Noncovariance of the dressed-metric approach in loop quantum cosmology

Martin Bojowald

Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, Pennsylvania 16802, USA

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The dressed-metric approach is shown to violate general covariance by demonstrating that it cannot have an off-shell completion in which the correct infinitesimal relations of space-time hypersurface deformations are realized. The main underlying reason—a separation of background degrees of freedom and modes of inhomogeneity that is incompatible with covariance—is shared with other approaches such as hybrid loop quantum cosmology.

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I. INTRODUCTION

The dressed-metric approach [1] is an attempt to extend modified Friedmann equations of loop quantum cosmology to perturbative inhomogeneity in order to describe structure formation. If any such proposal is to be consistent, it must respect general covariance in some form to guarantee that the equations are meaningful: If general covariance is violated, the theory is either plagued by spurious, unphysical degrees of freedom if one decides to impose a restricted number of covariance transformations (or none at all); or it is overconstrained if broken covariance transformations are imposed, which then identify physical solutions that are supposed to be distinct. A noncovariant modification of a covariant theory has either too many or too few propagating degrees of freedom, depending on how it is applied.

Since loop quantum cosmology [2] modifies the background dynamics of a homogeneous Universe, perturbative inhomogeneity is not guaranteed to obey covariance conditions. However, the dressed-metric approach assumes that classical observables and Hamiltonians can be used for inhomogeneity *without* modifications even while the background dynamics is modified such that it may allow a bounce, a crucial ingredient in some of the developed scenarios. In this paper, we provide the first analysis of covariance in the dressed-metric approach, pointing out several previously overlooked subtleties and ultimately reaching the conclusion that covariance is violated.

Several details of the technical implementation of the dressed-metric approach obscure the issue of covariance, which is perhaps the reason why this important issue has not been addressed yet. The approach postulates separate quantizations for an isotropic background space-time and inhomogeneous perturbations on it, even though the degrees of freedom of both ingredients are interrelated in any covariant setting that obtains background and perturbations from an expansion of a covariant theory. For instance, the limited covariance transformations that remain in a spatially homogeneous reduction of a covariant theory do not restrict the possible dynamics, which can be modified at will. Homogeneous background dynamics that can be obtained from some higher-curvature action, by contrast, is not arbitrary but subject to conditions that implicitly ensure its descendance from a covariant theory of this type. Once a covariant theory has been restricted to homogeneity, however, the dynamics can be modified consistently in the homogeneous setting, without any restrictions that would result from covariance or integrability conditions in an inhomogeneous theory. By separating the degrees of freedom into background and perturbations *before* implementing quantum modifications, and then leaving the perturbative degrees of freedom unmodified, the dressed-metric approach construes a setting in which the usual covariance conditions are relaxed. This observation does not directly imply that the approach violates covariance, but it shows that any analysis of covariance in this approach is subtle and must be performed in detail.

While covariance itself has not yet been analyzed in the dressed-metric approach, some transformations related to this condition have been discussed in the seminal papers. However, these transformations, like the implementation of degrees of freedom, act separately on background and perturbations and do not respect the interrelated nature of these degrees of freedom with respect to covariance. In particular, the dressed-metric approach replaces linear perturbations of metric and extrinsic curvature, or of other fields used in canonical gravity, with Bardeen potentials or curvature perturbations [3,4]. Since these variables are invariant with respect to small inhomogeneous coordinate transformations, they respect some partial form of covariance. The homogeneous background dynamics, meanwhile, is made invariant with respect to homogeneous time reparametrizations by using the method of deparametrization [5,6], formulating homogeneous evolution not with respect to a time coordinate but rather with respect to one of the dynamical fields of the theory, given by a free massless scalar. The resulting framework is formally consistent because no time coordinate is used explicitly, and spatial coordinates can be adapted to the homogeneous background. In this sense, the dressed-metric approach constructs a consistent quantum-field theory on a modified homogeneous space-time, but it does not show that fields and background can be part of a common covariant theory. Therefore, it is not clear whether it can rightfully be considered a description of cosmological evolution in quantum gravity, or of quantum space-time.

The treatment of transformations in the dressed-metric approach suffers from several old and new problems:

(1) While deparametrization eliminates the appearance of coordinate time, as applied in Ref. [1], it selects a specific reference scalar field as internal time (which has to be free and massless in order to play the role of a global measure of time). In models in which more than one choice of global internal time are available, quantum corrections in general imply inequivalent observables depending on which internal time is used [7–10]. Even if one does not refer to coordinate time, therefore, time reparametrization invariance is not guaranteed after quantization. This problem, which is being investigated with several methods—see for instance Refs. [7,8,11–17]—is not specific to the dressed-metric approach and will therefore be disregarded here.

(2) Bardeen potentials, in spite of one of their common names, are not gauge invariant [18,19]. They are invariant with respect to *small inhomogeneous* coordinate transformations in a perturbative setting, but they are no longer invariant if one or both of the two implied conditions, smallness and inhomogeneity, is violated. Curvature perturbations, which are available in the presence of a scalar matter field, are invariant, provided only that coordinate transformations are small and not necessarily inhomogeneous, but even this condition is not met by all transformations relevant for perturbative cosmology: While a first-order description of inhomogeneity need not consider higher than first-order transformations, it should include large *homogeneous* coordinate changes such as a transformation from proper time to conformal time. In the dressed-metric approach, homogeneous coordinate transformations are implemented by deparametrization for the background, separately from the inhomogeneous sector even though they act nontrivially on Bardeen potentials and curvature perturbations when they are large.

(3) A detailed analysis of space-time transformation in a four-dimensional or a canonical setting, presented in the next section, shows that background transformations and those acting on perturbations do not form a direct but rather a semidirect product. This important algebraic structure is violated by the separation of background and perturbation degrees of freedom imposed by the dressed-metric approach, which would be compatible only with a direct product. As a consequence, by its very construction the dressed-metric approach is unable to provide the correct off-shell structure required for a covariant theory of background and perturbations.

II. SPACE-TIME STRUCTURE

The perturbative form of covariance is somewhat different depending on whether one uses a formulation of tensor fields in four dimensions or a canonical description. However, both viewpoints lead to the same conclusion: that background and perturbative transformations form a semidirect product.

A. Four-dimensional formulation

Background coordinate transformations affect only time t and are generated by vector fields of the form $f(t)\partial/\partial t$ with an arbitrary function $f(t)$. Perturbative coordinate changes are generated by vector fields $\xi^\alpha\partial/\partial x^\alpha$ with four components ξ^α which are small in the sense that any products of multiple ξ^α's or of ξ^α with perturbative fields are ignored. Bardeen potentials and curvature perturbations are constructed by ensuring the ξ^α independence of suitable combinations of metric components, but they do not consider $f(t)$ (unless this function is small and may be considered a contribution to ξ^0).

1. Bardeen potentials and curvature perturbations

Specifically, we may transform metric components by inserting small coordinate changes $x^\alpha \mapsto x^\alpha + \xi^\alpha$ into the line element

$$\mathrm{d}s^2 = a^2\left(-(1+2\phi)\mathrm{d}\eta^2 + 2\partial_i B\mathrm{d}\eta\mathrm{d}x^i + \left((1-2\psi)\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\Delta\right)E\right)\mathrm{d}x^i\mathrm{d}x^j\right) \tag{1}$$

for linear scalar perturbations on a flat isotropic background, here using conformal time η and including only scalar modes. We distinguish between time transformations, $\eta \mapsto \eta + \xi^0$, and scalar spatial transformations, $x^i \mapsto x^i + \partial^i\xi$ with a scalar function ξ. In the first case, we denote a derivative with respect to η by a prime:

$$\mathrm{d}\eta^2 \mapsto \mathrm{d}\eta^2 + 2\xi^{0\prime}\mathrm{d}\eta^2 + 2\partial_i\xi^0\mathrm{d}\eta\mathrm{d}x^i \tag{2}$$

to first order in ξ^0, while $a(\eta)^2 \mapsto a(\eta)^2(1 + 2a'\xi^0/a)$. Rearranging the resulting line element to bring it back to the old form [Eq. (1)] but with adjusted scalar perturbations, we obtain the transformations

$$\begin{aligned} \phi &\mapsto \phi + \xi^{0\prime} + \frac{a'}{a}\xi^0, \\ \psi &\mapsto \psi - \frac{a'}{a}\xi^0, \\ B &\mapsto B - \xi^0, \end{aligned} \tag{3}$$

$$E \mapsto E. \tag{4}$$

Notice that the transformation of B follows only if $\partial_i\xi^0 \neq 0$ in Eq. (2) because the line element depends on $\partial_i B$ but not directly on B. Therefore, for spatially constant ξ^0, or a small background transformation, there is no need for B to change, in contrast to Eq. (3). In fact, the transformation of B is undetermined in this case, because $B \mapsto B - \alpha\xi^0$ would be consistent for any real α. This ambiguity is not relevant in the line element, which only depends on $\partial_i B$, but it implies an ambiguity in the Bardeen potentials, which depend directly on B and not just its spatial derivatives. Thus, we obtain a distinction between background and perturbation transformations even if both are small.

For small spatial transformations, we insert

$$\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \mapsto \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2\partial_i\xi'\mathrm{d}\eta\mathrm{d}x^i + 2\partial_i\partial_j\xi\mathrm{d}x^i\mathrm{d}x^j \tag{5}$$

into the line element and read off

$$\phi \mapsto \phi, \quad \psi \mapsto \psi, \quad B \mapsto B + \xi', \quad E \mapsto E + \xi. \tag{6}$$

Therefore, ϕ, ψ, and $B - E'$ are invariant with respect to spatial transformations. (Again, the transformation of B would be undetermined if $\partial_i\xi = 0$, but for spatial transformations we need $\partial_i\xi \neq 0$ in order to have a nontrivial $\xi_i = \partial_i\xi \neq 0$.) Since $B - E'$ changes to $B - E' - \xi^0$ by a time transformation, the combinations

$$\begin{aligned} \Phi &\coloneqq \phi + \frac{a'}{a}(B - E') + (B - E')', \\ \Psi &\coloneqq \psi - \frac{a'}{a}(B - E') \end{aligned} \tag{7}$$

are invariant, provided ξ^0 is not spatially constant.

If there is a matter scalar field, $\varphi = \bar{\varphi} + \delta\varphi$, its perturbation transforms by $\delta\varphi \mapsto \delta\varphi + \bar{\varphi}'\xi^0$. Therefore, one can obtain ξ^0-independent combinations, the curvature perturbations

$$\begin{aligned} \mathcal{R}_1 &= \psi + \frac{a'}{a\bar{\varphi}'}\delta\varphi, \\ \mathcal{R}_2 &= \phi - \frac{1}{2}\left(\frac{a}{a'}\right)'\psi - \frac{1}{\bar{\varphi}'}\left(\frac{a'}{a} - \frac{\bar{\varphi}''}{\bar{\varphi}'}\right)\delta\varphi + \frac{1}{2}\frac{a}{a'}\psi' - \frac{1}{2\bar{\varphi}'}\delta\varphi', \end{aligned} \tag{8}$$

without using B. These perturbations, unlike Bardeen potentials, are invariant also with respect to spatially constant ξ^0, but not with respect to large background transformations.

Formulating the dressed-metric approach using curvature perturbations instead of Bardeen potentials implies that we do not have to distinguish between small background transformations and perturbative transformations. However, there remain nontrivial large background transformations, hence the additional step of deparametrization in the approach. Large background transformations change curvature perturbations merely by reparametrizations, such as replacing $a'/(a\bar{\varphi}')$ with $\dot{a}/\dot{\bar{\varphi}}$ when transforming from conformal time to proper time. Formally, the approach therefore does take into account all relevant transformations. However, the way it does so violates the required off-shell structure of background and perturbative transformations. In algebraic terminology, the fact that background transformations do act on curvature perturbations means that the symmetries underlying background and perturbation transformations form a semidirect product, but not a direct product.

2. Algebraic structure

Background and perturbative transformations are not independent but are algebraically related. The commutator of two such transformations or of their generating vector fields, given by

$$\left[f(t)\frac{\partial}{\partial t}, \xi^\alpha\frac{\partial}{\partial x^\alpha}\right] = f\dot{\xi}^\alpha\frac{\partial}{\partial x^\alpha} - \dot{f}\xi^0\frac{\partial}{\partial t}, \tag{9}$$

is a perturbative transformation. Using pairs

$$(f, \xi^\alpha) \in \mathcal{V}_{\text{background}} \oplus \mathcal{V}_{\text{pert}} = \mathcal{V} \tag{10}$$

of background and perturbation vector fields, arranged by perturbative order to make the algebraic structure more clear, the combination of both types of transformations is therefore a semidirect product:

$$[(f_1, \xi_1^\alpha), (f_2, \xi_2^\alpha)] = (f_1\dot{f}_2 - f_2\dot{f}_1, \zeta^\alpha), \tag{11}$$

with

$$\zeta^\alpha = f_1\dot{\xi}_2^\alpha - f_2\dot{\xi}_1^\alpha - \delta_0^\alpha(\dot{f}_1\xi_2^0 - \dot{f}_2\xi_1^0) \tag{12}$$

depending on ξ_1 and ξ_2 as well as f_1 and f_2.

This bracket shows that $\mathcal{V}_{\rm background}$ is non-Abelian, with bracket $[f_1,f_2]_{\rm background}=f_1\dot{f}_2-f_2\dot{f}_1$, while $\mathcal{V}_{\rm pert}$ is Abelian, $[\xi_1^\alpha,\xi_2^\alpha]_{\rm pert}=0$. However, the full bracket in $\mathcal{V}$ has an extra term ζ^α, which can be written as $\zeta^\alpha=\phi(f_1)\xi_2^\alpha-\phi(f_2)\xi_1^\alpha$ with the homomorphism

$$\phi(f)\xi^\alpha=f\dot{\xi}^\alpha-\delta_0^\alpha\dot{f}\xi^0 \tag{13}$$

from $\mathcal{V}_{\rm background}$ to the derivations on $\mathcal{V}_{\rm pert}$. (It clearly maps to derivations, because $\mathcal{V}_{\rm pert}$ is Abelian. The homomorphism property can be shown by a direct calculation.) Therefore, the bracket on $\mathcal{V}$ can be written as

$$[(f_1,\xi_1^\alpha),(f_2,\xi_2^\alpha)]=([f_1,f_2]_{\rm background},[\xi_1^\alpha,\xi_2^\alpha]_{\rm pert}+\phi(f_1)\xi_2^\alpha-\phi(f_2)\xi_1^\alpha), \tag{14}$$

identifying

$$\mathcal{V}=\mathcal{V}_{\rm background}\ltimes_\phi\mathcal{V}_{\rm pert} \tag{15}$$

as the semidirect product of the Lie algebras $\mathcal{V}_{\rm background}$ and $\mathcal{V}_{\rm pert}$.

According to Eq. (14), both $\mathcal{V}_{\rm background}$ and $\mathcal{V}_{\rm pert}$ are subalgebras of $\mathcal{V}$, given by the restricted pairs $(f,0)$ and $(0,\xi^\alpha)$, respectively: Brackets $[(f_1,0),(f_2,0)]=([f_1,f_2]_{\rm background},0)$ and $[(0,\xi_1^\alpha),(0,\xi_2^\alpha)]=(0,[\xi_1^\alpha,\xi_2^\alpha]_{\rm pert})$ respect the restricted forms. However, in the full algebra, only the restricted form of $\mathcal{V}_{\rm pert}$ is respected, because for $f_1=0$, $[(0,\xi_1^\alpha),(f_2,\xi_2^\alpha)]=(0,[\xi_1^\alpha,\xi_2^\alpha]_{\rm pert}-\phi(f_2)\xi_1^\alpha)$ has a vanishing background field for any (f_2,ξ_2^α). (Thus, $\mathcal{V}_{\rm pert}$ is not only a subalgebra in $\mathcal{V}$ but also an ideal.) The restricted form of $\mathcal{V}_{\rm background}$ is not preserved in the full algebra, because for $\xi_1^\alpha=0$, $[(f_1,0),(f_2,\xi_2^\alpha)]=([f_1,f_2]_{\rm background},\phi(f_1)\xi_2^\alpha)$ is, generically, no longer of the restricted form. It is therefore impossible to separate background transformations from perturbation transformations.

Mathematically, the algebra $\mathcal{V}$ is not a direct product, which would have a version of the bracket (14) with $\phi=0$, but a semidirect product. If the product were direct, any representation of $\mathcal{V}$ would be a superposition of tensor products of a representation of $\mathcal{V}_{\rm background}$ and a representation of $\mathcal{V}_{\rm pert}$. In particular, spaces invariant under the transformations contained in $\mathcal{V}$ (which are trivial representations of the algebra) could be constructed from tensor products of spaces separately invariant under the transformations contained in $\mathcal{V}_{\rm background}$ and $\mathcal{V}_{\rm pert}$, respectively. This is what the dressed-metric approach assumes, using deparametrization to construct an invariant space for the transformations contained in $\mathcal{V}_{\rm background}$, and curvature perturbations to construct an invariant space for the transformations contained in $\mathcal{V}_{\rm pert}$. However, since the required product of transformations is semidirect but not direct, the approach implements incorrect transformation properties. For instance, this approach implies a vanishing bracket of a background transformation and a perturbation transformation, while Eq. (14) requires $[(f_1,0),(0,\xi_2^\alpha)]=(0,\phi(f_1)\xi_2^\alpha)$.

To summarize, we do not have a direct product that could be implemented by separate treatments of invariance, such as deparametrization for the background and curvature perturbations for the inhomogeneous fields. While the dressed-metric approach is formally consistent in that it eliminates the relevant transformations, it does so incorrectly by ignoring their interrelated off-shell nature. In the next section, we will demonstrate explicitly that there is no off-shell completion of the attempted invariance proposed by the dressed-metric approach, but first we review the off-shell structure in a canonical setting.

B. Canonical formulation

One might think that the canonical formulation should not have a nonzero commutator of background and perturbative transformations, because fields on a fixed spatial slice do not have any time dependence, such that the time derivatives on the right-hand side of Eq. (9) vanish. (Time dependence in canonical transformations is not explicit but is implemented by an additional term added to the usual constraints which depends on the momenta of lapse and shift and has coefficients given by initial values of time derivatives of the fields [20,21].) However, the canonical description must be equivalent to the four-dimensional formulation, and therefore it should give rise to a related semidirect product of background and perturbation transformations. The main mathematical difference is that canonical transformations form a Lie algebroid [22,23] rather than a Lie algebra.

1. Algebroid

Geometrically, the product remains semidirect, not because of time derivatives but because the canonical generators, given by constraints, refer to directions normal to spatial slices rather than time directions determined by a coordinate [24]. As a consequence, a perturbative inhomogeneous transformation changes the normal directions, such that a subsequent background transformation acquires new directions compared with one applied before the inhomogeneous transformation; see Fig. 1.

The specific commutator follows from a restriction of the full hypersurface deformation brackets of the Hamiltonian and diffeomorphism constraints, $H[N]$ and $D[M^a]$. We obtain background transformations by applying the Hamiltonian constraint to homogeneous lapse functions $\bar{N}$, while perturbative inhomogeneous constraints are obtained by specializing the Hamiltonian constraint to a small inhomogeneous perturbation, δN, and the diffeomorphism constraint to a small inhomogeneous vector field, δM^a. The leading

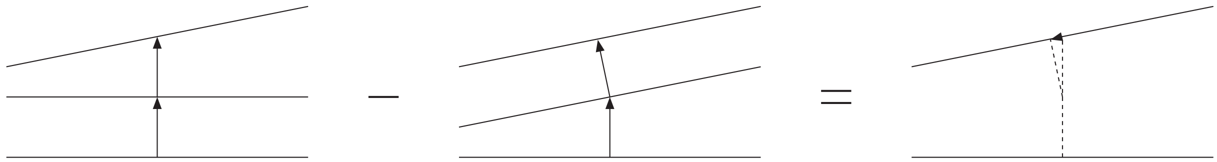

FIG. 1. Nonzero commutator of a homogeneous background transformation and a perturbative transformation (here, linear spatial dependence), equal to a nonzero spatial displacement.

perturbative expressions are then obtained by expanding the constraints $H[\delta N]$ and $D[\delta M^a]$ up to quadratic dependence on the fields, counting δN and δM^a as first-order contributions. The general bracket [25]

$$[H[N_1], H[N_2]] = D[q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)] \tag{16}$$

with the inverse spatial metric $q^{ab} = a^{-2}\delta^{ab}$ then turns into

$$[H[\bar{N}], H[\delta N]] = D[a^{-2}\bar{N}\partial^a\delta N] \tag{17}$$

with a nonzero right-hand side. More generally,

$$[H[\bar{N}_1 + \delta N_1], H[\bar{N}_2 + \delta N_2]] = D[a^{-2}(\bar{N}_1\partial^a\delta N_2 - \bar{N}_2\partial^a\delta N_1)], \tag{18}$$

while all brackets involving $D[\delta M^a]$ are zero to first perturbative order.

The brackets [Eq. (18)] are formulated for generators of symmetries labeled by five spatial functions, given by $\bar{N}$, δN, and the three components of δM^a. These functions correspond to the background function f and four components of the space-time vector field ξ^a encountered in the four-dimensional formulation. However, algebraically the bracket in Eq. (18) is quite different from Eq. (14) not only in its specific form, but also because the generator on the right-hand side depends on the scale factor a (or the spatial metric) in addition to the functions $\bar{N}$, δN, and δM^a that determine a transformation. In physics terminology, the bracket belongs to a Lie algebra with structure functions (depending on a), not structure constants. Mathematically, a well-defined algebraic object can be constructed by proposing that $\bar{N}$, δN, and δM^a are not only functions on space, but also functions of the metric. The bracket (18) is then meaningful because the factor of a^{-2} on the right-hand side is then no longer an external function but merely changes the dependence of $\bar{N}$, δN, and δM^a on a.

The notion of a Lie algebroid [26] formalizes this motivation. Because the functions $\bar{N}$, δN, and δM^a (now also depending on the spatial metric in addition to the spatial position) can be added for any fixed metric, they form a vector space. If the metric is allowed to vary, the functions $\bar{N}$, δN, and δM^a form a vector bundle over the space of metrics with five-dimensional fibers (not counting the spatial dependence of δN and δM^a). The bracket in Eq. (18) implies that sections of the vector bundle, picking one choice of $(\bar{N}, (\delta N, \delta M^a))$ for every metric, form a Lie algebra. In addition, the fibers are related to the tangent space of the space of metrics, because a choice of $(\bar{N}, (\delta N, \delta M^a))$ defines a space-time vector field and therefore a Lie derivative of the metric. The corresponding map from the fibers to the tangent space of the space of metrics is called the anchor map of the Lie algebroid, and it obeys certain relationships with the bracket; see Eq. (21) below.

In the canonical formulation, the background bracket is Abelian because $[H[\bar{N}_1], H[\bar{N}_2]] = 0$ for any spatially constant $\bar{N}_1$ and $\bar{N}_2$. The generators of perturbative inhomogeneous transformations, $H[\delta N_1]$ and $H[\delta N_2]$, also form an Abelian Lie algebra, because the right-hand side of Eq. (16) vanishes to the order considered here when both $N_1 = \delta N_1$ and $N_2 = \delta N_2$ are of first order. In particular, the individual brackets have structure constants and do not require a Lie-algebroid treatment. The Lie-algebroid structure of the full bracket (16) therefore seems to remain only in the nontrivial relation [Eq. (17)] between background and perturbation generators. It is nevertheless possible to interpret both background and perturbations as Lie algebroids, $\mathcal{E}_{\rm background}$ and $\mathcal{E}_{\rm pert}$, respectively, over the same base manifold $X_{\rm pert}$ of perturbed metrics. (Background metrics might seem sufficient for $\mathcal{E}_{\rm background}$, but using the same base manifold for $\mathcal{E}_{\rm background}$ and $\mathcal{E}_{\rm pert}$ is convenient for the construction of a semidirect product.)

With a base manifold of metrics, Eq. (18) determines the algebroid bracket only for constant sections—that is, $\bar{N}$ and $(\delta N, \delta M^a)$, which do not depend on the metric (while the perturbations δN and δM^a may always depend on the spatial position). If we allow metric-dependent functions, the Lie algebroid $\mathcal{E}_{\rm background}$ is no longer Abelian because the background part of the bracket (18) should then be generalized to

$$[H[\bar{N}_1], H[\bar{N}_2]] = H[\bar{N}_1\delta_n\bar{N}_2 - \bar{N}_2\delta_n\bar{N}_1], \tag{19}$$

where $\delta_n N = (\partial N/\partial q_{ab})\mathcal{L}_n q_{ab}$ is the normal derivative of N, constructed by the chain rule using the Lie derivative $\mathcal{L}_n$ along the vector field normal to hypersurfaces. (This extension can be derived from the Poisson bracket of Hamiltonian constraints with metric-dependent lapse functions.) The Lie algebroid $\mathcal{E}_{\rm pert}$ remains Abelian because the right-hand side of an equation analogous to Eq. (19) with $\bar{N}$ replaced by δN would be of second order.

The anchor map of a Lie algebroid $\mathcal{E}$, defined as

$$\rho:\Gamma(T\mathcal{E}) \to \Gamma(TX) \tag{20}$$

such that

$$[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 \tag{21}$$

for any $e_1, e_2 \in \Gamma(T\mathcal{E})$ and $f \in C^1(X)$, is necessarily zero for Abelian brackets—that is, for $\mathcal{E}_{\rm pert}$ in our case. The non-Abelian bracket of $\mathcal{E}_{\rm background}$ is compatible with the anchor map $\bar{N} \mapsto \delta q_{ab} = \bar{N}\mathcal{L}_n q_{ab}$. These two anchor maps are equivalent to the first-order perturbative content of the full anchor, given by $(N, M^a) \mapsto \delta q_{ab} = \mathcal{L}_{Nn+M} q_{ab}$ [22].

Abstractly, we denote elements in the fiber of the first Lie algebroid, $\mathcal{E}_{\rm background}$, simply by $\bar{N} \in \mathbb{R}$. Elements of fiber of the second Lie algebroid, $\mathcal{E}_{\rm pert}$ which is Abelian, are given by $(\delta N, \delta M^a)$, where δN and δM^a depend on the spatial position and therefore form infinite-dimensional fibers. The map

$$\psi(\bar{N})(\delta N, \delta M^a) = (0, a^{-2}\bar{N}\partial^a \delta N) \tag{22}$$

defines a Lie algebroid morphism from $\mathcal{E}_{\rm background}$ to the derivations on $\mathcal{E}_{\rm pert}$. (This map is well defined because background metrics, parametrized by the scale factor a, are included in both base manifolds. It maps to derivations because $\mathcal{E}_{\rm pert}$ is Abelian. In order to show the morphism property, note that $\delta_n \bar{N}$ is of first order, such that $\delta_n \bar{N}\partial^a \delta N \sim 0$ is of second order and therefore treated as zero.)

We can now combine the bracket (18) for nonzero background perturbations with the bracket (19) for metric-dependent background functions, writing them directly for the generators $(\bar{N}, (\delta N, \delta M^a))$. Also using vanishing brackets involving spatial deformations at the perturbative level, we obtain

$$\begin{aligned}&[(\bar{N}_1, (\delta N_1, \delta M_1^a)), (\bar{N}_2, (\delta N_2, \delta M_2^a))]\\ &\quad = ([\bar{N}_1, \bar{N}_2], \psi(\bar{N}_1)(\delta N_2, \delta M_2^a) - \psi(\bar{N}_2)(\delta N_1, \delta M_1^a)),\end{aligned} \tag{23}$$

where $[\bar{N}_1, \bar{N}_2] = \bar{N}_1 \delta_n \bar{N}_2 - \bar{N}_2 \delta_n \bar{N}_1$. [For instance, Eq. (18) is included in this equation if we set $\delta M_1^a = 0 = \delta M_2^a$ and assume metric-independent $\bar{N}_1$ and $\bar{N}_2$, such that $[\bar{N}_1, \bar{N}_2] = 0$. The bracket (19) is obtained if only $\bar{N}_1$ and $\bar{N}_2$ are nonzero.] The general form of the bracket (23) is the same as the bracket of a semidirect product of Lie algebroids defined in Ref. [27], where the analog of $\mathcal{E}_{\rm pert}$ (but not of $\mathcal{E}_{\rm background}$) is required to be Abelian in order to avoid obstructions. The general construction determines a semidirect product with an anchor map inherited directly from $\mathcal{E}_{\rm background}$, just as we have found here. We therefore have shown that

$$\mathcal{E} = \mathcal{E}_{\rm background} \ltimes_\psi \mathcal{E}_{\rm pert}. \tag{24}$$

Comparing with the four-dimensional perspective, although the precise algebraic structure of canonical transformations is rather different from that found in Sec. II A 2, the bracket of a semidirect product of background and perturbation transformations is obtained in both cases. The structure in the canonical approach is rather different than in the four-dimensional formulation, but it has the same implications for background and perturbation transformations. In particular, only perturbation transformations form an ideal in the full algebroid $\mathcal{E}$, while background transformations form a subalgebroid that is not an ideal. According to Eq. (23), a dressed-metric-like approach that implicitly assumes a direct product incorrectly implements brackets of the form $[(\bar{N}, (0,0)), (0, (\delta N, \delta M^a))] = (0, \psi(\bar{N})(\delta N, \delta M^a) = (0, (0, a^{-2}\bar{N}\partial^a \delta N))$, instead assuming a zero bracket.

2. Poisson structure

A formal derivation of the crucial equation (17) through Poisson brackets of phase-space representations of the hypersurface deformation generators shows the interplay of different perturbative orders in this result. Following the formalism developed in Refs. [28,29] or [30] for canonical perturbation theory in metric variables, we coordinatize the gravitational phase space in triad form, given by the components E^a_i of a densitized triad and the corresponding components of extrinsic curvature, K^i_a. With perturbative inhomogeneity, we write $E^a_i = p\delta^a_i + \delta E^a_i$ and $K^i_a = k\delta^i_a + \delta K^i_a$, where the background variables p and k depend only on time and their internal frame has been fixed by choosing the background fields to be proportional to the Kronecker delta.

Notice that δE^a_i and δK^i_a describe the full inhomogeneity contained in the perturbative treatment. That is, these fields will be assumed to be small compared with background fields, but not split into a hierarchy of the form $\delta f = \delta f^{(1)} + \delta f^{(2)} + \cdots$ of linear and higher-order perturbations. Such a hierarchy, which is often used in cosmology in order to obtain linear higher-order equations, is not possible in our context, because we will need a phase-space structure for inhomogeneity. Since individual orders in $\delta f = \delta f^{(1)} + \delta f^{(2)} + \cdots$ do not constitute independent degrees of freedom, they do not permit a phase-space structure and therefore cannot be used in our calculations. After deriving consistent Hamiltonians for some inhomogeneity δf, a hierarchy $\delta f^{(1)} + \delta f^{(2)} + \cdots$ could be introduced at the level of equations of motion, if desired.

For simplicity, we will assume that $p > 0$, fixing the orientation of space. The background variables can then be derived from the fields by integrating over a fixed spatial region $\mathcal{V}$ of coordinate volume $V_0 = \int_{\mathcal{V}} \mathrm{d}^3x$:

$$p = \frac{1}{V_0}\int_{\mathcal{V}} E^a_i \delta^i_a \mathrm{d}^3x, \qquad k = \frac{1}{V_0}\int_{\mathcal{V}} K^i_a \delta^a_i \mathrm{d}^3x. \tag{25}$$

In order to avoid double-counting the background variables, we impose linear second-class constraints

$$\int_{\mathcal{V}} \delta E^a_i \delta^i_a \mathrm{d}^3x = 0 = \int_{\mathcal{V}} \delta K^i_a \delta^a_i \mathrm{d}^3x \tag{26}$$

on the perturbation fields. With these conditions, we obtain the basic Poisson brackets

$$\{k, p\} = \frac{8\pi G}{3V_0},$$
$$\{\delta K^i_a(x), \delta E^b_j(y)\} = \delta^b_a \delta^i_j \left(\delta(x, y) - \frac{1}{V_0}\right). \tag{27}$$

(The subtraction of the constant $1/V_0$ refers to the Dirac bracket of fields subject to linear second-class constraints, but it will not contribute to the following calculations.)

For a spatially flat isotropic model in triad form, we have the background constraint

$$\bar{H} = -\frac{3V_0}{8\pi G}\sqrt{p}k^2, \tag{28}$$

the first-order constraint

$$H^{(1)}[\delta N] = \frac{1}{16\pi G}\int \mathrm{d}^3x \delta N \left(-4k\sqrt{p}\delta^c_j \delta K^j_c - \frac{k^2}{\sqrt{p}}\delta^j_c \delta E^c_j + \frac{2}{\sqrt{p}}\partial_c \partial^j \delta E^c_j\right), \tag{29}$$

and the second-order constraint

$$H^{(2)}[\bar{N}] = \frac{\bar{N}}{16\pi G}\int \mathrm{d}^3x \left(\sqrt{p}\delta K^j_c \delta K^k_d \delta^d_k \delta^c_j - \sqrt{p}(\delta K^j_c \delta^c_j)^2 - 2\frac{k}{\sqrt{p}}\delta E^c_j \delta K^j_c - \frac{k^2}{2p^{3/2}}\delta E^c_j \delta E^d_k \delta^k_c \delta^j_d + \frac{k^2}{4p^{3/2}}(\delta E^c_j \delta^c_j)^2 - \frac{1}{2p^{3/2}}\delta^{jk}(\partial_c \delta E^c_j)(\partial_d \delta E^d_k)\right). \tag{30}$$

Moreover, the first-order diffeomorphism constraint is

$$D[\delta M^c] = \frac{1}{8\pi G}\int_{\mathcal{V}} \mathrm{d}^3x \delta M^c (p\delta^d_k \partial_c \delta K^k_d - p\partial_j \delta K^j_c - k\delta^j_c \partial_d \delta E^d_j). \tag{31}$$

The background diffeomorphism constraint vanishes identically, and no second-order expression is required for our purposes.

Let us first consider only the background constraint, $\bar{N}\bar{H}$, and the first-order constraint, $H^{(1)}[\delta N]$, in the Poisson bracket

$$\{\bar{N}\bar{H}, H^{(1)}[\delta N]\} = \frac{1}{16\pi G}\int_{\mathcal{V}} \mathrm{d}^3x \bar{N}\delta N \left(2k^2\delta^c_j \delta K^j_c - 2\frac{k^3}{p}\delta^j_c \delta E^c_j + 2\frac{k}{p}\partial_c \partial^j \delta E^c_j\right). \tag{32}$$

It is easy to see that this bracket, which is a first-order expression, is not a linear combination of the available first-order constraints, $H^{(1)}[\delta N]$ and $D[\delta M^c]$. Therefore, if we combine only background and first-order constraints, we not only fail to produce the correct bracket [Eq. (17)] of perturbative hypersurface deformations, but worse, obtain an anomalous gauge system in which the constraint brackets do not close.

This problem can easily be solved by realizing that the second-order constraint $H^{(2)}[\bar{N}]$, while it can be ignored in the constraint equations imposed on first-order dynamics, should be included in the constraint brackets because its Poisson bracket with a first-order constraint is of first order. The second-order constraint therefore contributes to the first-order gauge flow relevant for a theory of first-order perturbations. Indeed, the Poisson bracket

$$\{H^{(2)}[\bar{N}], H^{(1)}[\delta N]\} = \frac{1}{32\pi G}\int_{\mathcal{V}} \mathrm{d}^3x \bar{N}\left(\delta N\left(-8k^2\delta^c_j \delta K^j_c + 4\frac{k^3}{p}\delta^j_c \delta E^c_j + 4\frac{k}{p}\partial_c \partial^j \delta E^c_j + 4k^2\delta^c_j \delta K^j_c\right) + 4\left(\delta K^j_c \partial_j \partial^c \delta N - \delta^c_j \delta K^j_c \delta^d_k \partial_d \partial^k \delta N - \frac{k}{p}\delta E^c_j \partial_c \partial^j \delta N\right)\right) \tag{33}$$

provides just the right terms for Eqs. (32) and (33) to combine into

$$\{\bar{N}\,\bar{H}+H^{(2)}[\bar{N}],H^{(1)}[\delta N]\} = \frac{1}{8\pi G}\int_{\mathcal{V}} \mathrm{d}^3x \frac{\bar{N}\partial^c\delta N}{p}(p\delta^d_k\partial_c\delta K^j_d - p\partial_j\delta K^j_c - k\delta^j_c\partial_d\delta E^d_j)$$
$$= D[p^{-1}\bar{N}\partial^c\delta N], \tag{34}$$

equivalent to Eq. (17).

With hindsight, the result of this rather technical calculation is not surprising if one only considers that a second-order constraint can generate a first-order gauge flow. Together with the general condition that all flows of the same order should be included on the same footing, it is clear that one cannot obtain an anomaly-free constrained system to first order if only the background and first-order constraints are included. In our following discussion, it will be useful to see the presented details of how this calculation works in order to rule out the specific proposal made in the dressed-metric approach.

The result [Eq. (34)] is closely related to our discussion of semidirect products because it can be considered a representation of Eq. (18) by specific phase-space functions $\bar{N}\,\bar{H}+H^{(2)}[\bar{N}]$ and $H^{(1)}[\delta N]$ instead of abstract generators $H[\bar{N}+\delta N]$. As the derivation shows, we need very specific relationships between the coefficients in $\bar{N}\,\bar{H}+H^{(2)}[\bar{N}]$ and $H^{(1)}[\delta N]$ for the nontrivial right-hand side to follow. Had the product of background and perturbation generators been direct, a simple representation of a single background Hamiltonian would have been sufficient, only required to commute with the first-order perturbation Hamiltonian. As already noticed, such a behavior has been implicitly assumed in Ref. [1], but it is erroneous on grounds of both the abstract reasoning of Secs. II A 2 and II B 1, and the specific representation considered here.

III. THE METRIC'S NEW CLOTHES

In Riemannian geometry, the metric $g_{\alpha\beta}$ is subject to the tensor transformation law such that the line element

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta \tag{35}$$

is invariant with respect to coordinate changes, $\mathrm{d}x^{\alpha'} = (\partial x^{\alpha'}/\partial x^\alpha)\mathrm{d}x^\alpha$. The line element therefore provides a coordinate-independent meaning of distances on which Riemannian geometry is based. In a geometrical field theory such as general relativity, this important condition on the metric is an off-shell property which cannot be tested if one restricts one's attention only to solutions of the canonical constraints or to Dirac or other observables.

If the theory is quantized canonically, coordinate transformations are unmodified, because the space-time coordinates x^α are not phase-space functions. (We ignore here the possibility that one might wish to modify the geometry in addition to canonically quantizing gravity—for instance, by making it noncommutative. Such a procedure would go beyond standard canonical quantization, and it is certainly not envisioned in Ref. [1].) Some of the components of $g_{\alpha\beta}$, however, represent phase-space degrees of freedom and may therefore be subject to quantum corrections not only in their dynamics but also in their behavior under gauge transformations. The covariance question in canonical quantizations of gravity therefore asks whether a quantum modified (or dressed) $\tilde{g}_{\alpha\beta}$ has *off-shell* transformations consistent with coordinate transformations. If this question is not answered in the affirmative, the standard interpretation of the metric through a line element is no longer available, demoting $\tilde{g}_{\alpha\beta}$ to a purely formal object without geometrical significance.

In Ref. [1], different versions of line elements have uncritically been introduced for modified metrics without asking the covariance question. In fact, since the formalism defined in Ref. [1] is purely on-shell, using deparametrization of the background dynamics together with Bardeen potentials or curvature perturbations, it is not amenable to a direct test of covariance. This lack of control on an important physical requirement may in itself present a good reason to discard the dressed metric.

It is possible to go even further and show that the modified dynamics used by the dressed-metric approach in order to obtain bouncing background solutions cannot represent on-shell solutions of a covariant off-shell theory. To do so, we use the canonical version of the tensor-transformation law dual to standard coordinate transformations, given by gauge generators subject to hypersurface deformation brackets. As we have already seen, perturbative inhomogeneity to first order requires us to use the Hamiltonian constraint up to second order because a second-order contribution may well generate a first-order flow. The dressed-metric approach is halfway aware of this important fact because it derives a dynamical flow using second-order generators, determining the dynamical vector field

$$X^\alpha_{\mathrm{Dyn}} = \Omega^{\alpha\beta}_o\partial_\beta S_o[N_{\mathrm{hom}}] + \Omega^{\alpha\beta}_1\partial_\beta S'_2[N_{\mathrm{hom}}] \tag{36}$$

in the notation of Ref. [1]. The generator S'_2 corresponds to our $H^{(2)}$, but it is written in terms of curvature perturbations, $T_{\vec{k}}$, and their momenta, $P_{\vec{k}}$, for tensor modes:

$$S_2'[a^3\ell^3/p_\phi] = \frac{1}{2}\sum_{\vec{k}}\left(4\frac{\kappa}{p_\phi}|P_{\vec{k}}|^2 + \frac{k^2}{4\kappa}\frac{a^4}{p_\phi}|T_{\vec{k}}|^2\right), \quad (37)$$

using the choice of lapse function, $N_{\rm hom} = a^3\ell^3/p_\phi$, preferred in Ref. [1]. Here, p_ϕ is the constant background momentum of the free, massless scalar field used for deparametrization, while $\kappa = 8\pi G$ and ℓ is a length parameter that is not relevant for our purposes. (The discussion for scalar modes is very similar to tensor modes once curvature perturbations are used. The only difference would be in the background functions, which are no longer given directly in terms of a by solutions of certain differential equations. Since our arguments in what follows do not depend on the specific form of background coefficients, the simpler case of tensor modes is sufficient.)

Quantization is then performed separately for S_o and S_2'. The background generator S_o, or our $\bar{H}$, is modified by loop quantization, replacing its quadratic momentum dependence in Eq. (28) with a bounded function. (The precise form of this modification does not matter for the arguments given below.) The perturbation part S_2', however, remains quadratic in momenta and has only slightly modified coefficients:

$$\tilde{S}_2'[a^3\ell^3/p_\phi] = \frac{1}{2}\sum_{\vec{k}}\Big(4\kappa\langle\hat{p}_\phi^{-1}\rangle|P_{\vec{k}}|^2 + \frac{k^2}{4\kappa}\langle\hat{p}_\phi^{-1/2}\hat{a}^4\hat{p}_\phi^{-1/2}\rangle|T_{\vec{k}}|^2\Big), \quad (38)$$

where background operators are reduced to (internal) time-dependent functions by taking expectation values in a background state. The same expectation values are then used to define a dressed metric in the proposed line element

$$\begin{aligned} \mathrm{d}\tilde{s}^2 &= \tilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b \\ &= -\ell^6\langle\hat{p}_\phi^{-1}\rangle^{1/2}\langle\hat{p}_\phi^{-1/2}\hat{a}^4\hat{p}_\phi^{-1/2}\rangle^{3/2}\mathrm{d}\phi^2 \\ &\quad + \langle\hat{p}_\phi^{-1}\rangle^{-1/2}\langle\hat{p}_\phi^{-1/2}\hat{a}^4\hat{p}_\phi^{-1/2}\rangle^{1/2}\mathrm{d}\vec{x}^2, \end{aligned} \quad (39)$$

such that the coefficients in Eq. (38) correspond to the classical expression if one were to use the dressed metric to compute it. [The proposal in Ref. [1] also includes a metric operator such that

$$\mathrm{d}\hat{s}^2 = \hat{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\ell^6\hat{p}_\phi^{-1}\hat{a}^6\hat{p}_\phi^{-1}\mathrm{d}\phi^2 + \hat{a}^2\mathrm{d}\vec{x}^2. \quad (40)$$

However, since geometrical procedures do not measure operators, this object does not have any well-defined meaning, other than that it produces Eq. (39) as a formal expectation value.]

The coefficients of the dressed metric are background functions and are therefore modified if one inserts solutions of the holonomy-modified background constraint. Moreover, there are state-dependent quantum corrections in these coefficients, defined through expectation values, which could be derived systematically in a moment expansion in the framework of effective canonical constraints; see for instance Refs. [31–33]. However, these two quantum corrections cannot counter modifications of the background constraint so as to produce the bracket (17), for the following reasons:

(1) The off-shell behavior of the metric does not depend on what kind of background solutions are entered, and therefore it does not know about holonomy modifications. For the off-shell behavior, relevant for covariance, coefficients in Eq. (39) depending on a and p_ϕ (and possibly their moments) are merely phase-space coordinates, just like the corresponding functions in the modified background constraint. The off-shell theory of the dressed-metric approach therefore corresponds to a system in which only the background constraint, $\bar{H}$, has been modified by using holonomies, but not the second-order constraint, $H^{(2)}$. Moreover, also the first-order constraint, $H^{(1)}$, is unmodified because Ref. [1] uses the classical curvature perturbations without modifications that would result if gauge transformations generated by $H^{(1)}$ were modified; see Refs. [29,34]. The bracket (33) then remains unchanged while Eq. (32) is modified, eliminating important cancellations that led to the combined result in Eq. (34). The dressed-metric approach functions by modifying only the background constraint, making it impossible to realize a valid version of the perturbative hypersurface deformation bracket [Eq. (17)].

(2) If moments of a state that result from a systematic semiclassical expansion of the expectation values in Eq. (38) were to counter the background modification, they would have to be fixed, severely restricting the class of quantum states that are allowed to propagate. Even if there were moments such that the bracket (17) could be closed after background modifications, the resulting mismatch of classical and quantum degrees of freedom would amount to an anomaly. (Recall that an anomaly in a constraint system implies that the system becomes overconstrained, imposing an additional constraint such as $\{\bar{N}\bar{H}, H^{(1)}[\delta N]\} = 0$ if the left-hand side is no longer zero on the solutions space of the original constraints.)

In addition to violating covariance, the dressed metric has the following problem: It depends on the ordering chosen for operators in the expectation value components. Moreover, for different background gauges, corresponding to different phase-space functions for the background lapse $\bar{N}$, different operator products appear, giving rise to different ordering ambiguities. Therefore, choosing a different background gauge in general results in an inequivalent dressed metric. Ordering issues can potentially be ignored if one uses sharply peaked states, such that fluctuation terms are negligible. Such an assumption is sometimes suggested by the dressed-metric approach, as in "one knows that there exist background quantum geometries Ψ_o which are *very* sharply peaked" (emphasis in Ref. [1]). However, this assumption is not justified in the Planck regime [35–37], where a dressed metric would be most relevant. (Claims as expressed in the quotation are sometimes applied even in the Planck regime, but they are based on the erroneous, and usually implicit, assumption that a large comoving volume may be assumed for a homogeneous background even at large curvature. This assumption is inconsistent with the generic behavior close to a spacelike singularity as described by the Belinskii-Khalatnikov-Lifshitz (BKL) scenario [38], according to which homogeneous dynamics may approximately be assumed at any point, but only in asymptotically small regions without a positive lower bound on the comoving volume. While the local classical dynamics does not depend on the size of the comoving volume, quantum fluctuations—and therefore the availability of sharply peaked states—do.)

IV. CONCLUSIONS

Covariance in canonical quantum gravity is a subtle issue. It requires a formulation of quantum effects such that the classical hypersurface deformation brackets [Eq. (16)] are obtained in the classical limit of the theory, while a closed, anomaly-free set of brackets is realized for nonzero $\hbar$ which vanishes when the constraints are solved but is not necessarily of the classical form. This statement includes two conditions, which cannot always both be met. For instance, a possible Abelianization of the bracket in some midisuperspace models [39,40] always leads to anomaly-free quantum constraints but even then is not guaranteed to be compatible with covariance [41,42]. Although such quantum theories in the latter case are formally consistent as quantizations of constrained systems, they cannot be interpreted as models of quantum space-time because there is no well-defined sense in which they are covariant.

As an alternative to realizations of the hypersurface deformation brackets, analog actions in space-time tensor form, such as certain scalar-tensor theories, have been proposed as a possible way to demonstrate covariance. However, while such analog actions may work in simple, isotropic models with a small number of degrees of freedom, in all known cases they fail to describe anisotropic models or perturbative inhomogeneity correctly. For instance, the Palatini-$f(R)$ model proposed in Ref. [43] claimed to show that loop quantum cosmology is covariant—is equivalent to a scalar-tensor theory with a nondynamical scalar [44] in which any correction to general relativity amounts to a simple cosmological constant in vacuum models. It therefore cannot possibly describe holonomy modifications in anisotropic vaccum models, ruling it out as a possible covariant version of loop quantum cosmology. More recent analog actions [45,46] based on mimetic gravity [47,48] again work in isotropic models but fail to describe anisotropies or perturbative inhomogeneity correctly [49–51].

As shown here, the dressed-metric approach also fails to provide a covariant version of perturbative inhomogeneity in loop quantum cosmology—in particular, in the presence of holonomy modifications of the background dynamics that may make it possible to have bouncing solutions. Although we have focused on the specific formulation described in Ref. [1] for technical details of the constructions, similar arguments apply to related (or precursor) formulations in Refs. [52,53] or the "hybrid" approach [54–56], which share with the dressed-metric approach the crucial feature of separating the background degrees of freedom from inhomogeneous modes, making it impossible to implement the key relation (17), which belongs to a semidirect product of Lie algebroids. Our derivations specifically show that such approaches cannot yield a covariant first-order perturbation theory. Since a theory that admits a perturbative treatment is covariant if and only if it is covariant at all orders, our results are sufficient to rule out the existence of any covariant theory that could complete dressed-metric approaches to higher orders of inhomogeneity: If terms violating covariance at first order were canceled by higher orders, the assumption that the theory permits a valid perturbative treatment would not be met. Only a completely nonperturbative theory might then be able to capture effects such as those imagined in a dressed-metric-like treatment, but since it would not allow an expansion by perturbative inhomogeneity, the equations of the dressed-metric approach would not approximate this theory in any way.

Our result adds to mounting evidence that models of loop quantum gravity cannot be covariant without drastic modifications of space-time structure; see also Refs. [57,58]. It is sometimes suggested that a noncovariant model which implements some quantum effects in an otherwise consistent way may be useful as a "first approximation" to a complicated formulation of cosmological dynamics in full quantum gravity. However, violating an important consistency condition such as covariance is not an approximation at all, because it usually gives rise to uncontrolled, spurious solutions that overshadow the relevant behavior, or to overconstrained dynamics. (See also Ref. [59] for a similar result in a different setting.) As an example, covariant versions of holonomy-modified models of loop quantum gravity, derived in Refs. [34,60–63], generically imply signature change at Planckian density. The would-be bounce is then a four-dimensional Euclidean region in which no deterministic evolution exists [64,65]. Nondeterministic behavior is an example for an effect that cannot be considered a small correction to modified but still deterministic dynamics, even if the modes used to determine the structure of space-time and propagation properties are perturbative.

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[1] I. Agulló, A. Ashtekar, and W. Nelson, Phys. Rev. D **87**, 043507 (2013).
[2] M. Bojowald, Living Rev. Relativity **11**, 4 (2008).
[3] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[4] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
[5] P. A. M. Dirac, Can. J. Math. **2**, 129 (1950).
[6] W. F. Blyth and C. J. Isham, Phys. Rev. D **11**, 768 (1975).
[7] P. Malkiewicz, Classical Quantum Gravity **29**, 075008 (2012).
[8] P. Malkiewicz, Classical Quantum Gravity **32**, 135004 (2015).
[9] M. Bojowald and T. Halnon, Phys. Rev. D **98**, 066001 (2018).
[10] S. Gielen and L. Menéndez-Pidal, arXiv:2005.05357.
[11] M. Bojowald, P. A. Höhn, and A. Tsobanjan, Classical Quantum Gravity **28**, 035006 (2011).
[12] M. Bojowald, P. A. Höhn, and A. Tsobanjan, Phys. Rev. D **83**, 125023 (2011).
[13] P. A. Höhn, E. Kubalova, and A. Tsobanjan, Phys. Rev. D **86**, 065014 (2012).
[14] F. Giacomini, A. Castro-Ruiz, and C. Brukner, Nat. Commun. **10**, 494 (2019).
[15] A. Vanrietvelde, P. A. Hoehn, F. Giacomini, and E. Castro-Ruiz, Quantum **4**, 225 (2020).
[16] A. Vanrietvelde, P. A. Hoehn, and F. Giacomini, arXiv: 1809.05093.
[17] P. A. Hoehn, A. R. H. Smith, and M. P. E. Lock, arXiv:1912 .00033.
[18] J. M. Stewart, Classical Quantum Gravity **7**, 1169 (1990).
[19] The relevance of this fact for the dressed-metric approach has been pointed out privately by Robert Brandenberger and Neil Turok.
[20] J. M. Pons, D. C. Salisbury, and L. C. Shepley, Phys. Rev. D **55**, 658 (1997).
[21] M. Bojowald, *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity* (Cambridge University Press, Cambridge, England, 2010).
[22] C. Blohmann, M. C. Barbosa Fernandes, and A. Weinstein, Commun. Contemp. Math. **15**, 1250061 (2013).
[23] J. M. Glowacki, arXiv:1905.03152.
[24] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), reprinted in Ref. [66].
[25] P. A. M. Dirac, Proc. R. Soc. A **246**, 333 (1958).
[26] J. Pradines, C. R. Acad. Sci. Paris Sér. A–B **264**, A245 (1967).
[27] E. Aldrovandi, U. Bruzzo, and V. Rubtsov, J. Algebra **505**, 456 (2018).
[28] M. Bojowald, G. Hossain, M. Kagan, and S. Shankaranarayanan, Phys. Rev. D **78**, 063547 (2008).
[29] M. Bojowald, G. Hossain, M. Kagan, and S. Shankaranarayanan, Phys. Rev. D **79**, 043505 (2009).
[30] D. Langlois, Classical Quantum Gravity **11**, 389 (1994).
[31] M. Bojowald and A. Skirzewski, Rev. Math. Phys. **18**, 713 (2006).
[32] M. Bojowald, B. Sandhöfer, A. Skirzewski, and A. Tsobanjan, Rev. Math. Phys. **21**, 111 (2009).
[33] M. Bojowald and A. Tsobanjan, Phys. Rev. D **80**, 125008 (2009).
[34] T. Cailleteau, L. Linsefors, and A. Barrau, Classical Quantum Gravity **31**, 125011 (2014).
[35] M. Bojowald, J. Cosmol. Astropart. Phys. 01 (2019) 026.
[36] M. Bojowald, Universe **5**, 44 (2019).
[37] M. Bojowald, Universe **6**, 36 (2020).
[38] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifschitz, Adv. Phys. **31**, 639 (1982).
[39] R. Gambini and J. Pullin, Phys. Rev. Lett. **110**, 211301 (2013).
[40] R. Gambini, J. Olmedo, and J. Pullin, Classical Quantum Gravity **31**, 095009 (2014).
[41] M. Bojowald, S. Brahma, and J. D. Reyes, Phys. Rev. D **92**, 045043 (2015).
[42] M. Bojowald and S. Brahma, Phys. Rev. D **92**, 065002 (2015).
[43] G. J. Olmo and P. Singh, J. Cosmol. Astropart. Phys. 01 (2009) 030.
[44] G. J. Olmo, Phys. Rev. Lett. **95**, 261102 (2005).
[45] N. Bodendorfer, A. Schäfer, and J. Schliemann, Phys. Rev. D **97**, 084057 (2018).
[46] D. Langlois, H. Liu, K. Noui, and E. Wilson-Ewing, Classical Quantum Gravity **34**, 225004 (2017).
[47] A. H. Chamseddine and V. Mukhanov, J. High Energy Phys. 11 (2013) 135.
[48] L. Sebastiani, S. Vagnozzi, and R. Myrzakulov, Adv. High Energy Phys. **2017**, 3156915 (2017).
[49] N. Bodendorfer, F. M. Mele, and J. Münch, Classical Quantum Gravity **35**, 225001 (2018).
[50] J. Haro, L. Aresté Saló, and S. Pan, Gen. Relativ. Gravit. **51**, 49 (2019).
[51] J. Ben Achour, F. Lamy, H. Liu, and K. Noui, J. Cosmol. Astropart. Phys. 05 (2018) 072.
[52] A. Ashtekar, W. Kaminski, and J. Lewandowski, Phys. Rev. D **79**, 064030 (2009).
[53] A. Dapor, J. Lewandowski, and J. Puchta, Phys. Rev. D **87**, 104038 (2013).
[54] M. Martín-Benito, L. J. Garay, and G. A. Mena Marugán, Phys. Rev. D **78**, 083516 (2008).

[55] L. J. Garay, M. Martín-Benito, and G. A. Mena Marugán, Phys. Rev. D **82**, 044048 (2010).
[56] M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Phys. Rev. D **88**, 044013 (2013).
[57] M. Bojowald, arXiv:1906.04650.
[58] M. Bojowald (to be published).
[59] S. Shankaranarayanan and M. Lubo, Phys. Rev. D **72**, 123513 (2005).
[60] J. D. Reyes, Ph. D. thesis, The Pennsylvania State University, 2009.
[61] M. Bojowald, J. D. Reyes, and R. Tibrewala, Phys. Rev. D **80**, 084002 (2009).
[62] M. Bojowald, G. M. Paily, and J. D. Reyes, Phys. Rev. D **90**, 025025 (2014).
[63] S. Brahma, Phys. Rev. D **91**, 124003 (2015).
[64] M. Bojowald and J. Mielczarek, J. Cosmol. Astropart. Phys. 08 (2015) 052.
[65] M. Bojowald, Front. Phys. **3**, 33 (2015).
[66] R. Arnowitt, S. Deser, and C. W. Misner, Gen. Relativ. Gravit. **40**, 1997 (2008).