Production of the predicted $K^*(4307)$ in B decays

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In this work we study the production of $K^*(4307)$ in B decays by determining the $J/\psi\pi^{+(0)}K^0$ and $J/\psi\pi^-K^+$ invariant mass distributions of the processes $B^+ \to J/\psi\pi^+\pi^0K^0$ and $B^+ \to J/\psi\pi^+\pi^-K^+$, respectively. Such $K^*(4307)$ has been recently predicted as a three-body state originating from the dynamics involved in the $KD\bar{D}^*$ system, with the KD subsystem forming the $D^*_{s0}(2317)$ in isospin 0, and the $D\bar{D}^*$ subsystem generating the $X(3872)$ in isospin 0 and the $Z_c(3900)$ in isospin 1. The hidden charm content of $K^*(4307)$ favors its decay to a state like $J/\psi\pi K$ and the study of B decays with these particles in their final states can constitute a way of finding experimental evidence for such an exotic vector meson, whose width—in spite of its large mass—is still quite narrow (around 18 MeV).

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I. INTRODUCTION

In recent years the B factories have become an unexpected and crucial source of experimental data useful for understanding the properties—as well as the discovery—of mesons whose nature seems to challenge the traditional quark model, especially those mesons/baryons with hidden or explicit charm quantum numbers. For instance, the states $D^*_{s0}(2317)$ and $D^*_{s1}(2460)$, observed for the first time in e^+e^- collisions [1,2], were also found in the study of $B \to \bar{D}D_s\pi$ and $B \to \bar{D}D^*_s\pi$ [3], with the particles $D_s\pi$ ($D^*_s\pi$) coming from the decay $D^*_{s0}(2317) \to D_s\pi$ [$D^*_{s1}(2460) \to D^*_s\pi$]. Such studies were crucial for confirming the quantum numbers of $D^*_{s0}(2317)$ and ruling out the possible spin-0 assignment for $D^*_{s1}(2460)$. The quantum numbers $J^{PC} = 1^{++}$ of $X(3872)$ were indeed confirmed by the LHCb Collaboration in the study of the decay $B^+ \to J/\psi\pi^+\pi^-K^+$, followed by $J/\psi \to \mu^+\mu^-$, where $X(3872)$ was found in the $\pi^+\pi^-J/\psi$ invariant mass distribution [4]. The same decay process, as well as the decays $B^+ \to J/\psi\pi^+\pi^0K^0$ and $B^- \to J/\psi\pi^-\pi^0K^0$, were previously investigated by the Belle [5–7] and *BABAR* collaborations [8,9] in the context of searching for $X(3872)$ and a possible charged partner. Along the same line, the study of the decay $B \to K\pi^\pm\psi'$ led to the claim by the Belle Collaboration of the existence of a state with a minimal tetraquark configuration, $Z^\pm(4430)$, in the $\pi^\pm\psi'$ invariant mass distribution [10,11]. Such a state has also been claimed by the LHCb Collaboration, which arrived at the conclusion that a highly significant $Z^-(4430) \to \psi'\pi^-$ is needed to describe the decay $B^0 \to \psi'\pi^-K^+$ [12]. The experimental observation of baryons with a minimal content of five quarks has also come from the decay of a baryon with bottom quantum number. In particular, charmonium pentaquark states were claimed by the LHCb Collaboration in the $J/\psi p$ invariant mass of the decay process $\Lambda^0_b \to J/\psi K^- p$ [13,14].

Interestingly, all of the above mentioned states share a property: the meson states can be interpreted as tetraquarks or as states obtained from the dynamics involved in two-meson systems, while the baryon states can be understood as pentaquarks or as states originating from meson-baryon systems (for some recent reviews on these topics see, for example, Refs. [15–20]). With the amount of data collected from B decays in recent years, it is natural to ask whether there could be signals for other kinds of exotic states, like those formed by the interaction of three hadrons, that is, a minimal configuration of six quarks in the case of mesons and seven in the case of baryons. In recent years, the formation of three-body bound states/resonances with hidden or explicit charm is being discussed [21–29], though an experimental investigation of these states is still not on the agendas of experimental facilities. Particularly interesting is the exotic K^* vector meson found in Refs. [22,25], a state with hidden charm, a mass around 4300 MeV, but still

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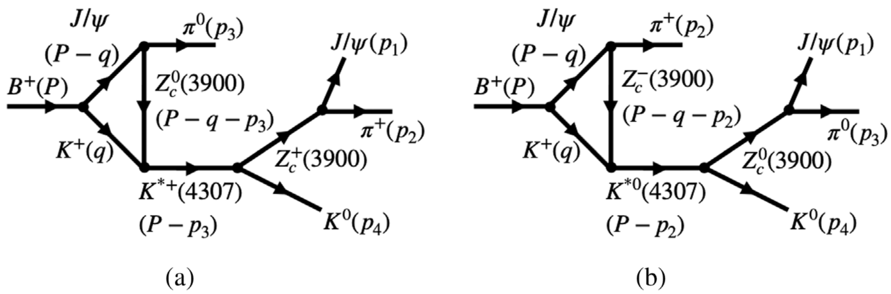

FIG. 1. Diagrammatical representation of the decay process $B^+ \to J/\psi\pi^+\pi^0K^0$ through the formation of $K^{*+(0)}(4307)$.

narrow, with a width of around 18 MeV [25]. As shown in Ref. [25], such a state arises from the dynamics involved in the $KD\bar{D}^*$ system when the interaction of the $D\bar{D}^*$ subsystem generates the $X(3872)$ in isospin 0 and the $Z_c(3900)$ in isospin 1. The $K^*(4307)$, with a dominant $KZ_c(3900)$ component in its wave function, can naturally decay to a final state formed by $KJ/\psi\pi$, with the J/ψ and π coming from the decay of $Z_c(3900)$. In this way, an experimental reconstruction of the $J/\psi\pi K$ invariant mass could confirm the existence of such an excited K^* state, and its narrow width would help to identify it. The fact that information on this invariant mass could be obtained from the existing experimental data on $B^\pm \to J/\psi\pi^\pm\pi^0K^0$ or $B^+ \to J/\psi\pi^+\pi^-K^+$ is especially motivating.

Conducting such experimental research could herald a whole new era of hunting for exotic mesons with strangeness, since the last excited state of a K/K^* observed experimentally according to the Particle Data Group is a kaon whose mass is around 3100 MeV [30]. There is then a vast energy region in which the formation of exotic K/K^* states has remained totally unexplored. Having this in mind, in this work we determine the branching ratio for the processes $B^+ \to J/\psi\pi^0\pi^+K^0$, through $B^+ \to \pi^{0(+)}K^{*+(0)}(4307) \to \pi^{0(+)}K^0Z_c^{+(0)}(3900) \to \pi^{0(+)}K^0J/\psi\pi^{+(0)}$, and $B^+ \to J/\psi\pi^+\pi^-K^+$, through $B^+ \to \pi^+K^{*0}(4307) \to \pi^+K^+Z_c^-(3900) \to \pi^+K^+J/\psi\pi^-$, and reconstruct the $J/\psi\pi^{+(0)}K^0$ and $J/\psi\pi^-K^+$ invariant mass distributions with the purpose of studying the $K^*(4307)$ signal in them.

II. FORMALISM

The decay process $B^+ \to J/\psi\pi^+\pi^0K^0$ proceeding through $K^*(4307)$ formation can be visualized diagrammatically as shown in Fig. 1, where the interaction between a K^+ and a $Z_c^{0(-)}(3900)$ generates the $K^{*+(0)}(4307)$ [25], which decays to $J/\psi\pi^{+(0)}K^0$. The nature of $Z_c(3900)$ is still under debate. Here, as done in Ref. [25], we follow the model of Ref. [31] where the state is generated from the interaction between $D\bar{D}^*$ and $J/\psi\pi$ within coupled channels as a weakly bound state of the $D\bar{D}^*$ system, with a finite width coming from its decay to the $J/\psi\pi$ channel. Due to the nature of $K^*(4307)$ and $Z_c(3900)$, the weak vertex $B^+ \to J/\psi K^+$ is the most favored for forming $Z_c(3900)$ and $K^*(4307)$. At the quark level, it involves internal emission of a W^+ via $\bar{b} \to \bar{c}$ ($W^+ \to c\bar{s}$) transitions, which are both Cabibbo favored (see Fig. 2). Based on the quantum chromodynamics factorization approach for nonleptonic B-meson decays [32], the amplitude related to the weak vertex shown in Fig. 2 can be written as

$$t_{B^+\to J/\psi K^+} = \frac{G_F}{\sqrt{2}}V_{cb}V_{cs}^*a_2\langle J/\psi|(\bar{c}c)_V|0\rangle\langle K|(\bar{b}s)_V|B\rangle, \tag{1}$$

where G_F is the Fermi coupling constant, V_{cb} and V_{cs}^* are elements of the Cabibbo-Kobayashi-Maskawa matrix, a_2 is an effective coupling constant, $\langle J/\psi|(\bar{c}c)_V|0\rangle$ is the factorized amplitude for the production of a J/ψ via the vector current $\bar{c}\gamma_\mu c$, and $\langle K|(\bar{b}s)_V|B\rangle$ represents the transition matrix element $B^+ \to K^+$. The amplitude $\langle J/\psi|(\bar{c}c)_V|0\rangle$ can be parametrized in terms of the decay constant $f_{J/\psi}$, the mass $m_{J/\psi}$, and the polarization vector $\epsilon_{J/\psi\mu}$ of the J/ψ vector meson as [33]

$$\langle J/\psi|(\bar{c}c)_V|0\rangle = \epsilon_{J/\psi\mu}m_{J/\psi}f_{J/\psi}, \tag{2}$$

while the transition matrix element $B^+(p) \to K^+(p')$ is given by [33]

$$\begin{aligned}\langle K^+(p')|(\bar{b}s)_V|B^+(p)\rangle &= \left[(p+p')_\mu - \frac{m_{B^+}^2 - m_{K^+}^2}{Q^2}Q_\mu\right]F_1(Q^2) \\ &\quad + \frac{m_{B^+}^2 - m_{K^+}^2}{Q^2}Q_\mu F_0(Q^2).\end{aligned} \tag{3}$$

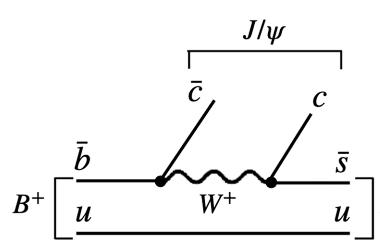

FIG. 2. Weak vertices involved in the decay of a B^+ into a J/ψ and a K^+.

In Eq. (3), $Q_\mu = (p - p')_\mu$, $F_1(Q^2)$, and $F_0(Q^2)$ correspond to form factors, which satisfy the condition $F_1(0) = F_0(0)$ [33], and m_{B^+} (m_{K^+}) is the mass of the B^+ (K^+) meson. The Q^2 dependence of these form factors can be written as [34]

$$F_i(Q^2) = \frac{F_i(0)}{1 - Q^2/m_{Pi}^2}, \tag{4}$$

where $i = 0$, 1, with m_{Pi} being the mass of the lowest-lying meson with the appropriate quantum numbers, i.e., $J^P = 0^+$ for F_0 ($m_P = 5890$ MeV) and 1^- for F_1 ($m_P = 5430$ MeV), and

$$F_1(0) = F_0(0) = 0.49 \pm 0.12. \tag{5}$$

Using Eq. (1), we can determine the branching ratio for the process $B^+ \to J/\psi K^+$ as

$$\mathrm{Br}(B^+ \to J/\psi K^+) = \frac{|\vec{p}_{\mathrm{CM}}|}{8\pi\Gamma_{B^+} m_{B^+}^2} \sum_\lambda |t_{B^+ \to J/\psi K^+}|^2, \tag{6}$$

where the symbol $\sum_\lambda$ indicates a sum over the polarizations of J/ψ, $|\vec{p}_{\mathrm{CM}}|$ is the center-of-mass momentum of the $J/\psi K^+$ system, Γ_{B^+} is the width of the B^+ meson, and, from Eqs. (1) and (3),

$$\sum_\lambda |t_{B^+\to J/\psi K^+}|^2 = \frac{G_F^2}{2}|V_{cb}|^2|V_{cs}|^2|a_2|^2 m_{J/\psi}^2 f_{J/\psi}^2 F_1^2(Q^2) \times \left[-(p+p')^2 + \frac{(m_{B^+}^2 - m_{K^+}^2)^2}{m_{J/\psi}^2}\right]. \tag{7}$$

Considering $a_2 = 0.21 \pm 0.02$ [33], $G_F = 1.166 \times 10^{-11}\ \mathrm{MeV}^{-2}$, $|V_{cs}| = 0.977 \pm 0.017$, $|V_{cb}| = (42.2 \pm 0.8) \times 10^{-3}$, $m_{J/\psi} = 3096.9 \pm 0.006$ MeV, $\Gamma_{B^+} = (4.01839 \pm 0.0098) \times 10^{-10}$ MeV [30], and $f_{J/\psi} = 405 \pm 14$ MeV [33], we get

$$\mathrm{Br}(B^+ \to J/\psi K^+) \simeq (0.83 \pm 0.32) \times 10^{-3}, \tag{8}$$

which is compatible with the measured branching ratio of [30]

$$\mathrm{Br}(B^+ \to J/\psi K^+)_{\mathrm{measured}} = (1.01 \pm 0.028) \times 10^{-3}. \tag{9}$$

Based on the above discussion, the dominant contribution from the weak vertex in the processes depicted in Fig. 1 can be written, for convenience, as

$$t_{B^+\to J/\psi K^+} = C_{B^+\to J/\psi K^+}(P+q)\cdot\epsilon_{J/\psi}(P-q), \tag{10}$$

where P^μ (q^μ) is the four-momentum of the B^+ (K^+), and the coefficient $C_{B^+\to J/\psi K^+}$, which corresponds to $\frac{G_F}{\sqrt{2}}|V_{cb}||V_{cs}||a_2| m_{J/\psi} f_{J/\psi} F_1$ of Eq. (7), is fixed to reproduce the observed branching ratio, i.e., Eq. (9), to better agree with the experimental finding,

$$C_{B^+\to J/\psi K^+} = (7.16 \pm 0.11) \times 10^{-8}. \tag{11}$$

Since $Z_c(3900)$ and $K^*(4307)$ couple to $J/\psi\pi$ and $KZ_c(3900)$, respectively, in the s partial wave [25,31], we can introduce the coupling constants $g_{Z_c\to J/\psi\pi}$ and $g_{K^*\to KZ_c}$ to describe the contribution from these vertices in Fig. 1. Such a contribution can be expressed (without specifying any particular charge for simplicity) in terms of the contraction between the polarization vectors of the particles involved as

$$\begin{aligned} t_{J/\psi\to\pi Z_c} &= g_{Zc\to J/\psi\pi}\epsilon_{J/\psi}\cdot\epsilon_{Z_c}, \\ t_{KZ_c\to K^*} &= g_{K^*\to KZ_c}\epsilon_{Z_c}\cdot\epsilon_{K^*}. \end{aligned} \tag{12}$$

In Eq. (12), the coupling constants $g_{Z_c\to J/\psi\pi}$ and $g_{K^*\to KZ_c}$ for a given charge can be obtained, respectively, from the residue of the $J/\psi\pi$ and KZ_c T matrices, recalling that in the vicinity of a pole the T matrix for the transition $i \to j$ can be written as

$$t_{ij} = \frac{g_i g_j}{s - s_R}, \tag{13}$$

where s_R corresponds to the pole position related to the resonance in the complex energy plane s and g_i (g_j) represents the coupling of the resonance to the channel i (j). In this way, the couplings can be calculated from the residue R_{ij} of t_{ij} as

$$g_j = \frac{R_{ij}}{\sqrt{R_{ii}}}. \tag{14}$$

The residue R_{ij} itself can be obtained via an integration of t_{ij} along a closed contour around the pole s_R. Alternatively, if M_R (Γ_R) is the mass (width) of the resonance, by considering a Breit-Wigner form for $|t_{ij}|^2$ in Eq. (13), i.e., $s_R = M_R^2 + iM_R\Gamma_R$, the couplings can also be determined from the T matrix on the real axis evaluated at the center-of-mass energy $\sqrt{s} = M_R$ as

$$g_j^2 = iM_R\Gamma_R t_{jj}(M_R). \tag{15}$$

With these ingredients, and considering the Feynman rules, the amplitudes associated with the diagrams in Fig. 1(a) can be written as

$$
\begin{aligned}
-it_{(a)} = &\int \frac{d^4q}{(2\pi)^4}(-it_{B^+\to J/\psi K^+})\frac{i}{(P-q)^2-m^2_{J/\psi}+i\epsilon}\frac{i}{q^2-m^2_{K^+}+i\epsilon}(-it_{J/\psi\to\pi^0 Z_c^0})\\
&\times\frac{i}{(P-p_3-q)^2-m^2_{Z_c^0}+i\epsilon}(-it_{K^+Z_c^0\to K^{*+}})\frac{i}{(P-p_3)^2-m^2_{K^{*+}(4307)}+i\epsilon}\\
&\times(-it_{K^{*+}\to K^0 Z_c^+})\frac{i}{(p_1+p_2)^2-m^2_{Z_c^+}+i\epsilon}(-it_{Z_c^+\to J/\psi\pi^+}),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
-it_{(b)} = &\int \frac{d^4q}{(2\pi)^4}(-it_{B^+\to J/\psi K^+})\frac{i}{(P-q)^2-m^2_{J/\psi}+i\epsilon}\frac{i}{q^2-m^2_{K^+}+i\epsilon}(-it_{J/\psi\to\pi^+ Z_c^-})\\
&\times\frac{i}{(P-p_2-q)^2-m^2_{Z_c^-}+i\epsilon}(-it_{K^+Z_c^-\to K^{*0}})\frac{i}{(P-p_2)^2-m^2_{K^{*0}(4307)}+i\epsilon}\\
&\times(-it_{K^{*0}\to K^0 Z_c^0})\frac{i}{(p_1+p_3)^2-m^2_{Z_c^0}+i\epsilon}(-it_{Z_c^0\to J/\psi\pi^0}).
\end{aligned}
\tag{17}
$$

Substituting Eqs. (10) and (12) into Eqs. (16) and (17) and summing over the polarizations of the internal particles present in the triangular loops, the amplitudes $t_{(a)}$ and $t_{(b)}$ can be written in a more compact form as

$$
\begin{aligned}
-it_{\mathcal{A}} = &\,C_{\mathcal{A}}\frac{1}{s_{3(\mathcal{A})}-m^2_{K^*(\mathcal{A})}+i\Gamma_{K^*(\mathcal{A})}m_{K^*(\mathcal{A})}}\frac{1}{s_{2(\mathcal{A})}-m^2_{Z_{\mathcal{A}}}+i\Gamma_{Z(\mathcal{A})}m_{Z(\mathcal{A})}}\\
&\times\Bigg[F_-\left\{P_\sigma-\frac{(P-p_{\mathcal{A}})_\sigma}{M^2_{Z(\mathcal{A})}}(P^2-p_{\mathcal{A}}\cdot P)\right\}I^{(0\mathcal{A})}+\left\{F_++\frac{F_-}{M^2_{Z(\mathcal{A})}}(P^2-p_{\mathcal{A}}\cdot P)\right\}I^{(1\mathcal{A})}_\sigma\\
&+\frac{(P-p_{\mathcal{A}})_\sigma}{M^2_{Z(\mathcal{A})}}\times\left\{F_+p^\mu_{\mathcal{A}}-2\frac{P^2}{m^2_{J/\psi}}P^\mu\right\}I^{(1\mathcal{A})}_\mu+\left\{\frac{P_\sigma}{m^2_{J/\psi}}+\frac{(P-p_{\mathcal{A}})_\sigma}{M^2_{Z(\mathcal{A})}}\right.\\
&\times\left.\left(1+\frac{p_{\mathcal{A}}\cdot P}{m^2_{J/\psi}}\right)\right\}I^{(2\mathcal{A})}-\frac{1}{M^2_{Z(\mathcal{A})}}\left\{F_+p^\mu_{\mathcal{A}}-2\frac{P^2}{m^2_{J/\psi}}P^\mu\right\}I^{(2\mathcal{A})}_{\sigma\mu}\\
&+\frac{(P-p_{\mathcal{A}})_\sigma}{M^2_{Z(\mathcal{A})}}\frac{(-p^\mu_{\mathcal{A}}+2P^\mu)}{m^2_{J/\psi}}I^{(3\mathcal{A})}_\mu-\left\{\frac{1}{m^2_{J/\psi}}+\frac{1}{M^2_{Z(\mathcal{A})}}\left(1+\frac{p_{\mathcal{A}}\cdot P}{m^2_{J/\psi}}\right)\right\}I^{(3\mathcal{A})}_\sigma\\
&-\frac{1}{M^2_{Z(\mathcal{A})}m^2_{J/\psi}}\{(P-p_{\mathcal{A}})_\sigma I^{(4\mathcal{A})}+(-p^\mu_{\mathcal{A}}+2P^\mu)I^{(4\mathcal{A})}_{\sigma\mu}-I^{(5\mathcal{A})}_\sigma\}\Bigg]\\
&\times\left[-g^{\sigma\rho}+\frac{(P-p_{\mathcal{A}})^\sigma(P-p_{\mathcal{A}})^\rho}{m^2_{K^*(\mathcal{A})}}\right]\left[-g_{\rho\beta}+\frac{(p_1+p_2)_\rho p_{2\beta}}{m^2_{Z(\mathcal{A})}}\right]\epsilon^\beta_{J/\psi}(p_1),
\end{aligned}
\tag{18}
$$

where the subscript $\mathcal{A}$ refers to the diagrams in Figs. 1(a) and 1(b), where $s_{3(\mathcal{A})}=s_{124}=(p_1+p_2+p_4)^2$ $[s_{134}=(p_1+p_3+p_4)^2]$, $s_{2(\mathcal{A})}=s_{12}=(p_1+p_2)^2$ $[s_{13}=(p_1+p_3)^2]$, $p_{\mathcal{A}}=p_3$ (p_2), $m_{Z(\mathcal{A})}=m_{Z_c^+}$ $(m_{Z_c^0})$, $M_{Z(\mathcal{A})}=m_{Z_c^0}$ $(m_{Z_c^-})$ is the mass of the Z_c particle involved in the triangular loops, and $m_{K^*(\mathcal{A})}=m_{K^{*+}(4307)}$ $[m_{K^{*0}(4307)}]$. To account for the propagation of the unstable particles Z_c and $K^*(4307)$ in the diagrams of Fig. 1 the corresponding $i\epsilon$'s present in the propagators of Eqs. (16) and (17) have been replaced by the product of the mass and width of the corresponding particle. In this way, in Eq. (18), $\Gamma_{Z(\mathcal{A})}=\Gamma_{Z_c^+}$ $(\Gamma_{Z_c^0})$, $\Gamma_{K^*(\mathcal{A})}=\Gamma_{K^{*+}(4307)}$ $[\Gamma_{K^{*0}(4307)}]$ for Fig. 1(a) [Fig. 1(b)]. The $F_\pm$ factor in Eq. (18) is given by

$$
F_\pm = 1\pm\frac{P^2}{m^2_{J/\psi}},
\tag{19}
$$

and the constant $C_{\mathcal{A}}$ corresponds to a product of the coupling constants involved in the different vertices shown in Fig. 1. To be precise,

$$
C_a=-C_b=\frac{\sqrt{2}}{3}C_{B^+\to J/\psi K^+}g^2_{Z_c\to(J/\psi\pi)_1}g^2_{K^*\to(KZ_c)_{1/2}},
\tag{20}
$$

where $g_{Z_c\to(J/\psi\pi)_1}$ is the coupling constant of $Z_c(3900)$ to the $J/\psi\pi$ system in isospin 1, and whose value is obtained from the model of Ref. [31], and $g_{K^*\to(KZ_c)_{1/2}}$ represents the coupling of $K^*(4307)$ to a $KZ_c(3900)$ system in isospin 1/2, which can be found from the model of Ref. [25]. The values used here are $g_{Z_c\to(J/\psi\pi)_1} = 3715$ MeV and $g_{K^*\to(KZ_c)_{1/2}} = 22143$ MeV [27]. The phase convention $|\pi^+\rangle = -|I=1, I_3=1\rangle$, $|K^-\rangle = -|I=1/2, I_3=-1/2\rangle$ has been used in our calculations. In this way, for example, by using Clebsch-Gordan coefficients

$$|KZ_c, I=1/2, I_3=-1/2\rangle = -\frac{1}{\sqrt{3}}|K^0 Z_c^0\rangle + \sqrt{\frac{2}{3}}|K^+ Z_c^-\rangle, \tag{21}$$

and then

$$g_{K^{*0}\to K^0 Z_c^0} = -\frac{1}{\sqrt{3}} g_{K^*\to KZ_c}, \qquad g_{K^{*0}\to K^+ Z_c^-} = \sqrt{\frac{2}{3}} g_{K^*\to KZ_c}. \tag{22}$$

In Eq. (18), $I^{(0\mathcal{A})}$, $I^{(1\mathcal{A})}_\alpha$, $I^{(2\mathcal{A})}_{\alpha\beta}$, $I^{(3\mathcal{A})}_\alpha$, $I^{(4\mathcal{A})}_{\alpha\beta}$, and $I^{(5\mathcal{A})}_\alpha$ are integrals defined as

$$\begin{aligned} I^{(0\mathcal{A})} &= \int \frac{d^4q}{(2\pi)^4}\frac{1}{\mathcal{D}(q,p_\mathcal{A})}, & I^{(1\mathcal{A})}_\alpha &= \int \frac{d^4q}{(2\pi)^4}\frac{q_\alpha}{\mathcal{D}(q,p_\mathcal{A})}, & I^{(2\mathcal{A})}_{\alpha\beta} &= \int \frac{d^4q}{(2\pi)^4}\frac{q_\alpha q_\beta}{\mathcal{D}(q,p_\mathcal{A})}, \\ I^{(3\mathcal{A})}_\alpha &= \int \frac{d^4q}{(2\pi)^4}\frac{q^2 q_\alpha}{\mathcal{D}(q,p_\mathcal{A})}, & I^{(4\mathcal{A})}_{\alpha\beta} &= \int \frac{d^4q}{(2\pi)^4}\frac{q^2 q_\alpha q_\beta}{\mathcal{D}(q,p_\mathcal{A})}, & I^{(5\mathcal{A})}_\alpha &= \int \frac{d^4q}{(2\pi)^4}\frac{q^4 q_\alpha}{\mathcal{D}(q,p_\mathcal{A})}, \end{aligned} \tag{23}$$

where

$$\mathcal{D}(q,p_\mathcal{A}) = [(P-q)^2 - m^2_{J/\psi} + i\epsilon][q^2 - m^2_{K^+} + i\epsilon] \times [(P-p_\mathcal{A}-q)^2 - M^2_{Z(\mathcal{A})} + i\epsilon]. \tag{24}$$

The integrals $I^{(2\mathcal{A})}$ and $I^{(4\mathcal{A})}$ in Eq. (18) correspond to the contraction of the metric tensor $g^{\alpha\beta}$ with the integrals $I^{(2\mathcal{A})}_{\alpha\beta}$ and $I^{(4\mathcal{A})}_{\alpha\beta}$ of Eq. (23), respectively. A way to continue with the calculation of these integrals consists of using the Passarino-Veltman decomposition of tensor integrals [35], which exploits the Lorentz covariance to write each of the integrals as a combination of the different Lorentz structures with some unknown coefficients. For example, the integral $I^{(1\mathcal{A})}_\alpha$ in Eq. (23) is a covariant tensor which can depend on the four-momenta P and $p_\mathcal{A}$. In this way, we can write

$$I^{(1\mathcal{A})}_\alpha = a^{(1\mathcal{A})}_1 P_\alpha + a^{(1\mathcal{A})}_2 p_{\mathcal{A}\alpha}, \tag{25}$$

where $a^{(1\mathcal{A})}_1$ and $a^{(1\mathcal{A})}_2$ are coefficients to be determined. Similarly,

$$\begin{aligned} I^{(2\mathcal{A})}_{\alpha\beta} &= a^{(2\mathcal{A})}_1 g_{\alpha\beta} + a^{(2\mathcal{A})}_2 P_\alpha P_\beta + a^{(2\mathcal{A})}_3 (P_\alpha p_{\mathcal{A}\beta} + P_\beta p_{\mathcal{A}\alpha}) \\ &\quad + a^{(2\mathcal{A})}_4 p_{\mathcal{A}\alpha} p_{\mathcal{A}\beta}, \\ I^{(3\mathcal{A})}_\alpha &= a^{(3\mathcal{A})}_1 P_\alpha + a^{(3\mathcal{A})}_2 p_{\mathcal{A}\alpha}, \\ I^{(4\mathcal{A})}_{\alpha\beta} &= a^{(4\mathcal{A})}_1 g_{\alpha\beta} + a^{(4\mathcal{A})}_2 P_\alpha P_\beta + a^{(4\mathcal{A})}_3 (P_\alpha p_{\mathcal{A}\beta} + P_\beta p_{\mathcal{A}\alpha}) \\ &\quad + a^{(4\mathcal{A})}_4 p_{\mathcal{A}\alpha} p_{\mathcal{A}\beta}, \\ I^{(5\mathcal{A})}_\alpha &= a^{(5\mathcal{A})}_1 P_\alpha + a^{(5\mathcal{A})}_2 p_{\mathcal{A}\alpha}, \end{aligned} \tag{26}$$

where we have used the fact that the tensor integrals $I^{(2\mathcal{A})}_{\alpha\beta}$ and $I^{(4\mathcal{A})}_{\alpha\beta}$ are symmetric under the interchange $\alpha \leftrightarrow \beta$, as can be seen from the definition in Eq. (23). Contracting the integrals in Eqs. (25) and (26) with the different Lorentz structures appearing in their decomposition, we can get a system of equations which permits the determination of the unknown a coefficients in terms of scalar integrals. For instance, using Eq. (25), we can write

$$\begin{aligned} P\cdot I^{(1\mathcal{A})} &= a^{(1\mathcal{A})}_1 P^2 + a^{(1\mathcal{A})}_2 P\cdot p_\mathcal{A}, \\ p_\mathcal{A}\cdot I^{(1\mathcal{A})} &= a^{(1\mathcal{A})}_1 P\cdot p_\mathcal{A} + a^{(1\mathcal{A})}_2 p^2_\mathcal{A}. \end{aligned} \tag{27}$$

By solving the system of equations (27), we find

$$\begin{aligned} a^{(1\mathcal{A})}_1 &= -\frac{p^2_\mathcal{A} P\cdot I^{(1\mathcal{A})} - (P\cdot p_\mathcal{A}) p_\mathcal{A}\cdot I^{(1\mathcal{A})}}{(P\cdot p_\mathcal{A})^2 - P^2 p^2_\mathcal{A}}, \\ a^{(1\mathcal{A})}_2 &= -\frac{P^2 p_\mathcal{A}\cdot I^{(1\mathcal{A})} - (P\cdot p_\mathcal{A}) P\cdot I^{(1\mathcal{A})}}{(P\cdot p_\mathcal{A})^2 - P^2 p^2_\mathcal{A}}, \end{aligned} \tag{28}$$

and the whole problem reduces to determining the scalar integrals $P\cdot I^{(1\mathcal{A})}$ and $p_\mathcal{A}\cdot I^{(1\mathcal{A})}$, which, from Eq. (23), are given by

$$\begin{aligned} P\cdot I^{(1\mathcal{A})} &= \int \frac{d^4q}{(2\pi)^4}\frac{P\cdot q}{\mathcal{D}(q,p_\mathcal{A})}, \\ p_\mathcal{A}\cdot I^{(1\mathcal{A})} &= \int \frac{d^4q}{(2\pi)^4}\frac{p_\mathcal{A}\cdot q}{\mathcal{D}(q,p_\mathcal{A})}. \end{aligned} \tag{29}$$

The next step consists in calculating the scalar integrals in Eq. (29), which we do in the rest frame of the decaying particle, i.e., $P^2 = m^2_{B^+}$ and $\vec{P} = \vec{0}$. To do this, it is convenient to realize that these integrals can be considered as particular cases of other more general integrals. For instance, if we define $\mathcal{I}^{(1\mathcal{A})}(a,b)$ as

$$\mathcal{I}^{(1\mathcal{A})}(a,b) = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{aq^0 + b|\vec{q}|\cos\theta}{\mathcal{D}(q,p_\mathcal{A})}, \quad (30)$$

we can write

$$P \cdot I^{(1\mathcal{A})} = \mathcal{I}^{(1\mathcal{A})}(P^0, 0), \quad p_\mathcal{A} \cdot I^{(1\mathcal{A})} = \mathcal{I}^{(1\mathcal{A})}(p^0_\mathcal{A}, -|\vec{p}_\mathcal{A}|). \quad (31)$$

After this, it is convenient to separate the temporal part in the denominator $\mathcal{D}(q,p_\mathcal{A})$ and write it as

$$\mathcal{D}(q,p_\mathcal{A}) = [(P^0 - q^0)^2 - \omega^2_{J/\psi}(\vec{q}) + i\epsilon][q^{02} - \omega^2_K(\vec{q}) + i\epsilon] \times [(P^0 - p^0_\mathcal{A} - q^0)^2 - \omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A} + \vec{q}) + i\epsilon], \quad (32)$$

where

$$\omega_{J/\psi}(\vec{q}) = \sqrt{\vec{q}^2 + m^2_{J/\psi}}, \quad \omega_K(\vec{q}) = \sqrt{\vec{q}^2 + m^2_{K^+}},$$
$$\omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A} + \vec{q}) = \sqrt{\vec{p}^2_\mathcal{A} + \vec{q}^2 + 2|\vec{p}_\mathcal{A}||\vec{q}|\cos\theta + M^2_{Z(\mathcal{A})}}, \quad (33)$$

with θ being the angle between $\vec{p}_\mathcal{A}$ and $\vec{q}$. The same procedure can be applied for the other coefficients appearing in Eq. (26), as shown in the Appendix. In all cases, the integration over the q^0 variable in Eq. (30) can be performed analytically using Cauchy's theorem such that we are left with integrals of the form

$$\mathcal{I}^{(k\mathcal{A})}(a,b,\ldots) = \int \frac{d^3q}{(2\pi)^3} \frac{\mathcal{N}^{(k\mathcal{A})}(a,b,\ldots)}{\mathcal{D}(\vec{q},p^0_\mathcal{A},\vec{p}_\mathcal{A})}, \quad (34)$$

with $k = 0, 1, 2, \ldots 4$ and

$$\begin{aligned}\mathcal{D}(\vec{q},p^0_\mathcal{A},\vec{p}_\mathcal{A}) = {} & 2\omega_{J/\psi}(\vec{q})\omega_K(\vec{q})\omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A}+\vec{q})[P^0 + \omega_{J/\psi}(\vec{q}) + \omega_K(\vec{q})] \\ & \times [p^0_\mathcal{A} + \omega_{J/\psi}(\vec{q}) + \omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A}+\vec{q})][P^0 - \omega_K(\vec{q}) - \omega_{J/\psi}(\vec{q}) + i\epsilon] \\ & \times \left[P^0 - p^0_\mathcal{A} - \omega_K(\vec{q}) - \omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A}+\vec{q}) + i\frac{\Gamma_{Z(\mathcal{A})}}{2}\right] \\ & \times \left[p^0_\mathcal{A} - \omega_{J/\psi}(\vec{q}) - \omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A}+\vec{q}) + i\frac{\Gamma_{Z(\mathcal{A})}}{2}\right] \\ & \times \left[p^0_\mathcal{A} - P^0 - \omega_K(\vec{q}) - \omega_{Z(\mathcal{A})}(\vec{p}_\mathcal{A}+\vec{q}) + i\frac{\Gamma_{Z(\mathcal{A})}}{2}\right]. \end{aligned} \quad (35)$$

In Eq. (35), a width $\Gamma_{Z(\mathcal{A})}$ of 28 MeV [30] has been considered for the $Z_c(3900)$ present in the triangular loops. The $\mathcal{N}^{(k\mathcal{A})}$ numerators in Eq. (34) and the expressions for the other a coefficients of Eq. (26) can be found in the Appendix.

The integration over d^3q of Eq. (34) is performed by using a cutoff $\Lambda \sim 700$ MeV for the modulus of the center-of-mass momentum of the $KZ_c(3900)$ system, $\vec{q}^*$, in the triangular loops, i.e.,

$$\int \frac{d^3q}{(2\pi)^3} \to \frac{1}{(2\pi)^2} \int_0^\infty d|\vec{q}||\vec{q}|^2 \int_{-1}^1 d\cos\theta\Theta(|\vec{q}^*| - \Lambda). \quad (36)$$

Such a value for Λ is related to the nature of $Z_c(3900)$, which (as mentioned earlier) is generated from the $D\bar{D}^*$ and coupled channels in isospin 1 and in the s partial wave [31], and, consequently, of $K^*(4307)$ [25] arising from the $KD\bar{D}^*$ interaction in the s partial wave when $D\bar{D}^*$ clusters as $Z_c(3900)$. The vectors $\vec{q}$ and $\vec{q}^*$ in Eq. (36) are related through a boost [36]

$$\vec{q}^* = \left[\left(\frac{E_{KZ(\mathcal{A})}}{m_{KZ(\mathcal{A})}} - 1\right)\frac{\vec{q}\cdot\vec{p}_\mathcal{A}}{\vec{p}^2_\mathcal{A}} + \frac{\omega_K}{m_{KZ(\mathcal{A})}}\right]\vec{p}_\mathcal{A} + \vec{q}, \quad (37)$$

where $m_{KZ(\mathcal{A})}$ is the invariant mass of the $KZ_c(3900)$ system in the triangular loop and $E_{KZ(\mathcal{A})} = \sqrt{m^2_{KZ(\mathcal{A})} + \vec{p}^2_\mathcal{A}}$ is its energy in the rest frame of the decaying particle.

The $D\bar{D}^*$ and coupled-channel interactions have been found to be attractive for both isospins: zero and one. A value of the cutoff in the range ~700–750 MeV was used in Ref. [37] to regularize the loops involved in the $D\bar{D}^*$ system and coupled channels in the isospin-0 configuration when solving the Bethe-Salpeter equation in its on-shell factorization form [38,39]. Such an interval of the cutoff was found to well reproduce the properties of $X(3872)$. In the isospin-1 sector, by considering a similar range for the cutoff, a pole that can be associated with the $Z_c(3900)$ was found in Ref. [31], and the experimental D^0D^{*-} and $D^+\bar{D}^{*0}$ invariant mass distributions of the process $e^+e^- \to \pi^\pm(D\bar{D})^\mp$ were well reproduced. To further understand the use of the same cutoff here as in Refs. [25,31,37], we recall that a quantum-mechanical description of the $D\bar{D}^*$ system in the s partial wave implies the use of a potential V when solving the Lippmann-Schwinger equation of the form [40]

$$V_{ij}(\vec{p},\vec{p}',\sqrt{s}) = v_{ij}(\sqrt{s})\Theta(\Lambda-|\vec{p}|)\Theta(\Lambda-|\vec{p}'|), \quad (38)$$

where $\vec{p}$ ($\vec{p}'$) stands for the center-of-mass momentum of the particles in the initial (final) channel and Λ is a cutoff. Such a potential leads to a solution for the T matrix given by

$$T_{ij}(\vec{p},\vec{p}',\sqrt{s}) = t_{ij}(\sqrt{s})\Theta(\Lambda-|\vec{p}|)\Theta(\Lambda-|\vec{p}'|). \quad (39)$$

The elements v_{ij} and t_{ij} of Eqs. (38) and (39) can be rearranged into matrices v and t, respectively, in the coupled-channel space such that t and v are also related through the Lippmann-Schwinger equation in its on-shell factorization form [38,39], i.e.,

$$t = v + vGt = [1-vG]^{-1}v, \quad (40)$$

where G is a diagonal matrix whose elements are the two-body loop functions in the nonrelativistic limit, i.e.,

$$G_{ii} = \int_0^\Lambda d^3q \frac{1}{E-m_i-M_i-\frac{\vec{q}^2}{2\mu_i}}, \quad (41)$$

where E is the relative energy (including the masses of the particles), and m_i and M_i are the masses of the particles involved in the loop and μ_i is the reduced mass of the system. In this way, the generation of resonances/bound states is contained in $t_{ij}(\sqrt{s})$ of Eq. (39) via Eq. (40) as far as the loop function G_{ii} of Eq. (41) is regularized with the same cutoff as the one appearing in Eq. (38). Close to the pole position, considering a Breit-Wigner form for $t_{ij}(\sqrt{s})$, Eq. (40) becomes

$$t_{ij}(\sqrt{s}) = \frac{g_i g_j}{s-M_R^2+iM_R\Gamma_R}, \quad (42)$$

which is analogous to Eq. (13). In this way, Eq. (39) can be written as

$$T_{ij}(\vec{p},\vec{p}',\sqrt{s}) = \frac{g_i g_j}{s-M_R^2+iM_R\Gamma_R}\Theta(\Lambda-|\vec{p}|)\Theta(\Lambda-|\vec{p}'|). \quad (43)$$

For the particular case of the $D\bar{D}^*$ and $J/\psi\pi$ coupled channels in isospin 1, in the vicinity of the $Z_c(3900)$, we can write the T matrix related to the $J/\psi\pi \to Z_c \to J/\psi\pi$ transition as

$$T_{J/\psi\pi}(\vec{p},\vec{p}',\sqrt{s}) = \frac{g^2_{Z_c\to(J/\psi\pi)_1}}{s-m^2_{Z_c}+im_{Z_c}\Gamma_{Z_c}}\Theta(\Lambda-|\vec{p}|)\Theta(\Lambda-|\vec{p}'|)\vec{\epsilon}_{J/\psi}(\vec{p})\cdot\vec{\epsilon}_{J/\psi}(\vec{p}'). \quad (44)$$

Similarly, for the KZ_c system in isospin $1/2$, in the vicinity of the $K^*(4307)$, the T matrix for the $KZ_c \to K^*(4307) \to KZ_c$ transition can be written as

$$T_{KZ_c}(\vec{p},\vec{p}',\sqrt{s}) = \frac{g^2_{K^*\to(KZ_c)_{1/2}}}{s-m^2_{K^*}+im_{K^*}\Gamma_{K^*}}\Theta(\Lambda-|\vec{p}|)\Theta(\Lambda-|\vec{p}'|)\vec{\epsilon}_{Z_c}(\vec{p})\cdot\vec{\epsilon}_{Z_c}(\vec{p}'). \quad (45)$$

As a consequence, each of the amplitudes of Eq. (12), which represent the covariant versions of the vertices $Z_cJ/\psi\pi$ and K^*KZ_c involved in Eqs. (44) and (45), respectively, when plugged into Eqs. (16) and Eqs. (17), is naturally accompanied by the cutoff used in the resolution of Eq. (40).

Finally, using the amplitude of Eq. (18), the decay width Γ for the process $B^+(P)\to J/\psi(p_1)\pi^+(p_2)\pi^0(p_3)K^0(p_4)$ can be obtained as

$$\Gamma = \frac{1}{2m_{B^+}}\int\frac{d^3p_1}{(2\pi)^3 2E_1(\vec{p}_1)}\int\frac{d^3p_2}{(2\pi)^3 2E_2(\vec{p}_2)}\int\frac{d^3p_3}{(2\pi)^3 2E_3(\vec{p}_3)}\int\frac{d^3p_4}{(2\pi)^3 2E_4(\vec{p}_4)} \times (2\pi)^4\delta^{(4)}(P-p_1-p_2-p_3-p_4)\sum_\lambda|t_a+t_b|^2. \quad (46)$$

Using the δ function of Eq. (46), and the relations

$$s_{124} = (p_1+p_2+p_4)^2 = (P-p_3)^2 = m^2_{B^+}+m_3^2-2m_{B^+}E_3,$$
$$s_{134} = (p_1+p_3+p_4)^2 = (P-p_2)^2 = m^2_{B^+}+m_2^2-2m_{B^+}E_2, \quad (47)$$

with $E_2 = \sqrt{\vec{p}_2^2 + m_2^2}$ ($E_3 = \sqrt{\vec{p}_3^2 + m_3^2}$) being the energy related to the particle with three-momentum $\vec{p}_2$ ($\vec{p}_3$) and mass m_2 (m_3) in the rest frame of the decaying particle, we can write Eq. (46) as

$$\Gamma = \frac{1}{(2\pi)^7 2^6 m_{B^+}^3} \int_{s_{124}^{\min}}^{s_{124}^{\max}} ds_{124} \int_{s_{134}^{\min}}^{s_{134}^{\max}} ds_{134} \int_{E_1^{\min}}^{E_1^{\max}} dE_1 \int_0^{2\pi} d\phi_1 \int_{-1}^{1} d\cos\theta_3 \int_0^{2\pi} d\phi_3 \times \frac{|\vec{p}_2||\vec{p}_3|}{|\vec{p}_2+\vec{p}_3|} \Theta(1-\cos^2\theta_1)\Theta(m_{B^+} - E_1 - E_2 - E_3) \sum_\lambda |t_a + t_b|^2, \tag{48}$$

with

$$s_{124}^{\min} = (m_1 + m_2 + m_4)^2, \qquad s_{124}^{\max} = (m_{B^+} - m_3)^2, \tag{49}$$

$$s_{134}^{\min} = (m_1 + m_3 + m_4)^2, \qquad s_{134}^{\max} = (m_{B^+} - m_2)^2, \tag{50}$$

$$E_1^{\min} = m_1, \qquad E_1^{\max} = \frac{m_{B^+}^2 + m_1^2 - s_{234}^{\min}}{2m_{B^+}}, \qquad s_{234}^{\min} = (m_2 + m_3 + m_4)^2, \tag{51}$$

and θ_1 is the angle between the vectors $\vec{p}_1$ and $\vec{p}_2 + \vec{p}_3$, which is fixed by the δ function of Eq. (46),

$$\cos\theta_1 = \frac{(m_{B^+} - E_1 - E_2 - E_3)^2 - \vec{p}_1^2 - m_4^2 - (\vec{p}_2 + \vec{p}_3)^2}{2|\vec{p}_1||\vec{p}_2 + \vec{p}_3|}. \tag{52}$$

The related Heaviside Θ function in Eq. (48) guarantees that $|\cos\theta_1| \le 1$, as it should be.

As mentioned in the Introduction, the decay $B^+ \to J/\psi\pi^+\pi^-K^+$ has been used for the experimental investigation of the properties of $X(3872)$. In this reaction, the reconstruction of the $J/\psi\pi^-K^+$ invariant mass distribution can also serve to investigate the properties of $K^*(4307)$. For the process $B^+ \to J/\psi\pi^+\pi^-K^+$, as can be seen in Fig. 3, the formation of $K^*(4307)$ is completely analogous to the one shown in Fig. 1(b), with the exception that the vertices $K^{*0}(4307) \to Z_c^0(3900)K^0 \to J/\psi\pi^0K^0$ should be replaced by $K^{*0}(4307) \to Z_c^-(3900)K^+ \to J/\psi\pi^-K^+$. Thus, the product of the coupling constants $g_{K^{*0}(4307)\to Z_c^0(3900)} g_{K^0 Z_c^0 \to J/\psi\pi^0}$ appearing in the amplitude related to the diagram in Fig. 1(b) should be substituted by $g_{K^{*0}(4307)\to K^+Z_c^-(3900)} g_{Z_c^- \to J/\psi\pi^-}$, which, by using the corresponding Clebsch-Gordan coefficients, is $\sqrt{2}$ times bigger than the former product. In this way, the calculation of the decay width for $B^+ \to J/\psi\pi^+\pi^-K^+$ and the determination of the $J/\psi\pi^-K^+$ invariant mass distribution is completely analogous to that for the reaction $B^+ \to J/\psi\pi^+\pi^0K^0$, but now we have a contribution from only one Feynman diagram instead of two (see Fig. 3), and the couplings (as explained above) are different.

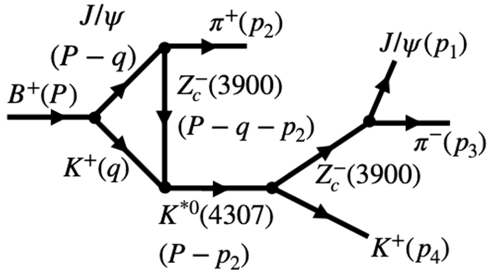

FIG. 3. Diagrammatical representation of the decay $B^+ \to J/\psi\pi^+\pi^-K^+$ through $K^*(4307)$ formation.

III. RESULTS

To obtain the $J/\psi\pi^{+(0)}K^0$ invariant mass distributions of the process $B^+ \to J/\psi\pi^+\pi^0K^0$, we have made use of Eq. (48) considering isospin average masses for those particles belonging to the same isospin multiplet. In such a case, there is no difference between the two invariant mass distributions. In Fig. 4 we show $d\Gamma/ds_{124}$ for the process $B^+ \to J/\psi\pi^+\pi^0K^0$ as a function of the invariant mass of the $J/\psi\pi^+K^0$ system, i.e., $\sqrt{s_{124}}$. The solid line in Fig. 4 represents the result obtained by using a cutoff Λ of 700 MeV to regularize the integrals in Eq. (23) for the center-of-mass momentum of the $K - Z_c$ system (see the Appendix for more details). As can be seen, a peak around 4307 MeV, with a width of 18 MeV, is observed in the distribution due to the formation of $K^*(4307)$, followed by an enhancement around the $K - Z_c(3900)$ threshold, a typical effect when triangular loops are involved in the determination of the amplitudes [36,41,42], as in our case. We also plot in Fig. 4 the contribution to $d\Gamma/ds_{124}$ originating from just the diagram in Fig. 1(b), which produces a background[1] (represented as a dashed line in Fig. 4). By integrating this distribution, we can get the

[1]Note that in the diagram in Fig. 1(b) the $K^*(4307)$ is formed in the s_{134} invariant mass.

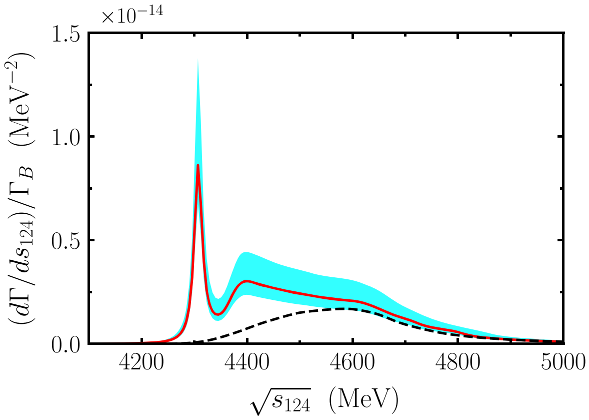

FIG. 4. Invariant mass distribution, divided by the full width of the B^+ meson, as a function of the invariant mass of the $J/\psi\pi^+K^0$ system, i.e., $\sqrt{s_{124}}$ in Fig. 1. The solid line corresponds to the result found with a cutoff Λ of 700 MeV. The dashed line represents the contribution to $d\Gamma/ds_{124}$ obtained from the diagram in Fig. 1(b). The band represents the uncertainty associated with $d\Gamma/ds_{124}$ when changing the cutoff in the range 700–750 MeV, changing the coupling in Eq. (11) inside the interval compatible with its error, and considering a 10% error for the couplings of $Z_c(3900)$ to the $J/\psi\pi$ system and that of $K^*(4307)$ to the $KZ_c(3900)$ system.

branching ratio for the process $B^+ \to \pi^{0(+)}K^{*+(0)}(4307) \to \pi^{0(+)}\,K^{0(+)}\,Z_c^{+(0)}\,(3900) \to \pi^{0(+)}\,K^0 J/\psi\pi^{+(0)}$, which is $\mathcal{BR} = 1.04 \times 10^{-8}$. We can also estimate the uncertainty related to this result. To do this, we vary the cutoff Λ in the range 700–750 MeV, as mentioned earlier, and vary the coupling in Eq. (11) in the range allowed by the related error. Further, we associate a 10% error to the coupling constants of $Z_c(3900)$ to the $J/\psi\pi$ system and of $K^*(4307)$ to the $KZ_c(3900)$ system. We then generate random numbers inside these intervals and calculate the mean value and the standard deviation for the branching ratio.

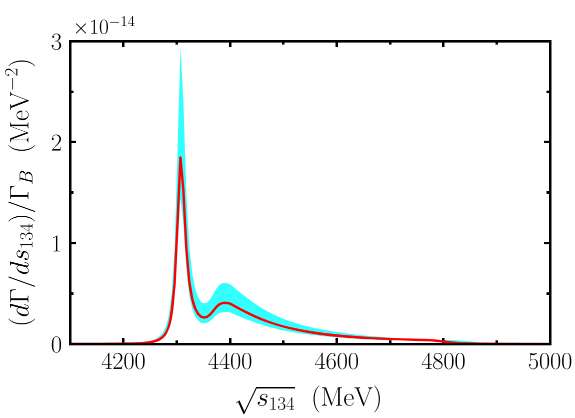

FIG. 5. Invariant mass distribution, divided by the full width of the B^+ meson, as a function of the invariant mass of the $J/\psi\pi^-K^+$ system, i.e., $\sqrt{s_{134}}$ in Fig. 3. The solid line and band have the same meaning as in Fig. 4.

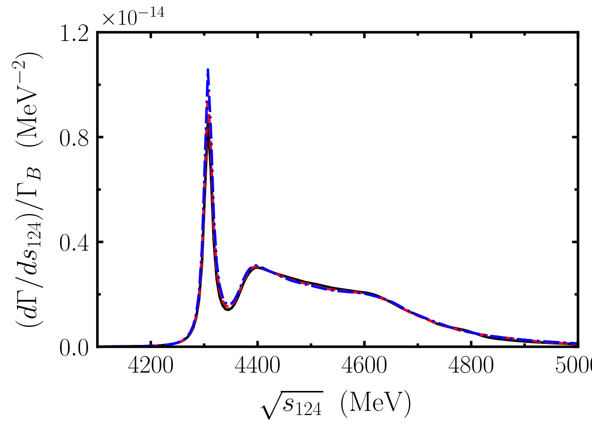

FIG. 6. Invariant mass distribution, divided by the full width of the B^+ meson, as a function of the invariant mass of the $J/\psi\pi^+K^0$ system, i.e., $\sqrt{s_{124}}$ in Fig. 1. The solid, dashed, and dash-dotted lines represent, respectively, the results found when regularizing the loop function in Eq. (34) with a Θ function, the exponential, or monopole form factors for the vertices involving molecular-type hadrons. The values $\Lambda = 700$ MeV, $\alpha \simeq 800$ MeV, and $\beta \simeq 900$ MeV have been used.

By doing this, we obtain the band shown in Fig. 4 and the estimated branching ratio becomes

$$\mathcal{BR} = (1.05 \pm 0.2) \times 10^{-8}. \tag{53}$$

In the case of the decay $B^+ \to J/\psi\pi^+\pi^-K^+$, the $d\Gamma/ds_{J/\psi\pi^-K^+}$ distribution is shown in Fig. 5 as a function of the $J/\psi\pi^-K^+$ invariant mass, i.e., $\sqrt{s_{134}}$ in Fig. 3. As can be seen, a peak structure related to the formation of $K^*(4307)$ is observed, together with an enhancement around 4400 MeV (as in Fig. 4) which is related to the threshold of the $K - Z_c$ system. The error band shown in the figure has been obtained in the same way as that of Fig. 4 and the solid line represents the result found with a cutoff of 700 MeV.

Alternatively to cutting the momentum integral when calculating Eq. (34), a form factor at each vertex of the triangular loop (shown in Figs. 1 and 3) could be introduced. In the case of the vertices involving molecular-type hadrons, typical form factor types considered in the literature [40,43] are exponential,

$$F(\vec{q}^{\,2}) = e^{-\frac{\vec{q}^{\,2}}{2\alpha^2}}, \tag{54}$$

and monopole,

$$F(\vec{q}^{\,2}) = \frac{\beta^2}{\beta^2 + \vec{q}^{\,2}}, \tag{55}$$

where $\alpha, \beta \sim 1000$ MeV. In this way, when regularizing the integral in Eq. (36), the Θ function would be substituted by a product of form factors.

In Fig. 6, we show the results obtained for the $d\Gamma/ds_{124}$ distribution by considering the exponential form factor of Eq. (54) with $\alpha \simeq 800$ MeV (dashed line) and by using the monopole form factor of Eq. (55) with $\beta \simeq 900$ MeV (dash-dotted line). The ratio α/β is chosen such that the area under the curve of $F^2(\vec{q}^2)$ versus $|\vec{q}|$ is the same, independently of the form factor used (exponential or monopole). As can be seen, some differences are found around the peak region of the distribution which gives an estimation of the uncertainties in the results due to different treatments of the ultraviolet divergences. Changes in the values of α and β lead to results which are compatible with the uncertainties shown in Fig. 4.

IV. CONCLUSION

By using isospin average masses between the members of the same multiplet, we have determined the $J/\psi\pi^{\pm,0}K^{+,0}$ invariant mass distributions of $B^+ \to J/\psi\pi^+\pi^0K^0$ and $B^+ \to J/\psi\pi^+\pi^-K^+$ with the purpose of analyzing the signal related to the formation of $K^*(4307)$. We found that the reconstruction of the $J/\psi\pi K$ invariant mass distributions for the reactions would exhibit the formation of the $K^*(4307)$, and the branching ratio determined for $B \to \pi K^*(4307) \to \pi K Z_c(3900) \to \pi K J/\psi\pi$ is $\sim 10^{-8}$. We hope that this calculation motivates the search for the $K^*(4307)$, formed as a consequence of the dynamics involved in the $KD\bar{D}^*$ system [22,25], by reconstructing the $J/\psi\pi K$ invariant mass distribution in $B \to J/\psi\pi\pi K$ reactions, for which experimental data are available.

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APPENDIX: DETERMINATION OF THE INTEGRALS IN EQ. (23)

The expression for the coefficients $a_i^{(3\mathcal{A})}$ and $a_i^{(5\mathcal{A})}$, with $i = 1, 2$, is analogous to that found in Eq. (28) for the $a_i^{(1\mathcal{A})}$ coefficients by replacing $P \cdot I^{(1\mathcal{A})} \to P \cdot I^{(3\mathcal{A})}$, $p_\mathcal{A} \cdot I^{(1\mathcal{A})} \to p_\mathcal{A} \cdot I^{(3\mathcal{A})}$ in the former case, and $P \cdot I^{(1\mathcal{A})} \to P \cdot I^{(5\mathcal{A})}$, and $p_\mathcal{A} \cdot I^{(1\mathcal{A})} \to p_\mathcal{A} \cdot I^{(5\mathcal{A})}$ in the latter, with

$$\begin{aligned} P \cdot I^{(3\mathcal{A})} &= \int \frac{d^4q}{(2\pi)^4} \frac{q^2(P\cdot q)}{\mathcal{D}(q, p_\mathcal{A})}, \qquad p_\mathcal{A} \cdot I^{(3\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^2(p_\mathcal{A}\cdot q)}{\mathcal{D}(q, p_\mathcal{A})}, \\ P \cdot I^{(5\mathcal{A})} &= \int \frac{d^4q}{(2\pi)^4} \frac{q^4(P\cdot q)}{\mathcal{D}(q, p_\mathcal{A})}, \qquad p_\mathcal{A} \cdot I^{(5\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^4(p_\mathcal{A}\cdot q)}{\mathcal{D}(q, p_\mathcal{A})}. \end{aligned} \tag{A1}$$

Proceeding in the same way as in the case of the $a_i^{(1\mathcal{A})}$ coefficients [see Eq. (28)], we have

$$\begin{aligned} a_1^{(2\mathcal{A})} &= \frac{1}{2[(P\cdot p_\mathcal{A})^2 - P^2 p_\mathcal{A}^2]}[P^2(p_\mathcal{A}\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} - p_\mathcal{A}^2 g\cdot I^{(2\mathcal{A})}) + (P\cdot p_\mathcal{A})^2 g\cdot I^{(2\mathcal{A})} \\ &\quad - 2(P\cdot p_\mathcal{A})P\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} + p_\mathcal{A}^2 P\cdot P\cdot I^{(2\mathcal{A})}], \\ a_2^{(2\mathcal{A})} &= \frac{1}{2[(P\cdot p_\mathcal{A})^2 - P^2 p_\mathcal{A}^2]^2}[p_\mathcal{A}^2\{P^2(p_\mathcal{A}\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} - p_\mathcal{A}^2 g\cdot I^{(2\mathcal{A})}) + 3p_\mathcal{A}^2 P\cdot P\cdot I^{(2\mathcal{A})}\} \\ &\quad + (P\cdot p_\mathcal{A})^2\{p_\mathcal{A}^2 g\cdot I^{(2\mathcal{A})} + 2p_\mathcal{A}\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})}\} - 6p_\mathcal{A}^2(P\cdot p_\mathcal{A})P\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})}], \\ a_3^{(2\mathcal{A})} &= \frac{1}{2[(P\cdot p_\mathcal{A})^2 - P^2 p_\mathcal{A}^2]^2}[P^2\{(P\cdot p_\mathcal{A})(p_\mathcal{A}^2 g\cdot I^{(2\mathcal{A})} - 3p_\mathcal{A}\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})}) \\ &\quad + 2p_\mathcal{A}^2 P\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})}\} - (P\cdot p_\mathcal{A})\{(P\cdot p_\mathcal{A})^2 g\cdot I^{(2\mathcal{A})} - 4(P\cdot p_\mathcal{A})P\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} \\ &\quad + 3p_\mathcal{A}^2 P\cdot P\cdot I^{(2\mathcal{A})}\}], \\ a_4^{(2\mathcal{A})} &= \frac{1}{2[(P\cdot p_\mathcal{A})^2 - P^2 p_\mathcal{A}^2]^2}[P^4\{3p_\mathcal{A}\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} - p_\mathcal{A}^2 g\cdot I^{(2\mathcal{A})}\} \\ &\quad + P^2\{(P\cdot p_\mathcal{A})^2 g\cdot I^{(2\mathcal{A})} - 6(P\cdot p_\mathcal{A})P\cdot p_\mathcal{A}\cdot I^{(2\mathcal{A})} + p_\mathcal{A}^2 P\cdot P\cdot I^{(2\mathcal{A})}\} \\ &\quad + 2(P\cdot p_\mathcal{A})^2 P\cdot P\cdot I^{(2\mathcal{A})}], \end{aligned} \tag{A2}$$

where

$$p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{(p_{\mathcal{A}} \cdot q)(p_{\mathcal{A}} \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})}, \qquad P \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{(P \cdot q)(p_{\mathcal{A}} \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})},$$
$$P \cdot P \cdot I^{(2\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{(P \cdot q)(P \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})}, \qquad g \cdot I^{(2\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^2}{\mathcal{D}(q, p_{\mathcal{A}})}. \tag{A3}$$

Analogously, for the coefficients $a_i^{(4\mathcal{A})}$, $i = 1, 2, \ldots, 4$, in Eq. (A2) we can simply replace the scalar integrals $p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})}$, $P \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})}$, $P \cdot P \cdot I^{(2\mathcal{A})}$, and $g \cdot I^{(2\mathcal{A})}$ by $p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})}$, $P \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})}$, $P \cdot P \cdot I^{(4\mathcal{A})}$, and $g \cdot I^{(4\mathcal{A})}$, respectively, with

$$p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^2(p_{\mathcal{A}} \cdot q)(p_{\mathcal{A}} \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})}, \qquad P \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^2(P \cdot q)(p_{\mathcal{A}} \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})},$$
$$P \cdot P \cdot I^{(4\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^2(P \cdot q)(P \cdot q)}{\mathcal{D}(q, p_{\mathcal{A}})}, \qquad g \cdot I^{(4\mathcal{A})} = \int \frac{d^4q}{(2\pi)^4} \frac{q^4}{\mathcal{D}(q, p_{\mathcal{A}})}. \tag{A4}$$

Similarly, by defining

$$\mathcal{I}^{(2\mathcal{A})}(a, b, c, d) = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{aq^{02} + b|\vec{q}|^2\cos^2\theta + cq^0|\vec{q}|\cos\theta + d|\vec{q}|^2}{\mathcal{D}(q, p_{\mathcal{A}})},$$
$$\mathcal{I}^{(3\mathcal{A})}(a, b, c, d) = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{aq^{03} + b|\vec{q}|q^{02}\cos\theta + cq^0|\vec{q}|^2 + d|\vec{q}|^3\cos\theta}{\mathcal{D}(q, p_{\mathcal{A}})},$$
$$\mathcal{I}^{(4\mathcal{A})}(a, b, c, d, e, f, g) = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\mathcal{D}(q, p_{\mathcal{A}})} [aq^{04} + b|\vec{q}|^2q^{02}\cos^2\theta + cq^{03}|\vec{q}|\cos\theta + dq^{02}|\vec{q}|^2 + e|\vec{q}|^4\cos^2\theta + fq^0|\vec{q}|^3\cos\theta + g|\vec{q}|^4],$$
$$\mathcal{I}^{(5\mathcal{A})}(a, b, c, d, e, f) = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\mathcal{D}(q, p_{\mathcal{A}})} [aq^{05} + bq^{04}|\vec{q}|\cos\theta + cq^{03}|\vec{q}|^2 + dq^{02}|\vec{q}|^3\cos\theta + eq^0|\vec{q}|^4 + f|\vec{q}|^5\cos\theta], \tag{A5}$$

we have

$$g \cdot I^{(2\mathcal{A})} = \mathcal{I}^{(2\mathcal{A})}(1, 0, 0, -1), \qquad P \cdot P \cdot I^{(2\mathcal{A})} = \mathcal{I}^{(2\mathcal{A})}(P^{02}, 0, 0, 0),$$
$$P \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})} = \mathcal{I}^{(2\mathcal{A})}(P^0 p^0_{\mathcal{A}}, 0, -P^0|\vec{p}_{\mathcal{A}}|, 0),$$
$$p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(2\mathcal{A})} = \mathcal{I}^{(2\mathcal{A})}(p^{02}_{\mathcal{A}}, |\vec{p}_{\mathcal{A}}|^2, -2p^0_{\mathcal{A}}|\vec{p}_{\mathcal{A}}|, 0),$$
$$P \cdot I^{(3\mathcal{A})} = \mathcal{I}^{(3\mathcal{A})}(P^0, 0, -P^0, 0), \qquad p_{\mathcal{A}} \cdot I^{(3\mathcal{A})} = \mathcal{I}^{(3\mathcal{A})}(p^0_{\mathcal{A}}, -|\vec{p}_{\mathcal{A}}|, -p^0_{\mathcal{A}}, |\vec{p}_{\mathcal{A}}|),$$
$$g \cdot I^{(4\mathcal{A})} = \mathcal{I}^{(4\mathcal{A})}(1, 0, 0, -2, 0, 0, 1), \qquad P \cdot P \cdot I^{(4\mathcal{A})} = \mathcal{I}^{(4\mathcal{A})}(P^{02}, 0, 0, -P^{02}, 0, 0, 0),$$
$$P \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})} = \mathcal{I}^{(4\mathcal{A})}(P^0 p^0_{\mathcal{A}}, 0, -P^0|\vec{p}_{\mathcal{A}}|, -P^0 p^0_{\mathcal{A}}, 0, P^0|\vec{p}_{\mathcal{A}}|, 0),$$
$$p_{\mathcal{A}} \cdot p_{\mathcal{A}} \cdot I^{(4\mathcal{A})} = \mathcal{I}^{(4\mathcal{A})}(p^{02}_{\mathcal{A}}, |\vec{p}_{\mathcal{A}}|^2, -2p^0_{\mathcal{A}}|\vec{p}_{\mathcal{A}}|, -p^{02}_{\mathcal{A}}, -|\vec{p}_{\mathcal{A}}|^2, 2p^0_{\mathcal{A}}|\vec{p}_{\mathcal{A}}|, 0),$$
$$P \cdot I^{(5\mathcal{A})} = \mathcal{I}^{(5\mathcal{A})}(P^0, 0, -2P^0, 0, P^0, 0),$$
$$p_{\mathcal{A}} \cdot I^{(5\mathcal{A})} = \mathcal{I}^{(5\mathcal{A})}(p^0_{\mathcal{A}}, -|\vec{p}_{\mathcal{A}}|, -2p^0_{\mathcal{A}}, 2|\vec{p}_{\mathcal{A}}|, p^0_{\mathcal{A}}, -|\vec{p}_{\mathcal{A}}|). \tag{A6}$$

We also define

$$\mathcal{I}^{(0\mathcal{A})} = \int \frac{dq^0}{(2\pi)} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\mathcal{D}(q, p_{\mathcal{A}})}, \tag{A7}$$

which coincides with the $I^{(0\mathcal{A})}$ integral in Eq. (23).

All of these integrals can be written in the same form as Eq. (34), with

$$\begin{aligned}
\mathcal{N}^{(0\mathcal{A})} &= -P^{02}\omega_K\omega_{J/\psi+Z(\mathcal{A})} + 2P^0 p^0_{\mathcal{A}}\omega_K\omega_{Z(\mathcal{A})} + \omega_{J/\psi+K}[\omega_{J/\psi+Z(\mathcal{A})} \\
&\quad \times \omega_{K+Z(\mathcal{A})}\omega_{J/\psi+K+Z(\mathcal{A})} - p^{02}_{\mathcal{A}}\omega_{Z(\mathcal{A})}], \\
\mathcal{N}^{(1\mathcal{A})}(a,b) &= a\omega_K f_1 + \cos\theta b|\vec{q}|\mathcal{N}^{(0\mathcal{A})}, \\
\mathcal{N}^{(2\mathcal{A})}(a,b,c,d) &= a\omega_K f_2 + |\vec{q}|[(\cos^2\theta b + d)|\vec{q}|\mathcal{N}^{(0\mathcal{A})} + \cos\theta c\omega_K f_1], \\
\mathcal{N}^{(3\mathcal{A})}(a,b,c,d) &= a\omega_K f_3 + |\vec{q}|[\cos\theta\{b\omega_K f_2 + d|\vec{q}|^2\mathcal{N}^{(0\mathcal{A})}\} + c|\vec{q}|\omega_K f_1], \\
\mathcal{N}^{(4\mathcal{A})}(a,b,c,d,e,f,g) &= a\omega_K f_4 + |\vec{q}|[\cos\theta\{|\vec{q}|(\cos\theta[b\omega_K f_2 + e|\vec{q}|^2\mathcal{N}^{(0\mathcal{A})}] + f|\vec{q}|\omega_K f_1) \\
&\quad + c\omega_K f_3\} + d|\vec{q}|\omega_K f_2 + g|\vec{q}|^3\mathcal{N}^{(0\mathcal{A})}], \\
\mathcal{N}^{(5\mathcal{A})}(a,b,c,d,e,f) &= a\omega_K f_5 + |\vec{q}|[\omega_K|\vec{q}|\{f_1|\vec{q}|^2 e + cf_3\} \\
&\quad + \{f\mathcal{N}^{(0\mathcal{A})}|\vec{q}|^4 + \omega_K(df_2|\vec{q}|^2 + bf_4)\}\cos\theta],
\end{aligned} \tag{A8}$$

where we have omitted the explicit dependence of the functions f_i, $i = 1, 2, \ldots, 5$ and ω_K, $\omega_{Z(\mathcal{A})}$, and $\omega_{J/\psi}$ [see Eq. (35)] on $\vec{q}$, $\vec{p}_{\mathcal{A}}$, and $p^0_{\mathcal{A}}$ for simplicity. In particular, by introducing

$$\begin{aligned}
\omega_{J/\psi+K} &= \omega_{J/\psi} + \omega_K, \qquad \omega_{J/\psi+Z(\mathcal{A})} = \omega_{J/\psi} + \omega_{Z(\mathcal{A})}, \\
\omega_{K+Z(\mathcal{A})} &= \omega_K + \omega_{Z(\mathcal{A})}, \qquad \omega_{J/\psi+K+Z(\mathcal{A})} = \omega_{J/\psi} + \omega_K + \omega_{Z(\mathcal{A})},
\end{aligned} \tag{A9}$$

the f_i functions in Eq. (A8) correspond to

$$\begin{aligned}
f_1(\vec{q}, p^0_{\mathcal{A}}, \vec{p}_{\mathcal{A}}) &= P^{02}p^0_{\mathcal{A}}\omega_{Z(\mathcal{A})} + \omega_{J/\psi+Z(\mathcal{A})}P^0(-P^{02} + P^0 p^0_{\mathcal{A}} + \omega_{J/\psi}[\omega_{J/\psi+Z(\mathcal{A})} + 2\omega_K] \\
&\quad + \omega^2_{K+Z(\mathcal{A})}) - P^0 p^{02}_{\mathcal{A}}\omega_{Z(\mathcal{A})} - p^0_{\mathcal{A}}\omega_{J/\psi}\omega_{J/\psi+K}[\omega_{J/\psi+K} + 2\omega_{Z(\mathcal{A})}], \\
f_2(\vec{q}, p^0_{\mathcal{A}}, \vec{p}_{\mathcal{A}}) &= \omega_{J/\psi+Z(\mathcal{A})}[-P^{02} + 2P^0 p^0_{\mathcal{A}} - p^{02}_{\mathcal{A}} + \omega^2_{J/\psi+K} + 2\omega_K\omega_{Z(\mathcal{A})} + \omega^2_{Z(\mathcal{A})}]P^{02} \\
&\quad - 2P^0 p^0_{\mathcal{A}}\omega_{J/\psi}[\omega_{J/\psi+K}\omega_{J/\psi+K+Z(\mathcal{A})} + \omega_{Z(\mathcal{A})}\omega_K] \\
&\quad + \omega_{J/\psi}\omega_{J/\psi+K}[\omega_{J/\psi+Z(\mathcal{A})}\{p^{02}_{\mathcal{A}} - \omega_{Z(\mathcal{A})}\omega_{K+Z(\mathcal{A})}\} + p^{02}_{\mathcal{A}}\omega_K], \\
f_3(\vec{q}, p^0_{\mathcal{A}}, \vec{p}_{\mathcal{A}}) &= -P^{05}\omega_{J/\psi+Z(\mathcal{A})} + P^{04}p^0_{\mathcal{A}}[2\omega_{J/\psi+Z(\mathcal{A})} + \omega_{J/\psi}] + P^{03}(\omega_{J/\psi+Z(\mathcal{A})} \\
&\quad \times [\omega^2_{J/\psi+K} + 2\omega_K\omega_{Z(\mathcal{A})} + \omega^2_{Z(\mathcal{A})}] - p^{02}_{\mathcal{A}}[\omega_{J/\psi+Z(\mathcal{A})} + 2\omega_{J/\psi}]) + P^{02}p^0_{\mathcal{A}}\omega_{J/\psi} \\
&\quad \times (p^{02}_{\mathcal{A}} - 2\omega_{Z(\mathcal{A})}[\omega_{J/\psi+K} + 2\omega_K] - 3\omega^2_{J/\psi+K} - \omega^2_{Z(\mathcal{A})}) + P^0\omega_{J/\psi} \\
&\quad \times (p^{02}_{\mathcal{A}}[\omega_{Z(\mathcal{A})}\{\omega_{J/\psi+K} + \omega_K\} + 3\omega^2_{J/\psi+K}] - \omega_{Z(\mathcal{A})}\omega_{J/\psi+Z(\mathcal{A})} \\
&\quad \times [2\omega_{J/\psi+K}\omega_{K+Z(\mathcal{A})} + \omega^2_K]) + p^0_{\mathcal{A}}\omega_{J/\psi}\omega_{J/\psi+K} \\
&\quad \times (\omega_{Z(\mathcal{A})}[\omega_{Z(\mathcal{A})}\omega_{J/\psi+K} + 2\omega_{J/\psi}\omega_K] - p^{02}_{\mathcal{A}}\omega_{J/\psi+K}),
\end{aligned}$$

$$\begin{aligned}
f_4(\vec{q}, p^0_{\mathcal{A}}, \vec{p}_{\mathcal{A}}) &= -P^{06}\omega_{J/\psi+Z(\mathcal{A})} + 2P^{05}p^0_{\mathcal{A}}(\omega_{J/\psi+Z(\mathcal{A})} + \omega_{J/\psi}) + P^{04}(\omega_{J/\psi+Z(\mathcal{A})} \\
&\quad \times [\omega_{Z(\mathcal{A})}\{\omega_{J/\psi+K} + \omega_K\} + \omega^2_{J/\psi+K} + \omega^2_{Z(\mathcal{A})}] - p^{02}_{\mathcal{A}}[\omega_{J/\psi+Z(\mathcal{A})} + 5\omega_{J/\psi}]) \\
&\quad + 4P^{03}p^0_{\mathcal{A}}\omega_{J/\psi}(p^{02}_{\mathcal{A}} - \omega_{Z(\mathcal{A})}[\omega_{J/\psi+K} + \omega_{K+Z(\mathcal{A})}] - \omega^2_{J/\psi+K}) \\
&\quad - P^{02}\omega_{J/\psi}(p^{04}_{\mathcal{A}} - 2p^{02}_{\mathcal{A}}[\omega_{Z(\mathcal{A})}\{\omega_{J/\psi+K} + \omega_{K+Z(\mathcal{A})}\} + 3\omega^2_{J/\psi+K}] \\
&\quad + \omega_{Z(\mathcal{A})}\omega_{J/\psi+Z(\mathcal{A})}[\omega_{J/\psi}\{\omega_{J/\psi+K} + 3\omega_K\} + 2\omega_K\{2\omega_{K+Z(\mathcal{A})} + \omega_K\} + \omega^2_{Z(\mathcal{A})}]) \\
&\quad + 2P^0 p^0_{\mathcal{A}}\omega_{J/\psi}(\omega_{Z(\mathcal{A})}[2\omega_{Z(\mathcal{A})}\omega^2_{J/\psi+K} + \omega_{J/\psi}\{\omega_{J/\psi+K} + \omega_K\}^2] - 2p^{02}_{\mathcal{A}}\omega^2_{J/\psi+K}) \\
&\quad + \omega_{J/\psi}\omega_{J/\psi+K}(p^{04}_3\omega_{J/\psi+K} - p^{02}_{\mathcal{A}}\omega_{Z(\mathcal{A})}[\omega^2_{J/\psi+K} + 2\omega_{Z(\mathcal{A})}\omega_{J/\psi+K} + \omega_{J/\psi}\omega_K] \\
&\quad + \omega_{Z(\mathcal{A})}\omega_{J/\psi+Z(\mathcal{A})}\omega_{K+Z(\mathcal{A})}[\omega_{Z(\mathcal{A})}\omega_{J/\psi+K} + \omega_{J/\psi}\omega_K]),
\end{aligned}$$

$$
\begin{aligned}
f_5(\vec{q}, p^0_{\mathcal{A}}, \vec{p}_{\mathcal{A}}) = & -\omega_{J/\psi+Z(\mathcal{A})} P^{07} + p^0_{\mathcal{A}}(2\omega_{J/\psi+Z(\mathcal{A})} + 3\omega_{J/\psi}) P^{06} + (\omega_{J/\psi+Z(\mathcal{A})}[\omega^2_{J/\psi} + \{2\omega_K + 3\omega_{Z(\mathcal{A})}\} \\
& \times \omega_{J/\psi} + \omega^2_{K+Z(\mathcal{A})}] - p^{02}_{\mathcal{A}}[\omega_{J/\psi+Z(\mathcal{A})} + 9\omega_{J/\psi}]) P^{05} + \omega_{J/\psi}(10 p^{03}_{\mathcal{A}} - \omega_{Z(\mathcal{A})} p^{02}_{\mathcal{A}} \\
& - 5[\omega_{J/\psi+K}\{\omega_{J/\psi+K} + 2\omega_{Z(\mathcal{A})}\} + 2\omega^2_{Z(\mathcal{A})}] p^0_{\mathcal{A}} + \omega_{Z(\mathcal{A})}\omega^2_{J/\psi+Z(\mathcal{A})}) P^{04} \\
& - \omega_{J/\psi}(5 p^{04}_{\mathcal{A}} - 2\omega_{Z(\mathcal{A})} p^{03}_{\mathcal{A}} + 2[-\omega_{J/\psi+K}\{5\omega_{J/\psi+K} + 4\omega_{Z(\mathcal{A})}\} - 4\omega^2_{Z(\mathcal{A})}] p^{02}_{\mathcal{A}} \\
& + 2\omega_{Z(\mathcal{A})}\omega^2_{J/\psi+Z(\mathcal{A})} p^0_{\mathcal{A}} + \omega_{Z(\mathcal{A})}\omega_{J/\psi+Z(\mathcal{A})}[\omega_{J/\psi}\{3\omega_{J/\psi} + \omega_{K+Z(\mathcal{A})} + 7\omega_K\} \\
& + 10\omega^2_K + \omega_{Z(\mathcal{A})}\{3\omega_{K+Z(\mathcal{A})} + 5\omega_K\}]) P^{03} - \omega_{J/\psi}(-p^{05}_{\mathcal{A}} + \omega_{Z(\mathcal{A})} p^{04}_{\mathcal{A}} \\
& + 2[\omega_{J/\psi+K}\{5\omega_{J/\psi+K} + \omega_{Z(\mathcal{A})}\} + \omega^2_{Z(\mathcal{A})}] p^{03}_{\mathcal{A}} - 2\omega_{Z(\mathcal{A})}[\omega_{J/\psi}\{\omega_{J/\psi} + \omega_{K+Z(\mathcal{A})}\} \\
& + \omega^2_{K+Z(\mathcal{A})} - \omega_K\omega_{Z(\mathcal{A})}] p^{02}_{\mathcal{A}} - \omega_{Z(\mathcal{A})}[\omega^2_{Z(\mathcal{A})}\{\omega_{Z(\mathcal{A})} + 2\omega_{J/\psi+K}\} \\
& + 2\{5\omega^2_{J/\psi} + 12\omega_K\omega_{J/\psi} + 5\omega^2_K\}\omega_{Z(\mathcal{A})} + 2\omega_{J/\psi}\{4\omega^2_{J/\psi} + 11\omega_K\omega_{J/\psi} + 10\omega^2_K\}] p^0_{\mathcal{A}} \\
& + \omega_{Z(\mathcal{A})}\omega^2_{J/\psi+Z(\mathcal{A})}[\omega^2_{J/\psi+K} + \omega^2_{K+Z(\mathcal{A})}]) P^{02} + \omega_{J/\psi}(\omega^2_{J/\psi+K}[5 p^{04}_{\mathcal{A}} \\
& - 2\omega_{Z(\mathcal{A})} p^{03}_{\mathcal{A}}] - \omega_{Z(\mathcal{A})}[7\omega^3_{J/\psi} + 20\omega_K\omega^2_{J/\psi} + 18\omega^2_K\omega_{J/\psi} + 4\omega^3_K + 8\omega^2_{J/\psi+K}\omega_{Z(\mathcal{A})}] p^{02}_{\mathcal{A}} \\
& + 2\omega^2_{J/\psi+K}\omega_{Z(\mathcal{A})}\omega^2_{J/\psi+Z(\mathcal{A})} p^0_{\mathcal{A}} + \omega_{Z(\mathcal{A})}\omega_{J/\psi+Z(\mathcal{A})}[\omega_{Z(\mathcal{A})}\{3\omega_{K+Z(\mathcal{A})} + \omega_K\}\omega^2_K \\
& + \omega_{J/\psi}\{\omega_{K+Z(\mathcal{A})} + \omega_{Z(\mathcal{A})}\}\{3\omega_{K+Z(\mathcal{A})} + \omega_K\}\omega_K + 3\omega^2_{J/\psi}\omega^2_{K+Z(\mathcal{A})}]) P^0 \\
& + \omega_{J/\psi}\omega_{J/\psi+K}(\omega_{J/\psi+K}[-p^{05}_{\mathcal{A}} + p^{04}_{\mathcal{A}}\omega_{Z(\mathcal{A})} + 2\omega_{Z(\mathcal{A})}\omega_{J/\psi+K+Z(\mathcal{A})} p^{03}_{\mathcal{A}} \\
& - \omega_{Z(\mathcal{A})}\{\omega_{J/\psi}(\omega_{J/\psi+Z(\mathcal{A})} + \omega_{Z(\mathcal{A})}) + \omega^2_K + 2\omega_{Z(\mathcal{A})}\omega_{K+Z(\mathcal{A})}\} p^{02}_{\mathcal{A}}] \\
& + \omega_{Z(\mathcal{A})}[-\omega_{Z(\mathcal{A})}\{\omega_{K+Z(\mathcal{A})} + \omega_K\}\{\omega_K\omega_{Z(\mathcal{A})} + \omega_{J/\psi}(\omega_{K+Z(\mathcal{A})} + \omega_K)\} \\
& - 2\omega^2_{J/\psi}\omega^2_{K+Z(\mathcal{A})}] p^0_{\mathcal{A}} + \omega_{J/\psi+K}\omega_{Z(\mathcal{A})}\omega^2_{J/\psi+Z(\mathcal{A})}\omega^2_{K+Z(\mathcal{A})}).
\end{aligned} \tag{A10}
$$

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