational self-force results. It has also allowed us to discover the hidden simplicity of binary dynamics through a deeper understanding of the mass-ratio dependence of perturbative results (see, notably Eqs. (9.2), (9.3), and (9.4), and the discussion of Sec. X).

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APPENDIX: HIGHER POST-MINKOWSKIAN LINKS BETWEEN THE SCATTERING ANGLE AND THE E-TYPE ENERGY-GAUGE EOB Q^E POTENTIAL

The terms on the right-hand side of Eq. (10.37) read

$$\chi_5^{\rm Schw}(\gamma) = \frac{1}{5(\gamma^2-1)^{5/2}}(1792\gamma^{10} - 5760\gamma^8 + 6720\gamma^6 - 3360\gamma^4 + 630\gamma^2 - 21), \tag{A1}$$

and

$$K[q_2,q_3,q_4] = q_2^2(\gamma)\frac{(2\gamma^2-1)}{(\gamma^2-1)^{1/2}} + \left[2q_3(\gamma)(\gamma^2-1)^{1/2} - \frac{2}{3}\frac{(64\gamma^6-120\gamma^4+60\gamma^2-5)}{(\gamma^2-1)^{3/2}}\right]q_2(\gamma) - 2q_3(\gamma)\frac{(8\gamma^4-8\gamma^2+1)}{(\gamma^2-1)^{1/2}} - \frac{4}{3}q_4(\gamma)\frac{(4\gamma^4-5\gamma^2+1)}{(\gamma^2-1)^{1/2}}. \tag{A2}$$

The explicit form of Eqs. (10.42) (where the 4PM-level term $q_{4,3}(\gamma)$ is considered as being known) is

$$\begin{aligned} q_{5,1}(\gamma) &= \frac{9(5\gamma^2-1)}{4(\gamma^2-1)}q_{3,2}(\gamma) + \frac{(4\gamma^2-1)}{(\gamma^2-1)}q_{4,3}(\gamma) + \frac{1}{16(\gamma^2-1)^3}(11160\gamma^8 - 20193\gamma^6 + 137 - 2323\gamma^2 + 11603\gamma^4), \\ q_{5,3}(\gamma) &= -\frac{9(5\gamma^2-1)}{4(\gamma^2-1)}q_{3,2}(\gamma) - \frac{(4\gamma^2-1)}{(\gamma^2-1)}q_{4,3}(\gamma). \end{aligned} \tag{A3}$$

At 6PM, the explicit links between the irreducible blocks of the scattering angle and the corresponding building blocks of the Q^E potential read

$$\begin{aligned} \chi_{6,3}(\gamma) &= -\frac{15}{128}\pi(-21+174\gamma^2-345\gamma^4)q_{3,2}(\gamma) - \frac{15}{128}\pi(-10+48\gamma^2-70\gamma^4)q_{4,3}(\gamma) \\ &\quad - \frac{15}{32}\pi(\gamma^2-1)^2 q_{6,3}(\gamma) - \frac{15}{128}\pi(-9+135\gamma^2-675\gamma^4+1125\gamma^6), \\ \chi_{6,5}(\gamma) &= -\frac{15}{32}\pi(\gamma^2-1)^2 q_{6,5}(\gamma). \end{aligned} \tag{A4}$$

At 7PM, we have the analogous links:

$$
\begin{aligned}
\chi_{7,2}(\gamma) &= \frac{4}{5}q_{3,2}^2(\gamma)(34\gamma^2-9)(\gamma^2-1)^{1/2} + \frac{1}{20(\gamma^2-1)^{3/2}}(+2685\gamma^8 - 7692\gamma^6 \\
&\quad + 4626\gamma^4 - 692\gamma^2 - 27)q_{3,2}(\gamma) - \frac{6}{5}q_{4,3}(\gamma)(34\gamma^2-9)(-1+5\gamma^2)(\gamma^2-1)^{1/2} \\
&\quad + \frac{2}{5}q_{5,2}(\gamma)(99\gamma^4-62\gamma^2+13)(\gamma^2-1)^{1/2} - \frac{8}{5}q_{7,2}(\gamma)(\gamma^2-1)^{5/2} \\
&\quad - \frac{3}{40(\gamma^2-1)^{3/2}}(-1+5\gamma^2)(60330\gamma^8 - 114477\gamma^6 + 65651\gamma^4 - 12773\gamma^2 + 669), \\
\chi_{7,4}(\gamma) &= -\frac{2}{5(\gamma^2-1)^{1/2}}q_{3,2}^2(\gamma)(34\gamma^2-9) + \frac{45}{4}q_{3,2}(\gamma)(25\gamma^4-10\gamma^2+1)(\gamma^2-1)^{1/2} \\
&\quad + \frac{6}{5}q_{4,3}(\gamma)(170\gamma^4-79\gamma^2+9)(\gamma^2-1)^{1/2} + \frac{2}{5}q_{5,4}(\gamma)(99\gamma^4-62\gamma^2+13)(\gamma^2-1)^{1/2} \\
&\quad - \frac{8}{5}q_{7,4}(\gamma)(\gamma^2-1)^{5/2}, \\
\chi_{7,6}(\gamma) &= -\frac{8}{5}q_{7,6}(\gamma)(\gamma^2-1)^{5/2}.
\end{aligned}
\tag{A5}
$$

The analogous 8PM links read:

$$
\begin{aligned}
\chi_{8,3}(\gamma) &= \frac{945}{128}\pi(\gamma^2-1)(-1+5\gamma^2)q_{3,2}^2(\gamma) \\
&\quad + \pi\left[\frac{35}{64}(\gamma^2-1)(47\gamma^2-11)q_{4,3}(\gamma) + \frac{35}{2048(\gamma^2-1)}(-18032\gamma^2 + 92698\gamma^4 + 889 + 40485\gamma^8 - 139080\gamma^6)\right]q_{3,2}(\gamma) \\
&\quad - \frac{35}{1024\pi(\gamma^2-1)}(65 + 2792\gamma^6 - 1590\gamma^4 + 301\gamma^8 - 32\gamma^2)q_{4,3}(\gamma) \\
&\quad + \frac{105}{256}\pi(\gamma^2-1)(-1+5\gamma^2)(47\gamma^2-11)q_{5,2}(\gamma) \\
&\quad + \frac{35}{64}\pi(33\gamma^4-19\gamma^2+4)(\gamma^2-1)q_{6,3}(\gamma) - \frac{35}{64}\pi(\gamma^2-1)^3 q_{8,3}(\gamma) \\
&\quad - \pi\frac{945}{8192(\gamma^2-1)}(-1+5\gamma^2)^2(185\gamma^6 - 3359\gamma^4 + 1627\gamma^2 - 85), \\
\chi_{8,5}(\gamma) &= -\pi\frac{945}{256}(\gamma^2-1)(-1+5\gamma^2)q_{3,2}(\gamma)^2 - \pi\frac{35}{128}(47\gamma^2-11)(\gamma^2-1)q_{4,3}(\gamma)q_{3,2}(\gamma) \\
&\quad + \pi\frac{2835}{512}(-1+5\gamma^2)^2(\gamma^2-1)q_{4,3}(\gamma) + \pi\frac{105}{256}(47\gamma^2-11)(-1+5\gamma^2)(\gamma^2-1)q_{5,4}(\gamma) \\
&\quad + \pi\frac{35}{64}(33\gamma^4-19\gamma^2+4)(\gamma^2-1)q_{6,5}(\gamma) - \pi\frac{35}{64}(\gamma^2-1)^3 q_{8,5}(\gamma), \\
\chi_{8,7}(\gamma) &= -\pi\frac{35}{64}(\gamma^2-1)^3 q_{8,7}(\gamma).
\end{aligned}
\tag{A6}
$$

[1] D. Bini, T. Damour, and A. Geralico, Novel Approach to Binary Dynamics: Application to the Fifth Post-Newtonian Level, Phys. Rev. Lett. **123**, 231104 (2019).

[2] D. Bini, T. Damour, and A. Geralico, Binary dynamics at the fifth and fifth-and-a-half post-Newtonian orders, arXiv: 2003.11891 [Phys. Rev. D (to be published)].

[3] J. A. Wheeler and R. P. Feynman, Classical electrodynamics in terms of direct interparticle action, Rev. Mod. Phys. **21**, 425 (1949).

[4] L. Infeld, Equations of motion in general relativity theory and the action principle, Rev. Mod. Phys. **29**, 398 (1957).

[5] T. Damour and G. Esposito-Farese, Testing gravity to second postNewtonian order: A Field theory approach, Phys. Rev. D **53**, 5541 (1996).

[6] L. Blanchet and T. Damour, Tail transported temporal correlations in the dynamics of a gravitating system, Phys. Rev. D **37**, 1410 (1988).

[7] L. Blanchet and T. Damour, Post-newtonian generation of gravitational waves, Ann. Inst. Henri Poincaré Phys. Theor. **50**, 377 (1989).

[8] T. Damour and B. R. Iyer, PostNewtonian generation of gravitational waves. 2. The spin moments, Ann. Inst. Henri Poincaré Phys. Theor. **54**, 115 (1991).

[9] L. Blanchet, On the multipole expansion of the gravitational field, Classical Quantum Gravity **15**, 1971 (1998).

[10] O. Poujade and L. Blanchet, Post-Newtonian approximation for isolated systems calculated by matched asymptotic expansions, Phys. Rev. D **65**, 124020 (2002).

[11] L. Blanchet and T. Damour, Radiative gravitational fields in general relativity I. general structure of the field outside the source, Phil. Trans. R. Soc. A **320**, 379 (1986).

[12] S. Foffa and R. Sturani, Tail terms in gravitational radiation reaction via effective field theory, Phys. Rev. D **87**, 044056 (2013).

[13] C. R. Galley, A. K. Leibovich, R. A. Porto, and A. Ross, Tail effect in gravitational radiation reaction: Time nonlocality and renormalization group evolution, Phys. Rev. D **93**, 124010 (2016).

[14] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[15] W. D. Goldberger and A. Ross, Gravitational radiative corrections from effective field theory, Phys. Rev. D **81**, 124015 (2010).

[16] A. Ross, Multipole expansion at the level of the action, Phys. Rev. D **85**, 125033 (2012).

[17] T. Damour, P. Jaranowski, and G. Schäfer, Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems, Phys. Rev. D **89**, 064058 (2014).

[18] L. Bernard, L. Blanchet, A. Bohé, G. Faye, and S. Marsat, Fokker action of nonspinning compact binaries at the fourth post-Newtonian approximation, Phys. Rev. D **93**, 084037 (2016).

[19] T. Marchand, L. Bernard, L. Blanchet, and G. Faye, Ambiguity-free completion of the equations of motion of compact binary systems at the fourth post-newtonian order, Phys. Rev. D **97**, 044023 (2018).

[20] S. Foffa and R. Sturani, Conservative dynamics of binary systems to fourth post-Newtonian order in the EFT approach I: Regularized Lagrangian, Phys. Rev. D **100**, 024047 (2019).

[21] S. Foffa, R. A. Porto, I. Rothstein, and R. Sturani, Conservative dynamics of binary systems to fourth Post-Newtonian order in the EFT approach II: Renormalized Lagrangian, Phys. Rev. D **100**, 024048 (2019).

[22] T. Damour, 2010 (to be published); cited in L. Barack, T. Damour, and N. Sago, Precession effect of the gravitational self-force in a Schwarzschild spacetime and the effective one-body formalism, Phys. Rev. D **82**, 084036 (2010), which quoted and used some combinations of the (4PN and 5PN) logarithmic contributions to the EOB potentials $A(u)$ and $\bar{D}(u)$.

[23] T. Damour, P. Jaranowski, and G. Schäfer, Fourth post-Newtonian effective one-body dynamics, Phys. Rev. D **91**, 084024 (2015).

[24] T. Damour, M. Soffel, and C. m. Xu, General relativistic celestial mechanics. 1. Method and definition of reference systems, Phys. Rev. D **43**, 3273 (1991).

[25] T. Damour, M. Soffel, and C. M. Xu, General relativistic celestial mechanics. 2. Translational equations of motion, Phys. Rev. D **45**, 1017 (1992).

[26] S. Foffa and R. Sturani, Hereditary terms at next-to-leading order in two-body gravitational dynamics, Phys. Rev. D **101**, 064033 (2020).

[27] L. Blanchet, S. Foffa, F. Larrouturou, and R. Sturani, Logarithmic tail contributions to the energy function of circular compact binaries, Phys. Rev. D **101**, 084045 (2020).

[28] M. Levi, Effective field theories of post-newtonian gravity: A comprehensive review, arXiv:1807.01699.

[29] T. Damour, Gravitational self force in a schwarzschild background and the effective one body formalism, Phys. Rev. D **81**, 024017 (2010).

[30] L. Blanchet, S. L. Detweiler, A. Le Tiec, and B. F. Whiting, High-order post-Newtonian fit of the gravitational self-force for circular orbits in the schwarzschild geometry, Phys. Rev. D **81**, 084033 (2010).

[31] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17**, 2 (2014).

[32] T. Damour and B. R. Iyer, Generation of gravitational waves: The PostNewtonian spin octupole moment, Classical Quantum Gravity **11**, 1353 (1994); Erratum, Classical Quantum Gravity **12**, 287 (1995).

[33] L. Blanchet, Second postNewtonian generation of gravitational radiation, Phys. Rev. D **51**, 2559 (1995).

[34] L. Blanchet, T. Damour, and B. R. Iyer, Gravitational waves from inspiralling compact binaries: Energy loss and wave form to second postNewtonian order, Phys. Rev. D **51**, 5360 (1995); Erratum, Phys. Rev. D **54**, 1860 (1996).

[35] T. Damour and N. Deruelle, Lagrangien généralisé du système de deux masses ponctuelles, à l'approximation post-post-newtonienne de la relativité générale, C.R. Acad. Sc. Paris, Série II **293**, 537 (1981).

[36] T. Damour, Problème des deux corps et freinage de rayonnement en relativité générale, C.R. Acad. Sc. Paris, Série II **294**, 1355 (1982).

[37] T. Damour and G. Schäfer, Redefinition of position variables and the reduction of higher order Lagrangians, J. Math. Phys. (N.Y.) **32**, 127 (1991).

[38] G. Schäfer, The gravitational quadrupole radiation reaction force and the canonical formalism of Adm, Ann. Phys. (N.Y.) **161**, 81 (1985).
[39] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, Tail effects in the 3PN gravitational wave energy flux of compact binaries in quasi-elliptical orbits, Phys. Rev. D **77**, 064034 (2008).
[40] T. Damour and G. Schäfer, Higher order relativistic periastron advances and binary pulsars, Nuovo Cimento Soc. Ital. Fis. **101B**, 127 (1988).
[41] G. Schäfer and N. Wex, Second post-Newtonian motion of compact binaries, Phys. Lett. A **174**, 196 (1993); Erratum, Phys. Lett. A **177**, 461 (1993).
[42] R. M. Memmesheimer, A. Gopakumar, and G. Schäfer, Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits, Phys. Rev. D **70**, 104011 (2004).
[43] T. Damour and N. Deruelle, Radiation reaction and angular momentum loss in small angle gravitational scattering, Phys. Lett. A **87**, 81 (1981).
[44] T. Damour, P. Jaranowski, and G. Schäfer, On the determination of the last stable orbit for circular general relativistic binaries at the third post-Newtonian approximation, Phys. Rev. D **62**, 084011 (2000).
[45] A. Le Tiec, L. Blanchet, and B. F. Whiting, The first law of binary black hole mechanics in general relativity and post-Newtonian theory, Phys. Rev. D **85**, 064039 (2012).
[46] E. Barausse, A. Buonanno, and A. Le Tiec, The complete non-spinning effective-one-body metric at linear order in the mass ratio, Phys. Rev. D **85**, 064010 (2012).
[47] A. Le Tiec, First law of mechanics for compact binaries on eccentric orbits, Phys. Rev. D **92**, 084021 (2015).
[48] S. L. Detweiler, A Consequence of the gravitational self-force for circular orbits of the Schwarzschild geometry, Phys. Rev. D **77**, 124026 (2008).
[49] L. Barack and N. Sago, Beyond the geodesic approximation: Conservative effects of the gravitational self-force in eccentric orbits around a Schwarzschild black hole, Phys. Rev. D **83**, 084023 (2011).
[50] D. Bini and T. Damour, Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation, Phys. Rev. D **87**, 121501 (2013).
[51] D. Bini and T. Damour, High-order post-Newtonian contributions to the two-body gravitational interaction potential from analytical gravitational self-force calculations, Phys. Rev. D **89**, 064063 (2014).
[52] D. Bini and T. Damour, Detweiler's gauge-invariant redshift variable: Analytic determination of the nine and nine-and-a-half post-Newtonian self-force contributions, Phys. Rev. D **91**, 064050 (2015).
[53] C. Kavanagh, A. C. Ottewill, and B. Wardell, Analytical high-order post-Newtonian expansions for extreme mass ratio binaries, Phys. Rev. D **92**, 084025 (2015).
[54] D. Bini, T. Damour, and A. Geralico, Confirming and improving post-Newtonian and effective-one-body results from self-force computations along eccentric orbits around a Schwarzschild black hole, Phys. Rev. D **93**, 064023 (2016).
[55] D. Bini, T. Damour, and A. Geralico, New gravitational self-force analytical results for eccentric orbits around a Schwarzschild black hole, Phys. Rev. D **93**, 104017 (2016).
[56] S. Hopper, C. Kavanagh, and A. C. Ottewill, Analytic self-force calculations in the post-Newtonian regime: Eccentric orbits on a Schwarzschild background, Phys. Rev. D **93**, 044010 (2016).
[57] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).
[58] T. Damour, Classical and quantum scattering in post-Minkowskian gravity, arXiv:1912.02139 [Phys. Rev. D (to be published)].
[59] A. Antonelli, C. Kavanagh, M. Khalil, J. Steinhoff, and J. Vines, Gravitational spin-orbit coupling through third-subleading post-newtonian order: From first-order self-force to arbitrary mass ratios, arXiv:2003.11391.
[60] C. Cheung, I. Z. Rothstein, and M. P. Solon, From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion, Phys. Rev. Lett. **121**, 251101 (2018).
[61] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon, and M. Zeng, Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order, Phys. Rev. Lett. **122**, 201603 (2019).
[62] Z. Bern, C. Cheung, R. Roiban, C. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206.
[63] J. Blümlein, A. Maier, P. Marquard, and G. Schäfer, Testing binary dynamics in gravity at the sixth post-Newtonian level, arXiv:2003.07145.
[64] C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams, arXiv:2003.08351.
[65] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, Energetics of two-body Hamiltonians in post-Minkowskian gravity, Phys. Rev. D **99**, 104004 (2019).
[66] T. Damour, Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory, Phys. Rev. D **94**, 104015 (2016).
[67] D. Bini and T. Damour, Gravitational scattering of two black holes at the fourth post-Newtonian approximation, Phys. Rev. D **96**, 064021 (2017).
[68] G. Kälin and R. A. Porto, From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist), J. High Energy Phys. 02 (2020) 120.
[69] A. Pound, B. Wardell, N. Warburton, and J. Miller, Second-Order Self-Force Calculation of Gravitational Binding Energy in Compact Binaries, Phys. Rev. Lett. **124**, 021101 (2020).