

Noncommutative black holes of various genera in the connection formalismMathew Schneider^{1,*} and Andrew DeBenedictis^{2,1,†}¹*Department of Physics, Simon Fraser University, 8888 University Drive,
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We consider black hole interiors of arbitrary genus number within the paradigm of noncommutative geometry. The study is performed in two ways: One way is a simple smearing of a matter distribution within the black hole. The resulting structure is often known in the literature as a “model inspired by noncommutative geometry.” The second method involves a more fundamental approach, in which the Hamiltonian formalism is utilized and a nontrivial Poisson bracket is introduced between the configuration degrees of freedom, as well as between the canonical momentum degrees of freedom. This is done in terms of connection variables instead of the more common ADM variables. Connection variables are utilized here since noncommutative effects are usually inspired from the quantum theory, and it is the connection variables that are used in some of the more promising modern theories of quantum gravity. We find that in the first study, the singularity of the black holes can easily be removed. In the second study, we find that introducing a nontrivial bracket between the connections (the configuration variables) may delay the singularity, but not necessarily eliminate it. However, by introducing a nontrivial bracket between the densitized triads (the canonical momentum variables), the singularity can generally be removed. In some cases, new horizons also appear due to the noncommutativity.

DOI: [10.1103/PhysRevD.102.024030](https://doi.org/10.1103/PhysRevD.102.024030)**I. INTRODUCTION**

The general theory of relativity has, to date, robustly passed a number of experimental tests. These tests are no longer limited to the arena of weak-field gravity, but also include, due to more recent gravitational wave detection events, strong field regimes such as black hole mergers [1]. As successful as general relativity is, there should be some way to reconcile the fundamental properties of matter fields sourcing gravity (which at the fundamental level are quantum in nature) with the gravitational field that the matter produces. This compatibility could come from a theory of quantum gravity. General relativity, however, possesses the fundamental symmetry of background independence, and this makes the theory difficult to quantize in traditional manners [2,3]. At the moment, there are a number of candidate theories of quantum gravity which are in various stages of development [4–7], although none can yet be seen as a complete theory of quantum gravitation. Because of this, it is useful at the classical level to attempt to glean what some effects of a quantum theory of gravity may be.

One issue that is believed to be resolved in a quantum gravity theory is that of the gravitational singularities predicted by various classical theories of gravity. The most

famous of these singularities reside in the realms of early-Universe cosmology and black hole interiors. It is the latter issue that we wish to discuss in this paper.

The fundamental mathematical object on which quantum theory is based is the nontrivial commutator between a system’s configuration variables and its associated canonical momentum variables. At the level of classical mechanics, this manifests itself as a nontrivial Poisson bracket. The field of noncommutative geometry augments this structure by introducing, in addition to the usual bracket between configuration momentum variables, a nontrivial bracket between configuration variables. At the level of usual quantum mechanics, this would be a nontrivial commutator of the form

$$[x^a, y^b] = i\epsilon^{ab}{}_c \theta^c, \quad (1)$$

where θ^c is a vector whose entries measure the amount of noncommutativity between the various coordinates. The bracket [Eq. (1)] of course implies an uncertainty relation between different coordinates, and it sets a limit on the amount of localization a particle may have. A measurement along one axis to high precision comes at the expense of losing some information along another axis. Hence, geometry in this sense really does become noncommutative.

It is natural to then further extend the theory to include a nontrivial bracket between the canonical momenta as

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$$[p_a, p_b] = i\epsilon_{ab}{}^c \beta_c, \quad (2)$$

leading to a similar uncertainty between the measurement of momenta in different principal directions.

Noncommutative quantum theories have been studied in various fields of physics. The original paper seems to be the pioneering work of Hartland Snyder [8], and since then, there has been much application of noncommutative geometry to theoretical physics. (See Refs. [9–13] and references therein.)

At the classical level, the new commutators should manifest themselves as an extension of the usual Poisson algebra of ordinary classical mechanics, leading to a type of “noncommutative classical mechanics.” In noncommutative mechanics, the usual Poisson algebra is deformed via the introduction of a deformed Moyal product. That is, the brackets of noncommutative mechanics are calculated via

$$\{f, g\} := f \star g - g \star f, \quad (3)$$

where the Moyal product here is defined as

$$(f \star g)(v) := \exp \left[\frac{1}{2} w^{ab} \partial_a \tilde{\partial}_b \right] f(v) g(\tilde{v})|_{\tilde{v}=v}. \quad (4)$$

Here operators with a tilde operate only on tilde coordinates, and un-tilded operators operate on un-tilded coordinates. In the end, the two sets of coordinates are made coincident. The matrix w^{ab} represents the deformed symplectic form

$$w^{ab} = \left[\begin{array}{c|c} \epsilon^{ab}{}_c \theta^c & -\delta^{ab} \\ \hline \delta^{ab} & \epsilon^{ab}{}_c \beta^c \end{array} \right]. \quad (5)$$

We note here that there is actually a further correction to the above symplectic form, but it is proportional to the product $\theta^a \beta^b$, and hence we ignore it, as both these parameters are assumed to be small [14]. It may be seen by explicit calculation that in the limit $\theta^c = 0 = \beta^c$, the expression in Eq. (4) yields the usual Poisson brackets of ordinary classical mechanics. Explicitly, Eqs. (3) and (4), using Eq. (5), yields

$$\{p_a, x^b\} = \delta_a^b, \quad (6a)$$

$$\{x^a, x^b\} = \epsilon^{ab}{}_c \theta^c, \quad (6b)$$

$$\{p_a, p_b\} = \epsilon_{ab}{}^c \beta_c. \quad (6c)$$

Reviews of noncommutative mechanics may be found in Refs. [14,15] and references therein.

The transition from particle mechanics to field theories is not necessarily straightforward, particularly in the realm of gravitation [16–21]. However, if one symmetry reduces the system to minisuperspace models, then it can be argued that

one augments the field Poisson algebra in a similar manner to what is done in the particle mechanics [22–25]. That is, in a minisuperspace model with fields ψ^a and corresponding canonical momenta π_b , we have

$$\{\pi_a(x), \psi^b(y)\} \propto \delta_a^b \delta(x-y), \quad (7a)$$

$$\{\psi^a(x), \psi^b(y)\} \propto \epsilon^{ab}{}_c \theta^c \delta(x-y), \quad (7b)$$

$$\{\pi_a(x), \pi_b(y)\} \propto \epsilon_{ab}{}^c \beta_c \delta(x-y). \quad (7c)$$

As the brackets are modified from the canonical ones, it is possible that such an algebraic deformation introduces an anomaly in the gravitational constraint algebra. In the symmetry-frozen homogeneous scenarios, variable deformations generally do not introduce such anomalies, as the algebra trivializes due to the vanishing of spatial derivatives and being able to globally set the shift vector to zero. The situation likely needs further study under algebraic deformations, but it is generally believed that at high energies, noncommutative effects would anyway alter the symmetry of the low-energy theory [26,27], so it is not clear if one should demand low-energy symmetries to hold in the regime where noncommutative effects become important. Still, one needs to be cautious in interpreting results in such potentially symmetry-broken theories. The general issue for the case of *variable* deformations is summarized in Ref. [28], and the situation with noncommuting coordinates, and their generalized noncommutativity via the Seiberg-Witten map technique and including generalizations to the Poincaré gauge gravity approach was studied in Ref. [29].

It is also interesting to note that aspects of spin can be viewed in the paradigm of noncommutative geometry [30], including within general relativity [31].

This manuscript is laid out as follows: In Sec. II, we analyze models where the black holes are supported by a smeared out distribution of material, which is sometimes performed in the literature as an approximation of noncommutative effects on the matter fields due to the nonlocalization that noncommutative geometry introduces. In Sec. III, the noncommutativity is manifestly included in the brackets of the Poisson algebra in the configuration and momentum variables of the gravitational Hamiltonian system. The study there is performed in the connection formalism, as this formalism is seen as a promising avenue to a theory of quantum gravity. Finally, we conclude with a brief summary of the findings.

II. SMEARING OF THE MATTER DISTRIBUTION

The method used here is often said to be “inspired by noncommutative geometry.” The idea here is quite simple and straightforward and mainly serves as a segue to the Hamiltonian analysis of the next section. For concreteness in setting up the problem and method, we will assume at the

moment that the black hole is a spherically symmetric one, but the ideas apply to all the types of metrics considered in this work. Consider the Einstein equations in mixed form:

$$R^\mu{}_\nu - \frac{1}{2}R\delta^\mu{}_\nu = 8\pi T^\mu{}_\nu. \quad (8)$$

If one restricts these equations to spherical symmetry by utilizing the line element

$$ds^2 = -\exp(\alpha(r, t))dt^2 + \exp(\beta(r, t))dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (9)$$

then Eq. (8) may be manipulated to yield the following solution, assuming the stress-energy tensor components $T^t{}_t$ and $T^r{}_r$ are free parameters [32–34]¹:

$$e^{-\beta} = 1 + \frac{8\pi}{r} \left[\int_{0_+}^r T^t{}_t(x, t)x^2 dx \right], \quad (10a)$$

$$e^\alpha = e^{-\beta} \left\{ \exp \left[h(t) + 8\pi \int_{0_+}^r [T^r{}_r(x, t) - T^t{}_t(x, t)]e^{\alpha(x, t)} x dx \right] \right\}, \quad (10b)$$

$$T^r{}_t := \frac{1}{r^2} \left[2f(t)\dot{f}(t) - \int_{0_+}^r T^t{}_{t,t}(x, t)x^2 dx \right], \quad (10c)$$

$$T^\theta{}_\theta \equiv T^\phi{}_\phi := \frac{r}{2} [T^r{}_{r,r} + T^t{}_{r,t}] + \left[1 + \frac{r}{4}\alpha_{,r} \right] T^r{}_r + \frac{r}{4}(\alpha + \beta)_{,t} T^t{}_r - \frac{r}{4}\alpha_{,r} T^t{}_t, \quad (10d)$$

where a comma denotes partial differentiation. Equation (10c) is defined from the $r-t$ Einstein equation, and (10d) is defined from the conservation law. Now, the Schwarzschild metric may be seen as a solution to the above equations with a “point mass” located at $r = 0$. That is, one may prescribe

$$T^t{}_t(r) = -\frac{M}{4\pi r^2}\delta(r), \quad T^r{}_r = 0. \quad (11)$$

It is straightforward to see, by inserting Eq. (11) into Eqs. (10a) and (10b), that the resulting metric functions e^α and e^β yield, after a trivial rescaling of the t coordinate, the famous Schwarzschild metric:

¹We cannot prescribe more than two functions, as Einstein’s spherically symmetric equations with matter are underdetermined by precisely two [33].

$$e^\alpha = \left(1 - \frac{2M}{r} \right) = e^{-\beta}.$$

In noncommutative-geometry-inspired models, one smears the matter distribution [Eq. (11)] on a scale proportional to the coordinate noncommutativity parameter, θ . The argument is that the matter is not completely localized due to the uncertainty principle between coordinates brought on by the noncommutativity. Such inspired models have been studied in Refs. [35–39] for spherical black holes without cosmological constant, and in Refs. [40,41] for rotating black holes. In Ref. [42], the relationship between inspired theories and the Voros product (instead of the Moyal one) has been explored. Studies of inspired models have been performed in Refs. [43–46] with respect to wormholes.

We wish to extend the study here to encompass black holes beyond spherical, both in shape and in topology. This is done for the reason of consistency. That is, one wishes to study if and how singularities are affected in as many scenarios as possible to determine how universal the noncommutative effects are. One can then make more general statements about noncommutativity. We also include a cosmological constant, since in four dimensions a cosmological constant is required for black holes of exotic topology [47–50].

As we are interested specifically in the singularity issue of black holes, we will be concentrating on the interior region. First, we wish to rewrite the line element [Eq. (9)] in a form more appropriate for the study of black hole interiors and various topologies. The form is as follows:

$$ds^2 = -e^{A(\tau)}d\tau^2 + e^{B(\tau)}dy^2 + \tau^2 d\varrho^2 + \tau^2 c_0 \sinh^2(\sqrt{d_0} \varrho) d\phi^2, \quad (12)$$

and it reflects the fact that the interior region is time dependent and that the exterior radial coordinate, r , is timelike in the interior region (we do not consider cases here with inner horizons).² We are considering time dependence only, due to the fact that we are smearing classical nonrotating *vacuum* black holes, save for the “point” source, whose corresponding interiors are also homogeneous. The constants c_0 and d_0 dictate the compatible topology of the spacetime’s two-dimensional subspaces. The various cases are as follows:

- (i) $d_0 = -1, c_0 = -1$: In this scenario, (ϱ, ϕ) submanifolds are spheres.
- (ii) $d_0 = 0, \lim_{d_0 \rightarrow 0} c_0 = \frac{1}{d_0}$: In this scenario, (ϱ, ϕ) submanifolds are tori (and the submanifolds for this case are intrinsically flat).

²In the coordinate chart of Eq. (12), the familiar Schwarzschild line element takes the form $ds^2 = -\frac{d\tau^2}{\tau} + \left(\frac{2M}{\tau} - 1\right)dy^2 + \tau^2 d\varrho^2 + \tau^2 \sin^2 \varrho d\phi^2$, with $\tau < 2M$.

- (iii) $d_0 = 1, c_0 = 1$: In this case, (ϱ, ϕ) submanifolds are surfaces of constant negative curvature of genus $g > 1$, depending on the identifications chosen. Such surfaces may be compact or not [50,51].

In the spherical case, such solutions are sometimes referred to in the literature as “ T -spheres” [52–54], and the time-dependent domain inside the event horizon is sometimes referred to as the “ T -domain” of the black hole.

Einstein’s equations for the line element in Eq. (12) yield the following general solution analogous to Eqs. (10a)–(10d), assuming here time dependence only:

$$e^{-A} = d_0 - \frac{8\pi}{\tau} \left[\int_{\tau_1}^{\tau} T^y{}_y(\tau') \tau'^2 d\tau' \right], \quad (13a)$$

$$e^B = e^{-A} \left\{ \exp \left[k_0 - 8\pi \int_{\tau_1}^{\tau} [T^r{}_\tau(\tau') - T^y{}_y(\tau')] e^{A(\tau')} \tau' d\tau' \right] \right\}, \quad (13b)$$

$$T^r{}_y = 0, \quad (13c)$$

$$T^{\varrho}{}_{\varrho} \equiv T^{\varphi}{}_{\varphi} := \frac{\tau}{2} T^r{}_{\tau, \tau} + \left[1 + \frac{\tau}{4} B_{, \tau} \right] T^r{}_\tau - \frac{\tau}{4} B_{, \tau} T^y{}_y. \quad (13d)$$

In the case of a “point” source [in the interior region $T^y{}_{y \text{ matter}}(\tau) = -\frac{M}{4\pi\tau^2} \delta(\tau)$], supplemented with the cosmological constant [$T^r{}_{\tau\Lambda} = T^y{}_{y\Lambda} = -\Lambda/(8\pi)$], the above solutions yield, after a rescaling of the y coordinate,

$$e^{-A(\tau)} = \left(d_0 + \frac{2M}{\tau} + \frac{\Lambda}{3} \tau^2 \right) = e^{B(\tau)}. \quad (14)$$

Such black hole solutions have been studied in detail in Refs. [47–50], and within quantum gravity theories in Refs. [55–58].

The noncommutative smearing is often performed via the implementation of replacing the “point” source with a Gaussian or Lorentzian whose characteristic width is of the scale of the noncommutativity parameter, θ . Without guidance from experiment, this is usually taken to be of the order of the Planck length. We consider here the following profile curves for $T^y{}_{y \text{ matter}}(\tau)$:

$$T^y{}_{y \text{ matter}}(\tau) = -\frac{M\sqrt{\theta}}{\pi^2(\tau^2 + \theta)^2}, \quad (15a)$$

$$T^y{}_{y \text{ matter}}(\tau) = -\frac{M}{(4\pi\theta)^{3/2}} e^{-\frac{\tau^2}{4\theta}}, \quad (15b)$$

the first profile being Lorentzian and the second Gaussian. Further, since the matter profile no longer vanishes

abruptly, we relate the above stress-energy components to their corresponding local energy densities via an equation of state, for which we take the polytropic form. That is, in both the Lorentzian and Gaussian scenarios we prescribe an energy density via

$$T^r{}_{\tau \text{ matter}}(\tau) = k(T^y{}_{y \text{ matter}}(\tau))^\gamma, \quad (16)$$

where k and γ are constants. This form works well in idealized studies of stellar structure [59–63] and seems a natural choice for the type of exotic “star” we are studying here.

By using Eqs. (15a) and (15b) in Eqs. (13a) and (13b), one arrives at the following analytical solutions:

$$e^{-A(\tau)} = d_0 + \frac{\Lambda}{3} \tau^2 + \frac{6M}{\pi\tau} \arctan(\tau/\sqrt{\theta}) - \frac{6M\sqrt{\theta}}{\pi(\tau^2 + \theta)} \quad (17)$$

for the Lorentzian case, and

$$e^{-A(\tau)} = d_0 + \frac{\Lambda}{3} \tau^2 + \frac{2M}{\tau} \operatorname{erf}(\tau/2\sqrt{\theta}) - \frac{2M}{\sqrt{\pi\theta}} e^{-\frac{\tau^2}{4\theta}} \quad (18)$$

for the Gaussian scenario. Solutions for $e^{B(\tau)}$ were also obtained, but being rather complicated and in terms of quadrature, are not displayed here.

Of particular interest here is to study the properties of what replaces the singularity of the commutative theory. To facilitate this, we calculate the components of the Riemann curvature tensor in an orthonormal (hatted) frame,

$$R_{\hat{\mu} \hat{\nu} \hat{\rho} \hat{\sigma}} = R_{\alpha\beta\gamma\delta} h_{\hat{\mu}}{}^\alpha h_{\hat{\nu}}{}^\beta h_{\hat{\rho}}{}^\gamma h_{\hat{\sigma}}{}^\delta. \quad (19)$$

Here, h represents the components of a local orthonormal tetrad. We choose specifically the tetrad coincident with the coordinate directions. That is, the tetrad is given by

$$h_{\hat{t}}{}^\mu = \frac{\delta_t{}^\mu}{e^{A(\tau)/2}}, \quad h_{\hat{y}}{}^\mu = \frac{\delta_y{}^\mu}{e^{B(\tau)/2}},$$

$$h_{\hat{\varrho}}{}^\mu = \frac{\delta_{\varrho}{}^\mu}{\tau}, \quad h_{\hat{\varphi}}{}^\mu = \frac{\delta_{\varphi}{}^\mu}{\tau|\sqrt{c_0} \sinh(\sqrt{d_0}\varrho)|}.$$

The resulting orthonormal Riemann components are rather lengthy and do not reveal much due to their complication. It is useful therefore to present the lowest-order terms in a series expansion about the commutative solution’s singular point ($\tau = 0$). Such an expansion yields the following components, plus those related by symmetries, for the Lorentzian case:

$$R_{\hat{t} \hat{y} \hat{t} \hat{y}} = -\frac{\Lambda}{3} + 4k\pi \left(\frac{M}{\pi^2 \theta^{3/2}} \right)^\gamma + \frac{4M}{3\pi\theta^{3/2}} + \mathcal{O}(\tau^2), \quad (20a)$$

$$R_{\hat{t} \hat{\varrho} \hat{t} \hat{\varrho}} = -\frac{\Lambda}{3} - \frac{8M}{3\pi\theta^{3/2}} + \frac{32M}{5\pi\theta^{5/2}} \tau^2 + \mathcal{O}(\tau^4), \quad (20b)$$

$$R_{\hat{y}\hat{\phi}\hat{y}\hat{\phi}} = \frac{\Lambda}{3} - 4k\pi \left(\frac{M}{\pi^2\theta^{3/2}} \right)^\gamma - \frac{4M}{3\pi\theta^{3/2}} + \frac{8k\pi\gamma}{\theta} \left(\frac{M}{\pi^2\theta^{3/2}} \right)^\gamma \tau^2 + \frac{8M}{5\pi\theta^{5/2}} \tau^2 + \mathcal{O}(\tau^4), \quad (20c)$$

$$R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} = \frac{\Lambda}{3} + \frac{8M}{3\pi\theta^{3/2}} - \frac{16M}{5\pi\theta^{5/2}} \tau^2 + \mathcal{O}(\tau^4), \quad (20d)$$

and for the Gaussian case:

$$R_{\hat{\tau}\hat{y}\hat{\tau}\hat{y}} = -\frac{\Lambda}{3} + 2k\pi \left(\frac{M}{(4\pi\theta)^{3/2}} \right)^\gamma + \frac{M}{12\sqrt{\pi}\theta^{3/2}} + \mathcal{O}(\tau^2), \quad (21a)$$

$$R_{\hat{\tau}\hat{\phi}\hat{\tau}\hat{\phi}} = \frac{\Lambda}{3} - \frac{M}{3\pi\theta^{3/2}} + \frac{M}{10\sqrt{\pi}\theta^{5/2}} \tau^2 + \mathcal{O}(\tau^4), \quad (21b)$$

$$R_{\hat{y}\hat{\phi}\hat{y}\hat{\phi}} = \frac{\Lambda}{3} - 4k\pi \left(\frac{M}{(4\pi\theta)^{3/2}} \right)^\gamma - \frac{M}{6\sqrt{\pi}\theta^{3/2}} + \frac{k\pi\gamma}{\theta} \left(\frac{M}{(4\pi\theta)^{3/2}} \right)^\gamma \tau^2 + \frac{M}{40\sqrt{\pi}\theta^{5/2}} + \mathcal{O}(\tau^4), \quad (21c)$$

$$R_{\hat{\phi}\hat{\phi}\hat{\phi}\hat{\phi}} = \frac{\Lambda}{3} + \frac{M}{3\pi\theta^{3/2}} - \frac{M}{20\sqrt{\pi}\theta^{5/2}} \tau^2 + \mathcal{O}(\tau^4). \quad (21d)$$

The results above are “universal” at $\tau = 0$ in the sense that the topological parameter, d_0 , does not contribute at zeroth order. This parameter comes in at order τ^2 in $R_{\hat{\tau}\hat{y}\hat{\tau}\hat{y}}$ (although this term is not shown due to its length), and at order τ^4 or higher in the other components.

It may be noted that none of the components in Eqs. (20a)–(21d) are singular for finite θ , and therefore the classical singularity present in the commutative theory is removed. We should point out here that this result is not really surprising. One has excised the singular distribution of Eq. (11) and replaced it with a smooth distribution. In the T -domain, this forced smearing is accompanied by the expected energy condition violations which circumvent the singularity theorems for such spacetimes. Therefore, at the level of noncommutative-geometry-inspired models of black holes, the noncommutativity introduces energy condition violation on scales set by the noncommutativity parameter θ .

We proceed next to a more rigorous analysis where the noncommutativity is truly manifest in the algebra of the field variables.

III. HAMILTONIAN EVOLUTION OF BLACK HOLE INTERIORS

In this section, we shall study the effects of noncommutativity by directly supplementing the usual Poisson

algebra of the gravitational Hamiltonian system with the additional structure on the configuration and momentum variables as in Eqs. (7a)–(7c). We work here in the connection variables consisting of the $su(2)$ connection, which we denote A^i_a , and its conjugate momentum, the densitized triad, denoted³ by E_i^a . These variables are chosen since they are the variables utilized in the theory of loop quantum gravity. At the quantum level, within the paradigm of noncommutative geometry, the commutator between the configuration variables is taken to be nontrivial. As well, one may also take the commutator between the conjugate momenta as nontrivial. Working “backwards” towards the corresponding classical theory, these nontrivial commutators should manifest themselves as nontrivial Poisson brackets.

It is generally accepted that loop quantum gravity puts forward a more promising approach towards a theory of quantum gravity than does the original Wheeler-DeWitt theory [2,64], which utilizes ADM variables. Therefore, in light of this, we choose to work in the variables of loop quantum gravity. With the modification of the Poisson brackets introduced by the extra noncommutativity, it is possible that working in these variables is a different theory than working in the corresponding noncommutative ADM theory.

In terms of cosmological studies, a number of interesting noncommutativity studies have been performed in Refs. [22,24,65–67] in standard variables.

A. Black holes in connection variables

Here we briefly review the mathematical structure of black holes in connection variables. In the connection variables which are utilized in the canonical formulation of loop quantum gravity, one begins with a $3+1$ decomposition of spacetime, where the metric is written in the usual way:

$$ds^2 = -N^2 d\tau^2 + q_{ab}[dx^a + N^a d\tau][dx^b + N^b d\tau], \quad (22)$$

with N being the lapse and N^a the shift vector. One then writes the resulting action in terms of the Ashtekar variables [68]. These variables comprise a generalized $su(2)$ connection A^i_a and a densitized triad E_i^a . The connection field plays the role of the configuration variable, and it is related to more familiar quantities as follows:

$$A^i_a = \Gamma^i_a + \gamma K^i_a, \quad (23)$$

where Γ^i_a is the “fiducial” spin connection

$$\Gamma^i_a := e^{ij} h_j^b (\partial_{[a} h^k_{b]}) + \delta^{k\ell} \delta_{pq} h_\ell^c h^p_a \partial_b h^q_c \quad (24)$$

³Indices i, j, k , etc., denote $su(2)$ indices; whereas indices a, b, c , etc., denote spatial indices.

and K^i_a is the densitized extrinsic curvature

$$K^i_a := \frac{1}{\sqrt{\det(E)}} K_{ab} E_j^b \delta^{ij}, \quad (25)$$

with K_{ab} being the usual extrinsic curvature of a $\tau = \text{const.}$ surface. The quantity γ is known as the Barbero-Immirzi parameter. In the classical theory, its value is arbitrary (though nonzero), but in the resulting quantum theory of loop quantum gravity, it must be set somehow. This is usually done by calculations of black hole entropy within the paradigm of loop quantum gravity and setting the result to one-quarter the area of the black hole [69–73].

The momentum variable, E_i^a , is related to the three-metric, q_{ab} , via

$$\det(q) q^{ab} = E_i^a E_j^b \delta^{ij}. \quad (26)$$

These two variables, A^i_a and E_i^a , are then the configuration and momentum variables, respectively, of the theory, subject to the Poisson algebra

$$\{A^i_a(x), E_j^b(y)\} = 8\pi\gamma \delta^i_j \delta_a^b \delta(x, y), \quad (27)$$

with other brackets equal to zero.

Via variation of the action with respect to the lapse and shift, one obtains the Hamiltonian (S) and diffeomorphism constraints (V_b):

$$S = \frac{E_i^a E_j^b}{\sqrt{\det(E)}} [\epsilon^{ijk} F^k_{ab} - 2(1 + \gamma^2) K^i_{[a} K^j_{b]}] = 0, \quad (28a)$$

$$V_b = E_i^a F^i_{ab} - (1 + \gamma^2) K^i_b G_i = 0, \quad (28b)$$

with $F^i_{ab} := \partial_a A^i_b - \partial_b A^i_a + \epsilon^i_{jk} A^j_a A^k_b$. The extrinsic curvature quantities in Eq. (28a) are replaced with the connection via Eq. (23) (the h_i^a in Γ^i_a being functions of E_i^a).

There is also the internal $SU(2)$ degree of freedom that can be fixed. (The metric, depending on the “square” of the densitized triad via Eq. (26), allows for $SU(2)$ rotations which preserve the metric.) This gauge can be fixed via the Gauss constraint:

$$G_i := \partial_a E_i^a + \epsilon_{ij}^k A^j_a E_k^a = 0. \quad (29)$$

At this stage, we need to choose an ansatz for the $su(2)$ connection and the densitized triad which is compatible with our geometries. An appropriate ansatz is provided by the following pair [55,57,74]:

$$\begin{aligned} A^i_a \tau_i dx^a &= a_3 \tau_y dy + (a_1 \tau_\varrho + a_2 \tau_\varphi) d\varrho \\ &+ (a_2 \tau_\varrho - a_1 \tau_\varphi) \sqrt{c_0} \sinh(\sqrt{d_0} \varrho) d\varphi \\ &+ \tau_y \sqrt{c_0} \sqrt{d_0} \cosh(\sqrt{d_0} \varrho) d\varphi, \end{aligned} \quad (30a)$$

$$\begin{aligned} E_i^a \tau^i \partial_a &= -E_3 \tau_y \sqrt{c_0} \sinh(\sqrt{d_0} \varrho) \frac{\partial}{\partial y} \\ &- (E_1 \tau_\varrho + E_2 \tau_\varphi) \sqrt{c_0} \sinh(\sqrt{d_0} \varrho) \frac{\partial}{\partial \varrho} \\ &+ (E_1 \tau_\varphi - E_2 \tau_\varrho) \frac{\partial}{\partial \varphi}, \end{aligned} \quad (30b)$$

where τ_i 's represent the $SU(2)$ generators. The functions a_I and E_I are functions of the interior time variable, τ , only.

In terms of Eq. (30b), using Eq. (26), a line-element of the form of Eq. (12) is written as

$$\begin{aligned} ds^2 &= -N^2 d\tau^2 + \frac{(E_1)^2 + (E_2)^2}{E_3} dy^2 + E_3 d\varrho^2 \\ &+ E_3 c_0 \sinh^2(\sqrt{d_0} \varrho) d\varphi^2. \end{aligned} \quad (31)$$

The Gauss constraint [Eq. (29)] for the cases considered here yields just one condition:

$$a_1 E_2 - a_2 E_1 = 0, \quad (32)$$

which we will satisfy here by choosing

$$a_1 = 0 = E_1. \quad (33)$$

The diffeomorphism constraint is automatically satisfied in these cases, leaving only the Hamiltonian (scalar) constraint. With the gauge fixing of Eq. (33), the resulting Hamiltonian constraint (supplemented with a cosmological constant term) may be written as

$$\begin{aligned} S &= -\frac{N\sqrt{c_0}}{2\sqrt{d_0} E_3 \gamma^2} [((a_2)^2 - \gamma^2 d_0) E_2 + 2E_3 a_2 a_3] \\ &+ \frac{N}{2\sqrt{E_3} \gamma^2} \Lambda E_2 E_3. \end{aligned} \quad (34)$$

It should be noted that when integrating the Hamiltonian density, the spatial variables have been integrated out, and therefore the above result contains an (arbitrary) area from the y and ϱ integrals. This is set equal to 1, and we show below that this does not spoil the Hamiltonian evolution of the system. We also should add here that in constructing the field strength tensor, no further noncommutativity was assumed beyond the usual commutation relations amongst the $su(2)$ generators. The action here acquires no corrections, as there are no Poisson brackets in its construction. A noncommutativity would introduce a potential ambiguity if one wished to study the true quantum theory (not what is

done here), where quantities in the Hamiltonian do not commute. This would be analogous to the usual factor-ordering ambiguity of going from a classical to a quantum theory.

At this stage, one has all the ingredients required to study the evolution of the interior regions of black holes. The evolution proceeds according to the usual Hamiltonian equations of motion $\dot{a}_2 = \{a_2, S\}$, etc., subject to the usual Poisson algebra between the configuration variables, a_I , and their corresponding canonical momenta, E_I .

B. Noncommutative evolution of black holes

Here we study the above gravitational system in connection variables, but where the usual Poisson algebra is augmented by the following brackets:

$$\{a_I, a_J\} = \epsilon_{IJ}\theta, \quad \{E_I, E_J\} = \epsilon_{IJ}\beta. \quad (35)$$

We study scenarios where either θ or β is zero, as well as those where neither parameter is zero. As this is a first study, we take the noncommutative parameters, θ and β , to be constants. However, it is possible that they can be modified in such a way that they depend on the metric properties (via the densitized triad) of the spacetime. This, for example, could arguably improve the theory by providing a natural way for the brackets to become less significant in low-curvature regions.

The resulting equations in the noncommutative case are too complex to find analytic solutions, so what we are solving here is a classic initial value problem. As such, initial conditions are required in order to study the evolution. We set initial conditions as follows: Note that the coordinate chart in use for the domain of Eq. (31) is $\tau < \tau_H$, where τ_H denotes the horizon value of τ and that the commutative solution's singular point is located at $\tau = 0$. The evolution is started far from the singular point, and relatively close to the horizon. We make the assumption that, far from the singular point, noncommutative effects should be small, as we know, for example, that the Schwarzschild solution is valid in moderately strong gravitational fields [75]. (In fact, commutative general relativity seems to hold well even in the strong field regime [75,76], so the noncommutative results here really are expected to be manifest only when one is approaching the scale of quantum gravity effects.) Therefore, on the initial time surface, which is far from the extremely strong field region, we set the values of the functions a_I and E_I to their general relativity values. That is, the following initial values are used:

$$a_{3\text{init}} = -\frac{\gamma}{2N} e^B \dot{B}, \quad a_{2\text{init}} = -\frac{\gamma}{N}, \quad (36a)$$

$$E_{3\text{init}} = \tau^2, \quad E_{2\text{init}} = e^{B/2} \tau, \quad (36b)$$

where the densitized triad components have been calculated via comparing Eqs. (31) and (12). The connection components have been calculated via a rather lengthy calculation utilizing Eqs. (23), (24), and (25), using metric (12)'s triad pulled back to a $\tau = \text{const.}$ hypersurface. The function B here is the commutative solution's value given by Eq. (14). The lapse, N , is generally arbitrary, but as we have set the coordinate system to be that of Eq. (14), we wish to use the time variable proportional to that used in Eq. (14). Therefore, we set the lapse equal to $\gamma^2 \sqrt{E_3}/E_2$, and at this stage one may proceed with the evolution.

The resulting "noncommutative" Hamilton equations of motion are given by

$$\dot{a}_2 = \frac{\sqrt{c_0} E_3 a_2 a_3}{\sqrt{d_0} (E_2)^2} - \theta \frac{2\sqrt{c_0} a_2 E_3}{\sqrt{d_0} E_2}, \quad (37a)$$

$$\begin{aligned} \dot{a}_3 = & -\frac{2\sqrt{c_0}}{\sqrt{d_0} E_2} \left[a_2 a_3 - \frac{E_2}{2} \Lambda \right] \\ & + \theta \frac{\sqrt{c_0}}{\sqrt{d_0} E_2} [a_2 E_2 + a_3 E_3], \end{aligned} \quad (37b)$$

$$\begin{aligned} \dot{E}_2 = & \frac{\sqrt{c_0}}{\sqrt{d_0} E_2} [a_2 E_2 + a_3 E_3] \\ & - \beta \frac{2\sqrt{c_0}}{\sqrt{d_0} E_2} \left[a_2 a_3 - \frac{E_2}{2} \Lambda \right], \end{aligned} \quad (37c)$$

$$\dot{E}_3 = \frac{2\sqrt{c_0} a_2 E_3}{\sqrt{d_0} E_2} - \beta \frac{\sqrt{c_0} E_3 a_2 a_3}{\sqrt{d_0} (E_2)^2}. \quad (37d)$$

As mentioned previously, these equations are generally too complex to solve numerically; hence, we illustrate the solutions below subject to the initial conditions provided by Eqs. (36a) and (36b).

1. Noncommutative connection only

Here we briefly summarize the results of the evolution of the above system subject to $\theta \neq 0$ and $\beta = 0$. That is, here the standard theory is augmented with a nontrivial configuration bracket only. For each of the three topological compatibilities (spherical, toroidal/cylindrical/planar, higher genus) the results are shown in Figs. 1–3.

The figures show a few, but not all, possible scenarios, and below, we summarize all cases. In none of the solutions can the results be evolved indefinitely. In some cases shown there is a true singularity present, with E_3 shrinking to zero, whereas in others it is a (curvature) finite solution (E_3 nonzero), but with a new horizon appearing (see figure captions for details).⁴ In general,

⁴The condition for an event horizon may seem peculiar here, but note that in the T -domain, the metric condition $g_{yy} \rightarrow 0_+$ is an equivalent statement to $g_{tt} \rightarrow 0_-$ (t being *exterior* time).

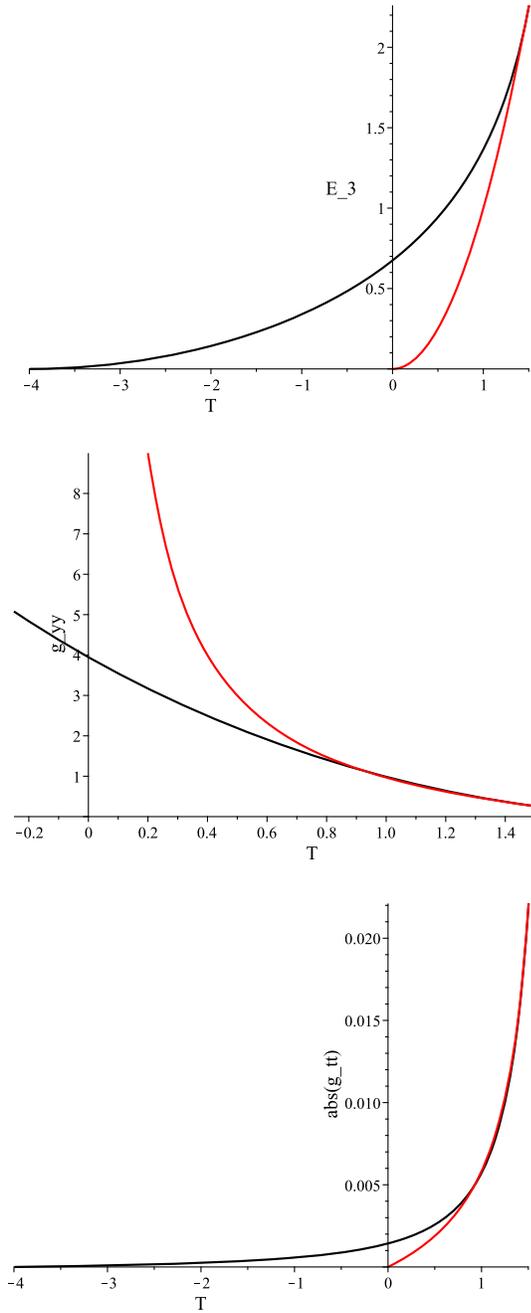


FIG. 1. $\theta \neq 0, \beta = 0$. Commutative (red) vs noncommutative (black) interior of a spherical black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces, governed by the value of E_3 , shrinks to zero beyond $\tau = 0$ in the noncommutative case, indicating a delay in the singularity. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.4$, and $\beta = 0$.

it is found that if $\beta = 0$, and the noncommutativity parameter θ is fairly large, one may eliminate the singularity, although in some cases an inner horizon results, and we are unable to

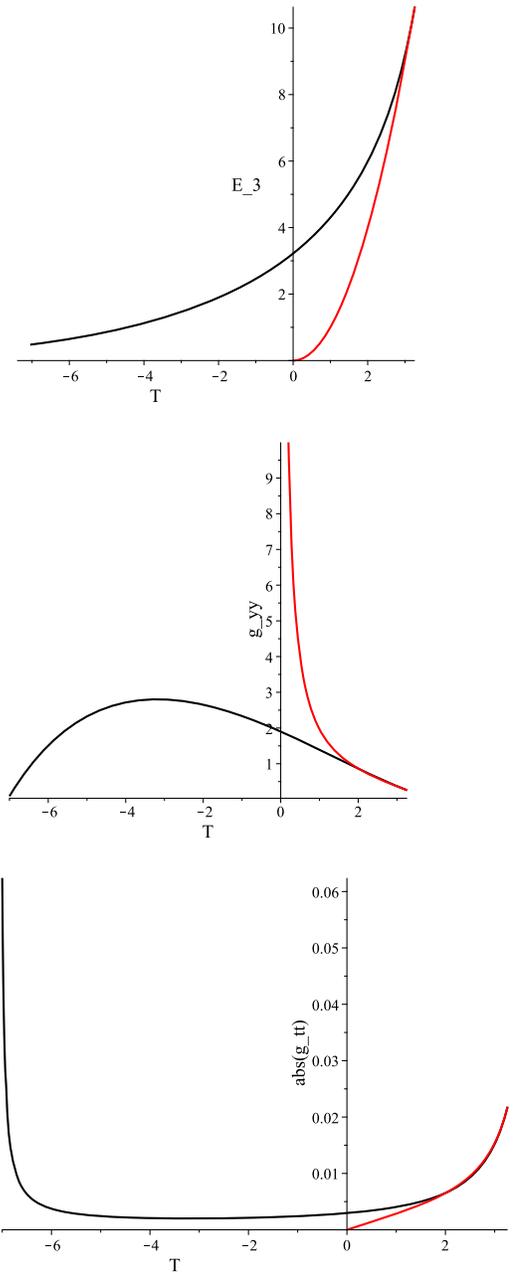


FIG. 2. $\theta \neq 0, \beta = 0$. Commutative (red) vs noncommutative (black) interior of a toroidal black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces in this particular case does *not* shrink to zero in the noncommutative case, indicating the removal of the singularity, but the evolution stops due to $g_{yy} \rightarrow 0$ and $|g_{\tau\tau}| \rightarrow \infty$, indicating the presence of another horizon. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.08$, and $\beta = 0$.

probe beyond that horizon to glean if there is singular structure hiding behind it. For small enough values of θ , the singularity is always present.

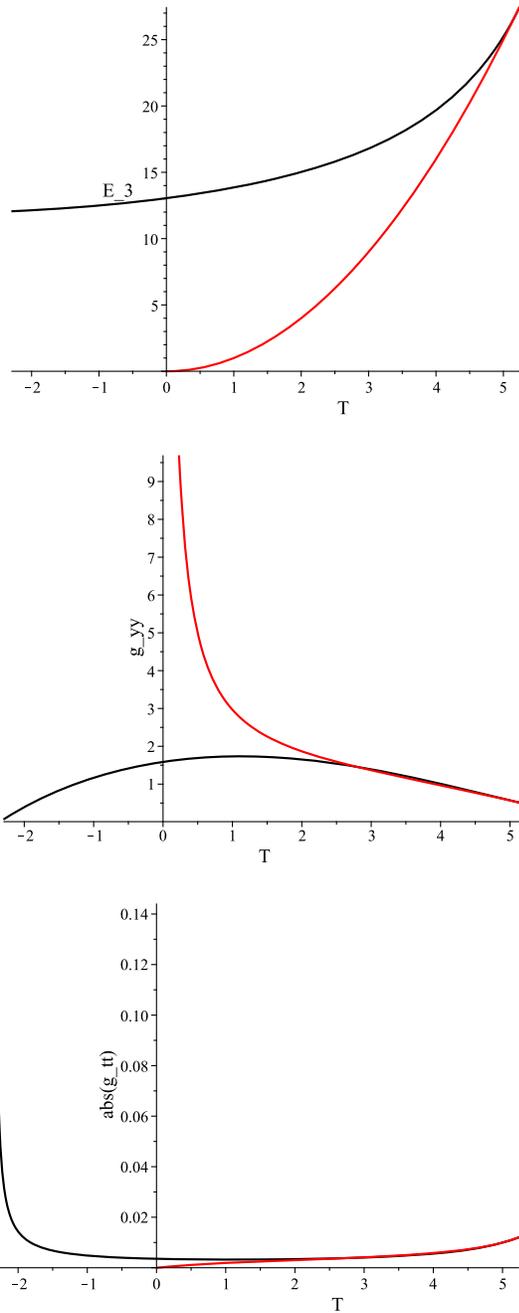


FIG. 3. $\theta \neq 0, \beta = 0$. Commutative (red) vs noncommutative (black) interior of a higher-genus black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces in this particular case does *not* shrink to zero in the noncommutative case, but the evolution stops due to $g_{yy} \rightarrow 0$ and $|g_{\tau\tau}| \rightarrow \infty$, indicating the presence of another horizon. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.06$, and $\beta = 0$.

2. Noncommutative triad only

For the case where the connection remains self-commutative but the triad becomes noncommutative, we present

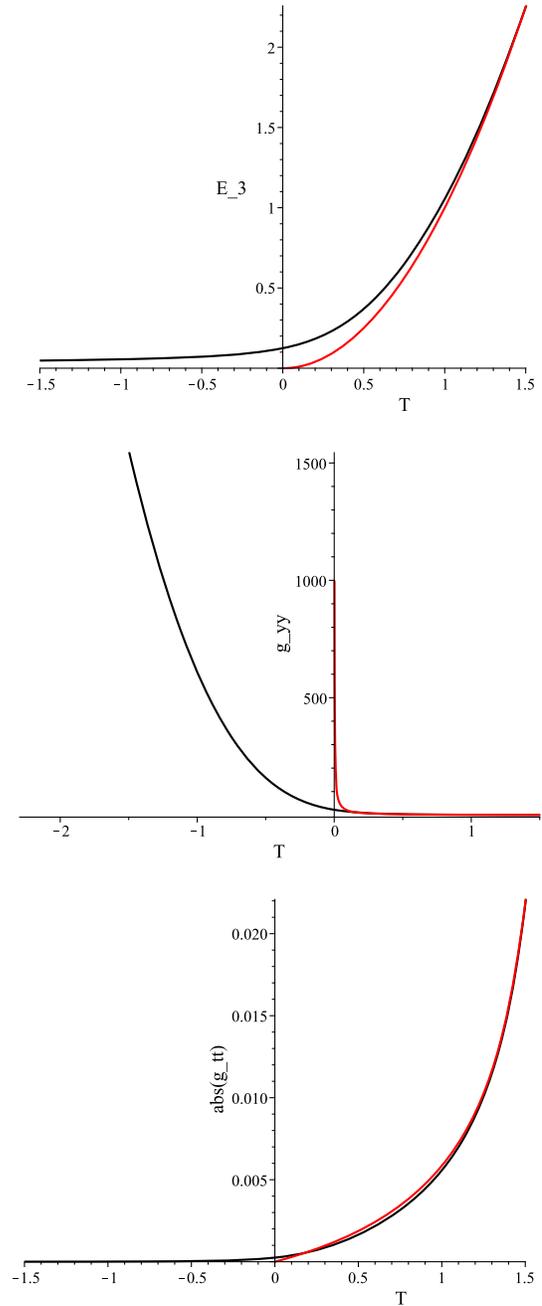


FIG. 4. $\theta = 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a spherical black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces, governed by the value of E_3 , does *not* shrink to zero in the noncommutative case, indicating removal of the singularity. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = 0$, and $\beta = -0.1$. There is no new horizon in this case.

the sets of scenarios in Figs. 4–6, along with a full summary at the end of the section.

It turns out that for both large and small nonzero values of β , the quantity E_3 asymptotes to a nonzero constant.

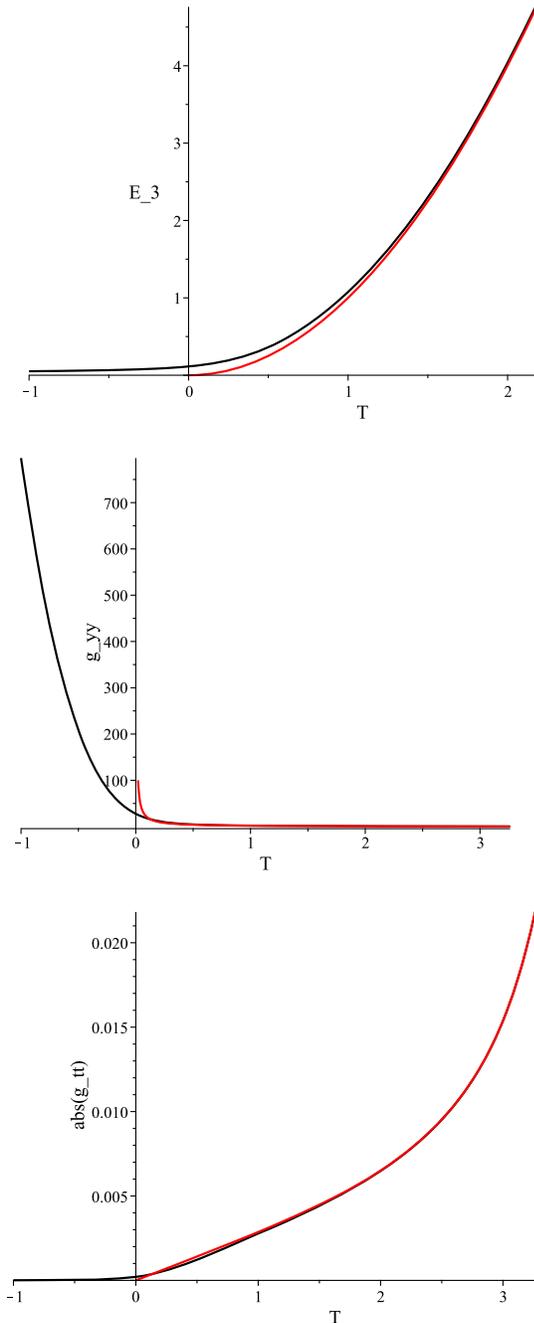


FIG. 5. $\theta = 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a toroidal black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{tt}|$. Notice from the top graph that the radius of the 2D subspaces does *not* shrink to zero in the noncommutative case, indicating removal of the singularity. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = 0$, and $\beta = -0.1$. There is no new horizon in this case.

The size of the 2D subspaces, governed by the value of E_3 , depends on the value of β , with larger β values yielding larger volumes. The situation here is somewhat reminiscent of what occurs in effective loop quantum gravity when

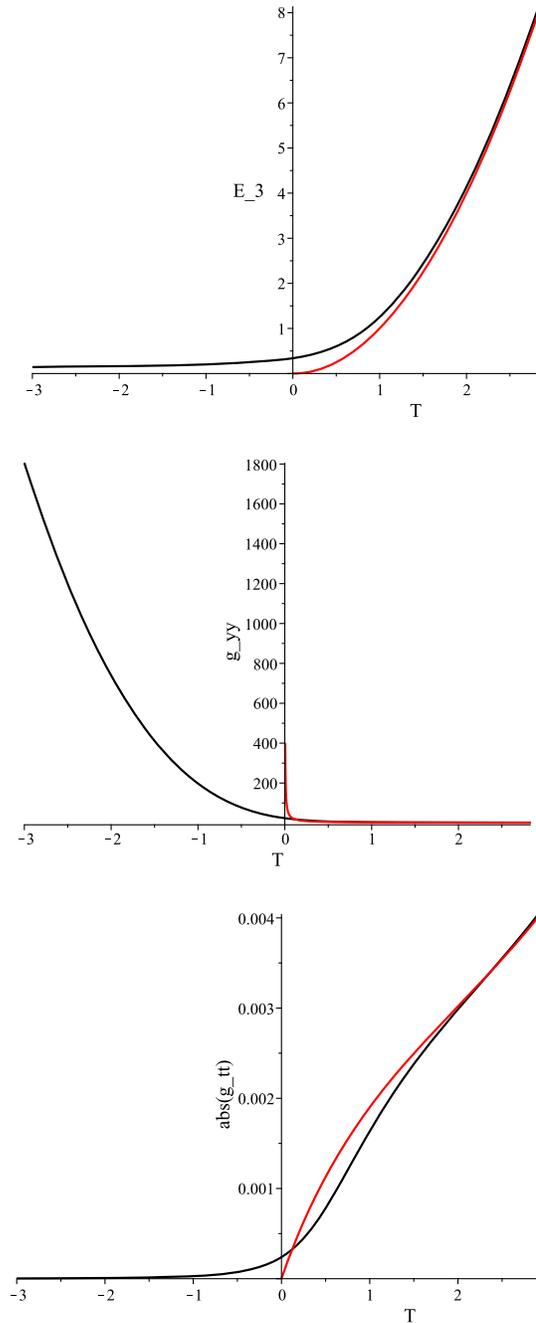


FIG. 6. $\theta = 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a higher-genus black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{tt}|$. Notice from the top graph that the radius of the 2D subspaces does *not* shrink to zero in the noncommutative case, indicating removal of the singularity. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = 0$, and $\beta = -0.5$. There is no new horizon in this case.

holonomy corrections are introduced, the main difference being that in the loop quantum gravity scenario the volume of the subspaces oscillates in a damped manner, asymptotically approaching a constant for large negative τ [56].

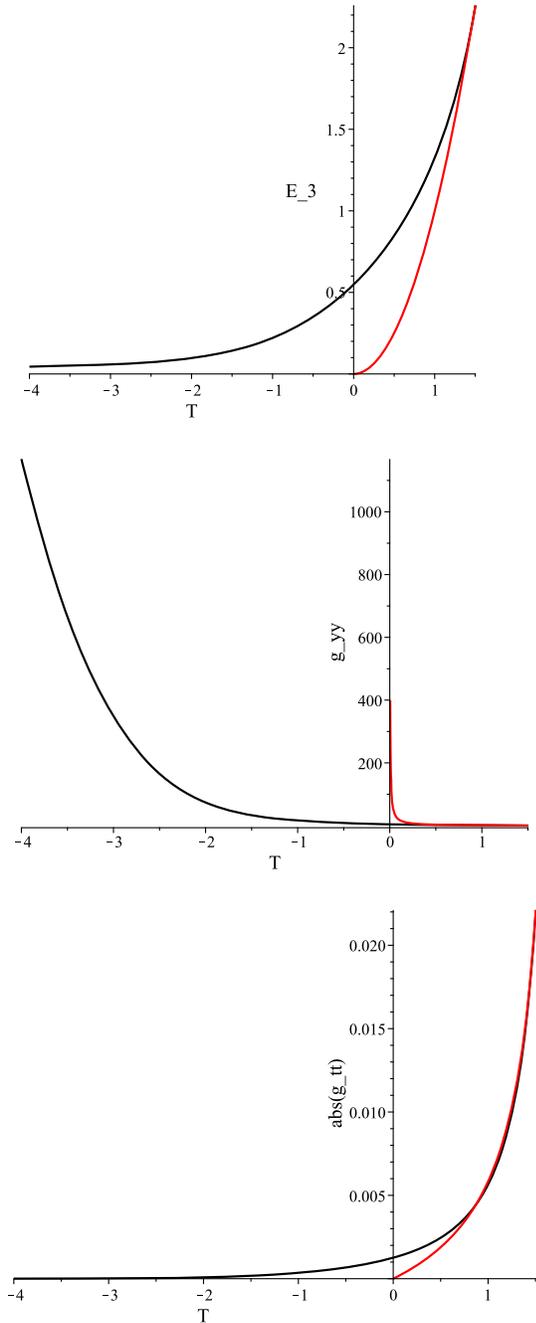


FIG. 7. $\theta \neq 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a spherical black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces, governed by the value of E_3 , does *not* shrink to zero in the noncommutative case, indicating removal of the singularity. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.3$, and $\beta = -0.1$. There is no new horizon in this case.

Although some appear small in the plots, no noncommutative metric component goes to zero in Figs. 4–6, and this remains true as long as $\beta \neq 0$.

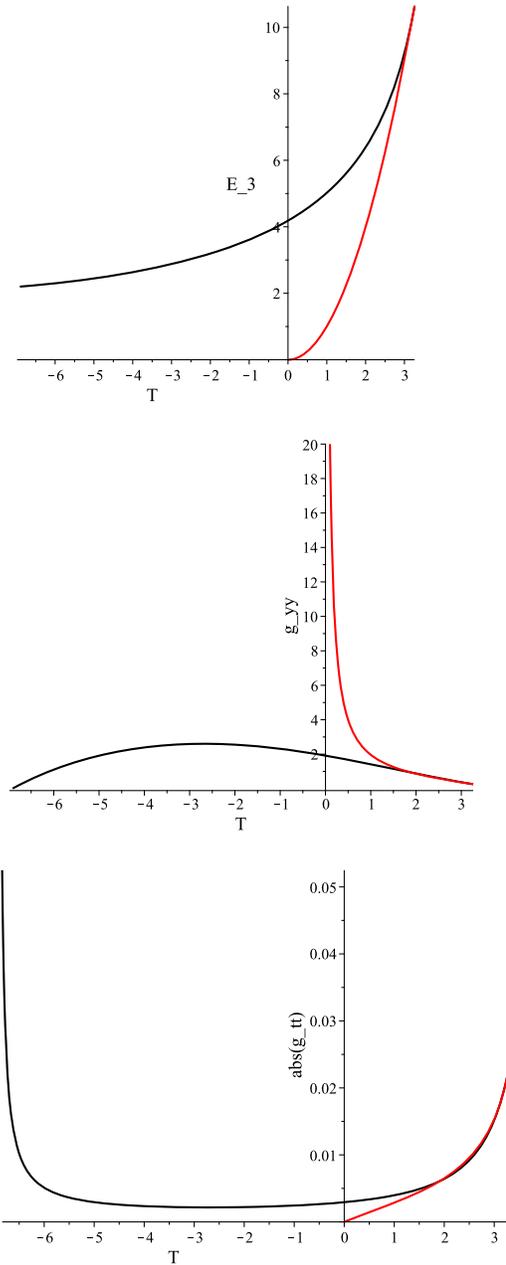


FIG. 8. $\theta \neq 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a toroidal black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces in this particular case does *not* shrink to zero in the noncommutative case, indicating the removal of the singularity, but the evolution stops due to $g_{yy} \rightarrow 0$ and $|g_{\tau\tau}| \rightarrow \infty$, indicating the presence of another horizon. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.1$, and $\beta = -0.05$.

3. Noncommutative connection and triad

Finally, we consider here the scenarios where all quantities possess a nontrivial Poisson bracket. Some representative results are shown in Figs. 7–9. (See figure captions for details.)

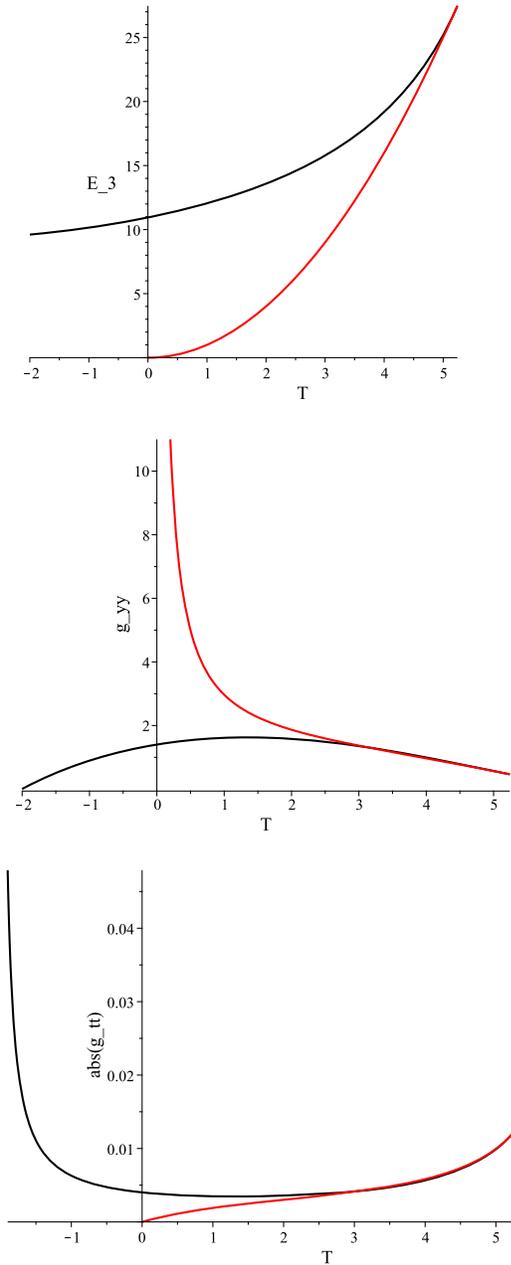


FIG. 9. $\theta \neq 0, \beta \neq 0$. Commutative (red) vs noncommutative (black) interior of a higher-genus black hole. *Top*: The triad component E_3 . *Middle*: The triad combination $(E_2)^2/E_3$, corresponding to g_{yy} . *Bottom*: N^2 , corresponding to $|g_{\tau\tau}|$. Notice from the top graph that the radius of the 2D subspaces in this particular case does *not* shrink to zero in the noncommutative case, indicating the removal of the singularity, but the evolution stops due to $g_{yy} \rightarrow 0$ and $|g_{\tau\tau}| \rightarrow \infty$, indicating the presence of another horizon. The parameters are $M = 1, \Lambda = -0.1, \gamma = 0.274, \theta = -0.05$, and $\beta = -0.1$.

The above analyses are somewhat complicated due to all the different possible scenarios. We therefore provide the following summary of all the possibilities:

(1) *Spherical topology*

- (a) $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ remove the singularity.

(b) $\theta = 0, \beta \neq 0$: Singularity is removed.

(c) $\theta \neq 0, \beta \neq 0$: Singularity is removed.

(2) *Toroidal topology*

(a) $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ introduce a new horizon.⁵

(b) $\theta = 0, \beta \neq 0$: Singularity is removed.

(c) $\theta \neq 0, \beta \neq 0$: Singularity is removed for large θ . For small θ a new horizon appears⁵.

(3) *Higher-genus topology*

(a) $\theta \neq 0, \beta = 0$: Small values of θ delay the singularity. Large values of θ introduce a new horizon⁵.

(b) $\theta = 0, \beta \neq 0$: Singularity is removed.

(c) $\theta \neq 0, \beta \neq 0$: Singularity is removed for large θ . For small θ a new horizon appears⁵.

IV. CONCLUDING REMARKS

In this manuscript, we studied the effects of noncommutative geometry on the interiors of black holes compatible with various topologies. The introduction of noncommutativity was performed in two ways. In the first part of the study, a smearing of the gravitating source was performed, mimicking the effects of the nonlocalization introduced by a nontrivial commutator between the space-time coordinates. It was found that this smearing was capable of removing the curvature singularity in all scenarios. This result, though, is not that surprising, as one has essentially forced a smoothness onto the system. The resulting matter system, known in the literature as “inspired by noncommutative geometry,” will violate the energy conditions in the T -domain, thus circumventing the results of the singularity theorems. However, it does hint at the fact that a possible resolution to the singularity issue lies in noncommutative geometry effects.

In the second part of the study, the Poisson algebra was directly altered by the introduction of a nontrivial bracket in (i) the configuration degrees of freedom only, (ii) the momentum degrees of freedom only, and (iii) both. It was found that for some cases the singularity is merely delayed, occurring later (earlier in coordinate time) than in the corresponding commutative scenario. However, in many cases, some rather interesting results emerge. Either the singularity is removed, or else a new inner horizon forms. In the case of a new horizon, the domain that we are able to study with the method here is also non-singular. Overall, the presence of the parameter β (nontrivial bracket between the triads) is more capable of singularity resolution than the parameter θ (nontrivial bracket between the connection). The results are summarized in at the end of Sec. III.

⁵The presence of the new horizon prevents us from determining whether there is singular structure beyond the second horizon.

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