Conjecture about the general cosmological solution of Einstein's equations

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We introduce consideration of a new factor, synchronization of spacetime mixmaster oscillations, that may play a simplifying role in understanding the nature of the general inhomogeneous cosmological solution to Einstein's equations. We conjecture that, on approach to a singularity, the interaction of spacetime mixmaster oscillations in different regions of an inhomogeneous universe can produce a synchronization of these oscillations through a coupling to their mean field in the way first demonstrated by the Kuramoto coupled oscillator model.

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I. INTRODUCTION

The search for an understanding of parts of the general inhomogeneous solution of the Einstein equations on approach to a cosmological singularity has been strongly influenced by the work of Belinskii, Khalatnikov, and Lifshitz (BKL) [1] on the evolution of the spatially homogeneous Bianchi type IX ("mixmaster") universe, first introduced by Misner [2]. The Einstein equations for this cosmological model are not integrable but clearly exhibit chaotic behavior in vacuum and with perfect fluids whose pressure and density obeys $p < \rho$, [3,4]. The smooth nonseparable invariant measure for its complete discrete dynamics has been solved exactly by Chernoff and Barrow [5] and simplifies to a double continued-fraction shift map [6,7]. The general behavior of the mixmaster model within the framework of spatially homogeneous cosmological solutions of Einstein's equations has led to claims that a general inhomogeneous solution looks like the spatially homogeneous type IX at the leading order in a series approximation, or in which the defining constants (four in vacuum) become independent slowly varying arbitrary functions of the three spatial variables. This proposal is not as straightforward as it sounds. We know that compact vacuum and radiation-dominated universes with Killing vectors, like Bianchi IX, are not linearization stable. This means that small perturbations around an exact spatially homogeneous solution of the defining equations are open dense in first-order series expansions that do not form the leading term of any series that converges to a true solution of those equations. This phenomenon is familiar in nonlinear dynamics [8,9]. As discussed in Ref. [10], Marsden, Fischer, Moncrief, and Arms [11–14] have proved that compact vacuum solutions of Einstein's equations with Killing symmetries have this subtle property, which is manifested in various other nonlinear systems. In fact, in the four-function space of the general cosmological solution, small open neighborhoods around the homogeneous type IX solution will be dense in spurious linearizations that are not approximations to a true inhomogeneous solution. The reason for this is that in the four-function space spanning cosmological vacuum solutions the points with Killing vectors are conical and so there are an infinite number of tangents that can be drawn through the conical point that represents the spatially homogeneous solution. Only those tangents that run down the sides of the cone correspond to linearizations of true solutions: the others form a dense set of spurious linearizations. The nonspurious perturbations must satisfy the Taub constraint [15] to ensure they lie down the sides of the cone. This situation has been examined in detail in the context of perturbations of the Einstein static universe in Refs. [16,17], although the latter paper makes no reference to linearization instability.

This situation alerts us to the possibility that the behavior of the inhomogeneous general solution might not just look like slightly different type IX models from place to place. Since the behavior of the type IX model is formally chaotic, it is hard to imagine how different locally chaotic regions are stitched together and remain so as the local oscillations in each get erratically out of synchronization. Other investigators have also drawn attention to additional unusual features, even in the homogeneous models, like "spikes" [18,19] caused by steep spatial gradients in other simpler inhomogeneous cosmologies, that were not part of the first BKL models. However, although these spikes do arise in many partial differential equations, they might not be generic [20].

Motivated by this situation, we propose one new phenomenon that might play a part in the general solution near Bianchi type IX and alleviate the "stitching problem" of joining different chaotic oscillatory regions,. This is described in Sec. II and is followed by concluding discussion and suggestions for further investigations in Sec. III.

II. SYNCHRONIZATION

The scenario suggested by BKL and adopted by other investigators is that on approach to the singularity different subregions of space will behave like separate type IX universes [21]. Let us imagine what might happen to one of those oscillating subregions on approach to the singularity. In vacuum, it will experience the collective oscillatory gravitational-wave perturbations of its neighboring regions. These perturbations might be expected to be out of phase and effectively random, but the strength of their effects on our single specimen subregion will grow in strength as the singularity is approached and each subregion responds to the mean field created by the oscillations of other regions.

This situation is familiar in many areas of science and has been well described by the famous Kuramoto model of synchronized oscillators [22–24]. The separate oscillators can become synchronized if the strength of coupling between different oscillators exceeds a critical value. They become synchronized because each responds to the mean field created by the oscillations. A simple familiar example is the hand-clapping of an audience. Clapping starts randomly, but if it strong enough, then soon everyone seems to be clapping in unison. A plethora of such examples are known, especially in the biological world. We propose that the same phenomenon occurs in an inhomogeneous general cosmological solution. Although different regions might seem like separate type IX universes, the coupling of their mixmaster oscillations enables them all to respond to the mean field and the oscillations should become synchronized. The critical coupling strength will inevitably be reached on approach to the singularity. Recall that on any open interval of mixmaster evolution around the singularity at $t = 0$ there are an infinite number of spacetime oscillations and they occur far faster than the rate of evolution of the mean volume (we ignore quantum gravitational effects).

The Bianchi IX vacuum equations in standard Hubble-normalized variables are [25–27]

$$N_1' = (q - 4\Sigma_+)N_1, \tag{1}$$

$$N_2' =(q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \tag{2}$$

$$N_3' =(q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \tag{3}$$

$$\Sigma_+' = -(2 - q)\Sigma_+ - 3S_+, \tag{4}$$

$$\Sigma_-' = -(2 - q)\Sigma_- - 3S_-, \tag{5}$$

where

$$q \equiv 2(\Sigma_+^2 + \Sigma_-^2), \tag{6}$$

$$S_+ \equiv \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)], \tag{7}$$

$$S_- \equiv \frac{\sqrt{3}}{2}[(N_3 - N_2)(N_1 - N_2 - N_3)], \tag{8}$$

with the constraint (the generalized vacuum Friedmann equation)

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}[N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_1N_3)] = 1. \tag{9}$$

Here, $\Sigma_+(\tau)$ and $\Sigma_-(\tau)$ are the dimensionless shear variables, $N_1(\tau)$, $N_2(\tau)$, and $N_3(\tau)$ define the Bianchi group structure and the anisotropic 3-curvature components. In Bianchi IX, the 3-curvature can change sign and is only positive when the dynamics are close to isotropy. The $'$ denotes differentiation with respect to a time coordinate τ, which is related to the comoving proper time by $dt/d\tau = 1/H$,where H is the mean Hubble expansion rate. Typically, $H \simeq 1/3t$ as Bianchi IX approaches the initial singularity at $t = 0$, but the ratios of the three expansion scale factors tend to infinity. Since $\tau \simeq \frac{1}{3}\ln(t)$, the initial singularity lies at $\tau = -\infty$. In the Friedmann universes, $\Sigma_+ = \Sigma_- = N_1 = N_2 = N_3 = 0$. However, note that this is not an exact solution of the constraint equation, (9), because there is no closed vacuum Friedmann universe. An axisymmetric, nonchaotic solution exists with $N_2 = N_3$ and $S_- = 0$, but this is not of interest for our discussion as it is nonoscillatory.

Using the constraint equation we have a 4-dimensional set of autonomous time-dependent ordinary differential oscillator equations of the form

$$x_i' = f(x_i), \qquad i = 1, \dots 4. \tag{10}$$

This simple form allows us to conjecture what the effects of interactions between different varieties of type IX dynamics in different places might be (in effect, allowing the Σ_+, Σ_- and N_1, N_2, N_3 to be both space and time variables).

We want to consider the effects of neighboring oscillatory regions on one particular subregion as the singularity is approached. The Kuramoto model imagines that N oscillatory cycles are interacting with a coupling strength that is the same for each pair of oscillatory regions. It creates the simplest possible setting for this problem and has turned out to have unexpectedly wide applications. The oscillators have natural frequencies, $\omega_j \epsilon(-\infty, \infty)$, and their phases are $\theta_j \epsilon[0, 2\pi]$. They are assumed to be coupled by the phase differences of the oscillators in the following simple way:

$$\frac{d\theta_j}{d\tau} = \omega_j + \frac{K}{N}\sum_{k=1}^{N} \sin\{\theta_k(\tau) - \theta_j(\tau)\}, \tag{11}$$

where the coupling constant is $K \geq 0$. The coherence of the phases, and the mean field created by the coupled oscillators, is conveniently measured by the complex order parameter, defined by

$$r\exp[i\psi] \equiv \frac{1}{N}\sum_{j=1}^{N}\exp[i\theta_j(\tau)], \tag{12}$$

where $|r|$ measures the degree of coherence: perfect synchronization occurs when $|r| = 1$ and perfect incoherence, with the θ_j uniformly distributed on $[0, 2\pi)$, occurs when $|r| = 0$; ψ is the average phase.

In terms of r, the Eqs. (11) now become

$$\frac{d\theta_j}{d\tau} = \omega_j + K\,\mathrm{Im}(r(\tau)\exp[i\theta_j(\tau)]). \tag{13}$$

This shows how the instantaneous mean field of all the oscillators leads to the evolution of the phases of the oscillators. The crucial effect of the phase couplings is that as K is allowed to increase a critical transition occurs when $K = K_{cr}$: when $K > K_{cr}$ the oscillator frequencies start to become synchronized by virtue of their common responses to the perturbations by the mean field, although their phases can be different. Kuramoto showed that

$$r = \sqrt{1 - \frac{K_c}{K}}, \tag{14}$$

for all $K > K_{cr}$ [28].

We suggest that a similar effect occurs in an inhomogeneous generalization of the type IX universe, if it exists. The effects of many separate local regions, each undergoing mixmaster oscillations, can synchonize the mixmaster oscillations of each. The effective coupling will always grow stronger on approach to the singularity at $\tau = -\infty$ and a critical coupling, K_{crit}, will always be passed so long as the dynamics are not cut off and replaced by a different quantum cosmological behavior at and below the Planck scale—only a finite number of mixmaster oscillations will then occur and K_{crit} will probably not be reached. On moving away from the singularity, the coupling will decline and synchronization will eventually break down. There is an indication that synchronized behavior will arise in the dynamics because in general the shear and Weyl curvature components will oscillate indefinitely on approach to the singularity [29]. Consider the dimensionless shear variables, Σ_- and Σ_+, in Eqs. (4) and (5) and put

$$\Sigma_+ = \rho\cos\theta, \qquad \Sigma_- = \rho\sin\theta. \tag{15}$$

Then, Eq. (9) determines the evolution of the new variable ρ, while Eqs. (4) and (5) show that the angular variable, θ, satisfies

$$\frac{d\theta}{d\tau} = \frac{3}{\rho}(S_+\sin\theta - S_-\cos\theta), \tag{16}$$

that is,

$$\frac{d\theta}{d\tau} = \frac{3}{\rho}\sqrt{S_+^2 + S_-^2}\sin(\theta - \psi), \tag{17}$$

where

$$\tan\psi = \frac{S_-}{S_+}. \tag{18}$$

A comparison with Eq. (11) shows that the interaction between the phases θ and ψ suggests a Kuramoto system of equations leading to synchronization. The behavior is more complicated than the basic Kuramoto model because the coupling is now τ dependent (see Ref. [30]).

III. DISCUSSION

Our discussion is of a possible toy model for couplings between different mixmaster oscillations. It is a possible effect that has not been considered previously in attempts to model an inhomogeneous BKL scenario by expansions around the spatially homogeneous mixmaster model. It is a picture that has proven to have very wide applicability in the study of interacting oscillators in ways that are not specific to individual details of the physics being modeled. We hope that it will stimulate investigations of new effects in the general cosmological solutions of the Einstein equations. There are several obvious features that can be made more realistic. The constants ω_j and K in Eq. (11) can be made time dependent or stochastic with external forcing. Studies of various time-dependent Kuramoto dynamics have been considered in Ref. [30]. A time delay might also be introduced to account for gravitational-wave propagations, as in Ref. [31] and a Hamiltonian formulation may be more suited for a general relativistic application, see [32]. We have discussed only the vacuum solution but the situation in models with pressure and density such that $p < \rho$ will be similar. In the $p = \rho$ case the chaotic oscillations always die out on approach to the singularity and the synchronization will probably never begin, and very soon end if it does. The mixmaster oscillations inside a long era where two scale factors oscillate approximate to periodic sine and cosine oscillations of the scale factors when the number of oscillations is very large, but becomes doubly periodic for smaller numbers of oscillations [33,34]. This will introduce other features of the oscillator couplings which will be reported on elsewhere.

In conclusion, we have introduced consideration of a new factor, synchronization of spacetime mixmaster oscillations, that may play a simplifying role in understanding the nature of the general inhomogeneous cosmological solution to Einstein's equations.

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