Geodesic motion in Bogoslovsky-Finsler spacetimes

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We study the free motion of a massive particle moving in the background of a Finslerian deformation of a plane gravitational wave in Einstein's general relativity. The deformation is a curved version of a one-parameter family of relativistic Finsler structures introduced by Bogoslovsky, which are invariant under a certain deformation of Cohen and Glashow's very special relativity group ISIM(2). The partially broken Carroll symmetry we derive using Baldwin-Jeffery-Rosen coordinates allows us to integrate the geodesics equations. The transverse coordinates of timelike Finsler geodesics are identical to those of the underlying plane gravitational wave for any value of the Bogoslovsky-Finsler parameter b. We then replace the underlying plane gravitational wave with a homogeneous pp-wave solution of the Einstein-Maxwell equations. We conclude by extending the theory to the Finsler-Friedmann-Lemaître model.

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I. INTRODUCTION

Our present fundamental physical theories are based on local Lorentz invariance, and hence on local isotropy. This leads naturally to the introduction of pseudo-Riemannian geometry and its associated metric tensor. It has long been known, however, that a "principle of relativity" can be made compatible with anisotropy by deforming the Lorentz group by the inclusion of dilations [1] (although the experiments of Hughes and Drever indicate that the anisotropy must be very weak [2]).

Currently, there is also a great deal of activity in exploring the astrophysics and cosmology of alternative gravitational theories based on standard Lorentzian geometry. Laboratory tests of local Lorentz invariance are very well developed and have reached impressive levels of precision.

Riemann himself envisaged more general geometries. An elegant construction combining these ideas was provided some time ago by Bogoslovsky [3,4] (for more recent accounts, see Ref. [5]). In what is now known as Finsler geometry, the line element is a general homogeneous function of degree 1 in displacements, rather than the square root of a quadratic form.

The theory proposed by Bogoslovsky, which is the main subject of interest of this paper, has turned out to be relevant for attempts to accommodate a proposal of Cohen and Glashow [1], accounting for weak CP violation in the standard model of particle physics, in the gravitational background [6,7].

The first significant application of Finsler geometry to physics is due to Randers [8], who pointed out that the world line of a particle of mass m and electric charge e extremizes the action

$$S_0 = \int \mathcal{L}_0 d\lambda = -\int m\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} + eA_\mu dx^\mu, \qquad (1.1)$$

where λ is an arbitrary parameter and A_μ is the electromagnetic potential. Randers applied this idea to Kaluza-Klein theory. Further studies followed [9–12]; it has also been applied to the gravitomagnetic effects occurring in stationary spacetimes [13]. For more recent work on Finsler spaces, see Refs. [14–24]. Null geodesics and causality are considered in particular in Ref. [24].

The aim of the present paper is to contribute to the physical applications of the Finslerian generalization of general relativity by exploring the motion of freely moving massive particles in the background of Bogoslovsky-Finsler deformations of plane gravitational waves and spatially flat Friedmann-Lemaître cosmologies.

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II. FINSLER SPACES

In this section, we shall briefly summarize some earlier results [9–12]. An excellent general reference to Finsler geometry used by these authors is Ref. [25], to which we refer the reader for more details of the general theory. If $F(x^\mu, y^\nu)$ is a Finsler function,[1] then it is assumed that $F^2(x,y)$ may be written such that [9]

$$F^2(x,y) = \mathcal{F}(x,y) g_{\mu\nu} y^\mu y^\nu, \tag{2.1}$$

where $\mathcal{F}(x,y)$ is a positive function which is positively homogeneous in the velocities. Moreover [9], if $H(x)_{\alpha_1,\alpha_2,\ldots\alpha_N}$ is a totally symmetric tensor of rank N which is covariantly constant with respect to the Levi-Civita covariant derivative of the metric $g_{\mu\nu}$, then if

$$\omega = g(x)_{\mu\nu} y^\mu y^\nu / (H_{\alpha_1,\alpha_2,\ldots,\alpha_N} y^{\alpha_1} y^{\alpha_2} \cdots y^{\alpha_N})^{2/N} \quad \text{and}$$
$$\mathcal{F}(x,y) = \mathcal{F}(\omega), \tag{2.2}$$

then the set of Finsler geodesics of F and the set of standard Riemannian geodesics of $g_{\mu\nu}$ coincide [see Eq. (20) in Ref. [9]]. Further aspects are considered in Refs. [22,23].

The case of Finsler pp-waves [15,16] occurs when H is a covariantly constant null covector and $g_{\mu\nu}$ is the metric of a pp-wave, a special case of which is a plane gravitational wave. This was in effect pointed out by Tavakol and Van den Bergh [12] in 1986 and elaborated and extended by Roxburgh in 1991. Bogoslovky's original flat Finsler metric [3,4] is a special case of their work, but no mention of Bogoslovsky is made in Refs. [9,11,12], and so one assumes that they were unaware of it.

We next recall some basic definitions and notation used in Refs. [9,11,12]. Given a Finsler function $F(x,y)$, one may define the *Finsler metric tensor*

$$f_{\mu\nu}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^\mu \partial y^\nu}, \tag{2.3}$$

which is homogeneous of degree 0 in y^μ. That is, $f_{\mu\nu}(x,y)$ depends only upon the *direction*. Differentiating the identity $F^2(x^\alpha, \lambda y^\mu) = \lambda^2 F(x^\alpha, y^\mu)$ twice with respect to λ implies that

$$f_{\mu\nu} y^\mu y^\nu = F^2(x,y). \tag{2.4}$$

The Finsler line element or arc length ds along a curve γ with tangent vector $y^\mu = \frac{dx^\mu}{d\lambda}$ is given by

$$ds^2 = F^2(x^\mu, dx^\mu) = f_{\mu\nu}(x,y) dx^\mu dx^\nu, \tag{2.5}$$

[1] See Refs. [9,11,12,25]; y^μ is a four-velocity, and (x^μ, y^μ) are local coordinates on TM, the tangent bundle of the spacetime manifold M.

and a Finsler geodesic is one for which $\delta \int_\gamma F(x, dx^\mu) = \delta \int_\gamma ds = 0$. The Euler-Lagrange equations are

$$\frac{d^2 x^\mu}{ds^2} + \gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{2.6}$$

where

$$\gamma_{\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial f_{\kappa\nu}}{\partial x^\mu} + \frac{\partial f_{\mu\kappa}}{\partial x^\nu} - \frac{\partial f_{\mu\nu}}{\partial x^\kappa}\right), \qquad \gamma^\mu_{\nu\kappa} = f^{\mu\sigma}\gamma_{\nu\sigma\kappa} \tag{2.7}$$

are the analogues of *Christoffel symbols of the first and second kind*, respectively. In deriving the Euler-Lagrange equations, one uses the fact that $y^\kappa \frac{\partial f_{\alpha\beta}}{\partial y^\kappa} = 0$ because $f_{\mu\nu}$ is homogeneous of degree 0 in y^μ. Evidently, under a change of parameter $s \to \lambda = \lambda(s)$, we have $\frac{d}{ds} = \lambda' \frac{d}{d\lambda}$, $f_{\mu\nu} \to f_{\mu\nu}$, since $f_{\mu\nu}$ is homogeneous of degree 0 in velocities. Thus, as in the standard Lorentzian situation,

$$\frac{d^2 x^\mu}{d\lambda^2} + \gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -\frac{\lambda''}{\lambda'}\frac{dx^\mu}{d\lambda}. \tag{2.8}$$

If $\lambda'' = 0$, λ is called an affine parameter, and in what follows, unless otherwise stated, λ will denote an affine parameter.

In Refs. [9,11,12], the quantities

$$G^\mu = \frac{1}{2}\gamma^\mu y^\nu y^\kappa, \qquad G^\mu_{\nu\kappa} = \frac{\partial^2 G^\mu}{\partial y^\nu \partial y^\kappa} \tag{2.9}$$

are introduced. Although in general $G^\mu_{\nu\kappa} \neq \gamma^\mu_{\nu\kappa}$, by virtue of the homogeneity of degree 0 of $\gamma^\mu_{\nu\kappa}$ in y^μ, one has

$$G^\mu_{\nu\kappa} y^\nu y^\kappa = \gamma^\mu_{\nu\kappa} y^\nu y^\kappa, \tag{2.10}$$

and therefore Euler-Lagrangian equations may be rewritten as

$$\frac{d^2 x^\mu}{d\lambda^2} + G^\mu_{\nu\kappa}\frac{dx^\nu}{d\lambda}\frac{dx^\kappa}{d\lambda} = 0. \tag{2.11}$$

In general, ${G^\mu}_{\nu\kappa}$ depends upon the direction [12]; a *Berwald-Finsler manifold* is one for which ${G^\mu}_{\nu\kappa}$ is independent of the direction—i.e.,

$$G^\mu_{\nu\kappa} = G^\mu_{\nu\kappa}(x). \tag{2.12}$$

The motivation for Refs. [9,11,12] came from a classic paper of Ehlers, Pirani, and Schild [26] examining the fundamental assumptions justifying the use of pseudo-Riemannian geometry adapted in Einstein's general relativity. Roughly speaking, the idea was that

(1) *The principle of the universality of free fall* endows spacetime $\mathcal{M}$ with a *projective structure*—that is, an

equivalence class of curves, $\gamma : \lambda \in \mathbb{R} \to \mathcal{M}$—up to reparametrization.

(2) *The principle of Einstein causality* endows spacetime with a causal structure such that light rays are determined by some connection.

They conjectured that the only way of achieving this was that freely falling particles and null rays follow the geodesics of a pseudo-Riemannian metric, and in the case of particles, that the curves carry a privileged parametrization given by proper time with respect to the pseudo-Riemannian metric along their paths in spacetime, whether freely falling or not.

In Refs. [11,12], Tavakol and Van den Bergh sought to show that one could pass to a Finsler structure as well, provided one assumes

(1) {Ai}

$$F^2(x, y) = e^{2\sigma(x,y)} g_{\mu\nu}(x) y^\mu y^\nu, \qquad (2.13)$$

where $g_{\mu\nu}(x)$ is a Lorentzian metric and $\sigma(x^\alpha, y^\mu)$ is homogeneous of degree 0 in y^μ.

This condition ensures that the conformal structures of the Finsler metric and the Lorentzian metric agree locally, in the spirit of Ref. [26]. It is pointed out in Ref. [12] that Eq. (2.13) is not equivalent to

$$f_{\mu\nu} = e^{2\sigma(x,y)} g_{\mu\nu}, \qquad (2.14)$$

because it this were true, then σ would only depend upon x, and hence $f_{\mu\nu}$ and $g_{\mu\nu}$ would be conformally related, citing Ref. [25].[2]

(2) {Aii}

$$G^\mu_{\nu\kappa} = \left\{ \begin{matrix} \mu \\ \nu\kappa \end{matrix} \right\}, \qquad (2.15)$$

where $\left\{ \begin{smallmatrix} \mu \\ \nu\kappa \end{smallmatrix} \right\}$ are the Christoffel symbols of the Lorentzian metric $g_{\mu\nu}$. This condition ensures that the projective structures of the Finsler structure $F(x, y)$ and the Lorentzian structure $g_{\mu\nu}$ agree locally, again in the spirit of Ref. [26].

Tavakol and Van den Bergh [12] claimed that the necessary and sufficient condition on $\sigma(x, y)$ is

$$\frac{\partial\sigma}{\partial x^\mu} - y^\nu \frac{\partial\sigma}{\partial y^\kappa} \left\{ \begin{matrix} \kappa \\ \mu\nu \end{matrix} \right\} = 0 \qquad (2.16)$$

and referred to it as the *metricity condition*. The name originates in the theory of the so-called *Cartan connection*. One defines

$$C_{\mu\nu\kappa} = \frac{1}{2} \frac{\partial f_{\mu\nu}}{\partial y^\kappa}, \qquad (2.17)$$

which is from Eq. (2.3), as totally symmetric in μ, ν, κ. Then one defines

$$\Gamma_{\mu\nu\kappa} = \gamma_{\mu\nu\kappa} - \left(C_{\sigma\kappa\nu} \frac{\partial G^\sigma}{\partial y^\mu} + C_{\sigma\kappa\mu} \frac{\partial G^\sigma}{\partial y^\nu} - C_{\mu\sigma\nu} \frac{\partial G^\sigma}{\partial y^\kappa} \right). \qquad (2.18)$$

Acting on a vector $W^\mu(x, y)$, the Cartan covariant derivative is defined by

$$\nabla^{\text{Cartan}}{}_{,\kappa} W^\mu = \frac{\partial W^\mu}{\partial x^\kappa} + -y^\sigma \Gamma^\lambda_{\sigma\kappa} \frac{\partial W^\mu}{\partial y^\kappa} + \Gamma^\mu_{\nu\sigma} W^\sigma \qquad (2.19)$$

and extended to tensors of arbitrary valence in the obvious way. The Cartan connection satisfies

$$\nabla^{\text{Cartan}}_\kappa f_{\mu\nu} = 0. \qquad (2.20)$$

This is equivalent to

$$\frac{\partial F^2}{\partial x^\kappa} - \frac{\partial F^2}{\partial y^\sigma} \frac{\partial G^\sigma}{\partial y^\kappa} = 0 \qquad (2.21)$$

and ensures that the norms of the vector remain constant under parallel transport along different routes. In Ref. [9], it is written as

$$\frac{\partial F}{\partial x^\kappa} - \frac{\partial F}{\partial y^\sigma} \frac{\partial G^\sigma}{\partial y^\kappa} = 0. \qquad (2.22)$$

In Ref. [12], it was suggested that

$$g_{\mu\nu} x^\mu dx^\nu = -2du dv + \alpha(u) dx^2 + \beta(u) dy^2, \qquad (2.23)$$

with $(x^1, x^2, x^3, x^4) = (x, y, u, v)$, which they call a plane wave, might lead to a solution. They find [in their Eq. (34)] that

$$\sigma = \sigma\left(\frac{\alpha \dot{x}^2 + \beta \dot{y}^2 - 2\dot{u}\,\dot{v}}{\dot{u}^2} \right), \qquad (2.24)$$

and they claim that it is indeed a solution.

The treatment of Ref. [9] starts with the helpful observation that the sums, products, and ratios of solutions are again solutions. Roxburgh investigated Lorentzian metrics with covariantly constant vector fields and pointed out that pp-waves are a special case.

III. BOGOSLOVSKY-FINSLER-F METRICS

Bogoslovsky's theory [3,4] was based on the Finsler line element such that the proper time τ along a future-directed *timelike* world line $x^\mu(\tau)$ in flat Minkowski spacetime is obtained by combining the Minkowski line element with what we call here the Bogoslovsky factor,

[2]In fact, Eq. (2.13) is obviously equivalent to Eq. (2.1), which is the form used by Ref. [9] (who appears to regard it as always true), although $g_{\mu\nu}$ is not necessarily unique.

$$d\tau = (-\eta_{\mu\nu} dx^\mu dx^\nu)^{\frac{1-b}{2}} (-\eta_{\mu\nu} l^\mu dx^\nu)^b, \tag{3.1}$$

where $0 \leq b < 1$ is a dimensionless constant, $\eta_{\mu\nu}$ is the flat Minkowski metric tensor (with a mainly positive signature), and l^μ is a constant future-directed null vector—i.e.,

$$\partial_\mu l^\nu = 0, \qquad \eta_{\mu\nu} l^\mu l^\nu = 0, \qquad l^0 > 0, \tag{3.2}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The Bogoslovsky factor makes Eq. (3.1) homogeneous of degree 1—i.e., Finslerian.

The parameter b introduces spatial anisotropy which might be relevant at the early stages of the Universe [27]. The constant b is very small by Hughes-Drever-type experiments [2]; Bogoslovsky argues that $b < 10^{-10}$ [28]. For $b = 0$, we recover the Minkowski proper time element, cf. Eq. (1.1) with $A_\mu = 0$.

Bogoslovsky's Finsler line element has an obvious generalization: in Eq. (3.1), one replaces $\eta_{\mu\nu}$ with a curved pseudo-Riemannian metric $g_{\mu\nu}(x)$ and l^μ with a future-directed null vector such that

$$\nabla_\mu l^\nu = 0, \qquad g_{\mu\nu} l^\mu l^\nu = 0, \tag{3.3}$$

where ∇_μ is the Levi-Civita connection of the Lorentzian metric $g_{\mu\nu}$. This idea has recently been explored in Refs. [15–17,21], where such spacetimes are called "Finsler pp-waves."

Such spacetimes are also referred to as Brinkman [29] or Bargmann [30] spacetimes. They admit Brinkmann coordinates $X^\mu = (V, U, X^i)$ such that

$$g_{\mu\nu} dX^\mu dX^\nu = 2dVdU + dX^i dX^i - 2H(X^i, U) dU^2, \tag{3.4}$$

where the spacetime dimension is $d+1$ and $i = 1, 2, \ldots, d-1$. $H(X^i, U)$ is an arbitrary, not identically vanishing function of its arguments.[3] U, V may be written as

$$V = X^- = \frac{1}{\sqrt{2}}(X^d - X^0), \qquad U = X^+ = \frac{1}{\sqrt{2}}(X^d + X^0). \tag{3.5}$$

We have $l^\mu \partial_\mu = -\partial_V$ so that $-g_{\mu\nu} l^\mu dX^\nu = dU$. Then the Finsler pp-line element is

$$\boxed{d\tau = (-g_{\mu\nu} dX^\mu dX^\nu)^{\frac{1-b}{2}} (-g_{\mu\nu} l^\mu dX^\nu)^b = (-g_{\mu\nu} dX^\mu dX^\nu)^{\frac{1-b}{2}} (dU)^b,} \tag{3.6}$$

where $g_{\mu\nu}$ is the pp-wave metric.

Returning to the pp-waves, we recall that the metric is Ricci flat if and only if $H(X^i, U)$ is a harmonic function of the coordinates X^i; it may, however, have arbitrary dependence upon U. It then represents a left-moving (i.e., in the negative X^d direction) gravitational wave such that X^i's are transverse to the direction of motion. The wave fronts $U =$ constant are null hypersurfaces, and the covariantly constant and hence Killing null vector field ∂_V lies in the wave fronts.

If, in addition, $H(X^i, U)$ is quadratic in the transverse coordinates, then we have a *plane gravitational wave*. If $d = 3$, which we assume from now on, then

$$-2H = \mathcal{A}_+(U)(X_1^2 - X_2^2) + \mathcal{A}_\times(U) 2X_1X_2 = K_{ij}(U) X^i X^j, \tag{3.7}$$

where $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$ are the amplitudes of the two plane polarization states.

For general $\mathcal{A}_+(U)$ and $\mathcal{A}_\times(U)$, there is a five-dimensional isometry group G_5 which acts multiply transitively on the three-dimensional wave fronts $U =$ constant [31,32]. This group is a subgroup of the six-dimensional Carroll group Carr(2) in three spacetime dimensions [33,34] in which the SO(2) subgroup is omitted [35].

The Carroll group Carr(2) may be regarded as a subgroup of the Poincaré group ISO(3, 1) defined by freezing out U-translations [34]; it acts on the null hyperplanes $U =$ constant. If we label the Killing vector fields of the Poincaré group as

$$P_\mu = \frac{\partial}{\partial X^\mu}, \qquad L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu, \tag{3.8}$$

then the Carroll group is generated by

$$P_- = \frac{\partial}{\partial V}, \qquad P_i = \frac{\partial}{\partial X^i}, \tag{3.9a}$$

$$L_{ij} = X_i P_j - X_j P_i, \qquad L_{-i} = X_- P_i - X_i P_- = U P_i - X_i P_-, \tag{3.9b}$$

$i = 1, 2$, and each hyperplane $U =$ const. is left variant. The generators in Eq. (3.9) are translations, whereas those in Eq. (3.9a) are planar rotation and boosts. Since $d = 3$, we may relabel the generators $L_{ij} = J$, and the $U - V$ boost

[3]Our choice of sign for g_{UV} has the advantage that raising and lowering the indices entails no minus signs—merely swapping U and V—and is consistent with our previous papers. It has, however, the consequence that if we choose a time orientation such that U increases to the future, then V decreases to the future. In other words, $\frac{\partial}{\partial U}$ is a future-directed null vector field and $\frac{\partial}{\partial V}$ is a past-directed vector field.

$$N_0 = L_{+-} = X_+ P_- - X_- P_+ = V P_- - U_- P_+, \quad \text{where}$$
$$P_+ = \frac{\partial}{\partial U}, \tag{3.10}$$

and find that the four generators N_0, J, L_{-i} generate a group which is abstractly isomorphic to the group SIM(2), the group of similarities—that is, dilations, rotations, and translations of the Euclidean plane $\mathbb{E}^2$. SIM(2) is the largest proper subgroup of the Lorentz group SO(3, 1). Adjoining the generators P_+, P_-, P_i gives rise to the eight generators of ISIM(2), which is a subgroup of the Poincaré group. The group ISIM(2) sacts multiply transitively on Minkowski spacetime $\mathbb{E}^{3,1}$.

It was suggested by Cohen and Glashow [1] that ISIM(2), which may be thought of as the subgroup of ISO(3, 1) leaving invariant a null direction, could explain weak *CP* violation while being compatible with tests of Lorentz invariance, since it would rule out *spurions*—that is, tensor vacuum expectation values.

In Ref. [36], it was pointed out that Ricci flat pp-waves are *strongly universal*. In particular, they have nonvanishing scalar invariants constructed from the Riemann tensor and as a consequence satisfy almost any set of covariant field equations. Quantum corrections to the metric vanish. Thus, this property may be thought of as the analogue for the proposed curved Bogoslovsky-Finsler structures with a Ricci flat metric $g_{\mu\nu}$ of Cohen and Glashow's "no spurions" condition.

In Ref. [6], an attempt was made to find a link with general relativity in which Minkowski spacetime $\mathbb{E}^{3,1}$ may be regarded as the coset ISO(3, 1)/SO(3, 1). The only two deformations of the Poincaré group led to the two de Sitter groups SO(4, 1) and SO(3, 2) for which translations act in a noncommutative fashion on the cosets' de Sitter spacetime $dS_4 = \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$ and anti–de Sitter spacetime $\mathrm{AdS}_4 = \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.

They therefore investigated the deformations of ISIM(2) and found that there exists a family of deformations depending upon two dimensionless parameters a and b. However, for all a and b, the translations P_+, P_-, P_i failed to commute. In general, the rotation J became a noncompact generator unless $a = 0$, leaving $\mathrm{DISIM}_b(2)$ depending on a dimensionless parameter b. They then observed that this is precisely the symmetry of Bogoslovsky's Finsler metric [Eq. (3.1)]. For a review of these ideas and their relation to the much earlier work of Voigt [37], the reader is directed to the recent review in Ref. [38]. For a recent discussion of Bogoslovsky-Finsler deformations in the light of the ideas of Segal, see Ref. [39].

In a recent paper (Ref. [15]), the authors have shown, among other things, that the Bogoslovsky-Finsler pp-waves enjoy the same universal properties with respect to generalizations of the Einstein equations to Finsler-Einstein equations as those in the pseudo-Riemannian case discussed in Ref. [36].

IV. GEODESICS

The geodesics of a Finsler metric with Finsler function $F(x^\mu, \dot{x}^\mu)$, where F is homogeneous of degree 1 in $\dot{x}^\mu$, are extrema of

$$I = \int F\left(x^\mu, \frac{dx^\mu}{d\lambda}\right) d\lambda. \tag{4.1}$$

In the case we are considering, we restrict our attention to future-directed timelike curves for which both $g_{\mu\nu} l^\mu \dot{x}^\mu$ and $-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ are strictly positive in order to ensure that F is real. For a particle of mass m, the action with respect to Lagrangians is

$$S_b = -m \int F d\lambda, \tag{4.2}$$

where F is the Bogoslovsky-Finsler-F line element [Eq. (3.6)]. The integral is independent of the parameter λ. Therefore, if $p_\mu = \frac{\partial(-mF)}{\partial \dot{x}^\mu}$, then $\mathcal{H} = p_\mu \dot{x}^\mu + mF$ is a constant of the motion. This is indeed true, but because $F(x^\mu, \dot{x}^\mu)$ is homogeneous of degree 1 in $\dot{x}^\mu$, one has $\dot{x}^\mu \frac{\partial F}{\partial \dot{x}^\mu} = F$, and consequently the constant vanishes identically. Standard Riemannian or Lorentzian metrics are, of course, a special case of this general fact.

Now we analyze the motion along geodesics in Bogoslovsky-Finsler plane gravitational waves, adapting the discussion for the standard Einstein case given in Refs. [35,40]. First, we find it convenient to pass to Baldwin-Jeffery-Rosen (BJR) coordinates $x^\mu = (v, u, x^i)$, defined by

$$X^i = P_{ij} x^j, \qquad U = u, \qquad V = v - \frac{1}{4}\frac{da_{ij}}{du} x^i x^j, \tag{4.3}$$

where $a \equiv (a_{ij}) = P^t P$, and the matrix P satisfies the matrix Sturm-Liouville equation

$$\frac{d^2 P}{du^2} = KP, \qquad P^t \frac{dP}{du} = \frac{dP^t}{du} P, \tag{4.4}$$

where $K = (K_{ij})$ is the profile in Brinkmann coordinates; see Eq. (3.7).

In BJR coordinates, we have

$$g_{\mu\nu} dx^\mu dx^\nu = 2du dv + a_{ij}(u) \dot{x}^i \dot{x}^j, \qquad l^\mu \frac{\partial}{\partial x^\mu} = -\frac{\partial}{\partial v}. \tag{4.5}$$

Here $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, where λ is an arbitrary parameter. The Lagrangian is proportional to the Bogoslovsky-Finsler function:

$$\boxed{\mathcal{L}_b = -mF, \quad \text{where } F = (-2\dot{u}\,\dot{v} - a_{ij}(u)\dot{x}^i \dot{x}^j)^{\frac{1}{2}(1-b)} (\dot{u})^b}. \tag{4.6}$$

For the curve to be timelike, we must have $\dot{u}\,\dot{v} < 0$ and $\dot{u} > 0$. Since the integral (4.1) is independent of the choice of the parameter λ, we are entitled to make the choice $\lambda = u$ and extremize

$$\int \left(-2\frac{dv}{du} - a_{ij}(u)\frac{dx^i}{du}\frac{dx^j}{du}\right)^{\frac{1}{2}(1-b)} du. \qquad (4.7)$$

With this choice of parametrization, the integrand of (4.7) is now no longer homogeneous in the velocities $\frac{dv}{du}$ and $\frac{dx^i}{du}$, but because a_{ij} depends on the "time" u, there is no conserved analogue of the quantity $\mathcal{H}$. The symmetry aspects will be further investigated in Sec. V.

Before analyzing the general case, we recall, for later comparison, some aspects of the geodesics of a pp-wave described by the square-root Lagrangian [Eq. (1.1)].

A. Geodesics in a *pp*-wave

Let us thus first consider a pp-wave written in BJR coordinates, whose geodesics are described by Eq. (1.1) with $A_\mu = 0$:

$$\mathcal{L}_0 = -m\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -m\sqrt{-a_{ij}(u)\dot{x}^i\dot{x}^j - 2\dot{u}\,\dot{v}}. \qquad (4.8)$$

The canonical momenta $p_\mu = \frac{\partial\mathcal{L}_0}{\partial\dot{x}^\mu}$ are

$$p_u = \frac{m\dot{v}}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}, \qquad p_i = \frac{ma_{ij}\dot{x}^j}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}, \qquad p_v = \frac{m\dot{u}}{\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}, \qquad (4.9)$$

of which p_i and p_v are constants of the motion, since $a_{ij} = a_{ij}(u)$. For a u-dependent profile, p_u is not conserved, though. The geodesic equations of motion are

$$\ddot{u} = \dot{u}\frac{d}{d\lambda}\ln\left(\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right), \qquad (4.10a)$$

$$\ddot{x}^i + \dot{u}a^{ij}a'_{jk}\dot{x}^k = \dot{x}^i\frac{d}{d\lambda}\ln\left(\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right), \qquad (4.10b)$$

$$\ddot{v} - \frac{1}{2}a'_{ij}\dot{x}^i\dot{x}^j = \dot{v}\frac{d}{d\lambda}\ln\left(\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\right), \qquad (4.10c)$$

where $a'_{ij} = \frac{da_{ij}}{du}$. Using the first equation, the two remaining ones simplify to[4]

$$\ddot{x}^i + \dot{u}a^{ij}a'_{jk}\dot{x}^k = \dot{x}^i\frac{\ddot{u}}{\dot{u}}, \qquad (4.11a)$$

$$\ddot{v} - \frac{1}{2}a'_{ij}\dot{x}^i\dot{x}^j = \dot{v}\frac{\ddot{u}}{\dot{u}}. \qquad (4.11b)$$

An ingenious way to solve these equations is to use the conserved quantities. We first define the constants of the motion by setting

$$P_i = \frac{p_i}{p_v} = \frac{a_{ij}\dot{x}^j}{\dot{u}}. \qquad (4.12)$$

The resulting first-order differential equation for x^i is solved at once as

$$x^i(u) = S^{ij}(u)P_j + x_0^i, \qquad (4.13)$$

where $S \equiv S^{ij}$ is the Souriau matrix [35], defined by

$$\frac{dS(u)}{du} = a^{-1}(u). \qquad (4.14)$$

p_v in Eq. (4.9) provides us in turn with a first-order equation for v:

$$\dot{v} = -\frac{1}{2}a^{ij}P_iP_j\dot{u} - \frac{1}{2}\mu_0^2\dot{u}, \quad \text{where } \mu_0 = \frac{m}{p_v}. \qquad (4.15)$$

This equation is then solved as

$$v = -\frac{1}{2}P_iP_jS^{ij}(u) - \frac{1}{2}\mu_0^2u + v_0. \qquad (4.16)$$

The transverse motion (4.13) is the same for all values of the mass m, which enters only the v motion (4.16) by a shift which is linear in u and proportional to the mass-quotient term μ_0 in Eq. (4.15), familiar from Ref. [41].

B. Finsler geodesics

Let us now consider the Bogoslovsky-Finsler-F Lagrangian $\mathcal{L}_b$ in Eq. (4.6). Its canonical momenta are

$$p_u = m(\dot{u})^{b-1}(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{-\frac{1+b}{2}}((1+b)\dot{u}\,\dot{v} + ba_{ij}\dot{x}^i\dot{x}^j), \qquad (4.17a)$$

$$p_i = m(1-b)(a_{ij}\dot{x}^j)\dot{u}^b(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{-\frac{1+b}{2}}, \qquad (4.17b)$$

$$p_v = m(1-b)\dot{u}^{b+1}(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{-\frac{1+b}{2}}. \qquad (4.17c)$$

p_i and p_v are constants of the motion as before, and we have the dispersion relation in Eq. (18) of Ref. [6]:

$$p^2 \equiv g^{\mu\nu}p_\mu p_\nu = -m^2(1-b^2)\dot{u}^{2b}(-2\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{-b}. \qquad (4.18)$$

[4]Choosing the affine parameter $\lambda = u$, the rhs would vanish.

The geodesic equations,

$$(b+1)\ddot{u} = \dot{u}\frac{d}{d\lambda}\ln(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{\frac{1+b}{2}}, \tag{4.19a}$$

$$\ddot{x}^i + \dot{u}a^{ij}a'_{jk}\dot{x}^k + b\frac{\ddot{u}}{\dot{u}}\dot{x}^i = \dot{x}^i\frac{d}{d\lambda}\ln(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{\frac{1+b}{2}}, \tag{4.19b}$$

$$\ddot{v} + \frac{3b-1}{2(1+b)}a'_{ij}\dot{x}^i\dot{x}^j + \frac{2b}{\dot{u}(1+b)}a_{ij}\dot{x}^i\dot{x}^j = \frac{1}{b+1}\left(\dot{v} + \frac{2b}{1+b}a_{ij}\dot{x}^i\dot{x}^j\right)\left(\frac{d}{d\lambda}\ln(-2\dot{u}\,\dot{v} - a_{ij}\dot{x}^i\dot{x}^j)^{\frac{1+b}{2}}\right), \tag{4.19c}$$

reduce to Eq. (4.10) when $b = 0$.

The remarkable fact is that using Eq. (4.19a), the two remaining equations become the *same* [Eq. (4.11)], as for the square root Lagrangian [Eq. (4.8)].

This does *not* imply identical solutions, though, as seen by solving the geodesics equations along the same lines as before. Setting once again $P_i = \frac{p_i}{p_v}$ provides us with the transverse motion

$$\boxed{x^i(\lambda) = S^{ij}(u(\lambda))P_j + x^i_0,} \tag{4.20}$$

which is again Eq. (4.13). Then, from Eq. (4.17b), we infer that

$$\dot{v} = -\frac{1}{2}(a^{ij}P_iP_j + \mu_b^2)\dot{u}, \quad \text{where } \mu_b = \left(\frac{m}{p_v}(1-b)\right)^{\frac{1}{1+b}}, \tag{4.21}$$

whose integration yields

$$\boxed{v = -\frac{1}{2}S^{ij}(u)P_iP_j - \frac{1}{2}\mu_b^2 u + v_0.} \tag{4.22}$$

Let us observe that this takes the same form as for Eq. (4.16)—however, with a new, b-dependent mass-quotient term, μ_b. For $b = 0$, the latter reduces to μ_0, and the massive Eq. (4.16) is recovered.

The family of pp-wave geodesics are given by Eqs. (4.13) and (4.16) and are labeled by the constants of integration P_i, x^i_0, v_0, and μ_0. The Finsler geodesics are given by Eqs. (4.20) and (4.22) and are labeled by the constants of integration P_i, x^i_0, v_0, and μ_b. It is clear that the two sets of geodesics are identical up to a b-dependent relabeling of the last constant of integration.

In the massless case $m = 0$ (photons), the b-dependent term drops out from Eq. (4.22). Letting $b \to 1$ turns off the mass-quotient term, $\mu_b \to 0$, and all geodesics behave as if they were massless, consistently with Eq. (4.18). See Fig. 1 in Sec. VII for an illustration.

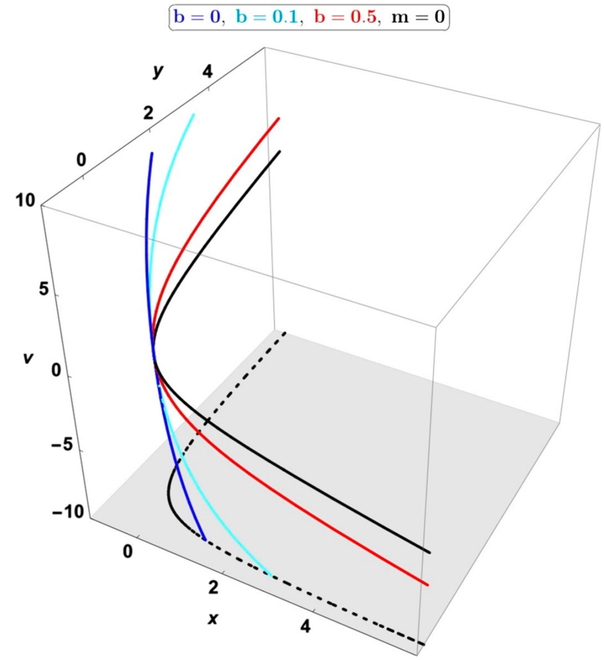


FIG. 1. Consistently with Eq. (4.13), the Bogoslovsky-Finsler geodesics project to the same curve in 2D transverse space for all values of the parameter b while their v coordinates differ, according to Eq. (4.22), in a b-dependent term, which is linear in retarded time, u. Experiments indicate that the anisotropy, and hence b, is very small. When $b \to 1$, the trajectory approaches the massless one (in heavy black), consistently with Eq. (4.22).

Another way to see the surprising identity of the geodesics is to consider the Euler-Lagrange equations

$$E_\mu = \frac{\partial\mathcal{L}_0}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}_0}{\partial\dot{x}^\mu}\right) = 0 \quad \text{and}$$
$$\tilde{E}_\mu = \frac{\partial\mathcal{L}_b}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}_b}{\partial\dot{x}^\mu}\right) = 0 \tag{4.23}$$

of the two Lagrangians $\mathcal{L}_0$ and $\mathcal{L}_b$ in Eqs. (4.8) and (4.6), respectively.

Both systems can be described by three independent equations, since the following identities hold: $\dot{x}^\mu E_\mu \equiv 0$, $\dot{x}^\mu\tilde{E}_\mu \equiv 0$. Then the combinations are as follows:

(1) For the first system:

$$\frac{(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}}{m}\left(\frac{2\dot{v}}{\dot{u}}E_v + \frac{\dot{x}^i}{\dot{u}}E_i\right) = 0, \tag{4.24}$$

$$\frac{(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}}{m}\left(\frac{\dot{x}^i}{\dot{u}}E_v - a^{ij}E_j\right) = 0. \tag{4.25}$$

(ii) For the second system:

$$\frac{(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{\frac{1+b}{2}}}{m}\left[\frac{1}{(1-b^2)\dot{u}^{b+2}}(2\dot{u}\,\dot{v}-ba_{ij}\dot{x}^i\dot{x}^j)\tilde{E}_v + \frac{\dot{x}^i}{(1-b)\dot{u}^{b+1}}\tilde{E}_i\right] = 0, \tag{4.26}$$

$$\frac{(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{\frac{1+b}{2}}}{m}\left[\frac{\dot{x}^i}{(1-b)\dot{u}^{b+1}}\tilde{E}_v + \frac{a^{ij}}{(b-1)\dot{u}^b}\tilde{E}_j\right] = 0, \tag{4.27}$$

(where i, $j = 1, 2$) yield both parts of Eq. (4.11).

V. PARTIALLY BROKEN CARROLL SYMMETRY

Generic plane gravitational waves are invariant under the same five-parameter group we denote by G_5 [31,32]. Expressed in BJR coordinates, G_5 is implemented as [35]

$$u \to u, \qquad \mathbf{x} \to \mathbf{x} + S(u)\mathbf{b} + \mathbf{c},$$
$$v \to v - \mathbf{b}\cdot\mathbf{x} - \frac{1}{2}\mathbf{b}\cdot S(u)\mathbf{b} + f, \tag{5.1}$$

where S is Souriau's matrix [Eq. (4.14)]. The 2-vectors $\mathbf{b}$ and $\mathbf{c}$ and f are constants, interpreted as boosts, and as transverse and vertical translations. These same transformations are isometries also for the Bogoslovsky-Finsler-F metric [Eq. (3.6)] because u is fixed, and the pp isometries leave the pp-wave metric—and hence their powers—invariant. The transformations in Eq. (5.1) are generated by the vector fields

$$B_i = S_{ij}(u)\partial_j - x_i\partial_v, \qquad \partial_i \quad \text{and} \quad \partial_v, \tag{5.2}$$

respectively. The only nonvanishing Lie bracket is

$$[\partial_i, B_j] = -\delta_{ij}\partial_v. \tag{5.3}$$

Rotations, generated by $L_{ij} = x_i\partial_j - x_j\partial_i$, are not symmetries in general.

The restriction of a pp-wave to the $u = 0$ hypersurface $\mathcal{C}_0$ carries a Carroll structure. The "vertical" coordinate v is interpreted as "Carrollian time" [33,34,42]. $\mathcal{C}_0$ is left invariant by the action (5.1), and the generators then satisfy the Carroll algebra in two space dimensions with rotations omitted [35]. Then Eq. (5.1) tells us how the Carroll group is implemented on any hypersurface $u = u_0 = \text{const}$.

In the flat case, $a_{ij} = \delta_{ij}$, we have further symmetries. In particular, adding the vector fields ∂_u, L_{ij}, and $L_{+-} = v\partial_v - u\partial_u$ yields the Lie algebra of an eight-parameter subgroup of the Poincaré group. We now have

$$[v\partial_v - u\partial_u, \partial_v] = -\partial_v, \tag{5.4}$$

and so the *direction* of the null Killing vector field ∂_v is preserved. The eight-dimensional group they generate is ISIM(2). Omitting the translations ∂_v, ∂_u, ∂_i gives SIM(2), the largest proper subgroup of the Lorentz group SO(3, 1). This is the symmetry of Cohen and Glashow's *very special relativity* [1].

Returning to the case of general pp-waves and their Bogoslovsky-Finsler-F version [Eq. (3.6)], we emphasize that the BJR matrix $a = (a_{ij})$, and thus the Souriau matrix S, depends on the pp-wave metric only, but *not* on the deformation parameter b. Therefore, the isometries in Eq. (5.1) act, for Eq. (3.6), exactly as for standard plane waves.

The invariance of the Bogoslovsky-Finsler-F model can be confirmed with respect to the partially broken Carroll group. The infinitesimal version of Eq. (5.1) is Y_{iso} in Eq. (5.2). The linear momenta in Eq. (4.17) are readily recovered; using Eq. (5.2) for boosts, we get in turn

$$k^i = p_v x^i - S_{ij}p_j, \tag{5.5}$$

just as for a gravitational wave [35]. Its conservation follows from Noether's theorem, and can also be confirmed by a direct calculation. The dependence on b is hidden in the momenta in Eq. (4.17). The initial position x_0^i in Eq. (4.13) is the conserved value of k^i.

For $b = 0$, the flat Bogoslovsky-Finsler model has one more isometry, identified with the $U - V$ boost $N_0 = L_{+-}$ in Eq. (3.10). For $b \neq 0$, this generator is broken but not entirely lost. Let us explain how this comes about.

As said above, the (rotationless) Carroll isometry group G_5 in Eq. (5.1) of the initial pp-wave remains a symmetry with identical generators for its Bogoslovsky-Finsler-F extension.

To see what happens to $U - V$ boosts, we start with the Minkowski metric, $\eta_{\mu\nu}dx^\mu dx^\nu = \delta_{ij}dx^i dx^j + 2du dv$. A $U - V$ boost, implemented as

$$u \to \lambda^{-1}u, \qquad x^i \to x^i, \qquad v \to \lambda v, \tag{5.6}$$

where $\lambda = \text{const} > 0$, is an isometry. Moreover, its b-dependent deformation of Eq. (5.6),

$$u \to \lambda^{b-1}u, \qquad x^i \to \lambda^b x^i, \qquad v \to \lambda^{b+1}v, \tag{5.7}$$

is readily seen to leave the Bogoslovsky-Finsler-F line element (3.1) invariant—although for $b \neq 0$ it is only a conformal transformation for the Minkowski metric, $\eta_{\mu\nu}dx^\mu dx^\nu \to \lambda^{2b}\eta_{\mu\nu}dx^\mu dx^\nu$, and not an isometry.[5] We record for later use that the b-deformed boost [Eq. (5.7)] is generated by

[5]Noting that $\lambda^{b-1} = (\lambda^b)^{\frac{b-1}{b}}$ shows that Eq. (5.7) has dynamical exponent $z = 1 - \frac{1}{b} < 0$, which corresponds to the conformal Galilei algebra labeled by z [43,44].

$$N_b = (b-1)u\partial_u + (b+1)v\partial_v + bx^i\partial_i. \quad (5.8)$$

Both Eqs. (5.6) and (5.7) leave the hypersurface $u = 0$ invariant, and extend the Carroll action (5.1). We note that the restriction to $\mathcal{C}_0$ of the deformed $U - V$ boost [Eq. (5.7)] scales the Carrollian time, $v \to \lambda^{b+1}v$. Therefore, it is only the *direction* of ∂_v (and not ∂_v itself) which is preserved—the isometry [Eq. (3.1)] is "chronoprojective" [40,45]:

$$\partial_v \to \lambda^{-1-b}\partial_v. \quad (5.9)$$

In the flat case, two more isometries—namely u-translations and rotations complete the algebra to one with eight parameters. With some abuse, we will still refer to G_5 extended with U-translations (but with no rotations) as "Carroll" for simplicity and denote it by G_6. Its further extension by $U - V$ boosts will be called chrono-Carroll [40] and denoted by G_7.

The Lie algebra structure is most easily checked by taking the commutators of the vector fields in Eqs. (3.9) and (3.10) and comparing with those given in Eq. (9) of Ref. [6], which gives the structure constants of the deformed group $\mathrm{DISIM}_b(2)$; those of $\mathrm{ISIM}(2)$ are obtained by setting $b = 0$.

Further insight is gained by decomposing the deformed $U - V$ boost generator N_b in Eq. (5.8) into the sum of the undeformed expression $N_0 = L_{+-}$ and a relativistic dilation D:

$$N_b = v\partial_v - u\partial u + b(u\partial_u + v\partial_v + x^i\partial_i) = N_0 + bD. \quad (5.10)$$

For $b \neq 0$, N_0 is broken, and it is only the above combination of $U - V$ boosts and dilations which is a symmetry—a situation familiar from gravitational plane waves [35,40,46].

It is instructive to see how this comes about. In the flat Minkowski case, $a_{ij} = \delta_{ij}$ and Eq. (4.17) yield

$$p_i = (\dot{x}_i/\dot{u})p_v \quad \text{and} \quad p_u = \frac{(1+b)\dot{u}\,\dot{v} - b\dot{x}^i\dot{x}_i}{(1-b)\dot{u}^2}p_v.$$

Then, for $\mathcal{D} = D^\mu p_\mu$ and $\mathcal{N}_0 = N_0^\mu p_\mu$, we have

$$\dot{\mathcal{D}} = \frac{(2\dot{u}\,\dot{v} + \dot{x}^i\dot{x}_i)}{(1-b)\dot{u}}p_v \quad \text{and}$$
$$\dot{\mathcal{N}}_0 = -b\frac{(2\dot{u}\,\dot{v} + \dot{x}^i\dot{x}_i)}{(1-b)\dot{u}}p_v = -b\dot{\mathcal{D}}, \quad (5.11)$$

so that the combination of the two expressions is conserved:

$$\dot{\mathcal{N}}_b = 0 \quad \text{for } \mathcal{N}_b = \mathcal{N}_0 + b\mathcal{D}. \quad (5.12)$$

Now we turn to the curved case. Let us consider a conformal transformation f of a pp-wave with metric $g_{\mu\nu}$:

$$f_\star g_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (5.13)$$

where $f_\star$ is the pullback map. This changes the "pp factor" in Eq. (4.5), as

$$(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}(1-b)} \to \Omega^{1-b}(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}(1-b)}. \quad (5.14)$$

The change can be compensated by the "B-F factor," though. Assuming that $f_\star l_\mu = \Omega^a l_\mu$ for some constant a yields

$$(l_\mu dx^\mu)^b(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}(1-b)} \to \Omega^{ab+1-b}(l_\mu dx^\mu)^b(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}(1-b)}. \quad (5.15)$$

In the undeformed case $b = 0$, this is a conformal transformation with conformal factor Ω; obtaining an isometry requires f to be an isometry for the pp-wave. This is consistent with our findings in the flat case for the $U - V$ boost [Eq. (5.6)].

For $b \neq 0$, we have another option. If the exponent of Ω in Eq. (5.15) vanishes,

$$ab = b - 1, \quad (5.16)$$

then we do get an isometry again. For the b-deformed $U - V$ boost in Eq. (5.7), we have $\Omega = \lambda^b$, consistently with Eq. (5.16).

In the case of plane gravitational waves, one drops the angular momentum J and ∂_U, and then the issue is what to do about N_b. Our only certainty so far is that the flat-space implementation [Eq. (5.8)] does not work.

The symmetry of the Bogoslovsky-Finsler-F model is in fact of the very special relativity (VSR) type—more precisely, a subgroup of the eight-parameter $\mathrm{DISIM}_b(2)$, where $0 < b < 1$ is a deformation parameter [7,43]. $\mathrm{DISIM}_b(2)$ is isomorphic to the conformal Galilei group with dynamical exponent [44]

$$z = 1 - \frac{1}{b}. \quad (5.17)$$

For $b \neq 0$, $U - V$ boosts (which are isometries for the Minkowski case) are deformed to Eq. (5.8), a combination of $U - V$ boosts and relativistic dilations.

One can be puzzled about whether the "deformation trick" can work also for a nontrivial profile. The answer is that it *might* work for a particular profile. Let us consider, for example, a pp-wave [Eq. (3.4)] written in Brinkmann coordinates with the (singular) profile

$$2H(X^i, U) = -\frac{K_{ij}^0}{U^2}X^iX^j, \qquad K_{ij}^0 = \text{const}. \quad (5.18)$$

This wave has a six-parameter isometry group [32,40,46,47]. It is, in particular, invariant under a $U - V$ boost [Eq. (5.6)]. Then we find that the deformed $U - V$ boost,

$$U \to \Lambda U, \qquad \boldsymbol{X} \to \Lambda^{\frac{b}{b-1}}\boldsymbol{X}, \qquad V \to \Lambda^{\frac{b+1}{b-1}}V, \tag{5.19}$$

leaves the Bogoslovsky-Finsler-F line element

$$ds_{BF} = \left(-2dUdV - d\boldsymbol{X}^2 - \frac{K^0_{ij}}{U^2}X^iX^j dU^2\right)^{\frac{1-b}{2}}(dU)^b \tag{5.20}$$

invariant. The usual $U - V$ boost is recovered for $b = 0$. Writing $\lambda = \Lambda^{\frac{b}{b-1}}$ shows, moreover, that when $b \neq 0$, the dynamical exponent is $z = -1 + \frac{1}{b}$, minus that in Eq. (5.17). Note that Eq. (5.19) is, once again, a conformal transformation of the pp-wave metric [Eqs. (3.4)–(5.18)] with conformal factor $\Omega^2 = \Lambda^{\frac{2b}{b-1}}$.

VI. PROLONGATION VECTORS AND SYMMETRIES

The connection of the aforementioned symmetries to integrals of the motion is established through Noether's first theorem [48]: each generator of any finite-dimensional Lie group of transformations which leaves the action form-invariant up to a surface term [49] is associated with a conserved quantity.

Consider, for example, a dynamical system with dependent and independent variables $x^\mu(\lambda)$ and λ, respectively. The most general point transformation one can have is

$$\Upsilon = \sigma(\lambda, x)\frac{\partial}{\partial\lambda} + Y^\mu(\lambda, x)\frac{\partial}{\partial x^\mu}, \tag{6.1}$$

where the coefficient $\sigma(\lambda, x)$ accounts for transformations which might also involve the parameter λ. This vector can be extended to the space of the first derivatives, $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$—i.e., we can consider the *first prolongation* of Υ, defined as [50,51]

$$pr^{(1)}\Upsilon = \Upsilon + \Phi^\mu\frac{\partial}{\partial\dot{x}^\mu}, \quad \text{where } \Phi^\mu = \frac{dY^\mu}{d\lambda} - \dot{x}^\mu\frac{d\sigma}{d\lambda}. \tag{6.2}$$

The coefficient Φ^μ here is to guarantee the correct transformation law for the derivatives. Given, for example, the generator (6.1), up to first order in the transformation parameter (say ϵ) we may write

$$\bar{\lambda} \sim \lambda + \epsilon\sigma(\lambda, x), \qquad \bar{x}^\mu \sim x^\mu + \epsilon Y^\mu(\lambda, x), \tag{6.3}$$

which furthermore implies

$$\frac{d\bar{x}^\mu}{d\bar{\lambda}} \sim \frac{d(x^\mu + \epsilon Y^\mu)}{d(\lambda + \epsilon\sigma)} \sim \left(\frac{dx^\mu}{d\lambda} + \epsilon\frac{dY^\mu}{d\lambda}\right)\left(1 - \epsilon\frac{d\sigma}{d\lambda}\right) \simeq \dot{x}^\mu + \epsilon\Phi^\mu. \tag{6.4}$$

With the use of the extended vector $pr^{(1)}\Upsilon$, the initial requirement of Noether's theorem, written as

$$\delta(Ld\lambda) = d\Sigma, \tag{6.5}$$

where $\Sigma = \Sigma(\lambda, x)$ is some function, can be cast in infinitesimal form as

$$pr^{(1)}\Upsilon(L) + L\frac{d\sigma}{d\lambda} = \frac{d\Sigma}{d\lambda}. \tag{6.6}$$

To an appropriate generator Υ and a function Σ satisfying the above relation, there corresponds a conserved quantity:

$$I = Y^\mu\frac{\partial L}{\partial\dot{x}^\mu} - \sigma\left(\dot{x}^\alpha\frac{\partial L}{\partial\dot{x}^\alpha} - L\right) - \Sigma. \tag{6.7}$$

The geodesic system is invariant under arbitrary changes of the parameter λ; therefore the inclusion of the coefficient σ in Eq. (6.1) does not contribute in the conservation law. As can be seen using Eq. (6.7), σ essentially multiplies the Hamiltonian, which is identically zero for Lagrangians which are homogeneous functions of degree 1 in the velocities. The coefficient σ plays a role instead in *Noether's second theorem* and an identity among the Euler-Lagrange equations of motion [52]. As a result, we may restrict ourselves to consider pure spacetime transformations generated by vectors $Y = Y^\alpha(x)\partial_\alpha$. Then the first prolongation becomes

$$pr^{(1)}Y = Y + \frac{dY^\alpha}{d\lambda}\frac{\partial}{\partial\dot{x}^\alpha} = Y^\alpha(x)\frac{\partial}{\partial x^\alpha} + \frac{\partial Y^\alpha}{\partial x^\beta}\dot{x}^\beta\frac{\partial}{\partial\dot{x}^\alpha}, \tag{6.8}$$

and Eqs. (6.6) and (6.7) reduce to

$$pr^{(1)}Y(L) = \frac{d\Sigma}{d\lambda}, \qquad I = Y^\mu\frac{\partial L}{\partial\dot{x}^\mu} - \Sigma. \tag{6.9}$$

If for a given spacetime vector Y, the relation $pr^{(1)}Y(L) = 0$ is satisfied (as for isometries of the geodesic system), then Σ is just a constant and can be omitted, thus having $\tilde{I} = I + \Sigma = Y^\mu\frac{\partial L}{\partial\dot{x}^\mu} = \text{const}$.

To illustrate the prolongation technique, we note that for a system in the background $g_{\mu\nu}$ [Eq. (4.5)] whose Lagrangian is L, the first prolongation of the isometries in Eq. (5.2),

$$Y_{\text{iso}} = (S^{ij}\beta_j + \gamma^i)\partial_i + (-\beta_i x^i + \varphi)\partial_v,$$

is

$$pr^{(1)}Y_{\text{iso}}(L) = \left(Y^\alpha_{\text{iso}}\partial_\alpha + \frac{\partial Y^\alpha_{\text{iso}}}{\partial x^\beta}\dot{x}^\beta\frac{\partial}{\partial\dot{x}^\alpha}\right)(L). \tag{6.10}$$

If the rhs is a total derivative, then we have a symmetry for the system.

Applying Eq. (6.10) first to the pp-wave Lagrangian

$$L_{pp} = \dot{u}\,\dot{v} + \frac{1}{2} a_{ij}\dot{x}^i\dot{x}^j \tag{6.11}$$

confirms that $Y_{\rm iso}$ is a symmetry for the pp-wave.

Next, for the Bogoslovsky-Finsler-F Lagrangian $\mathcal{L}_b$ in Eq. (4.6), we find that the rhs of Eq. (6.10) vanishes:

$$\begin{aligned} &pr^{(1)}Y_{\rm iso}(\mathcal{L}_b) \\ &\quad = (m(1-b)(-a_{ij}\dot{x}^i\dot{x}^j - 2\dot{u}\,\dot{v})^{-\frac{1+b}{2}}\dot{u}^b)pr^{(1)}Y_{\rm iso}(L_{pp}) \\ &\quad = 0, \end{aligned} \tag{6.12}$$

proving that the Carroll group (with broken rotations) generates symmetries also for the Bogoslovsky-Finsler metric. The conserved quantities listed in Sec. V are recovered using Eq. (6.9).

Turning now to $U - V$ boosts, we check first that for the *flat Minkowski metric*, the prolongation of the deformed boost N_b in Eq. (5.8) vanishes:

$$pr^{(1)}N_b(\mathcal{L}_0) = 0, \tag{6.13}$$

and thus generates the constant of the motion $\mathcal{N}_b$ in Eq. (5.12).

However, the same calculation carried out in the *curved* background $g_{\mu\nu}$ [Eq. (4.5)] yields instead

$$\begin{aligned} &pr^{(1)}N(\mathcal{L}_b) \\ &\quad = mu\dot{u}^b(b-1)^2\left(\frac{da_{ij}}{du}\dot{x}^i\dot{x}^j\right)(-a_{ij}\dot{x}^i\dot{x}^j - 2\dot{u}\,\dot{v})^{\frac{-1-b}{2}}. \end{aligned} \tag{6.14}$$

Consistently with what we said before, this vanishes for the flat metric $\eta_{\mu\nu}$. However, it is manifestly *not* a total derivative in general whenever $a = (a_{ij})$ is not a constant matrix.

VII. AN EINSTEIN-MAXWELL EXAMPLE

In this section, we treat the motion in the Bogoslovsky-Finsler deformation of a pp-wave which is not Ricci flat. It is

$$\begin{aligned} ds^2 &= (dX^1)^2 + (dX^2)^2 + 2dUdV \\ &\quad - \frac{\omega^2}{4}((X^1)^2 + (X^2)^2)dU^2. \end{aligned} \tag{7.1}$$

From Eq. (24.5) on p. 385 of Ref. [32], one learns that it belongs to a class first considered by Baldwin and Jeffery [53]. It is conformally flat and is an Einstein-Maxwell solution with a covariantly constant null Maxwell field. From the Bargmann point of view, this metric describes an isotropic harmonic oscillator in the plane with frequency ω [30]. The kinematic group arising from the null reduction is the Newton-Hooke group [54]. Because the metric (7.1) is of the form of Eq. (3.4) with

$$-2H = K_{ij}X^iX^j, \tag{7.2}$$

where K_{ij} is nondegenerate and independent of U, it is also a Cahen-Wallach symmetric space [55–58]. Following the procedure outlined in Sec. IV, the metric (7.1) can presented in the BJR form. We set $a = P^tP$, where

$$P = \begin{bmatrix} (1-\sin(\omega u))^{1/2}\cos\phi & -(1+\sin(\omega u))^{1/2}\sin\phi \\ (1-\sin(\omega u))^{1/2}\sin\phi & (1+\sin(\omega u))^{1/2}\cos\phi \end{bmatrix} \tag{7.3}$$

is a solution of the Sturm-Liouville equation (4.4) with diagonal profile $K = -\frac{\omega^2}{4}{\rm Id}$. Then, using Eq. (4.3), we end up with

$$ds^2 = (1-\sin(\omega u))dx^2 + (1+\sin(\omega u))dy^2 + 2dudv, \tag{7.4}$$

which has $a = P^TP = {\rm diag}(1-\sin(\omega u), 1+\sin(\omega u))$. On p. 386 of Ref. [32], this result is ascribed to Brdička [59]. Equation (7.4) shows that the $U - V$ boost symmetry is manifestly broken.

The Souriau matrix is found by integrating the inverse of (a_{ij}), cf. Eq. (4.14),

$$S(u) = \frac{1}{\omega}\begin{bmatrix} \tan\left(\frac{\omega u}{2} + \frac{\pi}{4}\right) + C_1 & 0 \\ 0 & \tan\left(\frac{\omega u}{2} - \frac{\pi}{4}\right) + C_2 \end{bmatrix}, \tag{7.5}$$

where $C_{1,2}$ are integration constants. Choosing $u_0 = 0$ yields $C_1 = -1$ and $C_2 = 1$. The trajectories (4.13)–(4.22) for different values of b are depicted in Fig. 1.

We mention that the profile of the metric (7.1) is U-independent, and therefore U-translation, $U \to U + \epsilon$, is an additional isometry. This carries over trivially to its Finslerized line element [Eq. (3.6)], since both the pp-wave metric and the "Bogoslovsky-Finsler-F factor" are invariant.

VIII. BOGOSLOVSKY-FINSLER-FRIEDMANN-LEMAÎTRE MODEL

In this section, we shall describe a simple extension of Bogoslovsky's theory to take into account the expansion of the Universe. For some previous work, see Refs. [60–63]. In contrast to our work, these authors consider only Finsler metrics which share the isotropy and spatial homogeneity of Friedmann-Lemaître models. This necessarily excludes the use of a null vector field.

A. The ΛCDM model

The simplest standard model consistent with current observational data is the spatially flat Friedmann-Lemaître model with metric

$$g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2 d\mathbf{x}^2, \tag{8.1}$$

where $a = a(t)$ and $\mathbf{x} = (x, y, z)$. The "scale factor" $a(t)$ is determined by the Einstein equations once the matter content has been specified. The favored ΛCDM model has

$$a(t) = \sinh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{4}t\right), \tag{8.2}$$

which enjoys the remarkable property that the jerk equals 1:

$$j = a^2\left(\frac{da}{dt}\right)^{-3}\frac{d^3a}{dt^3} = 1. \tag{8.3}$$

See Ref. [64] for details and original references.

Here we shall leave the precise form of $a(t)$ unspecified. The coordinate t is called *cosmic time*. The spatial coordinate $\mathbf{x}$ is usually said to be *comoving*, since the world lines of the cosmic fluid have constant $\mathbf{x}$. Two events simultaneous with respect to constant time—i.e., with $x_1^\mu = (0, \mathbf{x}_1)$ and $x_2^\mu = (0, \mathbf{x}_2)$—have a time-dependent proper separation $a(t)(\mathbf{x}_1 - \mathbf{x}_2)$.

B. The choice of null vector field

The vector field

$$l^\mu \frac{\partial}{\partial x^\mu} = g(t)\frac{1}{\sqrt{2}}\left(a\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right), \tag{8.4}$$

where $g(t)$ is a nonvanishing arbitrary function is past directed and null but is neither covariantly constant nor Killing, as it can be checked by a tedious calculation. The associated one-form is

$$l_\mu dx^\mu = -a^2 g\frac{1}{\sqrt{2}}\left(\frac{1}{a}dt + dz\right). \tag{8.5}$$

If $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, then

$$\mathcal{L} = -m(a^2 g)^b\left(\frac{1}{\sqrt{2}}(a(t)^{-1}\dot{t} + \dot{z})\right)^b(\dot{t}^2 - a^2\dot{\mathbf{x}}^2)^{\frac{1}{2}(1-b)} \tag{8.6}$$

is a possible Bogoslovsky-Finsler-type Lagrangian for a particle of mass m. It admits three commuting symmetries generated by $\frac{\partial}{\partial\mathbf{x}}$, and hence three conserved momenta $\boldsymbol{p} = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{x}}}$.

If $b = 0$, then Eq. (8.6) is the standard action for a freely moving particle in a flat isotropic Friedmann-Lemaître universe.

C. Hubble friction

A notable feature of the free motion of a massive particle moving in a Friedmann-Lemaître universe is *Hubble friction*. The conserved momenta are

$$\boldsymbol{p} = ma^2\frac{d\mathbf{x}}{d\tau}, \quad \text{where } d\tau = \sqrt{1 - \left(a\frac{d\mathbf{x}}{dt}\right)^2}dt. \tag{8.7}$$

Here $d\tau$ is the increment of proper time along the world line of a particle. The four-velocity of the particle with respect to the local inertial reference frame $\frac{\partial}{\partial t}, \frac{\partial}{a(t)\partial\mathbf{x}}$ is

$$\boldsymbol{u} = a(t)\frac{d\mathbf{x}}{d\tau}, \quad \text{whence} \quad \boldsymbol{u} = \frac{\boldsymbol{p}}{ma(t)}. \tag{8.8}$$

One may also define a velocity $\boldsymbol{v}$ measured in units of cosmic time t, $\boldsymbol{v} = a(t)\frac{d\mathbf{x}}{dt}$, so that

$$d\tau = \sqrt{1 - \boldsymbol{v}^2}dt, \qquad \boldsymbol{u} = \frac{\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2}}, \quad \text{and} \quad \boldsymbol{v} = \frac{\boldsymbol{p}}{m}\frac{1}{\sqrt{a^2 + \frac{\boldsymbol{p}^2}{m^2}}}. \tag{8.9}$$

Hence,

$$\boxed{\frac{d\mathbf{x}}{dt} = \frac{\boldsymbol{p}}{a\sqrt{m^2a^2 + \boldsymbol{p}^2}}}. \tag{8.10}$$

Thus, in an expanding phase in which $a(t)$ increases with time, both $\boldsymbol{v}$ and $\boldsymbol{u}$ decrease with time. However, as a consequence of isotropy, their directions remain constant. The fact that we have three conserved momenta and the constraint

$$\left(\frac{dt}{d\tau}\right)^2 = 1 + a^2\left(\frac{d\mathbf{x}}{d\tau}\right)^2 \tag{8.11}$$

implies that the system of geodesics is completely integrable. In fact,

$$d\mathbf{x} = \frac{\boldsymbol{p}}{ma^2}\frac{dt}{\sqrt{1 + (\frac{\boldsymbol{p}}{ma})^2}} \quad \text{and} \quad d\tau = \frac{dt}{\sqrt{1 + (\frac{\boldsymbol{p}}{am})^2}}. \tag{8.12}$$

D. Conformal flatness

Before proceeding further, we recall that the Friedmann-Lemaître metric (8.1) is conformally flat, as becomes clear if we define *conformal time* η by

$$\eta(t) = \int^t \frac{d\tilde{t}}{a(\tilde{t})}, \tag{8.13}$$

where the lower limit is left unspecified for the time being. In terms of cosmic time, the Friedmann-Lemaître metric (8.1) becomes

$$g_{\mu\nu}dx^\mu dx^\nu = a^2\{-d\eta^2 + dz + dx^i dx^i\} = a^2\{2du dv + dx^i dx^i\}, \tag{8.14}$$

where a^2 is regarded as a function of conformal time η, and we introduce the light-cone coordinates

$$u = \frac{z+\eta}{\sqrt{2}}, \qquad v = \frac{z-\eta}{\sqrt{2}}. \tag{8.15}$$

From Eq. (8.5), we learn that $l_\mu dx^\mu = -a^2 g du$, whence our Bogoslovsky-Finsler-F-ized line element is

$$ds = a^{1+b} g^b (-2du dv - dx^i dx^i)^{\frac{1}{2}(1-b)} (du)^b. \tag{8.16}$$

This Lagrangian would yield Bogoslovsky's original flat spacetime model, provided we choose $g(t)$ such that

$$f = a^{1+b} g^b = 1. \tag{8.17}$$

The only freedom with this model would be to introduce an arbitrary factor

$$\mathcal{L}_f = f(\eta)(-2\dot{u}\,\dot{v} - \dot{x}^i \dot{x}^i)^{\frac{1}{2}(1-b)} (\dot{u})^b, \tag{8.18}$$

which amounts to saying that the mass depends upon cosmic time.

This situation is the same as in the ordinary spatially flat Friedmann-Lemaître cosmology for which $b = 0$. We can either say the Universe is expanding, but our rulers are constructed from massive particles, all of whose masses are constant in cosmic time t, or that the Universe is time independent, but the rulers all change with the same time dependence. In that case, the phenomenon of Hubble friction would be ascribed not to the expansion of the Universe but to masses getting heavier.

E. Redshifting

If we adopt Eq. (8.18) then light rays move along straight lines in $(\eta, \mathbf{x})$ coordinates. Emitters and observers (e.g., galaxies and astronomers) are usually held to be at rest in these coordinates.

Suppose the observer is at the origin at $(\eta_0, 0, 0, 0)$ and receives light rays from a galaxy at (η_e, x_e, y_e, z_e) so that the duration of emission in conformal time is $d\eta_e$, and the duration of the corresponding observation is $d\eta_e$; then

$$d\eta_e = d\eta_0. \tag{8.19}$$

Then the emitted and observed proper times are $d\tau_e = f(\eta_e)d\eta_e$, $d\tau_0 = f(\eta_0)d\eta_0$, and so the redshift is

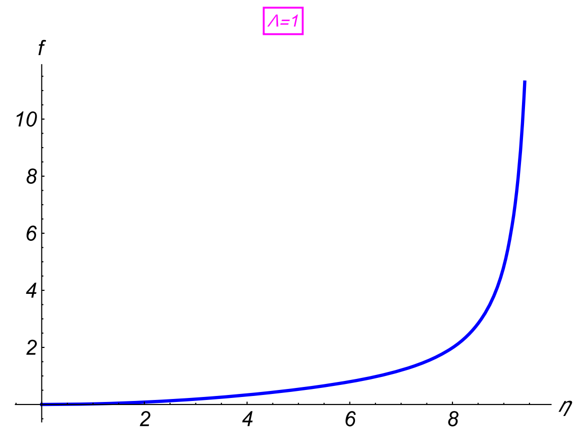


FIG. 2. The conformal factor [Eq. (8.21b)] of the Friedmann-Lemaître model [Eq. (8.1)], expressed as a function of the conformal time η, obtained by numerical integration of Eq. (8.21).

$$1 + z = d\tau_0/d\tau_e = f(\eta_0)/f(\eta_e). \tag{8.20}$$

Thus, if the Universe is expanding—that is, if $f' > 0$—then the signal received is redshifted, and contrariwise if the Universe is contracting—that is, if $f' < 0$.

Note that under these assumptions, the emitted light from all galaxies at the same conformal time will be redshifted in the same way. That is, *the redshift should be isotropic*.

F. A possible choice for $f(\eta)$

As mentioned earlier, our observed Universe is well described by a scale factor $a(t)$ given by Eq. (8.2). Applying the Einstein equations to the Friedmann-Lemaître metric [Eq. (8.1)], one finds that it is supported by a pressure-free fluid (some of it visible and some of it not—so-called dark matter) and a positive cosmological constant term Λ often called dark energy. Near the big bang—i.e., for small t—$a(t) \propto t^{2/3}$, because the Λ term is negligible. This is the Einsten–de Sitter model. At late times, $a(t) \propto \exp\sqrt{\Lambda/3}t$, which exhibits cosmic acceleration. This is de Sitter spacetime.

From Eq. (8.13), choosing Eq. (8.2) and setting $ag = 1$ in Eq. (8.16), we have

$$\eta(t) = \int_0^t \sinh^{-\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{4}\tilde{t}\right) d\tilde{t}, \tag{8.21a}$$

$$f(\eta) = a(t) = \sinh^{\frac{2}{3}}\left(\frac{\sqrt{3\Lambda}}{4}t\right). \tag{8.21b}$$

This step depends only on the scale factor a in Eq. (8.1) and does not involve the deformation parameter b. See Fig. 2. It is worth noting that conformal time as a function of cosmic time is bounded from above—as happens for de Sitter space, to which our spacetime tends when $t \to \infty$.

IX. BOGOSLOVSKY-FINSLER-FRIEDMANN-LEMAÎTRE GEODESICS

Written in coordinates (η, x, y, z), the Lagrangian (8.18) is

$$\mathcal{L}_f = -mf(\eta)\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^b(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{\frac{1}{2}(1-b)}, \tag{9.1}$$

providing us with the momenta

$$p_x = m(1-b)f(\eta)\dot x\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^b(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{-\frac{1}{2}(1+b)}, \tag{9.2a}$$

$$p_y = m(1-b)f(\eta)\dot y\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^b(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{-\frac{1}{2}(1+b)}, \tag{9.2b}$$

$$p_z = m(1-b)f(\eta)\dot z\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^b(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{-\frac{1}{2}(1+b)} - \frac{b}{\sqrt{2}}mf(\eta)\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^{-1+b}(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{\frac{1}{2}(1-b)}, \tag{9.2c}$$

$$p_\eta = -mf(\eta)\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^{b-1}(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{-\frac{1}{2}(1+b)} \times\left((1-b)\dot\eta\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)+\frac{b}{\sqrt{2}}(\dot\eta^2-\dot x^2+\dot y^2-\dot z^2)\right). \tag{9.2d}$$

Evidently, the three momenta p_x, p_y, p_z are conserved. Moreover, since

$$\frac{p_y}{p_x} = \frac{dy}{dx}, \tag{9.3}$$

the projections of the geodesics onto the transverse x-y plane are straight lines. Choosing the proper time as a parameter, $\lambda = \tau$, one has the constraint

$$f(\eta)\left(\frac{\dot\eta+\dot z}{\sqrt{2}}\right)^b(\dot\eta^2-\dot x^2-\dot y^2-\dot z^2)^{\frac{1}{2}(1-b)} = 1, \tag{9.4}$$

which we may rewrite in terms of conformal time, η, as an equation for τ as

$$\tau' = f(\eta)\left(\frac{1+z'}{\sqrt{2}}\right)^b(1-(x')^2-(y')^2-(z')^2)^{\frac{1}{2}(1-b)}, \tag{9.5}$$

where $(x', y', z') = (\frac{dx}{d\eta}, \frac{dy}{d\eta}, \frac{dz}{d\eta})$.

If $f' = 0$, then $p_\mu/f(\eta)$ is independent of η, leaving us with the same straight-line motion at constant velocity as for the flat Bogoslovsky spacetime.

As we have seen above, even if $f' \neq 0$, the projections of the motion on the $x-y$ plane are straight lines, although not with constant speed with respect to the conformal time η. The speeds of the projections onto the x-z and y-z planes are also not at constant η speed, but they are not straight lines either. Over conformal η times that are short compared with $\frac{f}{f'}$, they are approximately straight lines with slopes given by $\frac{p_x}{mf(\eta)}$ but over longer time periods, the speeds and directions change, reflecting precisely the effects of Hubble friction.

The geodesics are conveniently studied by switching to conformal time, η. Introducing

$$\dot u = \frac{\dot\eta+\dot z}{\sqrt{2}} \quad\text{and}\quad \dot w = \dot\eta^2-\dot x^2-\dot y^2-\dot z^2 \tag{9.6}$$

in place of $\dot\eta$ and $\dot z$, Eqs. (9.2a), (9.2b), and (9.2c) become

$$m(b-1)f(\eta)\dot x\dot u^b + p_x\dot w^{\frac{b+1}{2}} = 0, \tag{9.7a}$$

$$m(b-1)f(\eta)\dot y\dot u^b + p_y\dot w^{\frac{b+1}{2}} = 0, \tag{9.7b}$$

$$\frac{mf(\eta)}{2\sqrt{2}}\dot u^{b-1}\dot w^{\frac{1}{2}(-b-1)}(2(b-1)\dot u^2+(b+1)\dot w - (b-1)(\dot x^2+\dot y^2)) + p_z = 0. \tag{9.7c}$$

The first two equations imply identical evolution:

$$\dot x = \frac{\dot u^{-b}\dot w^{\frac{b+1}{2}}}{m(1-b)f(\eta)}p_x, \qquad \dot y = \frac{\dot u^{-b}\dot w^{\frac{b+1}{2}}}{m(1-b)f(\eta)}p_y, \tag{9.8}$$

which confirms once again that the transverse projection is a straight line, owing to $\dot x/\dot y = p_x/p_y = \text{const}$. With their help, Eq. (9.7c) becomes

$$-\sqrt{2}(b-1)m^2f(\eta)^2\dot u^{2b}(2(b-1)\dot u^2+(b+1)\dot w) - 4(b-1)mp_zf(\eta)\dot u^{b+1}\dot w^{\frac{b+1}{2}} + \sqrt{2}(p_x^2+p_y^2)\dot w^{b+1} = 0. \tag{9.9}$$

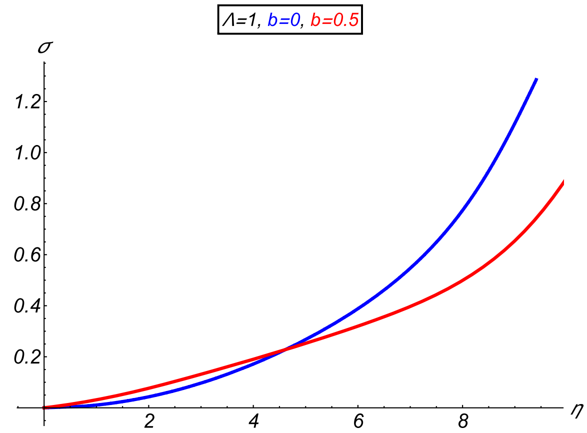
By reparametrizing $w(\lambda)$ as

$$w(\lambda) = \int\sigma(\lambda)^2\dot u(\lambda)^2d\lambda, \tag{9.10}$$

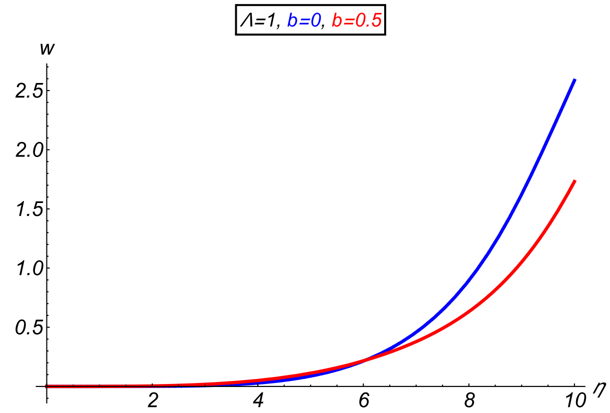
where $\sigma(\lambda)$ is a new function that we introduce, Eq. (9.9) reduces from a differential one to an algebraic

$$-\sqrt{2}(b-1)m^2f(\eta)^2((b+1)\sigma(\lambda)^2+2(b-1)) - 4(b-1)mf(\eta)p_z\sigma(\lambda)^{b+1} + \sqrt{2}(p_x^2+p_y^2)\sigma(\lambda)^{2(b+1)} = 0. \tag{9.11}$$

For $b = 0$, this is simply quadratic in $\sigma(\lambda)$, but for $b \neq 0$, it is not trivial to solve it for σ, cf. Fig. 3(a). However, as it is



(a)



(b)

FIG. 3. (a) $\sigma(\eta)$ and (b) $w(\eta)$ in Eq. (9.10) plotted for blue $\mathbf{b}=\mathbf{0}$ and for red $\mathbf{b}=\mathbf{0.5}$.

quadratic in $f(\eta)$, the inverse problem [which amounts to choosing $\sigma(\lambda)$ to find the corresponding $f(\lambda)$] still works. The functions $\sigma(\eta)$ and $w(\eta)$ are plotted in Fig. 3. Using Eqs. (9.6), (9.10), and (9.8) we get

$$\dot{\eta}=\frac{\dot{u}}{2\sqrt{2}}\left(2+\sigma^{2}+\frac{(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}{(b-1)^{2}m^{2}f(\eta)^{2}}\right), \tag{9.12a}$$

$$\dot{x}=\dot{u}\frac{\sigma^{b+1}}{(1-b)mf(\eta)}p_{x}, \tag{9.12b}$$

$$\dot{y}=\dot{u}\frac{\sigma^{b+1}}{(1-b)mf(\eta)}p_{y}, \tag{9.12c}$$

$$\dot{z}=\frac{\dot{u}}{2\sqrt{2}}\left(2-\sigma^{2}-\frac{(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}{(b-1)^{2}m^{2}f(\eta)^{2}}\right), \tag{9.12d}$$

together with the algebraic constraint between $\sigma(\lambda)$ and $f(\eta(\lambda))$ in Eq. (9.11).

The joint system can be shown to satisfy the Euler-Lagrange equations.

The $u(\lambda)$ that remains unspecified in Eq. (9.12) and disappears from Eq. (9.11) serves as a gauge parameter [by seeing the ratios of derivatives that are being formed in Eq. (9.12)], for which we can simply set $u(\lambda)=\lambda$. So, in this time-gauge, $\eta(\lambda)+z(\lambda)=\sqrt{2}\lambda$, which is compatible with Eqs. (9.12a) and (9.12d), as seen above. We thus have, in the "conformal time gauge,"

$$\frac{dx}{d\eta}=\frac{2\sqrt{2}(1-b)m\sigma^{b+1}f(\eta)}{(1-b)^{2}m^{2}(\sigma^{2}+2)f(\eta)^{2}+(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}p_{x}, \tag{9.13a}$$

$$\frac{dy}{d\eta}=\frac{2\sqrt{2}(1-b)m\sigma^{b+1}f(\eta)}{(1-b)^{2}m^{2}(\sigma^{2}+2)f(\eta)^{2}+(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}p_{y}, \tag{9.13b}$$

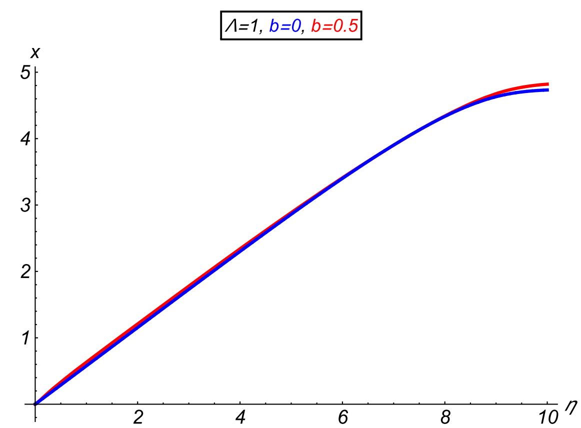


FIG. 4. For blue $\mathbf{b}=\mathbf{0}$, all trajectories follow straight lines and have identical evolution. For red $\mathbf{b}=\mathbf{0.5}$, $z(\eta)$ becomes different from the transverse trajectories $[x(\eta),\ y(\eta)]$ consistently with Eq. (9.13), as shown in Fig. 5.

$$\frac{dz}{d\eta}=\frac{2-\sigma^{2}-\frac{(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}{(1-b)^{2}m^{2}f(\eta)^{2}}}{2+\sigma^{2}+\frac{(p_{x}^{2}+p_{y}^{2})\sigma^{2(b+1)}}{(1-b)^{2}m^{2}f(\eta)^{2}}}. \tag{9.13c}$$

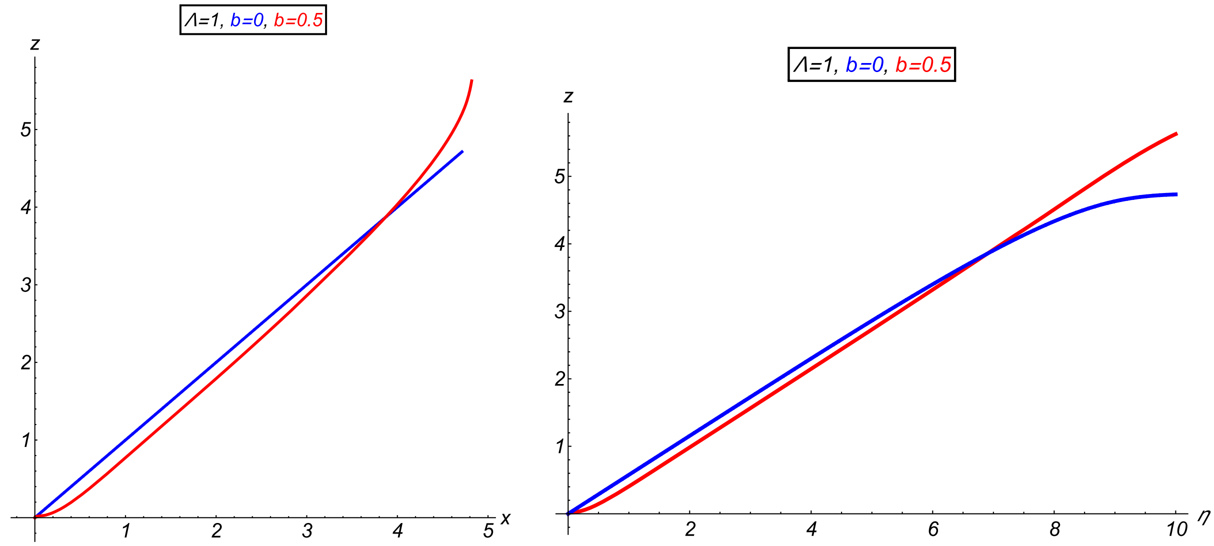
A. The Friedmann-Lemaître case $b=0$

If $b=0$, the algebraic relation (9.11) is quadratic and can be solved for σ:

$$\sigma(\eta)=\pm\frac{\sqrt{2}mf(\eta)}{\sqrt{m^{2}f(\eta)^{2}+\boldsymbol{p}^{2}}\pm p_{z}}, \tag{9.14}$$

as shown by the blue line in Fig. 3(a) for the upper sign.[6] In terms of conformal time η,

[6]Choosing the lower sign would amount to an overall sign change when that of p_z is also reversed.



(a) (b)

FIG. 5. For red $b > 0$, the motion in the x-z plane is not more along a straight line (as it is for blue $\mathbf{b} = \mathbf{0}$). The Hubble friction slows down the z motion for blue $\mathbf{b} = \mathbf{0}$, but not when red $b > 0$.

$$\boxed{\frac{d\mathbf{x}}{d\eta} = \pm\frac{\boldsymbol{p}}{\sqrt{m^2 f(\eta)^2 + \boldsymbol{p}^2}},} \tag{9.15}$$

to be compared with Eq. (8.10). The consistency with the equations in Sec. VIII C follows from

$$d\tau = \frac{mf^2 d\eta}{\sqrt{m^2 f^2 + \boldsymbol{p}^2}}, \qquad d\mathbf{x} = \frac{d\eta}{\sqrt{m^2 f^2 + \boldsymbol{p}^2}}\boldsymbol{p}$$
$$\Rightarrow \boxed{\frac{d\mathbf{x}}{d\tau} = \frac{1}{mf^2}\boldsymbol{p}.} \tag{9.16}$$

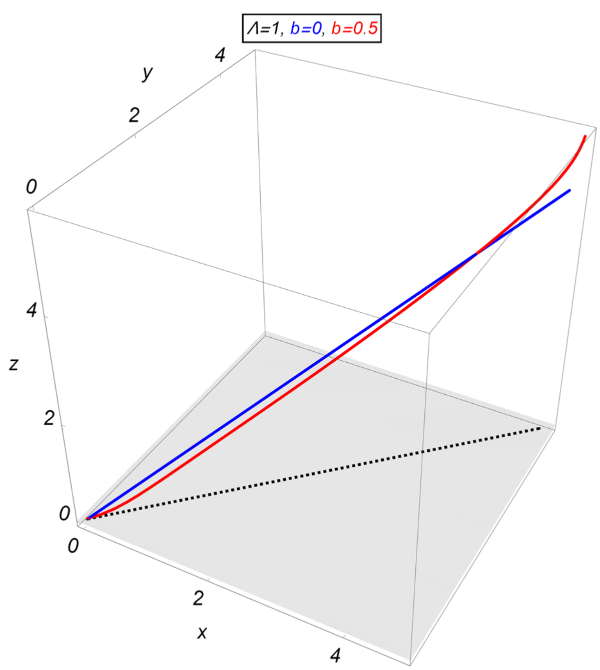


FIG. 6. For the Friedmann-Lemaître model for blue $\mathbf{b} = \mathbf{0}$, the 3D trajectory is a straight line. For the Bogoslovsky-Finsler-F modification for red $\mathbf{b} = \mathbf{0.5}$, however, while the projection to the $x - y$ plane is still along a straight line, the z component becomes curved, consistently with Fig. 5.

We could not obtain analytical expressions; however, using the numerically calculated values of $f(\eta)$ (see Fig. 2) allows us to plot $x(\eta)$ by solving Eq. (9.15), as shown in Fig. 4.[7]

The $b = 0$ case nicely illustrates *Hubble Friction*: all trajectories slow down and ultimately come to rest. For $b \neq 0$ it seems that this happens only for the transverse motion but not for the motion in the z-direction, see Fig. 6. The slowing down in the transverse case is plausible from Eqs. (9.12b) and (9.12c).

X. CONCLUSION

In this paper, motivated by work by Bogoslovsky [3–5,7], and by that of Tavakol and Van den Bergh, and Roxburgh [9,11,12], and by Cohen and Glashow [1], and more recently by others [14–21], we have studied the free motion of a massive particle moving in a one-parameter family of Finslerian deformations of a plane gravitational wave. By free motion, we mean that it extremizes the proper time along its timelike world line. Finslerian proper time is measured by replacing the usual square-root integrand

[7]The two signs in Eq. (9.14) can be compensated by $\boldsymbol{p} \to -\boldsymbol{p}$, implying an overall sign change. In what follows, the upper sign will be chosen.

$$\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \text{ with } \left(-g_{\mu\nu}l^\mu\frac{dx^\nu}{d\lambda}\right)^b\left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}(1-b)},$$

where l^μ is a null vector field and b is a dimensionless constant.

In earlier work, we have shown that because of the five-dimensional isometry group of plane gravitational waves, the motion of the usual timelike geodesics is completely integrable.

In the present paper, we have shown that the five-dimensional partially broken Carroll symmetry group G_5 remains a symmetry of our Finslerian line element, provided we choose the null vector field l^μ to be the covariantly constant null vector of the underlying gravitational wave. As a consequence, we find that that not only is the free motion completely integrable, but it differs only in that the "vertical" coordinate v involves in turn a b-dependent term, which is linear in the retarded time coordinate u. The motion in the transverse directions is unchanged. The situation is analogous to what happens for massive vs massless geodesics in a pp-wave [41].

The symmetry of the Bogoslovsky-Finsler-F model is in fact that of the very special relativity (VSR) type; in the Minkowski case it is the eight-parameter $\mathrm{DISIM}_b(2)$ [7,39,43]. The clue is to deform a $U-V$ boost N_0 to N_b as in Eq. (5.8). The trick works for certain nontrivial profiles, as for the U^{-2} discussed at the end of Sec. V.

We have also examined the free motion of a Finslerian deformation of a homogeneous pp-wave, which is an Einstein-Maxwell solution. The resulting spacetime is a Cahen-Wallach symmetric space [55] and arises in a wide variety of physical applications, and whose null reduction in the fashion of Eisenhart and Duval *et al.* [30] is a simple harmonic oscillator with a Newton-Hooke-type symmetry. Here again, the free motion is qualitatively independent of the deformation parameter b.

We have also studied a simple anisotropic cosmological model based on that of Friedmann and Lemaître with vanishing spatial curvature. Because the latter is conformally flat, the motion of massive particles is equivalent to motion in flat Bogoslovsky spacetime, except that all masses become time dependent with identical time dependence.

Although our present Universe shows little sign of anisotropy of the sort that arises in Bogoslovsky-Finsler metrics, that may not have been true earlier in the history of the Universe, since the absence of anisotropy now is usually ascribed to a rapid phase of inflation during which the scale factor of the Universe increased by a factor of perhaps 60 e-folds. It is of interest, therefore, to study geodesics in Bogoslovsky-Finsler deformations of Friedmann-Lemaître metrics.

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