$T\bar{T}$ deformation of the compactified boson and its interpretation in lattice gauge theory

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(Received 4 March 2020; accepted 18 June 2020; published 8 July 2020)

We study the effective string description of spacelike Polyakov loop correlators at finite temperature with the goal of describing the behavior of the spacelike string tension in the vicinity of the deconfinement transition. To this end we construct the partition function of the Nambu-Goto effective string theory in presence of a compact transverse direction of length L equal to the inverse temperature. We then show that, under particular conditions, our result can be interpreted as the partition function of the $T\bar{T}$ deformation of the 2d quantum field theory describing a compactified bosonic field and that this mapping allows a deeper insight on the behavior of the spacelike observables of the theory. In particular we show, by imposing that the spectrum of the model obeys the inviscid Burgers equation, that the $T\bar{T}$ deformations follow well-defined trajectories in the parameter space (σ, L) of the model, where σ is the string tension, which are characterized by a constant value of the dimensionless compactification radius $\rho = L\sqrt{\sigma/2\pi}$. We discuss the potential usefulness of these results for studying the spacelike string tension of the underlying lattice gauge theory and its behavior across the deconfinement transition.

DOI: 10.1103/PhysRevD.102.014504

I. INTRODUCTION

An interesting open issue in lattice gauge theory is to understand and model the behavior of the so-called "spacelike string tension" [1–16] across the deconfinement transition. The spacelike string tension is extracted from the correlator of spacelike Polyakov loops, i.e., Polyakov loops which lay in a spacelike plane, orthogonal to the compact time direction, whose size coincides with the inverse finite temperature of the theory. Due to their spacelike nature these Polyakov loops do not play the role of the order parameter of deconfinement and the spacelike [13–16] string tension extracted from them is different from the actual string tension of the model, which is instead extracted from timelike Polyakov loop correlators. At low temperature the two string tensions coincide but as the temperature increases they behave differently [1–5]. The ordinary string tension decreases as the deconfinement temperature is approached and vanishes at the deconfinement point, while the spacelike one remains constant and then increases in the deconfined phase [1–3]. The physical reason for this behavior is that the correlator of two spacelike Polyakov loops describes quarks moving in a finite temperature environment. It can be shown that what we called spacelike string tension is related to the screening masses in hot QCD [6–12], and thus it does not vanish in the deconfined phase.

Despite the fact that it can be measured very precisely with Montecarlo simulations, a satisfactory modelization of the behavior of the spacelike string tension is still lacking. To partially fill this gap we construct in this paper the effective string description of the correlator of two spacelike Polyakov loops assuming a Nambu-Goto form for the effective string action. This essentially amounts to extend the known effective string results to the case in which one of the transverse degree of freedom of the model—representing the Euclidean time direction—is compact. It turns out that the natural setting to understand the behavior of this partition function, and thus model the spacelike string behavior is in terms of a $T\bar{T}$ perturbation of the free theory of a compact boson. This mapping imposes a well-defined relation between the string tension σ and the finite

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temperature $1/L$ of the model thus allowing to take the continuum limit in a physically meaningful way.

$T\bar{T}$ perturbations of 2d quantum field theories (QFT) [17–36] attracted a lot of interest in the past few years mainly due to the fact that they are at the crossroad of several different research lines. They are related to the effective string models of the Nambu-Goto type [27–30] and, as such, can be used to describe the infrared properties of extended objects such as Wilson loops or Polyakov loops correlators in the confining regime of lattice gauge theories (LGTs) [37–39]. They are a remarkable example of an irrelevant perturbation of a 2d QFT which is integrable and whose spectrum can be easily obtained as the solution of an integrable differential equation, the inviscid Burgers equation [17–19,23–25]. They have a deep connection with gravitylike theories [24,26,28,29] and can be understood, in the framework of the AdS/CFT correspondence, as a perturbation which changes the boundary conditions of the bulk fields on AdS_3 (for a recent review see [36]). In this paper we shall concentrate in particular on the first of these lines, looking at our effective string partition function as the $T\bar{T}$ perturbation of the 2d CFT of a compactified free boson.

Let us recall a few important features of this relation. The simplest possible effective string model, compatible with the Lorentz invariance of the underlying d dimensional LGT is the Nambu-Goto model which assumes the string action $S_{\rm NG}$ to be proportional to the area spanned by the string world sheet and is parametrized by the string tension σ. It can be shown that the Nambu Goto action in the physical gauge is equivalent to the $T\bar{T}$ perturbation of the 2d conformal field theory (CFT) of $d-2$ free bosons, the transverse degrees of freedom of a string propagating in a d-dimensional target space [27]. If we denote by $S_{\rm cl}$ the "classical" contribution to the action, proportional to the area of the minimal string surface, and by S_0 the action of the $d-2$ free bosons representing its transverse fluctuations, one has

$$S_{\rm NG} = S_{\rm cl} + S_0 - t\int d^2\xi T\bar{T}, \tag{1.1}$$

where, with this choice of sign, the perturbing parameter t is related to the string tension by $\sigma = 1/(2t)$ (see Sec. III for more details).

A lot of information on the spectrum of $T\bar{T}$ perturbed models can be obtained thanks to the fact that the evolution of the energy levels as a function of t obeys an inviscid Burgers equation. Also many properties of the partition functions of such models are under control [20,28,29,32]. In particular, when the perturbation parameter t is positive within the sign convention employed in Eq. (1.1) it can be shown that the perturbed partition function is unique and that it satisfies a differential equation originating from the Burgers equation for the energy levels [20]; this allows us to determine many of its properties, both at the perturbative and at the nonperturbative level in t. In this respect, the effective string theory approach, which allows us to construct in an explicit way the partition function of the perturbed model for any value of the perturbing parameter, represents a perfect laboratory to test the above approaches and, indeed, another aim of our paper is also to extend the range of explicitly known partition functions to the $T\bar{T}$ perturbation of the CFT of a compactified boson. This CFT is particularly interesting because it is doubly perturbed. It admits a marginal perturbation, parametrized by the compactification radius ρ which moves the theory along the critical line in the $c=1$ plane [40], where c is the central charge of the CFT. This corresponds to moving the theory along the critical low temperature phase of the XY model, which is the most important statistical mechanics realization of the compactified boson universality class. There is however also the irrelevant $T\bar{T}$ perturbation. We shall see that the model shows a range of interesting different behaviors due to the interplay between the two perturbing parameters ρ and t.

A final remark on the choice of the boundary conditions. For the LGT applications, the most natural choice of boundary conditions corresponds to a cylindrical world sheet with Dirichlet boundary conditions along the Euclidean time direction—or along a compact spacelike directions in our case—which can be immediately mapped into the correlator of two Polyakov loops and hence to the interquark potential. From the CFT point of view instead the most natural choice is the torus geometry which allows us to use modular transformations to study the nature and the location of singular points in the perturbing parameter and to relate among them these singular behaviors. We shall discuss both cases in the following, constructing both the torus and the cylinder partition functions.

This paper is organized as follows. In Sec. II we shall explicitly construct, both for the torus and the cylinder geometry, the Nambu-Goto effective string model describing the confining regime of a d dimensional lattice gauge theory in which one of the $d-2$ transverse directions is compactified. At the end of the section we shall discuss the LGT interpretation of our result. In Sec. III we discuss the interpretation of our effective string theory results as the $T\bar{T}$ perturbation of a compactified boson and obtain the relation imposed by integrability between the string tension σ and the finite temperature $1/L$ of the model. Finally, Sec. IV will be devoted to a discussion of potential applications of our results and to some concluding remarks.

II. PARTITION FUNCTIONS OF THE NAMBU-GOTO EFFECTIVE STRING MODEL

In this section we review some aspects of the effective string description of the confining regime of gauge theories, focusing on two observables, the correlator of Polyakov loops and the interface. The first-order approach to the

Nambu-Goto (NG) string allows us to explicitly obtain the exact form of the corresponding partition functions [37,38]. We work here in a compactified setup, not yet considered in the previous literature.

A. The effective string setup

The starting point of the effective string description of the interquark potential is to model the latter in terms of a string partition function. One can, for instance, evaluate the interquark potential by using the expectation value $\langle P^\dagger(R)P(0)\rangle$ of a pair of Polyakov loops separated by a distance R in the spatial direction x_1. Each Polyalov loop is the line traced in the direction x_0 by an external static quark. The line is closed because this direction is compact: $x_0 \sim x_0 + L_0$. If this direction is interpreted as the Euclidean time direction, L_0 represents the inverse temperature of the system and we are dealing with timelike Polyakov loops. As we anticipated in the Introduction, we will later take the point of view in which the 0th direction is to be considered a spatial one. In the confining phase, the chromomagnetic flux sourced by the external quarks is squeezed into a one-dimensional string stretched between the two quarks. This open string describes, in its evolution along the temperature direction x_0, a surface $\mathcal{M}$ with the topology of a cylinder. In Fig. 1, on the left, we outlined in gray the surface $\mathcal{M}$ with the minimal area, calling it $\mathcal{M}_0$. We denote by

$$\mathcal{A} = L_0 R, \qquad u = L_0/R, \tag{2.1}$$

the area of $\mathcal{M}_0$ and the ratio of its sides.

The surface $\mathcal{M}$ can be described parametrically as

$$x^\mu = X^\mu(\xi^0, \xi^1), \quad \mu = 0, 1, \ldots d-1, \tag{2.2}$$

where the adimensional parameters ξ^α with $\alpha = 0,\ 1$, usually referred to as world sheet coordinates, live on a reference cylindrical surface Σ. The functions $X^\mu(\xi^0, \xi^1)$ represent in this language the embedding in the physical space of the world sheet. The effective string theory approach aims at taking into account the contributions of all possible surfaces $\mathcal{M}$ by path-integrating over these maps with the prescribed “cylinder” boundary conditions, namely fixed (Dirichlet) along the spacelike directions and periodic along the temperature direction,

$$\langle P^\dagger(R)P(0)\rangle = \int_{\mathrm{cyl}} [\mathcal{D}X] \mathrm{e}^{-S_{\mathrm{eff}}[X]} \equiv \hat{\mathcal{Z}}_{\mathrm{cyl}}(L_0, R). \tag{2.3}$$

An effective string model is characterized by the choice of the string action $S_{\mathrm{eff}}[X]$.

Let us note that the cylinder $\mathcal{M}$ can also be seen, in a dual way, as the world sheet swept out by a closed string emitted by a Polyakov loop (to be represented by a boundary state in the closed string Hilbert space) and reabsorbed by the other. In this setup, it is natural to parameterize the cylinder $\mathcal{M}_0$ by its area $\mathcal{A}$ and the ratio $v = 1/u = R/L_0$. We will use this interpretation later.

Another observable which can be described by an effective string is the interface free energy $F_{\mathrm{int}}(L_0, L_1)$ of suitably chosen dual models [38]. This situation is depicted on the right in Fig. 1. Besides the direction x_0 (which in this context is usually interpreted as a spatial coordinate), at least one of the other directions, x_1 in our case, is compact with length L_1. In this case the surface $\mathcal{M}$ has the topology of a torus, and we will have

$$F_{\mathrm{int}}(L_0, L_1) = \int_{\mathrm{torus}} [\mathcal{D}X] \mathrm{e}^{-S_{\mathrm{eff}}[X]} \equiv \hat{\mathcal{Z}}_{\mathrm{torus}}(L_0, L_1), \tag{2.4}$$

where the maps $X^\mu(\xi^0, \xi^1)$ from a reference torus Σ to space-time describe parametrically $\mathcal{M}$. The quantities,

$$\mathcal{A} = L_0 L_1, \qquad u = L_0/L_1, \tag{2.5}$$

represent respectively the imaginary parts of the Kähler modulus (the area) and of the complex structure modulus for the minumal surface $\mathcal{M}_0$.

Compact transverse directions— In the following we shall consider the situation in which one of the transverse directions is compact, with size L, generalizing the results of [37,38]. This generalization is interesting in its application to the description of lattice gauge theories, for the reasons described in the Introduction and discussed more in detail in Sec. II F. It is interesting also within the reinterpretation of the NG string as a $T\bar{T}$ deformation, to be discussed in Sec. III, in particular because of the interplay

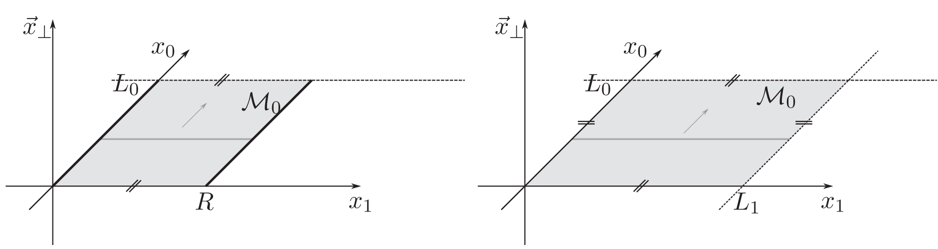

FIG. 1. Effective string description of the correlator of two Polyalov loops (left) and of an interface (right). See the text for a detailed explanation.

between the marginal deformation parametrized by the transverse compactification radius and the relevant $T\bar{T}$ deformation. Specifically, we will consider the direction x_2 to be compact. Note that, even if it is not presented here, the generalization to more than one compact transverse directions is straightforward.

B. The Nambu-Goto model

The action for the effective string model must be consistent with the Lorentz invariance of the underlying gauge theory. The simplest choice is the Nambu-Goto model, for which the string action $S_{\rm eff}$—which we will call $S_{\rm NG}$—is proportional to the area of $\mathcal{M}$,

$$S_{\rm NG} = \sigma \int_\Sigma d^2\xi \sqrt{\det g}, \quad \text{where } g_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha}\frac{\partial X^\mu}{\partial \xi^\beta} \tag{2.6}$$

is the induced metric on the reference world sheet surface Σ. Moreover, σ is the string tension, which has dimension $(\text{length})^{-2}$.

The actual effective string model which describes a gauge theory in its confining regime differs from the NG one. However, as shown in [41,42] (and in [43] for the boundary corrections) deviations from the NG expression are strongly constrained by Lorentz invariance; they are expected to start at order $1/R^6$ in a large distance expansion where R represents the smallest relevant length in the geometry under consideration—for instance, the Polyakov loop separation when we are in a low-temperature regime in which the length L_0 of the loops themselves is much bigger that R; for a review see for instance [44]. In this respect the Nambu–Goto action can be considered as a sort of "mean field" approximation of the actual confining string and indeed, as expected for a mean field approximation, it gives the same answer for the confining potential of any gauge theory. Determining the higher order terms in the confining string action of a given gauge theory is a most interesting and important open issue. One may hope to find some indications on the nature and size of the higher order corrections by studying the discrepancies between lattice simulations of gauge theories and Nambu-Goto predictions. Actually, the gauge fixed Nambu-Goto action fits remarkably well the lattice data for different gauge theories, which means that deviations are rather small. In the last few years, thanks to the improvement in the precision of lattice simulations, a few signatures of these higher order corrections have been detected in the 3d gauge Ising model [45] and in the 3d SU(N) gauge theories [46,47]. The results of these papers indicate that these corrections are enhanced in the high temperature regime, i.e., when the size L_0 of the lattice in the direction of the Polyakov loops is slightly above the deconfinement temperature, still in the confining regime but as short as possible and represents the smallest scale in the game.

Let us go back to the Nambu-Goto action (2.6). Its reparametrization and Weyl invariances require a gauge choice.

The physical gauge— A standard choice is the so called "physical gauge," in which

$$x^0 = X^0 = L_0\xi^0, \qquad x^1 = X^1 = R\xi^1. \tag{2.7}$$

Here we are referring to the cylinder setup, but the analogous choice can be taken also in the interface setup. In this gauge, the surface $\mathcal{M}$ is described by expressing the $d-2$ transverse coordinates X^i with $i = 2, \ldots d-1$ as functions of the coordinates x^0, x^1, i.e., by giving the height profile of the surface over the minimal surface $\mathcal{M}_0$ which takes over the role of the reference world sheet Σ. In the physical gauge the determinant of the induced metric reduces to

$$\mathcal{A}^{-2}\det g = 1 + \left(\frac{\partial\vec{X}}{\partial x^0}\right)^2 + \left(\frac{\partial\vec{X}}{\partial x^1}\right)^2 + \left(\frac{\partial\vec{X}}{\partial x^0}\right)^2\left(\frac{\partial\vec{X}}{\partial x^1}\right)^2 - \left(\frac{\partial\vec{X}}{\partial x^0}\cdot\frac{\partial\vec{X}}{\partial x^1}\right)^2, \tag{2.8}$$

where we used the notation $\vec{X} = \{X^i\}$. Inserting this into Eq. (2.6) we get

$$S_{\rm NG} = S_{\rm cl} + S_0[\vec{X}] + S_1[\vec{X}] + \ldots, \tag{2.9}$$

where $S_{\rm cl} = \sigma L_0 R$ is the usual "classical" area term, while $S_0[\vec{X}]$ is simply the action of the two–dimensional conformal field theory of $(d-2)$ free bosons living on $\mathcal{M}_0$,

$$S_0[\vec{X}] = \frac{\sigma}{2}\int_{\mathcal{M}_0} dx^0dx^1\,\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\vec{X}}{\partial x^\alpha}. \tag{2.10}$$

The remaining terms in the expansion on the rhs of Eq. (2.9) represent a perturbation of this free bosonic theory. In particular at the next to leading order we have

$$S_1[X] = \sigma\int_{\mathcal{M}_0} dx^0dx^1\left[\frac{1}{8}\left(\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\vec{X}}{\partial x^\alpha}\right)^2 - \frac{1}{2}\left(\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\vec{X}}{\partial x^\beta}\right)\left(\frac{\partial\vec{X}}{\partial x^\alpha}\cdot\frac{\partial\vec{X}}{\partial x^\beta}\right)\right]. \tag{2.11}$$

Actually, as it is well known [48], the full NG action can be seen as a $T\bar{T}$ deformation of the free bosonic theory. In this perspective, Eq. (2.11) encodes just the first order in this perturbation; however, it already fixes the relation between the parameter, usually called t, of the $T\bar{T}$ perturbation and the string tension σ. This will be discussed in Sec. III.

If we rescale the fields by setting $\vec{X} = \vec{\phi}/\sqrt{\sigma}$, the Nambu-Goto action takes the form,

$$S_{\rm NG} = \sigma\mathcal{A}\cdot\int_\Sigma d^2\xi\sqrt{1+\frac{1}{\sigma\mathcal{A}}\left(\frac{\partial\vec{\phi}}{\partial\xi^\alpha}\cdot\frac{\partial\vec{\phi}}{\partial\xi^\alpha}\right)}. \qquad (2.12)$$

This expression makes it explicit that the above expansion is actually an expansion in powers of $1/(\sigma\mathcal{A}) = 1/(\sigma L_0 R)$. In particular, a part from the classical area term $S_{\rm cl}$, the free Gaussian action is recovered in the $\sigma L_0 R \to \infty$ limit. This is exactly the long string limit in the lattice gauge theory context in which, as it is well known, the effective string contribution reduces to the so-called "Luscher term" [49,50] that corresponds to the leading term of the partition function of $(d-2)$ noninteracting bosons.

C. First-order formulation

A remarkable feature of the Nambu-Goto model is that the functional integral on the transverse degrees of freedom $\vec{X}(\xi^0,\xi^1)$ can be performed explicitly using the Polyakov trick for any choice of the boundary conditions; see [37–39] for the application to the present context (see also [41] for a derivation of the NG partition function using a different approach). One starts from a first-order reformulation of the NG model, in which the action is

$$S = \frac{\sigma}{2}\int_\Sigma d^2\xi\sqrt{\det h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu, \qquad (2.13)$$

with $\mu = 0,\ldots d-1$. If one integrates out the independent world sheet metric h, its equation of motion (e.o.m). identifies it with the induced metric g of Eq. (2.6), and one retrieves the NG model. For each fixed topology of the world sheet, however, one can use the reparametrization and Weyl invariance of Eq. (2.13) to bring $h_{\alpha\beta}$ to a reference form $e^\phi\hat{h}_{\alpha\beta}$. The scale factor e^ϕ decouples at the classical level; at the quantum level, due to an anomaly, this is true only in the critical dimension $d = 26$. However, the effect of the anomaly is known to become irrelevant at large physical distances, for instance, for R large in the cylinder case [51]. In principle its effect should be captured by adding to the NG action suitable higher order terms in a large distance expansion. Such kind of corrections of the same type of those that, as we remarked in Sec. II B, are expected to differentiate the effective string models for specific gauge theories from the NG model. For this reason our results, based on the first-order Polyakov reformulation of the NG model should be only considered as a large distance effective description of the actual confining string.

If Σ is a cylinder or a torus, we can choose the so-called conformal gauge, fixing $\hat{h}_{\alpha\beta} = \delta_{\alpha\beta}$ and realizing Σ as a rectangle with one or two couples of opposite sides identified. The action of the model reduces then to the free action describing the CFT of d bosons plus the action $S_{\rm gh}$ for the ghost-antighost system that arise from the Jacobian of the gauge-fixing procedure,

$$S = \frac{\sigma}{2}\int_\Sigma d^2\xi\partial^\alpha X^\mu\partial_\alpha\beta X_\mu + S_{\rm gh}. \qquad (2.14)$$

In both the cylinder case and the torus case there is a residual Teichmüller parameter which we cannot change by means of conformal rescalings and which has to be integrated over. This integration is the remnant of the path-integral over the independent world sheet metric h.

We will now consider separately in detail these two cases.

D. The interface case (torus geometry)

Let us now start from the case in which the world sheet Σ is a torus, as appropriate for the one-loop partition function of closed strings. The closed string world sheet is periodic; in our conventions, $(\xi^0,\xi^1)\sim(\xi^0,\xi^1+\pi)$. The complex structure of the world sheet, $\tau = \tau_1 + i\tau_2$ with $\tau_2 \geq 0$ defines a second identification: $(\xi^0,\xi^1)\sim(\xi^0+\pi\tau_1,\xi^1+\pi\tau_2)$. It represents a Teichmüller parameter and has to be integrated over. In fact, the string partition function has the form,

$$\begin{aligned}\mathcal{Z}_{\rm torus} &= \int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2}Z^{(d)}(\tau,\bar{\tau})Z_{\rm gh}(\tau,\bar{\tau})\\ &= \int_{\mathcal{F}}\frac{d^2\tau}{4\tau_2^2}Z^{(d)}(\tau,\bar{\tau})\tau_2 Z_{\rm gh}(\tau,\bar{\tau}),\end{aligned} \qquad (2.15)$$

where $\mathcal{F}$ is the fundamental region of the Teichmüller space with respect to the action of the modular group PSL$(2,\mathbb{Z})$ that maps τ to τ', with

$$\tau' = \frac{a\tau+b}{c\tau+d}, \quad a,b,c,d\in\mathbb{Z}, \quad ad-bc=1. \qquad (2.16)$$

The parameter τ' describes an equivalent torus. The integral in Eq. (2.15) is limited to the fundamental cell $\mathcal{F}$ to avoid redundancy, but this is consistent only if the integrand is modular invariant. In the second step in Eq. (2.15) we have singled out the measure $d^2\tau/\tau_2^2$ which, according to Eq. (2.16), is modular invariant by itself.

Moreover, $Z^{(d)}(\tau,\bar{\tau})$ is the CFT partition function of d bosonic fields X^μ, and $Z^{\rm gh}(\tau,\bar{\tau})$ that of the ghost system, both computed on a fixed world sheet of complex structure τ.

Let us describe the CFT partition functions appearing in (2.15), taking into account that in our setup three of the target space coordinates are compact,

$$x^a \sim x^a + L^a, \quad a = 0,1,2, \qquad (2.17)$$

while the remaining $d-3$ ones we take for simplicity to be noncompact (but it is straightforward to compactify some of them).

Compact boson— The partition function for a boson field X with compactification length L can be easily derived in an operatorial formalism, in which the coordinate ξ^0 on Σ plays the rôle of (Euclidean) world sheet time. Since the world sheet coordinate ξ^1 is periodic, the free field $X(\xi^0, \xi^1)$ can have non-trivial winding $w \in \mathbb{Z}$ defined by

$$X(\xi^0, \xi^1 + \pi) = X(\xi^0, \xi^1) + wL. \tag{2.18}$$

In the wth winding sector, its expansion comprises left-moving and right-moving oscillators plus zero modes,

$$X(\xi^0, \xi^1) = \hat{x} + \frac{\hat{p}}{\pi\sigma}\xi^0 + \frac{wL}{\pi}\xi^1 + \frac{\mathrm{i}}{\sqrt{4\pi\sigma}}\sum_{k\neq 0}\left(\frac{\alpha_k}{k}\mathrm{e}^{-2\mathrm{i}k\xi} + \frac{\bar{\alpha}^i_k}{k}\mathrm{e}^{-2\mathrm{i}k\bar{\xi}}\right), \tag{2.19}$$

where we introduced $\xi = \xi^0 + \mathrm{i}\xi^1$ and $\bar{\xi} = \xi^0 - \mathrm{i}\xi^1$. Since the target space is compact, the spectrum of the momentum operator $\hat{p}$ is quantized: $p = 2\pi n/L$, with $n \in \mathbb{Z}$.

The partition function is given by

$$Z(\tau, \bar{\tau}; L) = \mathrm{Tr}(q^{\mathcal{L}_0 - \frac{1}{24}}\bar{q}^{\bar{\mathcal{L}}_0 - \frac{1}{24}}), \tag{2.20}$$

where $q = \exp(2\pi\mathrm{i}\tau)$ while the *0*th Virasoro operators $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$ are the zero modes of the holomorphic (left-moving) and antiholomorphic stress-energy tensor. One has

$$\mathcal{L}_0 = \frac{1}{8\pi\sigma}\left(\frac{2\pi n}{L} + w\sigma L\right)^2 + \sum_{k=1}^{\infty} N_k,$$
$$\bar{\mathcal{L}}_0 = \frac{1}{8\pi\sigma}\left(\frac{2\pi n}{L} - w\sigma L\right)^2 + \sum_{k=1}^{\infty} \bar{N}_k, \tag{2.21}$$

where N_k and $\bar{N}_k$ are the number operators for the kth left-moving and right-moving oscillator systems. These Virasoro operators are simply related to the Hamiltonian and the angular momentum by $H = \mathcal{L}_0 + \bar{\mathcal{L}}_0 - 1/12$ and $J = \mathcal{L}_0 - \bar{\mathcal{L}}_0$. The evaluation of the trace is straight–forward, and the result is

$$Z(\tau, \bar{\tau}; L) = \frac{1}{\eta(q)}\frac{1}{\eta(\bar{q})}\Gamma(\tau, \bar{\tau}; L). \tag{2.22}$$

The factor $1/\eta(q)\eta(\bar{q})$, with the Dedekind eta-function being defined in Eq. (B1), arises from the trace over the Fock spaces of the oscillator nonzero modes. The other factor comes from the sum over the zero-mode quantum numbers and reads

$$\Gamma(\tau, \bar{\tau}; L) = \sum_{n,w\in\mathbb{Z}} q^{\frac{1}{8\pi\sigma}(\frac{2\pi n}{L} + w\sigma L)^2}\bar{q}^{\frac{1}{8\pi\sigma}(\frac{2\pi n}{L} - w\sigma L)^2}$$
$$= \sqrt{\frac{\sigma}{2\pi\tau_2}}L\sum_{m,w\in\mathbb{Z}}\mathrm{e}^{-\frac{\sigma L^2}{2\tau_2}|m-\tau w|^2}, \tag{2.23}$$

where in the second step we used the Poisson resummation formula (B8). The sum over m, w represents the sum over classical solutions of the field X which, beside Eq. (2.18), also have a nontrivial wrapping along the compact propagation direction,

$$X(\xi^0 + 2\pi\tau_2, \xi^1 + 2\pi\tau_1) = X(\xi^0, \xi^1) + mL. \tag{2.24}$$

The partition function $Z(\tau, \bar{\tau}; L)$ is invariant under the modular transformations (2.16). This is evident using the second expression of $\Gamma(\tau, \bar{\tau}; L)$ in Eq. (2.23), taking into account that $\sqrt{\tau_2}\eta(q)\eta(\bar{q})$ is modular invariant; see Eq. (B7), and so is the sum over m and w, in which the effect of the modular transformation (2.16) is to replace m, w with

$$m' = dm + bw, \qquad w' = aw + cm. \tag{2.25}$$

$Z(\tau, \bar{\tau}; L)$ also displays the so-called T-duality, namely it is invariant under

$$L \to \frac{2\pi}{\sigma L}, \tag{2.26}$$

as it is clear from the structure of the sum in the first expression of Γ in Eq. (2.23).

Noncompact boson— In this case the field X has no winding, so we have to set $w = 0$ in the expansion (2.19) and in the expression of the Virasoro operators, Eq. (2.21). Moreover the momentum eigenvalue p is not quantized and, in computing the partition function trace, the sum over n is replaced by a Gaussian integration over p. The final result, proportional to the regularized volume V of the target space for X, is

$$Z_{\mathrm{n.c.}}(\tau, \bar{\tau}) = \frac{1}{\eta(q)}\frac{1}{\eta(\bar{q})}\sqrt{\frac{\sigma}{2\pi\tau_2}}V \tag{2.27}$$

and is modular invariant.

The ghost partition function— As is well known, the partition function for the ghost system exactly cancels the nonzero mode contributions of two bosonic fields,

$$Z_{\mathrm{gh}}(\tau, \bar{\tau}) = \eta^2(q)\eta^2(\bar{q}). \tag{2.28}$$

The combination $\tau_2 Z_{\mathrm{gh}}(\tau, \bar{\tau})$ is therefore modular invariant.

The string partition function— Let us go back to Eq. (2.15). Taking into account Eqs. (2.17) and (2.22), we have

$$Z^{(d)}(\tau,\bar{\tau}) = Z(\tau,\bar{\tau};L_0)Z(\tau,\bar{\tau};L_1)Z(\tau,\bar{\tau};L)[Z_{\text{n.c.}}(\tau,\bar{\tau})]^{d-3}, \tag{2.29}$$

where we denoted the compactification radius in the direction x^2 by L. Using this expression and Eq. (2.28) we have the explicit integral expression of the full string partition function $\mathcal{Z}_{\text{torus}}$ of Eq. (2.15), and we see that the integrand is indeed modular invariant. However, to recover in our first-order formulation the quantity $\hat{\mathcal{Z}}_{\text{torus}}$ that gives the effective string description of the interface free energy, see Eq. (2.4) and Fig. 1; we have to extract the contributions of the embeddings in which the world sheet Σ covers once the torus target space of sides L_0, L_1.

To do so, one can consider the contribution of the zero-modes in the directions 0 and 1 and group the terms corresponding to the possible values of m^a and w^a, with $a = 0, 1$, into orbits of the modular group. Following [52], we restrict to specific representatives in each orbits, while correspondingly enlarging the τ integration region. At this point, we extract the contributions mentioned above by taking $m^0 = 1$, $w^0 = 0$, $w^1 = 1$ and summing over m^1 or, equivalently, undoing the Poisson resummation, over n^1. The details are given in Appendix B. With this choice, we obtain

$$\hat{\mathcal{Z}}_{\text{torus}} = \frac{V_{d-3}L_0}{4}\left(\frac{\sigma}{2\pi}\right)^{\frac{d-2}{2}}\int_0^\infty \frac{d\tau_2}{(\tau_2)^{1+\frac{d-2}{2}}}\mathrm{e}^{-\frac{\sigma L_0^2}{2\tau_2}}\int_{-1/2}^{1/2}d\tau_1\left(\frac{1}{\eta(q)\eta(\bar{q})}\right)^{d-2} \times \sum_{n_0,n_2,w_2\in\mathbb{Z}} q^{\frac{1}{8\pi\sigma}\left[\left(\frac{2\pi n_1}{L_1}+\sigma L_1\right)^2+\left(\frac{2\pi n_2}{L}+w_2\sigma L\right)^2\right]}\bar{q}^{\frac{1}{8\pi\sigma}\left[\left(\frac{2\pi n_1}{L_1}-\sigma L_1\right)^2+\left(\frac{2\pi n_2}{L}-w_2\sigma L\right)^2\right]}, \tag{2.30}$$

with V_{d-3} being the volume of the noncompact target space directions. Expanding in series the Dedekind eta functions according to Eq. (B2) we obtain

$$\hat{\mathcal{Z}}_{\text{torus}} = \frac{V_{d-3}L_0}{4}\left(\frac{\sigma}{2\pi}\right)^{\frac{d-2}{2}}\sum_{k,k'=0}^{\infty}c_k c_{k'}\sum_{n_1,n_2,w_2\in\mathbb{Z}}\int_{-1/2}^{1/2}d\tau_1 \mathrm{e}^{2\pi\mathrm{i}\tau_1(k-k'+n_1+n_2w_2)} \times \int_0^\infty \frac{d\tau_2}{\tau_2^{1+\frac{d-2}{2}}}\times \mathrm{e}^{-\tau_2\left(\frac{\sigma L_1^2}{2}+\frac{\sigma L^2 w_2^2}{2}+\frac{2\pi^2 n_1^2}{\sigma L_1^2}+\frac{2\pi^2 n_2^2}{\sigma L^2}+2\pi(k+k'-\frac{d-2}{12})\right)-\frac{1}{\tau_2}\frac{\sigma L_0^2}{2}}. \tag{2.31}$$

The integration over τ_1 produces the Kronecker delta function $\delta_{k-k'+n_1+n_2m_2}$, while the integration over τ_2 can be carried out by means of the formula in Eq. (B9). The final result is

$$\hat{\mathcal{Z}}_{\text{torus}} = \frac{V_{d-3}L_0}{2}\left(\frac{\sigma}{2\pi}\right)^{\frac{d-2}{2}}\sum_{k,k'=0}^{\infty}c_k c_{k'}\sum_{n_1,n_2,w_2\in\mathbb{Z}}\delta_{k-k'+n_1+n_2w_2}\left(\frac{\mathcal{E}}{u}\right)^{\frac{d-2}{2}}K_{\frac{d-2}{2}}(\sigma\mathcal{A}\mathcal{E}), \tag{2.32}$$

with

$$\mathcal{E} = \sqrt{1+\frac{4\pi u}{\sigma\mathcal{A}}\left(k+k'-\frac{d-2}{12}\right)+\frac{4\pi^2u^2n_1^2}{(\sigma\mathcal{A})^2}+\frac{4\pi^2un_2^2}{\sigma^2\mathcal{A}L^2}+\frac{uL^2w_2^2}{\mathcal{A}}}. \tag{2.33}$$

Considering the $L_0 \to \infty$ limit of Eq. (2.32), we extract from the asymptotic behavior of the Bessel function the energy spectrum. Writing $\exp(-\sigma\mathcal{A}\mathcal{E}) = \exp(-L_0E)$, and remembering that $u/\mathcal{A} = 1/L_1^2$ is independent of L_0, we find

$$E = \sigma L_1\mathcal{E}, \tag{2.34}$$

which represents the mass of a closed string state with left- and right-moving occupation numbers k and k' and quantum numbers n_1, n_2, w_2 subject to the level-matching condition $k - k' + n_1 + n_2w_2 = 0$.

As a consistency check, let us consider the limit $L \to \infty$ in which all transverse directions are noncompact. Following the derivation in Appendix B one sees that in this limit only the trivial winding $w_2 = 0$ contributes, so that the δ function reduces to imposing $n_0 = k' - k$. Moreover, the KK sum over the discrete momentum n_2 is replaced by a Gaussian integral, whose result modifies the final integration over the Teichmüller modulus τ. In this way, one recovers the result of [38],

$$\hat{\mathcal{Z}}_{\text{torus}} = \frac{V_{d-2}L_0}{2}\left(\frac{\sigma}{2\pi}\right)^{\frac{d-1}{2}}\sum_{k,k'=0}^{\infty}c_k c_{k'}\left(\frac{\mathcal{E}}{u}\right)^{\frac{d-1}{2}}K_{\frac{d-1}{2}}(\sigma\mathcal{A}\mathcal{E}), \tag{2.35}$$

where V_{d-2} denotes the transverse volume and

$$\mathcal{E} = \sqrt{1 + \frac{4\pi u}{\sigma\mathcal{A}}\left(k + k' - \frac{d-2}{12}\right) + \frac{4\pi^2 u^2 (k-k')^2}{(\sigma\mathcal{A})^2}}. \tag{2.36}$$

As mentioned above, the partition function (2.32) enjoys the T-duality symmetry,

$$L \to \frac{2\pi}{\sigma L}. \tag{2.37}$$

It would be interesting to explore in lattice simulations[1] the consequences of this duality. For instance it implies the existence of a "minimal" compactification radius given by the self–dual point,

$$L_c = \sqrt{\frac{2\pi}{\sigma}}. \tag{2.38}$$

E. Polyalov loop setup

Again, to compute the string partition function we use the first order setup. The world sheet Σ is a cylinder and corresponds to the one-loop partition function of open strings. In our conventions, the range of the spacelike world sheet coordinate is $\xi^1 \in [0, \pi]$. The length of the cylinder Σ is a real Teichmüller parameter, which we call t. We have to integrate over this Teichmüller parameter, and the string partition function is

$$\mathcal{Z}_{\rm cyl} = \int_0^\infty \frac{dt}{2t} Z^{(d)}(t) Z_{\rm gh}(t), \tag{2.39}$$

where $Z^{(d)}(t)$ is the CFT cylinder partition function of d bosonic fields X^μ and $Z_{\rm gh}(t)$ that of the ghost system, both computed on a fixed cylinder of parameter t.

Open string channel— As depicted in Fig. 1, the open string is attached to the two Polyakov loops, which span the direction x^0. This means that the embedding field $X^0(\xi^0, \xi^1)$ has free (Neumann-Neumann) boundary conditions at both end points, while $X^1(\xi^0, \xi^1)$ has fixed (Dirichlet-Dirichlet) boundary conditions, with one end point fixed in 0 and the other one in R. The field $X^2(\xi^0, \xi^1)$ has DD boundary conditions, but can wind w_2 times around the compact coordinate x^2. The fields $X^i(\xi^0, \xi^1)$, for $i = 3, \ldots, d$ are DD, with both end points in 0. Their mode expansion is therefore the following:

$$\begin{aligned}
X^0(\xi^0, \xi^1) &= \hat{x}^0 + \frac{\hat{p}_0}{\sigma\pi}\xi^0 + \frac{\mathrm{i}}{\sqrt{4\pi\sigma}} \sum_{k\neq 0} \frac{\alpha_k^0}{k} \mathrm{e}^{-\mathrm{i}k\xi^0} \cos k\xi^1, \\
X^1(\xi^0, \xi^1) &= \frac{R}{\pi}\xi^1 - \frac{1}{\sqrt{4\pi\sigma}} \sum_{k\neq 0} \frac{\alpha_k^1}{k} \mathrm{e}^{-\mathrm{i}k\xi^0} \cos k\xi^1, \\
X^2(\xi^0, \xi^1) &= \frac{w_2 L}{\pi}\xi^1 - \frac{1}{\sqrt{4\pi\sigma}} \sum_{k\neq 0} \frac{\alpha_k^2}{k} \mathrm{e}^{-\mathrm{i}k\xi^0} \cos k\xi^1, \\
X^i(\xi^0, \xi^1) &= -\frac{1}{\sqrt{4\pi\sigma}} \sum_{k\neq 0} \frac{\alpha_k^i}{k} \mathrm{e}^{-\mathrm{i}k\xi^0} \cos k\xi^1, \quad i = 3, \ldots, d-1.
\end{aligned} \tag{2.40}$$

Note that the fields with DD boundary conditions (b.c.'s) do not possess zero modes. Since the target space direction x^0 is compact with length L_0, the spectrum of the momentum operator $\hat{p}_0$ is discrete: $p_0 = 2\pi n_0/L_0$, with $n_0 \in \mathbb{Z}$. The cylinder CFT partition function for these fields is given by

$$Z^{(d)}(t) = \mathrm{Tr}(q^{\mathcal{L}_0 - \frac{d}{24}}), \tag{2.41}$$

where we $q = \exp(-2\pi t)$ and the Virasoro zero-mode operator reads

$$\mathcal{L}_0 = \frac{2\pi}{\sigma}\frac{n_0^2}{L_0^2} + \frac{\sigma R^2}{2\pi} + \frac{\sigma L^2 w_2^2}{2\pi} + N^{(d)}. \tag{2.42}$$

Here N^d is the total number operator for all the nonzero mode oscillators of the d bosonic fields. The trace in Eq. (2.41) receives contributions from the nonzero mode oscillators and from the zero-mode sector and is found to be given by

$$\begin{aligned}
Z^{(d)}(t) &= \left(\frac{1}{\eta(q)}\right)^d \sum_{n_0, w_2 \in \mathbb{Z}} \mathrm{e}^{-2\pi t\left(\frac{2\pi n_0^2}{\sigma L_0^2} + \frac{\sigma(R^2 + w_2^2 L^2)}{2\pi}\right)} \\
&= \left(\frac{1}{\eta(q)}\right)^d \sqrt{\frac{\sigma}{4\pi t}} L_0 \sum_{m_0, w_2 \in \mathbb{Z}} \mathrm{e}^{-\frac{\sigma L_0^2}{4t} m_0^2 - t\sigma(R^2 + w_2^2 L^2)},
\end{aligned} \tag{2.43}$$

where in the second step we performed a Poisson resummation; see Eq. (B8). As in the closed string case, the ghost partition function cancels exactly the non-zero-mode contributions of two bosonic fields: $Z_{\rm gh}(t) = \eta^2(q)$.

To recover in our first-order string partition function (2.39) the quantity $\hat{\mathcal{Z}}_{\rm cyl}$ that, according to Eq. (2.3), describes the Polyakov loop correlator, we have to select $m_0 = 1$, so that the target space cylinder $\mathcal{M}$ is covered exactly once. Altogether we get

[1]Notice that the LGT realization of the torus topology is rather nontrivial and is possible only using duality. For instance the simplest possible realization is in the 3d gauge Ising model and is given by the free energy of interfaces in the dual Ising spin model [53]. More complex realizations require for instance the study of the so called 't Hooft loops in non-Abelian LGTs [54].

$$\hat{\mathcal{Z}}_{\rm cyl} = \sqrt{\frac{\sigma}{4\pi}} L_0 \int_0^\infty \frac{dt}{2t^{\frac{3}{2}}} \left(\frac{1}{\eta(q)}\right)^{d-2} \sum_{w_2\in\mathbb{Z}} \mathrm{e}^{-\frac{\sigma L_0^2}{4t} - t\sigma(R^2+w_2^2L^2)}$$
$$= \sqrt{\frac{\sigma}{4\pi}} L_0 \sum_{k=0}^\infty c_k \sum_{w_2\in\mathbb{Z}} \int_0^\infty \frac{dt}{2t^{\frac{3}{2}}} \mathrm{e}^{-\frac{\sigma L_0^2}{4t} - t[\sigma(R^2+w_2^2L^2+2\pi(k-\frac{d-2}{24})]}. \quad (2.44)$$

where in the second line we expanded in series $\eta(q)$ according to Eq. (B2).

We can perform the integral over t using Eq. (B10), obtaining the simple expression,

$$\hat{\mathcal{Z}}_{\rm cyl} = \sum_{k=0}^\infty c_k \sum_{w_2\in\mathbb{Z}} \mathrm{e}^{-L_0\mathcal{E}_{\rm op}}. \quad (2.45)$$

where

$$\mathcal{E}_{\rm op} = \sigma\tilde{R}\sqrt{1+\frac{2\pi}{\sigma\tilde{R}^2}\left(k-\frac{d-2}{24}\right)}, \quad (2.46)$$

and

$$\tilde{R} = \sqrt{R^2+w_2^2L^2}. \quad (2.47)$$

This corresponds to the result of [37] in which the Polyakov loops distance R is replaced by $\tilde{R}$, accounting for the effects of the further compactification in the transverse direction x^2.

Closed string channel— On the other hand, the cylindrical world sheet Σ can also be viewed, interchanging the role of the world sheet (w.s). coordinates ξ^0 and ξ^1, as the tree level propagation of a closed string between two boundary states representing the Polyakov loops. This interpretation is made explicit rewriting the first line of Eq. (2.44) in terms of the variable $s = 1/t$. Taking into account that under this transformation we have

$$\eta(\mathrm{e}^{-2\pi/s}) = \sqrt{s}\eta(\mathrm{e}^{-2\pi s}); \quad (2.48)$$

see Eq. (B7), and expanding now in series $\eta(\mathrm{e}^{-2\pi s})$ we get

$$\mathcal{Z}_{\rm cyl} = \sqrt{\frac{\sigma}{4\pi}} L_0 \sum_{k=0}^\infty c_k$$
$$\times \sum_{w_2\in\mathbb{Z}} \int_0^\infty \frac{ds}{2s} s^{\frac{3-d}{2}} \mathrm{e}^{-s\left(\frac{\sigma L_0^2}{4}+2\pi(k-\frac{d-2}{24})\right)-\frac{\sigma}{s}(R^2+w_2^2L^2)}. \quad (2.49)$$

Performing the integral over s with the help of Eq. (B9) one gets

$$\hat{\mathcal{Z}}_{\rm cyl} = \sqrt{\frac{\sigma}{4\pi}} L_0 \sum_{k=0}^\infty c_k \sum_{w_2\in\mathbb{Z}} \left(\frac{\mathcal{E}_{\rm cl}}{2\tilde{v}}\right)^{\frac{d-3}{2}} K_{\frac{d-3}{2}}(\sigma\tilde{\mathcal{A}}\mathcal{E}_{\rm cl}), \quad (2.50)$$

where

$$\tilde{\mathcal{A}} = L_0\tilde{R}, \qquad \tilde{v} = \tilde{R}/L_0, \quad (2.51)$$

with $\tilde{R}$ given by Eq. (2.47), and

$$\mathcal{E}_{\rm cl} = \sqrt{1+\frac{8\pi}{\sigma L_0^2}\left(k-\frac{d-2}{24}\right)}. \quad (2.52)$$

In the $L\to\infty$ limit, due to the exponential asymptotic behavior of the modified Bessel functions, the sum over w_2 is dominated by the term $w_2 = 0$ and all the higher values of w_2 are exponentially depressed. We recover again the result of [37],

$$\hat{\mathcal{Z}}_{\rm cyl} = \sqrt{\frac{\sigma}{4\pi}} L_0 \sum_{k=0}^\infty c_k \left(\frac{\mathcal{E}_{\rm cl}}{2u}\right)^{\frac{d-3}{2}} K_{\frac{d-3}{2}}(\sigma\mathcal{A}\mathcal{E}_{\rm cl}), \quad (2.53)$$

with $\mathcal{E}$ still given by Eq. (2.52) and $u = R/L_0$.

As we mentioned, in the cylinder setup the closed string propagates between two boundary states that represent the Polyakov loops in the closed string Hilbert space. In the stringy language the Polyakov loops are D0-branes; see the remark after Eq. (2.40). Namely, they are $(0+1)$–dimensional objects, extended in the direction x^0, on which open strings can end. In presence of D0-branes, the T-duality $L\to 2\pi/(L\sigma)$ does not map the theory to itself; rather it maps it to a theory in which the D0-branes become D1-branes, i.e., two-dimensional objects extended also in the direction x^2. When we lower L below L_c, we can reexpress the cylinder partition function in terms of the theory with D1-branes and with the dual compactification length $2\pi/(L\sigma)$, or we can remain within the D0–brane description. The length scale L_c looses the meaning of a minimum compactification length. This fact has deep consequences in the LGT framework since it allows us to decrease “ad libitum” the compactification radius thus allowing to study in great detail, as we shall see in the next section, the properties of the chromoelectric flux tube.

F. Relation to the spacelike string tension in LGT

In the lattice gauge theory context, the results discussed in the previous section have a natural application in the study of the so-called “spacelike” string tension [1–16]. In this framework the compact transverse direction is identified with the inverse temperature of the theory. In this setting the Polyakov loops whose correlator we are studying lay in a spacelike plane of the lattice and do not represent the order parameter of deconfinement.

As we mentioned in the Introduction the string tension extracted from such correlators is different from the actual string tension of the model (which is instead extracted from timelike Polyakov loop correlators) and is denoted as "spacelike" string tension to avoid confusion. At low temperature, i.e., for large values of L in our setting, the two string tensions coincide but as the temperature increases they behave in a different way [1–5]. The ordinary string tension decreases as the deconfinement temperature is approached and vanishes at the deconfinement point, while the spacelike one remains constant and instead of vanishing it increases in the deconfined phase [1–3]. The physical reason for this behavior is that the correlator of two spacelike Polyakov loops describes quarks moving in a finite temperature environment, and it can be shown that what we call "spacelike string tension" is actually related to the screening masses in hot QCD [6–12], and thus it does not vanish in the deconfined phase. The precise temperature T_c at which this change of behavior of the spacelike string tension occurs is not known. It is near the deconfinement point, but there is no physical reason to expect that it should coincide with it. Moreover it is not clear from the simulations if this change of behavior indicates an actual critical point or simply a crossover.

The effective string theory description discussed above can be used to shed light on these issues. To this end however it is mandatory to take in a meaningful way the continuum limit, i.e., to relate σ and the finite temperature $1/L$ to well-defined physical observables in the continuum limit. In the present setting this is much less obvious than for the ordinary string tension. In the standard case, for timelike Polyakov loop correlators, it is easy to identify the deconfinement phase transition because the Polyakov loop is itself an order parameter. This allows us to set the scale of the model, typically by measuring T_c in units of the zero temperature string tension σ, and from this to set the temperature of the model by simply measuring it[2] in units of T_c. In our case the situation is the other way around, we have no insight on the location of the (putative) phase transition, and there is in principle no obvious way to relate the inverse temperature L with the spacelike string tension σ and thus give a physical scale in the continuum to measure and relate temperatures and masses. We shall see in the next section that describing the model in terms of a $T\bar{T}$ perturbation allows us to make relevant progress in this direction.

[2]All these steps are usually performed using numerical estimates of T_c and σ extracted from Monte Carlo simulations, but it is interesting to notice that the same analysis can be performed within the framework of the effective string description, without resorting to any Monte Carlo simulation and nevertheless with an impressive agreement with the numerical results, by using the Olesen relation [55]. This relation essentially amounts to identify the finite temperature deconfinement transition of the LGT with the Hagedorn transition of the effective string model [56], which relates T_c and σ as $T_c = \sqrt{\frac{3\sigma}{(d-2)\pi}}$.

III. $T\bar{T}$ PERTURBATION OF A COMPACTIFIED BOSON

In this section we reconsider the previous computation from a different perspective. As we mentioned in the Introduction—and as we will recall very briefly below—the NG string in a d-dimensional target space represents the $T\bar{T}$ perturbation of the theory of $d-2$ free bosons. The spectrum of a $T\bar{T}$–deformed theory can be obtained from the unperturbed spectrum through a differential equation of the Burger's type. Also the explicit expression of the partition function satisfies differential constraints and could in principle be determined, but this is in practice not so trivial. In particular, the $T\bar{T}$ deformed partition function of compactified bosons was not yet written down. But this is exactly what we obtained, using the NG formulation, in the previous section. In fact we will show that the NG theory with one compact transverse direction can be described as the $T\bar{T}$ deformation of the free bosonic theory, with one field being compact. Remarkably, we shall also show that the $T\bar{T}$ deformation induces well defined trajectories in the parameter space of the model and can be performed only imposing a constant value of a suitable dimensionless combination of the compactification radius and the string tension of the model.

A. The NG theory as a $T\bar{T}$ perturbation

We noticed in Sec. II B that the next to leading terms in the derivative expansion of the NG action in the physical gauge, Eq. (2.9), can be understood as a perturbation of the free bosonic action. It can be shown [27] that this is actually an integrable perturbation and that it coincides with the $T\bar{T}$ perturbation of the free action $S_0[X]$. In fact, the Nambu-Goto action can be rewritten as

$$S_{\mathrm{NG}} = S_{\mathrm{cl}} + S_0[X] - \frac{1}{2\sigma}\int d^2\xi T\bar{T}. \tag{3.1}$$

The perturbing operator which appears in the above equation is the energy momentum tensor of the deformed theory, and thus its explicit form must be evaluated order by order in the $1/\sigma$ expansion.

The normalization of the $T\bar{T}$ term above, by comparison with Eq. (1.1), relates the $T\bar{T}$ perturbation parameter t to the string tension σ as follows:

$$t = \frac{1}{2\sigma}. \tag{3.2}$$

This normalization can be determined [27] considering the first order term in the expansion, namely the one in Eq. (2.11). In this case the energy-momentum tensor of the free-field theory (2.10) is simply given by

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\delta_{\alpha\beta}(\partial^\gamma X \cdot \partial_\gamma X). \tag{3.3}$$

It is easy to see that, up to the next to leading term, Eq. (2.9) can be rewritten as

$$S_{\rm NG} = S_{\rm cl} + S_0[X] - \frac{\sigma}{4}\int d^2\xi T_{\alpha\beta}T^{\alpha\beta} + \dots \tag{3.4}$$

If we now rewrite the perturbing term using chiral components,[3]

$$T = -\sigma(T_{11} - iT_{12}), \qquad \bar{T} = -\sigma(T_{11} + iT_{12}), \tag{3.5}$$

we end up with the normalization appearing in Eq. (3.1).

The above analysis can be extended to all order in the perturbing parameter [18,19] and induces a set of constraints on the spectrum of the theory and in particular on the dependence of the energy levels on the perturbing parameter t. Let us assume that the theory is defined on a two-dimensional manifold with a compact spacelike direction of size L_1. The spatial momentum $P(L_1)$ of any state is quantized in unities of $2\pi/L_1$ and is preserved along the perturbation. The energy $E(L_1, t)$ of the state depends in general both on L_1 and on the perturbing parameter t. Remarkably enough the constraints alluded to above can be summarized in the requirement that the energy levels satisfy the following inhomogeneous Burgers equation [18,19]:

$$\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial(E^2 - P^2)}{\partial L_1}. \tag{3.6}$$

The CFT of a free noncompact boson— It is useful for the following analysis to look at the solution of this equation in the case of a 2d free bosonic theory (see [18,19,23] for a detailed derivation). A quantum state of the unperturbed theory is characterized by the left- and right-moving oscillation numbers k and k'; its unperturbed energy and momentum are given by

$$E(0, L_1) = \frac{2\pi}{L_1}\left(k + k' - \frac{1}{12}\right) \equiv \frac{2\pi}{L_1}\varepsilon,$$
$$P(L_1) = \frac{2\pi}{L_1}(k - k') \equiv \frac{2\pi}{L_1}p. \tag{3.7}$$

The subscript (0) means "unperturbed value," and we introduced $\varepsilon = k + k' - 1/12$ and $p = k - k'$ to avoid typographical clutter in the following formulas. The general solution to (3.6) with these initial conditions is

$$E(t, L_1) = \frac{L_1}{2t}\left(-1 + \sqrt{1 + \frac{8\pi t}{L_1^2}\varepsilon + \frac{16\pi^2 t^2}{L_1^4}p^2}\right). \tag{3.8}$$

In our case however there is an additional subtlety. Since we are interested in particular in mapping the perturbed spectrum onto the Nambu Goto one we must also keep into account the possible presence of an additional "classical" energy term—the term denoted as $S_{\rm cl}$ in Eq. (3.1). One has to start in this case with the unperturbed spectrum,

$$E(0, L_1) = \frac{2\pi}{L_1}\varepsilon + F_0 L_1, \tag{3.9}$$

and solving the Burgers equation following [18,23] one finds

$$E(t, L_1) = \tilde{F}L_1 + \frac{L_1}{2\tilde{t}}\left(-1 + \sqrt{1 + \frac{8\pi\tilde{t}}{L_1^2}\varepsilon + \frac{16\pi^2\tilde{t}^2}{L_1^4}p^2}\right), \tag{3.10}$$

where

$$\tilde{t} = t(1 - tF_0), \qquad \tilde{F} = \frac{F_0}{1 - tF_0}. \tag{3.11}$$

If we assume[4] $\tilde{t} = 1/(2\sigma)$ and $\tilde{F} = \sigma$, the resulting energy spectrum the Nambu-Goto string model, namely,

$$E(t, L_1) = \sigma L_1 \mathcal{E}(t, L_1) = \sigma L_1\sqrt{1 + \frac{4\pi}{\sigma L_1}\varepsilon + \frac{4\pi^2}{(\sigma L_1)^2}p^2}. \tag{3.12}$$

coincides with the one obtained for the Nambu-Goto string model in $d = 3$ with a single transverse direction taken to be noncompact; see Eqs. (2.34) and (2.36).

These results, and the constraints imposed by the Burgers equation will play a major role in the following analysis, in which we consider the presence of a compactified transverse dimension in the model. We will first discuss the CFT of a free compactified boson which

[3]Notice that this is not the standard normalization of the energy momentum tensors which would instead require an additional factor π: $T = -\pi\sigma(T_{11} - iT_{12})$, $\bar{T} = -\pi\sigma(T_{11} + iT_{12})$ so as to obey the standard Operator Product Expansion (OPE) relation $T(z)T(w) = \frac{d-2}{2}\frac{1}{(z-w)^4} + \cdots$ with $z = \xi^0 + {\rm i}\xi^1$. We chose the normalization of Eq. (3.5) to conform with the standard notations of the $T\bar{T}$ literature.

[4]Notice that as a consequence of these assumptions the string tension of the unperturbed model does not coincide any more with the Nambu-Goto one [57]; in fact, one has $F_0 = \sigma/2$ and $t = 1/\sigma$. This fact has no direct relevance for the present analysis, but it might be important when using these $T\bar{T}$ deformed models to describe the confining regime of lattice gauge theories [57]. We plan to address this issue in a future work.

represents the unperturbed CFT in such a setting. We shall then obtain its $T\bar{T}$ deformation in Sec. III C from the NG partition function derived earlier.

B. The CFT of a free compactified boson

Let us consider the free bosonic theory with the action in Eq. (2.10), defined on the two-dimensional manifold $\mathcal{M}_0$ which represents, in the effective string perpective, the minimal surface swept out in target space by the string. Let us now assume that a bosonic field X, which in the setup of Sec. II we took to be X_2, obeys the compactification condition,

$$X \sim X + L. \tag{3.13}$$

This changes drastically the behavior of the 2d theory of the field X. In particular, a marginal operator appears in the spectrum and, as a consequence, a whole line of critical points. The action is given by Eq. (2.10), restricted to the single field X. The coupling constant[5] σ allows us to tune the model along the critical line and to introduce the dimensionless compactification radius,

$$\rho = \sqrt{\frac{\sigma}{2\pi}}L. \tag{3.14}$$

The CFT of a compactified boson is characterized by a rich spectrum of primary operators $O_{n,w}$ whose scaling dimensions are labeled by two indices n and w (which in the XY model label the "spin" and "vortex" sectors respectively) and depend on ρ as follows[6]:

$$h_{n,w}(\rho) = \frac{1}{2}\left(\frac{n^2}{\rho^2} + w^2\rho^2\right). \tag{3.15}$$

These weights exhibit a $\rho \to 1/\rho$ "duality" symmetry which exchanges spin and vortex sectors. In the string theory language, this is known as T-duality; see Eq. (2.26).

This CFT is well defined for any value of ρ but for some choices of ρ additional symmetries emerge which make the theory particularly interesting. In particular for $\rho = 1$—i.e., at the self–dual point—one has the level 1 SU(2) WZW model. For $\rho = \sqrt{2}$ one finds the CFT of a free Dirac fermion, and finally for $\rho = 2$ one has the famous Kosterliz-Thouless critical point. There are several lattice realization of this CFT. The most important ones are the XY model, which is defined for $\rho > 2$ (or equivalently $\rho < 1/2$) and the SOS model, which can be shown to be equivalent to the well-known six vertex model and which is defined for all values of ρ.

[5]Often in the literature the coupling constant is defined as $g = 2\pi\sigma$.

[6]See for instance [40] for further details, but note that the compactification radius r of [40] is equivalent to our $\rho/\sqrt{2}$.

The toroidal partition function corresponds to the euclidean path integral of this model, with the action (2.10), when the base manifold $\mathcal{M}_0$ is a torus; this is the situation considered in the effective string description of the interface free energy; see Sec. II D. Let us denote by τ_0 the complex structure modulus of $\mathcal{M}_0$; in the case considered in Sec. II A; see Eq. (2.1), we simply have

$$\tau_0 = \mathrm{i}u = \mathrm{i}L_0/L_1, \tag{3.16}$$

as we consider a straight torus, but the following formulas hold also when the modulus τ_0 has a real part as well. We reviewed the computation of the partition function for the compact boson [40] in Sec. II D, with the result given Eq. (2.22), which we rewrite here in terms of the adimensional compactification parameter ρ and of the modulus τ_0 through the quantity,

$$q_0 = \exp(2\pi\mathrm{i}\tau_0) = \exp(-2\pi u). \tag{3.17}$$

Thus we have[7]

$$\begin{aligned} Z_{(0)}(\rho) &= \frac{1}{\eta(q_0)}\frac{1}{\eta(\bar{q}_0)}\sum_{n,w\in\mathbb{Z}} q_0^{\frac{1}{4}(\frac{n}{\rho}+w\rho)^2}\bar{q}_0^{\frac{1}{4}(\frac{n}{\rho}-w\rho)^2} \\ &= \sum_{n,w\in\mathbb{Z}}\sum_{k,k'=0}^{\infty} p_k p_{k'} \mathrm{e}^{-2\pi u(\frac{1}{2}(\frac{n^2}{\rho^2}+w^2\rho^2)+k+k'-\frac{1}{12})}, \end{aligned} \tag{3.18}$$

where in the second step we have used Eq. (B2). Writing for each term in the sum the exponent as $-L_0E(0, L_1)$ this corresponds to the energy spectrum,

$$E(0, L_1) = \frac{2\pi}{L_1}\left(h_{n,w}(\rho) + k + k' - \frac{1}{12}\right) \equiv \frac{2\pi}{L_1}\tilde{\varepsilon}, \tag{3.19}$$

where, in each sector labeled by n and w we have, on top of the contribution of Eq. (3.15), that of right-moving and left-moving oscillation modes. We use the notation $E(0, L_1)$ to stress that this spectrum is at zero $T\bar{T}$ perturbation. Let us note that these states also have momentum,

$$P(L_1) = \frac{2\pi}{L_1}(k - k' + nw) \equiv \frac{2\pi}{L_1}\tilde{p}. \tag{3.20}$$

With respect to the non-compact free boson spectrum of Eq. (3.7) we simply replace ε and p with $\tilde{\varepsilon}$ and $\tilde{p}$ which depend also on the additional quantnotableum numbers n, w.

C. The partition function of the $T\bar{T}$ perturbed theory

It is natural to expect the relation between $T\bar{T}$ perturbed models and the Nambu-Goto effective action to hold also in

[7]We denote this partition function as $Z_{(0)}$ to remark that it corresponds to the free case in absence of $T\bar{T}$ deformation.

presence of a transverse compactified dimension. The Nambu-Goto model interface partition function computed in Sec. II, with $d=3$ and with the single transverse direction being compact, should therefore represent the partition function of the $T\bar{T}$ deformation of a free compact boson. Let us consider the result of Eq. (2.32) for $d=3$. Using Eq. (B11) and renaming for simplicity n and w the integers n_2 and w_2 appearing in Eq. (2.32) we get[8]

$$\hat{\mathcal{Z}}_{\rm torus} = \frac{1}{4}\sum_{n_1,n,w\in\mathbb{Z}}\sum_{k,k'=0}^{\infty} p_k p_{k'}\delta_{n_1+k-k'+nw}\mathrm{e}^{-\sigma L_0 L_1 \mathcal{E}}, \qquad (3.21)$$

where

$$\mathcal{E} = \sqrt{1+\frac{4\pi}{\sigma L_1^2}\left(k+k'-\frac{1}{12}\right)+\frac{4\pi^2 n_1^2}{\sigma^2 L_1^4}+\frac{4\pi^2 n^2}{\sigma^2 L_1^2 L^2}+\frac{w^2 L^2}{L_1^2}}. \qquad (3.22)$$

To see that this expression corresponds to energy levels $E=\sigma L_1\mathcal{E}$ which satisfy the Burgers' equation we have to rewrite it in terms of the dimensionless compactification parameter ρ introduced in Eq. (3.14), setting

$$L=\sqrt{\frac{2\pi}{\sigma}}\rho. \qquad (3.23)$$

Moreover, the Kronecker delta in Eq. (3.21) identifies (minus) the integer n_1 with the momentum $\tilde{p}$ appearing in Eq. (3.20),

$$-n_1 = k-k'+nw=\tilde{p}. \qquad (3.24)$$

In this way we obtain

$$\sigma L_1\mathcal{E} = \sigma L_1\sqrt{1+\frac{4\pi}{\sigma L_1^2}\left[\frac{1}{2}\left(\frac{n^2}{\rho^2}+w^2\rho^2\right)+k+k'-\frac{1}{12}\right]+\frac{4\pi^2 n_1^2}{\sigma^2 L_1^4}} = \sigma L_1\sqrt{1+\frac{4\pi}{\sigma L_1^2}\tilde{\varepsilon}+\frac{4\pi^2}{\sigma^2 L_1^4}\tilde{p}^2}, \qquad (3.25)$$

where $\tilde{\varepsilon}$ was defined in Eq. (3.19) and $\tilde{p}$ in Eq. (3.20). We see that this expression takes the form of Eq. (3.12), namely that of the energy levels $E(t,L_1)$ of the $T\bar{T}$ perturbation of a theory that at $t=0$ has the energy levels and momenta characterized by the energy $\tilde{\varepsilon}$ and the momentum $\tilde{p}$. This unperturbed theory is exactly the free compactified boson.

In other words, the Nambu-Goto model with one compact transverse dimension can indeed be interpreted as the $T\bar{T}$ deformation of the free compactified boson, and the torus NG partition function (3.21) provides the partition function for this deformed theory. Let us remark however that it is crucial to use Eq. (3.23) before mapping the string tension σ to the perturbing parameter t. In the 2d space of couplings (σ,L) the $T\bar{T}$ perturbation defines trajectories at fixed ρ and thus sets a well-defined relation—modulated by the marginal parameter ρ—between the compactification scale L_2 and the perturbation parameter σ. This is exactly the relation that we were looking for and that will allow us in the next section to take a sensible continuum limit of our lattice observables.

IV. IMPLICATIONS FOR LGTS

Let us now come back to the result of Sec. II E on the expectation value the Polyakov loops correlator in presence of a compactified transverse dimension. Within the Nambu-Goto effective string description this expectation value is represented by the cylinder partition function,

$$\langle P^\dagger(R)P(0)\rangle \equiv \hat{\mathcal{Z}}_{\rm cyl}, \qquad (4.1)$$

and we have obtained exact expressions of $\hat{\mathcal{Z}}_{\rm cyl}$ both in the open string channel, Eq. (2.45), and in the closed string channel, Eq. (2.50). These exact results could be compared with numerical simulations of this observable in lattice gauge theories to investigate the extent of validity of the effective Nambu-Goto description. In the following, we will focus on the $d=3$ case for simplicity.

In particular, if we consider the closed string channel expression and assume a large distance R between the two Polyakov loops, the exact result simplifies and takes a form which displays clearly the effects of the compactification. For large R the amplitude is dominated by the exchange of the lowest-lying closed string state, the so-called tachyon. To be more explicit let us introduce, extending what we did for the transverse compactification scale in Eq. (3.14), the dimensionless quantities,

$$r=\sqrt{\frac{\sigma}{2\pi}}R, \qquad \rho=\sqrt{\frac{\sigma}{2\pi}}L, \qquad l_0=\sqrt{\frac{\sigma}{2\pi}}L_0. \qquad (4.2)$$

In terms of these quantities we can single out different interesting regimes. For instance, choosing

$$r\sim l_0 \gg \rho, \qquad (4.3)$$

[8]The coefficients c_k appearing in Eq. (2.32) reduce, for $d=3$, to the numbers p_k of partitions of the integer k appearing in Eq. (3.18).

the effect of the compactification radius is emphasized. Testing in this regime our predictions against the ρ dependence of lattice results one could investigate the possible presence of a critical value ρ_c (see below).

Another interesting limit is the one in which

$$r \gg \rho \gg l_0. \qquad (4.4)$$

In this limit, the smallest length scale is the length of the Polyakov loops and, as we mentioned in Sec. II B, this is the situation in which one expects to find the largest deviations of lattice data with respect to the NG predictions. Thus, starting from a precise prediction of our first-order NG model for this compactified situation, we may hope to use the comparison with the corresponding lattice results to gain further insight into the higher order corrections to the confining string action. To this end it is mandatory to isolate precisely the corrections due to the compactification radius ρ from the remaining terms. We shall show below that this is indeed possible and thus that this particular geometry is perfectly suited to identify subtle correction terms in the confining string action.

In the $r \gg \rho \gg l_0$ limit, the arguments of the Bessel functions appearing in Eq. (2.50) are large. Thus the Bessel functions themselves have a decaying exponential behavior, see Eq. (B12), and the terms with $k > 0$ are exponentially suppressed with respect to the term with $k = 0$. Note that also increasing w yields contributions which are suppressed, but less steeply than increasing k. We will thus consider the lowest values of $|w|$ to account for the effect of a large but finite compactificaton parameter ρ.

We remain with

$$\langle P^\dagger(R)P(0)\rangle = \frac{l_0}{\sqrt{2}} \sum_{w\in\mathbb{Z}} K_0(\sigma\tilde{\mathcal{A}}\mathcal{E}_0) \sim \frac{l_0}{\sqrt{2}} \sum_{w\in\mathbb{Z}} \sqrt{\frac{\pi}{2\sigma\tilde{\mathcal{A}}\mathcal{E}_0}}\, \mathrm{e}^{-\sigma\tilde{\mathcal{A}}\mathcal{E}_0}. \qquad (4.5)$$

Here $\mathcal{E}_0$ corresponds to the expression in Eq. (2.52) with $d = 3$ and $k = 0$, namely to

$$\mathcal{E}_0 = \sqrt{1 - \frac{1}{6 l_0^2}}. \qquad (4.6)$$

Note that we have to assume $l_0 > 1/\sqrt{6}$, that is $L_0 > \sqrt{\pi/3\sigma}$ to avoid the tachyon singularity in which $\mathcal{E}_0$ vanishes. Moreover from Eqs. (2.51), (2.47) and (4.2) we have

$$\sigma\tilde{\mathcal{A}} = 2\pi l_0 r \sqrt{1 + \frac{\rho^2}{r^2} w^2} \sim 2\pi l_0 r \left(1 + \frac{w^2}{2}\frac{\rho^2}{r^2} + \ldots\right), \qquad (4.7)$$

where in the second step we took into account Eq. (4.3).

Let us keep in Eq. (4.5) the contributions of the lowest winding numbers, $w = 0$ and $w = \pm 1$. With a straightforward expansion in the small ratio ρ/r we get

$$\langle P^\dagger(R)P(0)\rangle \sim \sqrt{\frac{l_0}{8r\mathcal{E}_0}}\, \mathrm{e}^{-2\pi l_0 r \mathcal{E}_0} \left(1 + 2\left(1 - \frac{1}{4}\frac{\rho^2}{r^2}\right) \mathrm{e}^{-\pi l_0 \mathcal{E}_0 \frac{\rho^2}{r}}\right). \qquad (4.8)$$

We see that the correction associated to the transverse dimension has a very peculiar $1/R$ dependence which should make it accessible to numerical simulations. It should be possible, either using the exact expressions or the approximation we just described, to compare efficiently numerical simulations of the Polyakov correlator in lattice gauge theories with a compact transverse direction with the Nambu-Goto prediction; we plan to do so in a future work.

Let us go back to the regime introduced in Eq. (4.3). One expects, decreasing the size ρ of the compact direction to reach a critical value ρ_c where the NG predictions cease to describe correctly the data. A reliable identification of ρ_c would represent an important piece of evidence with respect to some existing conjectures about the behavior of this observable in lattice gauge theories. Let us discuss a couple of specific issues.

It was proposed a few years ago by Meyer [16] that, at the critical value of the compact direction, the theory undergoes a dimensional reduction which could be described as a Kosterlitz-Thouless (KT) phase transition. The idea behind this conjecture is to consider the topologically nontrivial windings of the flux tube around the compactified transverse dimension as vortices in the world sheet of the effective string which condense in the vacuum at the point at which dimensional reduction occurs. In the normalizations that we introduced in Sec. III B for the $c = 1$ CFT of a compact boson, see Eq. (3.15) and footnote 6, the KT point corresponds to $\rho = 2$. If the ρ_c value at which our NG result for the Polyakov loop correlators ceases to agree with the lattice data turns out to be close to 2, it would support this conjecture. As a side remark let us notice that our analysis suggests that, if the conjecture holds, we should better expect the dimensional reduction point to be represented by the $T\bar{T}$ deformation of a KT point. Such a critical point would represent the first physical realization of a $T\bar{T}$ deformed Kosterlitz-Thouless transition.

Another important aspect that could be better understood following a precise determination of ρ_c is related to the effects of the intrinsic width[9] of the flux tube. One expects that the effective string picture holds only for values of the transverse dimension larger than this scale. This is indeed

[9]For an introduction to the intrinsic width of the flux tube in the confining regime of lattice gauge theories see for instance [58] and the references therein.

the main difference between the real flux tube of a confining gauge theory and the Nambu-Goto string which instead has a (classically) vanishing thickness[10]

The correlator of spacelike Polyakov loops is a perfect tool to identify and measure this intrinsic width. In fact by increasing enough the temperature, i.e., decreasing the transverse size L, at some point it will reach the intrinsic width of the flux tube. This point can be easily recognized because, once this threshold is reached, if one keep increasing the temperature, i.e., if one starts to "squeeze" the flux tube below its intrinsic width, the spacelike string tension starts to increase [1–3]. This phenomenon is understood in the effective string framework as due to the increase in the flux density within the flux tube due to its squeezing [1–5]. With our analysis we may associate a value ρ_c to this threshold and perform a well-defined continuum limit for this quantity. It would be very interesting to see if this value is universal or if it depends on the particular gauge theory under study and if it is related to the KT transition mentioned above or to some other special point along the ρ line.

ACKNOWLEDGMENTS

We thank R. Conti, F. Gliozzi, and R. Tateo for many useful discussions and suggestions. The work of M. B. is partially supported by the MIUR PRIN Contract No. 2015 MP2CX4 "Non-perturbative Aspects Of Gauge Theories And Strings."

APPENDIX A: DETAILS ON THE DERIVATION OF THE PARTITION FUNCTIONS

Let us consider the zero-mode factor in the directions 0 and 1, and let us write it as

$$\Gamma(\tau,\bar{\tau};L_0)\Gamma(\tau,\bar{\tau};L_1)=\frac{\sigma L_0 L_1}{2\pi\tau_2}\Gamma^{(2)}(\tau,\bar{\tau};L_0,L_1). \quad \text{(A1)}$$

Starting from the last expression in Eq. (2.23) and using the manipulations introduced in [52] we can write

$$\Gamma^{(2)}(\tau,\bar{\tau};L_0,L_1) = \sum_A \mathrm{e}^{\sigma L_0 L_1 \det A}\exp\left(-\frac{\sigma L_0^2}{2\tau_2}\left|(1,\mathrm{i}L_1/L_0)A\begin{pmatrix}\tau\\1\end{pmatrix}\right|^2\right), \quad \text{(A2)}$$

where A is the two-by-two integer matrix,

$$A=\begin{pmatrix} w^0 & m^0\\ w^1 & m^1\end{pmatrix}. \quad \text{(A3)}$$

A modular transformation $\tau\to\tau'$ as in Eq. (2.16) is easily seen to be equivalent to

$$A\to A'=A\begin{pmatrix} a & b\\ c & d\end{pmatrix}; \quad \text{(A4)}$$

this amounts to the transformation (2.25) on both m^0, w^0 and m^1, w^1. The space of the matrices A is partitioned into orbits of this action, under which

$$\det A = w^0 m^1 - w^1 m^0 \quad \text{(A5)}$$

is invariant. Given the meaning of the m's and w's as wrapping numbers, $\det A$ is the number of times the string wraps the target space torus in the direction 0 and 1. In Eq. (2.15) $\Gamma^{(2)}$, multiplied by other modular invariant factors, is integrated over the fundamental cell $\mathcal{F}$. Equivalently, as argued in [52], we can integrate τ over the entire upper half plane but limit the sum over the matrices A to one representative for each orbit. To reproduce the interface free energy we have to pick up the orbit with $\det A=1$, a representative of which is simply $A=\mathbf{1}$. We can also restrict the integration over τ to the fundamental cell with respect to the subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ given by the translations $\tau\to\tau+b$ only, namely the periodic strip $-1/2\geq\tau_1\geq 1/2$, and sum in $\Gamma^{(2)}$ over all matrices of the form,

$$A=\begin{pmatrix} 1 & m^0\\ 0 & 1\end{pmatrix}. \quad \text{(A6)}$$

APPENDIX B: USEFUL FORMULAS

We list here miscellaneous definitions, properties and formulas that we use in the main text.

Dedekind's η-function is defined as

$$\eta(q)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \quad \text{(B1)}$$

It can be expanded in powers of q as follows:

$$[\eta(q)]^{-1}=\sum_{k=0}^{\infty}p_k q^{k-\frac{1}{24}}, \quad \text{(B2)}$$

where p_k is the number of partitions of the integer k. We will use the following generalization of this expansion:

$$[\eta(q)]^{-(d-2)}=\sum_{k=0}^{\infty}c_k q^{k-\frac{d-2}{24}}. \quad \text{(B3)}$$

[10]The quantity which is usually called in the LGT context "width of the Nambu-Goto string", which diverges logarithmically with the interquark distance, is only due to quantum fluctuation [59] and should not be confused with the *intrinsic* width that we discuss here. The actual width of the flux tube is the sum of the two.

Under a modular transformation $\tau \to \tau'$ with

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \tag{B4}$$

one has

$$\tau_2 \to \frac{\tau_2}{|c\tau + b|^2}, \tag{B5}$$

while the following quantities are invariant:

$$\frac{d^2\tau}{\tau_2^2}, \qquad \sqrt{\tau_2}\eta(q)\eta(\bar{q}), \tag{B6}$$

where $q = \exp(2\pi \mathrm{i}\tau)$. Under the S modular transformation $\tau \to -1/\tau$, in particular, we have

$$\eta(\mathrm{e}^{-2\pi\mathrm{i}/\tau}) = \sqrt{-\mathrm{i}\tau}\eta(\mathrm{e}^{2\pi\mathrm{i}\tau}). \tag{B7}$$

The Poisson resummation formula states that

$$\sum_{n,w\in\mathbb{Z}} \exp\left(-\pi a n^2 + 2\pi \mathrm{i} b n\right) = a^{-\frac{1}{2}} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{\pi(m-b)^2}{a}\right). \tag{B8}$$

In the main text we make use of the following integral:

$$\int_0^\infty \frac{dt}{t^{1+\gamma}} \mathrm{e}^{-\alpha^2 t - \frac{\beta^2}{t}} = 2\left(\frac{|\alpha|}{|\beta|}\right)^\gamma K_\gamma(2|\alpha||\beta|). \tag{B9}$$

In the case $\gamma = 1/2$ this reduces simply to

$$\int_0^\infty \frac{dt}{t^{3/2}} \mathrm{e}^{-\alpha^2 t - \frac{\beta^2}{t}} = \frac{\sqrt{\pi}}{|\beta|} \mathrm{e}^{-2|\alpha||\beta|}, \tag{B10}$$

in accord with the relation,

$$K_{1/2}(x) = \frac{1}{2}\sqrt{\frac{2\pi}{x}} \mathrm{e}^{-x}. \tag{B11}$$

The asymptotic expansion of the functions $K_\alpha(z)$ is of the form,

$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z}\left(1 + \frac{4\alpha^2 - 1}{8z} + \ldots\right). \tag{B12}$$

[1] L. Karkkainen, P. Lacock, D. E. Miller, B. Petersson, and T. Reisz, Space-like Wilson loops at finite temperature, Phys. Lett. B **312**, 173 (1993).
[2] G. S. Bali, J. Fingberg, U. M. Heller, F. Karsch, and K. Schilling, The Spatial String Tension in the Deconfined Phase of the (3 + 1)-Dimensional SU(2) Gauge Theory, Phys. Rev. Lett. **71**, 3059 (1993).
[3] F. Karsch, E. Laermann, and M. Lutgemeier, Three-dimensional SU(3) gauge theory and the spatial string tension of the (3 + 1)-dimensional finite temperature SU(3) gauge theory, Phys. Lett. B **346**, 94 (1995).
[4] M. Caselle, R. Fiore, F. Gliozzi, P. Guaita, and S. Vinti, On the behavior of spatial Wilson loops in the high temperature phase of LGT, Nucl. Phys. **B422**, 397 (1994).
[5] M. Caselle, F. Gliozzi, and S. Vinti, On the relation between the width of the flux tube and T_c^{-1} in lattice gauge theories, Nucl. Phys. B, Proc. Suppl. **34**, 263 (1994).
[6] V. Koch, On the temperature dependence of correlation functions in the space-like direction in hot QCD, Phys. Rev. D **49**, 6063 (1994).
[7] S. Ejiri, Monopoles and spatial string tension in the high temperature phase of SU(2) QCD, Phys. Lett. B **376**, 163 (1996).
[8] T. Sekiguchi and K. Ishiguro, Abelian spatial string tension in finite temperature SU(2) gauge theory, Int. J. Mod. Phys. A **31**, 1650149 (2016).
[9] Y. Schroder and M. Laine, Spatial string tension revisited, Proc. Sci., LAT2005 (**2006**) 180.
[10] J. Liddle (RBC-Bielefeld Collaboration), The spatial string tension and dimensional reduction in QCD, PoS LATTICE2007 (**2007**) 204.
[11] M. Cheng *et al.*, The spatial string tension and dimensional reduction in QCD, Phys. Rev. D **78**, 034506 (2008).
[12] Y. Maezawa, N. Ukita, S. Aoki, S. Ejiri, T. Hatsuda, N. Ishii, and K. Kanaya (WHOT-QCD Collaboration), Heavy-quark free energy, debye mass, and spatial string tension at finite temperature in two flavor lattice QCD with Wilson quark action, Phys. Rev. D **75**, 074501 (2007).
[13] J. Alanen, K. Kajantie, and V. Suur-Uski, Spatial string tension of finite temperature QCD matter in gauge/gravity duality, Phys. Rev. D **80**, 075017 (2009).
[14] O. Andreev and V. I. Zakharov, The spatial string tension, thermal phase transition, and AdS/QCD, Phys. Lett. B **645**, 437 (2007).
[15] O. Andreev, The spatial string tension in the deconfined phase of SU(N) gauge theory and gauge/string duality, Phys. Lett. B **659**, 416 (2008).

[16] H. B. Meyer, Vortices on the worldsheet of the QCD string, Nucl. Phys. **B724**, 432 (2005).
[17] A. B. Zamolodchikov, Expectation value of composite field T anti-T in two-dimensional quantum field theory, arXiv:hep-th/0401146.
[18] F. A. Smirnov and A. B. Zamolodchikov, On space of integrable quantum field theories, Nucl. Phys. **B915**, 363 (2017).
[19] A. Cavaglià, S. Negro, I. M. Szécsényi, and R. Tateo, $T\bar{T}$-deformed 2D quantum field theories, J. High Energy Phys. 10 (2016) 112.
[20] O. Aharony, S. Datta, A. Giveon, Y. Jiang, and D. Kutasov, Modular invariance and uniqueness of $T\bar{T}$ deformed CFT, J. High Energy Phys. 01 (2019) 086.
[21] O. Aharony and T. Vaknin, The $T\bar{T}$ deformation at large central charge, J. High Energy Phys. 05 (2018) 166.
[22] A. Hashimoto and D. Kutasov, $T\bar{T}, J\bar{T}, T\bar{J}$ partition sums from string theory, J. High Energy Phys. 02 (2020) 080.
[23] R. Conti, L. Iannella, S. Negro, and R. Tateo, Generalised Born-Infeld models, Lax operators and the $\mathrm{T\bar{T}}$ perturbation, J. High Energy Phys. 11 (2018) 007.
[24] R. Conti, S. Negro, and R. Tateo, The $\mathrm{T\bar{T}}$ perturbation and its geometric interpretation, J. High Energy Phys. 02 (2019) 085.
[25] R. Conti, S. Negro, and R. Tateo, Conserved currents and $\mathrm{T\bar{T}}_s$ irrelevant deformations of 2D integrable field theories, arXiv:1904.09141.
[26] J. Cardy, The $T\bar{T}$ deformation of quantum field theory as random geometry, J. High Energy Phys. 10 (2018) 186.
[27] M. Caselle, D. Fioravanti, F. Gliozzi, and R. Tateo, Quantisation of the effective string with TBA, J. High Energy Phys. 07 (2013) 071.
[28] S. Dubovsky, V. Gorbenko, and M. Mirbabayi, Asymptotic fragility, near AdS_2 holography and $T\bar{T}$, J. High Energy Phys. 09 (2017) 136.
[29] S. Dubovsky, V. Gorbenko, and G. Hernández-Chifflet, $T\bar{T}$ partition function from topological gravity, J. High Energy Phys. 09 (2018) 158.
[30] C. Chen, P. Conkey, S. Dubovsky, and G. Hernández-Chifflet, Undressing confining flux tubes with $T\bar{T}$, Phys. Rev. D **98**, 114024 (2018).
[31] M. Taylor, TT deformations in general dimensions, arXiv:1805.10287.
[32] S. Datta and Y. Jiang, $T\bar{T}$ deformed partition functions, J. High Energy Phys. 08 (2018) 106.
[33] L. Santilli and M. Tierz, Large N phase transition in $T\bar{T}$ -deformed 2d Yang-Mills theory on the sphere, J. High Energy Phys. 01 (2019) 054.
[34] S. Frolov, TTbar deformation and the light-cone gauge, arXiv:1905.07946.
[35] J. Cardy, $T\bar{T}$ deformation of correlation functions, J. High Energy Phys. 12 (2019) 160.
[36] Y. Jiang, Lectures on solvable irrelevant deformations of 2d quantum field theory, arXiv:1904.13376.
[37] M. Billo and M. Caselle, Polyakov loop correlators from D0-brane interactions in bosonic string theory, J. High Energy Phys. 07 (2005) 038.
[38] M. Billo, M. Caselle, and L. Ferro, The partition function of interfaces from the Nambu-Goto effective string theory, J. High Energy Phys. 02 (2006) 070.
[39] M. Billo, M. Caselle, and R. Pellegrini, New numerical results and novel effective string predictions for Wilson loops, J. High Energy Phys. 01 (2012) 104.
[40] P. H. Ginsparg, Applied conformal field theory, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France* (North Holland, 1988), pp. 1–168.
[41] M. Luscher and P. Weisz, String excitation energies in SU(N) gauge theories beyond the free-string approximation, J. High Energy Phys. 07 (2004) 014.
[42] O. Aharony and Z. Komargodski, The effective theory of long strings, J. High Energy Phys. 05 (2013) 118.
[43] M. Billo, M. Caselle, F. Gliozzi, M. Meineri, and R. Pellegrini, The Lorentz-invariant boundary action of the confining string and its universal contribution to the inter-quark potential, J. High Energy Phys. 05 (2012) 130.
[44] B. B. Brandt and M. Meineri, Effective string description of confining flux tubes, Int. J. Mod. Phys. A **31**, 1643001 (2016).
[45] M. Caselle, G. Costagliola, A. Nada, M. Panero, and A. Toniato, Jarzynski's theorem for lattice gauge theory, Phys. Rev. D **94**, 034503 (2016).
[46] A. Athenodorou and M. Teper, Closed flux tubes in higher representations and their string description in $\mathrm{D}=2+1$ SU(N) gauge theories, J. High Energy Phys. 06 (2013) 053.
[47] A. Athenodorou and M. Teper, Closed flux tubes in $\mathrm{D}=2+1$ SU(N) gauge theories: Dynamics and effective string description, J. High Energy Phys. 10 (2016) 093.
[48] M. Caselle, D. Fioravanti, F. Gliozzi, and R. Tateo, Quantisation of the effective string with TBA, J. High Energy Phys. 07 (2013) 071.
[49] M. Luscher, K. Symanzik, and P. Weisz, Anomalies of the free loop wave equation in the WKB approximation, Nucl. Phys. **B173**, 365 (1980).
[50] M. Luscher, Symmetry breaking aspects of the roughening transition in gauge theories, Nucl. Phys. **B180**, 317 (1981).
[51] P. Olesen, Strings and QCD, Phys. Lett. **160B**, 144 (1985).
[52] L. J. Dixon, V. Kaplunovsky, and J. Louis, Moduli dependence of string loop corrections to gauge coupling constants, Nucl. Phys. **B355**, 649 (1991).
[53] M. Caselle, R. Fiore, F. Gliozzi, M. Hasenbusch, K. Pinn, and S. Vinti, Rough interfaces beyond the Gaussian approximation, Nucl. Phys. **B432**, 590 (1994).
[54] P. de Forcrand, M. D'Elia, and M. Pepe, A Study of the 't Hooft Loop in SU(2) Yang-Mills Theory, Phys. Rev. Lett. **86**, 1438 (2001).
[55] P. Olesen, Strings, Tachyons and deconfinement, Phys. Lett. **160B**, 408 (1985).
[56] M. Caselle, A. Nada, and M. Panero, Hagedorn spectrum and thermodynamics of SU(2) and SU(3) Yang-Mills theories, J. High Energy Phys. 07 (2015) 143; Erratum, 11 (2017) 16.
[57] F. Gliozzi, *Poincaré Invariance versus T-Tbar Deformation in Effective String Theory, Seminar at the Simons Program: Hadronic Strings and Large-N Field Theory, Copenhagen* (2019).
[58] M. Caselle and P. Grinza, On the intrinsic width of the chromoelectric flux tube in finite temperature LGTs, J. High Energy Phys. 11 (2012) 174.
[59] M. Luscher, G. Munster, and P. Weisz, How thick are chromoelectric flux tubes?, Nucl. Phys. **B180**, 1 (1981).