Minimally coupled scalar fields as imperfect fluids

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We revisit the issue of the fluid description of minimally coupled scalar fields. While in a cosmological setup the interpretation of a time-evolving scalar field as a perfect fluid is well understood; the situation is more intricate when the scalar field is static but has a spatial gradient, a situation motivated by black hole perturbations in scalar-tensor theories. Then the scalar field is interpreted as either a particular imperfect fluid of type I or a superposition of a pair of leftgoing (incoming) and rightgoing (outgoing) null dusts with a perfect fluid. Finally, when the scalar gradient is null, it is equivalent to an imperfect fluid of type II, degenerating into null dust when the energy conditions are imposed. We also propose the suitable action in terms of the fluid pressure components for each case and discuss the variational principle for a generic class of minimally coupled scalar fields.

I. INTRODUCTION

Scalar fields recurrently show up in modern gravitational physics either as generating inflation in the early universe, emerging from dimensional reduction of higher-order theories, as models of dark matter and dark energy, or as additions to the tensorial sector in modified gravitational models representing low-energy approximations of the sought for quantum gravity theory.

At a classical level the most general scalar-tensor theory with, at most, second-order dynamics both for the scalar and the tensor (hence avoiding Ostrogradski instabilities) was proposed by Horndeski [1] and rediscovered later in the context of generalized Galileons [2]. In certain higher-order theories the degrees of freedom still evolve according to second-order dynamics, as the analysis of cosmological perturbations in an effective field theory (EFT) approach has proved [3,4]. With cosmological symmetries the scalar is purely time dependent, hence (provided its gradient never vanishes and it is timelike) it can be promoted to a time coordinate (unitary gauge).

Odd sector perturbations of spherically symmetric static black holes in generic scalar-tensor theories were also discussed in the EFT framework [5]. Instead of the Arnowitt–Deser–Misner (ADM) decomposition explored in the cosmological case, here a similar $2 + 1 + 1$ decomposition [6,7] of the 4-metric $\tilde{g}_{ab}$ turned useful:

$$\tilde{g}_{ab} = -n_a n_b + m_a m_b + g_{ab}, \qquad (1)$$

with g_{ab} a 2-metric on a surface with spherical topology and its normals satisfying

$$-n_a n^a = m_a m^a = 1,$$
$$n_a m^a = n^a g_{ab} = m^a g_{ab} = 0. \qquad (2)$$

In this case the scalar is static but has a radial spacelike gradient. If the latter is nowhere vanishing, the scalar can emulate a radial coordinate (radial unitary gauge). A nonorthogonal $2 + 1 + 1$ decomposition was recently worked out allowing for an unambiguous gauge choice [8], the closest to the Regge–Wheeler gauge of general relativity, paving the road for the discussion of the even sector perturbations in an EFT approach of spherically symmetric static black holes.

Gravitational dynamics is obtained by varying the action both with respect to the (inverse) metric tensor and scalar. At first order the respective equations are generalizations of the Einstein and Klein–Gordon equations. Second-order variations provide the evolutions of perturbations. When other matter fields are present, their dynamics arise from similar matter field variations.

The coupling of the scalar to the tensor sector is frame dependent. Horndeski theories are naturally written up in Jordan frame. In this case the diffeomorphism-invariant action,

$$S_G(g^{ab}, \phi) + S_\phi(g^{ab}, \phi) + S_M(g^{ab}, \psi), \qquad (3)$$

exhibits minimal coupling of matter fields ψ to the metric; however, the coupling of the scalar field ϕ is minimal only in S_ϕ, not in S_G. Due to diffeomorphism invariance of the action and the matter equations of motion, the energy-momentum tensor of matter,

$$\tilde{T}_{ab}^M = -\tilde{T}_{ab}^\phi = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{ab}}, \qquad (4)$$

has a vanishing covariant 4-divergence [9], nevertheless, the energy-momentum tensor of the scalar field (as defined

in terms of the variation of S_ϕ with respect to the inverse metric) does not obey such a conservation law. The scalar, however, belongs to the gravitational sector for the tensorial part of which no proper energy-momentum tensor can be defined either. Generalized Brans–Dicke type theories have been studied in this framework [10,11] and the imperfect fluid description of a different effective energy-momentum tensor worked out. The relation between the two types of energy-momentum tensors has been discussed in Ref. [12].

By contrast, in the Einstein frame obtained by a convenient conformal rescaling of the metric, the scalar becomes minimally coupled while the matter sources cease to be coupled minimally to the metric. In this case the energy-momentum tensor of matter has no vanishing covariant 4-divergence, while the energy-momentum tensor of the scalar field has. If no other matter source is present, but the scalar is treated as matter, the Einstein frame is more natural. When the scalar field is coupled minimally in the Jordan frame, obviously the two frames and the corresponding energy-momentum tensor definitions for the scalar coincide.

In the simplest case of general relativity and minimally coupled Klein–Gordon scalar field with timelike gradient $\tilde{\nabla}_a\phi$ (where $\tilde{\nabla}$ is the connection compatible with the metric $\tilde{g}_{ab}$), the energy-momentum tensor of the scalar has been shown to mimic a perfect fluid [13],

$$\tilde{T}_{ab}^{PF} = \rho^{PF} n_a n_b + p^{PF}(m_a m_b + g_{ab}), \qquad (5)$$

with energy density

$$\rho^{PF} = -\frac{1}{2}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi + V(\phi), \qquad (6)$$

isotropic pressure

$$p^{PF} = -\frac{1}{2}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi - V(\phi), \qquad (7)$$

and fluid 4-velocity

$$n_a = \frac{\tilde{\nabla}_a\phi}{\sqrt{-\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi}}. \qquad (8)$$

For perfect fluids, both $L_1 = p^{PF}$ and $L_2 = -\rho^{PF}$ (regarded as functions of particle number density and entropy per particle) are valid Lagrangians [14], differing only by surface terms. However, when the density and pressure become functions of the scalar field, Faraoni has proven [15] that the equations of motion of the scalar-tensor theory are reproduced solely from the Lagrangian (7). Related to the perfect fluid interpretation, Ref. [16] proved that for a finite period of time a shift-invariant scalar field accurately describes the potential flow of an isentropic perfect fluid.

The case of a scalar with spatial gradient was deemed nonphysical in Ref. [15] and discussed only briefly (a sign flip being overlooked in the $3 + 1$ decomposition with respect to the scalar gradient). The correct expressions for the scalar energy density and isotropic pressure were given by Ref. [17], noting that the perfect fluid interpretation does not hold for a comoving, but rather for a tachyonic observer, when the scalar gradient is spacelike.

From a timelike observer point of view it is more natural to consider a $2 + 1 + 1$ fluid decomposition with respect to both the timelike observer and a preferred spacelike direction,

$$\tilde{T}_{ab}^{IPF} = \rho n_a n_b + p_r m_a m_b + p_t g_{ab}, \qquad (9)$$

similar to the metric decomposition (1). Such an imperfect fluid could have different radial and tangential pressures p_r and p_t. (They are dubbed radial and tangential for simplicity—the decomposition also applies for different scenarios.) In Sec. II, we explore this approach describing all cases in terms of an imperfect fluid, regardless of whether the gradient of the scalar is timelike, spacelike, or null. In the latter case we need to add heat flow too. The energy conditions will also be discussed here; also, we interpret the scalars with radial gradients as a sum of perfect fluid, incoming and outgoing null dust. The case of null gradient reduces to a null dust after the energy conditions are imposed. Finally, we revisit the issue of the proper action for a wide class of minimally coupled scalar fields in Sec. III. We summarize our findings in the Conclusion.

Throughout the paper we assume $16\pi G = 1 = c$.

II. KLEIN–GORDON SCALAR FIELD IN GENERAL RELATIVITY AS IMPERFECT FLUID

A. Timelike scalar field gradient

By adding the Klein–Gordon Lagrangian (7) to the Einstein–Hilbert action, (inverse) metric variation yields the energy momentum tensor

$$\tilde{T}_{ab}^{KG} = \tilde{\nabla}_a\phi\tilde{\nabla}_b\phi - \tilde{g}_{ab}\left[\frac{1}{2}\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi + V(\phi)\right] \quad (10)$$

for the scalar field. For a timelike scalar gradient, as discussed before, this mimics the perfect fluid (5)–(7) [13,15].

The energy-momentum tensor being diagonal (type I) for both timelike and spacelike scalar field gradients, the energy conditions translate to $\rho \geq 0$, $\rho + p_\alpha \geq 0$ (weak), $\rho \geq 0$, $|p_\alpha| \leq \rho$ (dominant), and $\rho + p_\alpha \geq 0$, $\rho + \sum_\alpha p_\alpha \geq 0$ (strong). For timelike scalar gradient, these imply $\rho \geq 0$ (thus $-\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq -2V(\phi)$) for the weak energy condition, $V(\phi) \geq 0$ (then the weak energy condition $\rho \geq 0$ is fulfilled automatically) for the dominant energy condition, and finally, $-\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq V(\phi)$ for the strong energy condition. All energy conditions hold for $0 \leq V(\phi) \leq -\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi$.

B. Spacelike scalar field gradient

For a spacelike $\tilde{\nabla}_a\phi$, the energy momentum is rather of the form of an imperfect fluid (9). The scalar field gradient then is associated to the spatial vector,

$$m_a = \frac{\tilde{\nabla}_a\phi}{\sqrt{\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi}}. \qquad (11)$$

Then $T_{ab}m^am^b$ gives the radial pressure

$$p_r = \frac{1}{2}\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi - V(\phi), \qquad (12)$$

while the tangential pressure and energy density are identified as

$$p_t = -\rho = -\frac{1}{2}\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi - V(\phi). \qquad (13)$$

For spacelike scalar gradient, the energy conditions are as follows: $\rho \geq 0$ (thus $\tilde{\nabla}_c\phi\tilde{\nabla}^c\phi \geq -2V(\phi)$) for the weak energy condition, $V(\phi) \geq 0$ (then the weak energy condition $\rho \geq 0$ is fulfilled automatically) for the dominant energy condition, and finally, $V(\phi) \leq 0$ for the strong energy condition. All energy conditions hold only for vanishing potential.

By imposing all energy conditions,

$$\rho = p_r = -p_t = \frac{1}{2}\tilde{\nabla}_a\phi\tilde{\nabla}^a\phi > 0 \qquad (14)$$

emerges, representing an imperfect fluid with radial pressure equaling its energy density and tangential tension of the same magnitude.

Two equivalent interpretations will be discussed below.

1. Perfect fluid as seen by a tachyonic observer

The energy density (8) and isotropic pressure (5) of Ref. [17], emerging from the perfect fluid interpretation of the scalar by a tachionic observer,

$$\tilde{T}_{ab}^{tach} = (\rho^{tach} + P^{tach})m_am_b + P^{tach}\tilde{g}_{ab}, \qquad (15)$$

relate to the radial and tangential pressures of the anisotropic fluid as

$$\rho^{tach} = p_r - 2p_t, \qquad P^{tach} = p_t. \qquad (16)$$

This interpretation advanced in Ref. [17] is less attractive due to the nonexistence of tachyonic observers.

2. Incoming and outgoing radiation fields superposed on a perfect fluid

The imperfect energy-momentum tensor of the scalar field with spatial gradient,

$$\tilde{T}_{ab}^{IPF} = -p_tn_an_b + p_rm_am_b + p_tg_{ab}, \qquad (17)$$

and with p_r and p_t given by Eqs. (12) and (13), can be rewritten as a sum of a perfect fluid (with energy density $-p_r$ and isotropic pressure p_t)

$$\tilde{T}_{ab}^{(1)} = -p_rn_an_b + p_tm_am_b + p_tg_{ab}, \qquad (18)$$

and two null dusts (with the same energy density $p_r - p_t$):

$$\tilde{T}_{ab}^{(2)} = (p_r - p_t)k_ak_b, \qquad \tilde{T}_{ab}^{(3)} = (p_r - p_t)l_al_b. \qquad (19)$$

The null dusts propagate in the null directions

$$k_a = \frac{n_a + m_a}{\sqrt{2}}, \qquad l_a = \frac{n_a - m_a}{\sqrt{2}}, \qquad (20)$$

which span a pseudoorthonormal basis obeying

$$k_ak^a = l_al^a = 0, \qquad k_al^a = -1,$$
$$k^ag_{ab} = l^ag_{ab} = 0. \qquad (21)$$

The null dusts represent leftgoing (incoming for spherical symmetry) and rightgoing (outgoing) radiation fields. Such null dusts add up to an anisotropic fluid with no tangential pressures [18], a scenario explored for describing a static superposition of incoming and outgoing radiations under spherical symmetry [19]. A similar dynamical construction under spatial homogeneity yielded a Kantowski–Sachs type homogeneous universe filled by a two component radiation evolving from an initial singularity to a final one [20]. Switching to ghost radiation streams (negative energy densities), the corresponding solutions represented wormholes [21], naked singularities, or open universes [22]. These all emerged in a dilatonic approach as solutions for the massless scalar field minimally coupled to the spherically reduced Einstein–Hilbert gravity. In our case the addition of a third perfect fluid is necessary to account for the tangential pressure.

By imposing all energy conditions, the perfect fluid will have negative energy density (ghost fluid) and negative isotropic pressure (tension), both equaling $-\rho$, while the energy density of both null dusts simplifies to 2ρ.

C. Null scalar field gradient

Finally, we discuss the case when $\tilde{\nabla}_a\phi$ is null. Then we rather decompose the metric into the pseudoorthonormal basis:

$$\tilde{g}_{ab} = -2k_{(a}l_{b)} + g_{ab}, \tag{22}$$

consistent with the metric decomposition (1).

Next, we associate k_a with the scalar field as

$$k_a = \frac{\tilde{\nabla}_a \phi}{\sqrt{2}}. \tag{23}$$

Then the energy-momentum tensor (10) simplifies:

$$\tilde{T}_{ab}^{KG} = 2k_a k_b - \tilde{g}_{ab} V(\phi), \tag{24}$$

while in the (n^a, m^a) basis it reads

$$\tilde{T}_{ab}^{KG} = n_a n_b + 2n_{(a}m_{b)} + m_a m_b$$
$$+ (n_a n_b - m_a m_b - g_{ab})V(\phi), \tag{25}$$

or in matrix form

$$\tilde{T}_{ab}^{KG} = \begin{pmatrix} 1 + V(\phi) & 1 & \\ 1 & 1 - V(\phi) & \\ & & -g_{ab}V(\phi) \end{pmatrix}. \tag{26}$$

This energy-momentum tensor is of type II according to the classification of Ref. [23], as it should be, due to the double eigenvector k^a of the energy-momentum tensor (24).

For the weak energy condition, the tangential pressures should be positive, hence $V(\phi) \leq 0$, but on the other hand the energy density should be ≥ 1, thus $V(\phi) \geq 0$. Hence, the weak energy condition holds only for $V(\phi) = 0$. A similar conclusion stems from imposing either the dominant or the strong energy condition. With $V(\phi) = 0$ the scalar field becomes a massless radiation field (null dust). As shown in Ref. [24], and transparent from Eq. (26), with $V(\phi) = 0$ the null dust can also be perceived as an imperfect fluid with energy density, radial pressure, and heat flow: all of them equal.

III. LAGRANGIAN DESCRIPTION AS IMPERFECT FLUID OF A MINIMALLY COUPLED GENERIC SCALAR FIELD

The proof in Ref. [15] that the correct Lagrangian, in the case of timelike scalar field gradient is the isotropic pressure (7), combined with the anisotropic fluid description applying for the case of a spatial gradient, implies that the Lagrangian in the latter case is the tangential pressure (13). The same expression vanishes in the case of a scalar with null gradient obeying the energy conditions. This case will be discussed in the more generic framework below.

The result that the isotropic pressure of a perfect fluid mimicking the scalar field with timelike gradient qualifies as Lagrangian applies to a much wider class of minimally coupled scalar-tensor theories:

$$L = L(X, \phi), \tag{27}$$

with

$$X = \frac{1}{2} \tilde{\nabla}_a \phi \tilde{\nabla}^a \phi. \tag{28}$$

(For the Klein–Gordon field $L = X - V$.) Variation with respect to ϕ gives the dynamics of the scalar field:

$$\nabla_a (L_X \tilde{\nabla}^a \phi) = L_\phi(\phi). \tag{29}$$

Variation with respect to the (inverse) metric results in

$$\delta S_\phi = \int dx^4 \left[L(X, \phi) \delta \sqrt{-\tilde{g}} + \frac{1}{2} \sqrt{-\tilde{g}} L_X \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi \delta \tilde{g}^{ab} \right]$$
$$= \int dx^4 \frac{\sqrt{-\tilde{g}}}{2} [-L(X, \phi) \tilde{g}_{ab} + L_X \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi] \delta \tilde{g}^{ab} \tag{30}$$

(the subscript X on L denoting partial derivative with respect to X). The energy-momentum tensor arises as

$$\tilde{T}_{ab}^\phi = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S}{\delta \tilde{g}^{ab}} = -L_X \tilde{\nabla}_a \phi \tilde{\nabla}_b \phi + \tilde{g}_{ab} L(X, \phi). \tag{31}$$

A. Timelike scalar field gradient

For a timelike scalar field gradient $\tilde{\nabla}_a \phi$, a 4-velocity n^a can be associated with the scalar field through Eq. (8), and the metric decomposed in the manner (1). This yields the energy-momentum tensor

$$\tilde{T}_{ab}^\phi = (2XL_X - L)n_a n_b + (m_a m_b + g_{ab})L(X, \phi), \tag{32}$$

in a form of a perfect fluid with energy density

$$\rho = 2XL_X - L, \tag{33}$$

and isotropic pressure

$$p = L. \tag{34}$$

Thus, the Lagrangian density is but the pressure of the fluid.

B. Spacelike scalar field gradient

For a spacelike scalar field gradient $\tilde{\nabla}_a \phi$ associated to the vector m^a through Eq. (11), the energy momentum tensor rather becomes the imperfect fluid

$$\tilde{T}_{ab}^\phi = (L - 2XL_X)m_a m_b + (-n_a n_b + g_{ab})L(X, \phi), \tag{35}$$

with energy density

$$\rho = -L, \tag{36}$$

and pressure components

$$p_r = L - 2XL_X, \quad (37)$$

$$p_t = -\rho = L. \quad (38)$$

Thus, in this case the Lagrangian density is $p_t = -\rho$, extending the result established for the Klein–Gordon field to a generic minimally coupled scalar.

C. Null scalar field gradient

For a null scalar field gradient $\tilde{\nabla}_a\phi$ associated to the vector k^a through Eq. (23), $X = k^a k_a = 0$ holds, hence

$$\tilde{T}^\phi_{ab} = -2L_X(\phi)k_a k_b + \tilde{g}_{ab}L(\phi). \quad (39)$$

A similar analysis to the one presented in Subsection II C shows that this is of type II, and the energy conditions are satisfied for $L(\phi) = 0$, rendering the scalar to a null dust.[1]

The diffeomorphism invariance of the scalar energy-momentum tensor (27) implies $\tilde{\nabla}^a \tilde{T}^\phi_{ab} = 0$, which (due to $\tilde{\nabla}^a L_X(\phi) \propto k^a$ and $\tilde{\nabla}_b L(\phi) = \sqrt{2}L_\phi(\phi)k_b$) leads to a geodesic equation,

$$k^a\tilde{\nabla}_a k_b = \left(\frac{L_\phi(\phi)}{\sqrt{2}L_X(\phi)} - \tilde{\nabla}_a k^a\right)k_b. \quad (40)$$

On the other hand, the scalar dynamical equation (29) implies

$$\tilde{\nabla}_a k^a = \frac{L_\phi(\phi)}{\sqrt{2}L_X}, \quad (41)$$

hence, the geodesic is affinely parametrized:

$$k^a\tilde{\nabla}_a k_b = 0. \quad (42)$$

Thus, the null gradient of the minimally coupled generic scalar field (27) obeys an affinely parametrized geodesic equation, similarly as found for the Klein–Gordon field in Ref. [25].

IV. CONCLUDING REMARKS

We have revisited the fluid description of a minimally coupled scalar field to gravity. For scalar fields with timelike gradient (in situations motivated by cosmological considerations), the fluid is perfect, both for the Klein–Gordon field and for a quite generic scalar with

Lagrangian (27) with the equation of state provided by the scalar field itself. The isotropic pressure serves as Lagrangian. However, the scalar field has a spatial gradient when discussing spherically symmetric static black hole solutions and their stability in scalar-tensor gravity theories. It has been known in the literature that sometimes the scalar behaves as a radiation field.

Hence, the cases of scalar fields with spatial or null gradients require special attention. In both cases the fluid corresponding to the scalar field is imperfect but simple enough due to the minimal coupling. In the spatial case the energy-momentum tensor is diagonal of type I. The tangential pressure serves as Lagrangian and (being its opposite) it also determines the energy density. The radial pressure is different. We have shown that such an energy-momentum tensor can be equally interpreted as a super-position of a perfect fluid and a pair of leftgoing (incoming) and rightgoing (outgoing) radiation streams represented by null dusts.

In the null case the energy-momentum tensor is of type II, representing an imperfect fluid with different energy density, radial and tangential pressures, and also heat flow.

We discussed the restrictions imposed by the energy conditions in all three cases. In particular, in the spacelike and null cases the energy conditions switch off the potential, hence the mass of the scalar. In the null case the scalar field degenerates into a null dust.

It is well known that massless particles forming a null dust follow null geodesics. In general, these geodesics are not affinely parametrized. The freedom to rescale null vectors while the energy density of null dust is rescaled, such to preserve the form of the null dust energy-momentum tensor, allows for the achievement of divergenceless of the null vector, hence the affine parametrization [26]. By imposing the energy conditions, the minimally coupled scalar field becomes a null dust, already affinely parametrized.

Remarkably, even without energy conditions fulfilled, a generic minimally coupled scalar field with null gradient still evolves along affinely parametrized null geodesics.

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[1]The strategy to follow here is to insert $X = 0$ and $L(\phi) = 0$ only in the equations derived from the variational principle, rather than into the Lagrangian (27). This is how a nonvanishing $L_X(\phi)$ enters the equations, however, $L_\phi(\phi) = 0$ holds.

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