$\Lambda(1405)$ as a $\bar{K}N$ Feshbach resonance in the Skyrme model

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We describe the $\Lambda(1405)$ hyperon as a Feshbach resonance of a $\bar{K}N$ quasibound state coupled by a decaying channel of $\pi\Sigma$ in the Skyrme model. A weakly bound $\bar{K}N$ state is generated in the laboratory frame, while the Σ hyperon as a strongly bound state of $\bar{K}N$ in the intrinsic frame. We obtain a coupling of $\bar{K}N$ and $\pi\Sigma$ channels by computing a baryon matrix element of the axial current. This coupling enables the decay of the $\bar{K}N$ bound state to $\pi\Sigma$. It is shown that the Skyrme model supports the $\Lambda(1405)$ as a narrow Feshbach resonance.

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I. INTRODUCTION

The negative parity state of the hyperon of the lowest mass, $\Lambda(1405)$, has brought many discussions over half a century, because its properties are not easily explained by the standard quark model [1]. For example, its excitation energy of about 300 MeV above the ground state Λ hyperon with mass 1116 MeV is considerably smaller than the other light flavored baryons of typical excitation energy about 600 MeV, i.e., $N(1535) - N(940) \sim 600$ MeV. In fact, before the quark model became popular, Dalitz and Tuan analyzed the antikaon and nucleon ($\bar{K}N$) scattering data and suggested the existence of a quasibound state of $\bar{K}N$ corresponding to $\Lambda(1405)$ [2,3]. To support such a bound state, the interaction between $\bar{K}$ and N must be sufficiently attractive. The $\bar{K}N$ bound state turns into a resonance state by coupling with the open $\pi\Sigma$ channel of lower mass. The resonances formed in this way are called Feshbach resonances, the mechanism of which is a realization of a general many-body quantum systems [4,5].

Employing the mass of $\Lambda(1405)$ at the nominal value of 1405 MeV, a $\bar{K}N$ potential was proposed to reproduce the mass in Refs. [6,7] and applied to few-body systems of $\bar{K}$ and a few nucleons, resulting in unexpectedly deeply bound states. On the other hand, a chiral model for the $\bar{K}N$ was developed, which predicted a less attractive interaction that is still sufficient to generate a loosely bound $\bar{K}N$ state with a mass spectrum of $\Lambda(1405)$ being consistent with experimental data [8]. In contrast with the former approach, the chiral model does not generate deeply bound strange nuclei. Moreover, a unique feature of the chiral models is that it generates two-pole structure for $\Lambda(1405)$; one is of $\bar{K}N$ origin while the other $\pi\Sigma$ origin [9–11]. The one of $\pi\Sigma$ origin locates at a deep imaginary region on the complex energy plane, resulting in a broad background structure in the spectrum.

These different natures originate from the uncertainties in the basic interaction. The phenomenological interaction is determined by the nominal mass of the $\Lambda(1405)$. The structure of the interaction such as the ranges and strengths depends much on the data employed. The chiral model that is based on spontaneous breaking of chiral symmetry of QCD still contains parameters for renormalization or subtraction. In both methods, parameters are adjusted to reproduce the existing data for the $\Lambda(1405)$.

Observing this situation, we have developed an alternative approach in the Skyrme model [12,13]. It is a nonlinear field theory with chiral symmetry for mesons, where baryons emerge as solitons [14–21]. The model has been shown to be successful, at least qualitatively, for meson and baryon spectroscopy and their interactions. The advantage of this model is that once the two parameters are fixed from meson properties, the dynamics of baryons are determined without additional parameters. In this manner we expect that we better discuss exotic phenomena such as high-density matter with knowing the origin of the dynamics. This is the reason that we employ the Skyrme model in the present study.

In our previous publications [12,13], we have investigated the $\bar{K}N$ interaction in the Skyrme model using an analogous method to the bound state approach by Callan and Klebanov [22,23]. Their method is formulated following the $1/N_c$ expansion with the collective quantization of solitons and was shown to be successful for the descriptions of the ground state Λ and Σ hyperons. An interesting observation is that the $\bar{K}$ is strongly bound to the hedgehog soliton in its rest frame (intrinsic frame), and consequently $\bar{K}$ is interpreted as a strange quark with spin 1/2 when quantized. Their method corresponds in many-body physics to the projection after variation, or the strong coupling scheme [24]. In our approach, observing that the $\bar{K}$ in $\Lambda(1405)$ is weakly bound to the nucleon, we have proposed an alternative method, that is the method of projection before variation, or the weak coupling scheme. Setting the two parameters at suitable values, the pion decay constant and the Skyrme parameter, it has been shown that the $\bar{K}$ feels an attractive interaction from the nucleon and is bound with a binding energy of order 10 MeV, which is identified with $\Lambda(1405)$. Another interesting feature is that when the $\bar{K}N$ interaction is expressed in the form of a local potential, it exhibits an attractive pocket at medium distances supplemented by a repulsion at short distances. These features would influence on the properties of high-density matter with kaons.

In this paper, we introduce a coupling of $\bar{K}N$ to $\pi\Sigma$ to enable the $\bar{K}N$ bound state to decay and investigate whether the bound $\bar{K}N$ state survives as a Feshbach resonance. In terms of the low-energy method of chiral symmetry, the one-pion emission decay is computed by the baryon matrix element of the axial current. Details of technical issues in performing such a computation are developed.

II. ACTIONS AND ANSATZ

Let us start with the SU(3) Skyrme model action given by [21]

$$\Gamma = \int d^4x \left\{ \frac{1}{16} F_\pi^2 \mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{1}{32e^2} \mathrm{tr}[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2 + L_{SB} \right\} + \Gamma_{WZ}. \tag{1}$$

The first and second terms are the original Skyrme model actions and the third term is the symmetry-breaking term due to finite masses of the pseudoscalar mesons,

$$L_{SB} = \frac{1}{48} F_\pi^2 (m_\pi^2 + 2m_K^2) \mathrm{tr}(U + U^\dagger - 2) + \frac{\sqrt{3}}{24} F_\pi^2 (m_\pi^2 - m_K^2) \mathrm{tr}[\lambda_8 (U + U^\dagger)]. \tag{2}$$

In this paper, we treat the pion as a massless particle while the kaon as a massive one. The last term in Eq. (1) is the contribution of the chiral anomaly called the Wess-Zumino-Witten (WZW) action given by [18,19]

$$\Gamma_{WZ} = \frac{iN_c}{240\pi^2} \int d^5x \epsilon^{\mu\nu\alpha\beta\gamma} \mathrm{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\alpha U)(U^\dagger \partial_\beta U)(U^\dagger \partial_\gamma U)], \tag{3}$$

with N_c the number of colors, $N_c = 3$. In (1), the only apparent SU(3)-breaking term is the kaon mass term of finite $m_K \neq 0$. In general, different meson decay constants are also the source of the breaking, $F_\pi \neq F_K$. One way to take it into account is discussed in, for instance, [25]. We will come back to this point when we discuss numerical results later.

The $\bar{K}N$ system is described by employing an ansatz [22]

$$U(x) = \xi(x) U_K(x) \xi(x), \tag{4}$$

where $\xi(x)$ is for the pion field embedded in the upper 2×2 components,

$$\xi(x) = \begin{pmatrix} \sqrt{U(x)} & 0 \\ 0 & 1 \end{pmatrix}, \qquad U(x) = \exp[2i\boldsymbol{\tau} \cdot \boldsymbol{\pi}(x)/F_\pi], \tag{5}$$

with $F_\pi \sim 186$ MeV the pion decay constant, and U_K for the kaon field defined by

$$U_K(x) = \exp\left[\frac{2\sqrt{2}i}{F_\pi} \begin{pmatrix} 0 & K(x) \\ K^\dagger(x) & 0 \end{pmatrix}\right],$$
$$K(x) = \begin{pmatrix} K^+(x) \\ K^0(x) \end{pmatrix}. \tag{6}$$

The Skyrme model describes the nucleon as solitons of the pion field. The model accommodates a static classical solution with a specific symmetry, that is called the hedgehog solution:

$$U_H(\boldsymbol{x}) = \exp[i\boldsymbol{\tau} \cdot \hat{\boldsymbol{x}} F(r)], \tag{7}$$

where $F(r)$ is a soliton profile function of radius $r \equiv |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$. Such a classical solution does not correspond to the physical nucleons with spin and isospin quantum numbers. They are generated in the collective coordinate method, where the variables for spin and isospin rotations of the hedgehog solution are quantized. Therefore, the nucleon is regarded as a rotating hedgehog:

$$U_H(\boldsymbol{x}) \to A(t) U_H(\boldsymbol{x}) A(t)^\dagger. \tag{8}$$

Due to the symmetry of the hedgehog solution, rotations in spin and isospin spaces are related leading to the constraint of equal spin (J) and isospin (I) values, $J = I$.

In the present study for the decay $\Lambda(1405) \to \pi\Sigma$, we need the kaon field that plays dual roles. One is for $\Lambda(1405)$ where the physical $\bar{K}$ of isospin 1/2 is bound to the nucleon, the rotating hedgehog in the laboratory frame. Hence we have proposed an ansatz [12]:

$$U_{\rm EH}(x) = A(t)\xi_H A(t)^\dagger U_K A(t)\xi_H A(t)^\dagger, \qquad (9)$$

which is used for the construction of the $\Lambda(1405)$. The other is for Σ where the $\bar{K}$ is bound to the hedgehog soliton in its intrinsic rest frame. The total configuration of the hedgehog soliton with a bound $\bar{K}$ is then rotated simultaneously:

$$U_{\rm CK}(x) = A(t)\xi_H U_K \xi_H A(t)^\dagger, \qquad (10)$$

where the subscript CK is from Callan-Klebanov [22]. This equation can be written also

$$U_{\rm CK} = (A\xi_H A^\dagger)(AU_K A^\dagger)(A\xi_H A^\dagger), \qquad (11)$$

which explicitly indicates that the hedgehog and kaon are rotating in the same way by the rotation matrix $A(t)$. In terms of the two-component isospinor the kaon field is rotated as

$$K \to A(t)K. \qquad (12)$$

In short, the difference between (9) and (10) [or (11)] is in the quantization of the kaon, while the quantization of the nucleon is the same. In (9) the kaon is physical while in (10) the kaon behaves like a strange quark after quantization [22].

At this point we would like to discuss general features of meson fields beyond static classical solutions. The kaon field introduced here is a quantum fluctuation or vibration of order N_c^0 in the $1/N_c$ expansion of QCD. Here N_c is the number of colors and the classical solutions are of order N_c^1. The collective rotation $A(t)$ is then of order $1/N_c$ as is characterized by the rotation velocity $\Omega \sim A^\dagger \dot{A}$. In this regard, the nature of the kaon field here differs from meson fields that were discussed many times in extended soliton models with vector mesons [26–28]. There, some components of the vector mesons appear at the classical level of order N_c such as the time component of the ω meson, while others induced by rotations such as spatial components of the ω meson of order $1/N_c$. These mesons are regarded as mean fields around the slowly rotating hedgehog.

Now Callan-Klebanov's method for Λ and Σ follows the scheme of $1/N_c$ expansion, first obtain the hedgehog soliton, next find a kaon bound state and then rotate the system of the soliton and bound kaon. In our ansatz (9) for $\Lambda(1405)$, the order in the last two steps is reversed based on the following physical intuition. The two different schemes are understood by comparing the time for the rotating hedgehog to turn around once, Δt_H, and the time of the bound kaon to go around the soliton (nucleon) once, Δt_K. The time Δt_H for the nucleon of spin $J = 1/2$ is estimated if we know the angular velocity Ω of the rotating hedgehog for the nucleon by $\Delta t_H \sim 2\pi/\Omega$. Using the relation $J = \mathcal{I}\Omega = 1/2$ and the moment of inertia value $\mathcal{I} \sim 1$ fm of the rotating hedgehog, we estimate $\Omega \sim 1/2\ {\rm fm}^{-1}$ and hence $\Delta t_H \sim 10$ fm. The time Δt_K for $\Lambda(1405)$ is estimated by using a typical binding energy of the $\bar{K}$ that is of order 10 MeV, while that for Σ is estimated by using a typical binding energy of order 100 MeV. We find the relation

$$\Delta t_H < \Delta t_K \sim \text{several ten fm} \qquad (13)$$

for the $\bar{K}$ of $\Lambda(1405)$ (the $\bar{K}$ goes around more slowly than the hedgehog rotates), implying that the $\bar{K}$ is treated as a particle moving around the rotating hedgehog in the laboratory frame. On the other hand, we find

$$\Delta t_H > \Delta t_K \sim \text{a few fm} \qquad (14)$$

for the $\bar{K}$ of Σ (the $\bar{K}$ goes around faster than the hedgehog rotation), implying that the $\bar{K}$ is treated as a particle moving around the static hedgehog in the intrinsic (rotating) frame.

III. COUPLING TO THE $\pi\Sigma$ CHANNEL

A. Definitions

The decay of $\Lambda(1405) \to \pi\Sigma$ is regarded as a baryon transition accompanied by one-pion emission, which is described by the amplitude

$$\langle \pi\Sigma | \mathcal{L}_{\rm int} | \Lambda(1405) \rangle. \qquad (15)$$

To the leading order of chiral expansion in powers of small momentum, the interaction Lagrangian with one pion $\mathcal{L}_{\rm int}$ is written as

$$\mathcal{L}_{\rm int} = \frac{2}{F_\pi}\partial_\mu \pi^a J_5^{\mu,a}. \qquad (16)$$

The isospin axial current $J_5^{\mu,a}$ with the isospin index a is the one with the one-pion pole term subtracted and is computed in the Skyrme model in the present study. We note that it is normalized in accordance with the isospin; for instance, for the effective interaction with the nucleon, $J_5^{\mu,a} \to \bar{\psi}_N \gamma_\mu \gamma_5 (\tau^a/2)\psi_N$. For the transition of $\Lambda(1405) \to \pi\Sigma$ we need the isovector axial current in the form

$$J_5^{\mu,a} \to \bar{\psi}_\Sigma^a \gamma_\mu \psi_{\Lambda(1405)}, \qquad (17)$$

where ψ_Σ^a and $\psi_{\Lambda(1405)}$ are the Dirac spinors for Σ and $\Lambda(1405)$, respectively, with a an isospin index for Σ. Here

γ_5 is not needed due to the negative parity of $\Lambda(1405)$. The baryon matrix element computed in the Skyrme model is then identified with the coupling constant for the effective Lagrangian

$$\mathcal{L}_{\Lambda(1405)\to\pi\Sigma} = g_{\Lambda(1405)\pi\Sigma}\frac{2}{F_\pi}\partial^\mu\pi^a\bar{\psi}^a_\Sigma\gamma_\mu\psi_{\Lambda(1405)}. \quad (18)$$

The coupling constant $g_{\Lambda(1405)\pi\Sigma}$ is then defined to be the matrix element

$$g_{\Lambda(1405)\pi\Sigma} = \langle\Sigma^0|J^{5,3}_\mu|\Lambda(1405)\rangle, \quad (19)$$

the estimation of which is the main purpose of the present paper.

B. The axial current

The axial current is derived from the action Eq. (1) as the Noether current associated with the axial transformation:

$$U \to g_A U g_A, \qquad g_A = e^{i\boldsymbol{\theta}\cdot\boldsymbol{\lambda}/2}, \quad (20)$$

where $\boldsymbol{\lambda} = \lambda^a$ $(a = 1, 2, 3, \ldots, 8)$ and $\boldsymbol{\theta}$ are the Gell-Mann matrices and SU(3) parameters, respectively. The result is $[x = (t, \boldsymbol{x})]$

$$\begin{aligned} J^{\mu,a}_5(x) &= \frac{iF_\pi^2}{16}\mathrm{tr}[\lambda^a(R^\mu - L^\mu)] \\ &\quad + \frac{i}{16e^2}\mathrm{tr}[\lambda^a\{[R^\nu,[R_\nu,R^\mu]] - [L^\nu,[L_\nu,L^\mu]]\}] \\ &\quad - \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{\lambda^a}{2}(L_\nu L_\alpha L_\beta + R_\nu R_\alpha R_\beta)\right], \end{aligned} \quad (21)$$

where $R_\mu = U\partial_\mu U^\dagger$ and $L_\mu = U^\dagger\partial_\mu U$. The current here is regarded as an operator acting on the quantized soliton states written in terms of the collective coordinates of rotations and on the second-quantized states of the kaon as we will see below.

Substituting the ansatz (4) for (21), and expanding in powers of the kaon field K up to the second order, we find

$$J^{\mu,a}_5 = J^{\mu,a,(0)}_5 + J^{\mu,a,(2)}_5 + \mathcal{O}(K^3), \quad (22)$$

where superscripts (0) and (2) stand for the order of the kaon field. For our purpose, we need the second-order term $J^{\mu,a,(2)}_5$ which contains two kaon fields, K and $K^\dagger$. Moreover, in the nonrelativistic approximation that we employ for baryons, the time component $\mu = 0$ is dominant. The explicit form of the relevant piece of the first term of (21) is

$$J^{0,a}_5(x) = \frac{i}{4}\mathrm{tr}(\xi^\dagger\tau^a\xi - \xi\tau^a\xi^\dagger)(K\dot{K}^\dagger - \dot{K}K^\dagger). \quad (23)$$

The computation of the second and third terms of (21) is tedious, but possible and is given in Appendix A. As anticipated in the previous section, the dual roles of the kaon fields in (23) are implemented by identifying one of the K's in (23) with that for $\Lambda(1405)$ and the other one for Σ, when computing the matrix element $\langle\Sigma^0|J^{5,a=3}_{\mu=0}|\Lambda(1405)\rangle$. Explicitly, we follow the relation

$$\begin{aligned} K &\to A(t)K_{\mathrm{CK}} \quad \text{for } \Sigma, \\ K^\dagger &\to K^\dagger_{\mathrm{EH}} \quad \text{for } \Lambda(1405). \end{aligned} \quad (24)$$

The presence of collective coordinate $A(t)$ in the first equation is inferred from (12) and is regarded as a coordinate operator.

Following the standard method for field quantization, the kaon fields are expanded in terms of a complete set of wave functions with the corresponding creation or annihilation operators as their coefficients. The field $K^\dagger_{\mathrm{EH}}$ is regarded as an annihilation operator for the antikaon for $\Lambda(1405)$ and is expanded by the wave functions in the laboratory frame:

$$K^\dagger_{\mathrm{EH}}(t,\boldsymbol{x}) = \phi^\dagger_{K^-}(t,\boldsymbol{x})a_{K^-} + \phi^\dagger_{\bar{K}^0}(t,\boldsymbol{x})a_{\bar{K}^0} + \cdots, \quad (25)$$

where ϕ's and a's are the wave functions and the corresponding annihilation operators, respectively. Here we have shown only the terms of the lowest s wave for the antikaon that are necessary for our purpose:

$$\begin{aligned} \phi^\dagger_{K^-}(t,\boldsymbol{x}) &= (1, \quad 0)\frac{1}{\sqrt{4\pi}}k^*(r)e^{+iE_{\mathrm{EH}}t}, \\ \phi^\dagger_{\bar{K}^0}(t,\boldsymbol{x}) &= (0, \quad -1)\frac{1}{\sqrt{4\pi}}k^*(r)e^{+iE_{\mathrm{EH}}t}, \end{aligned} \quad (26)$$

where $k(r)$ is the s-wave radial function of the antikaon bound to the nucleon with E_{EH} being the corresponding energy including its rest mass. The minus sign in the second component for $\phi^\dagger_{\bar{K}^0}$ reflects the proper isospin transformation of $\bar{K}$.

For Σ, the kaon is bound to the hedgehog with quantum numbers of the grand spin, the sum of isospin and orbital angular momentum, $T = I + L$. As discussed in Ref. [22], such a bound kaon is interpreted as a strange quark in p wave. Therefore,

$$K_{\mathrm{CK}}(t,\boldsymbol{x}) = \phi_{s\uparrow}(t,\boldsymbol{x})a^\dagger_{s\uparrow} + \phi_{s\downarrow}(t,\boldsymbol{x})a^\dagger_{s\downarrow} + \cdots, \quad (27)$$

with

$$\begin{aligned} \phi_{s\uparrow}(t,\boldsymbol{x}) &= -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\begin{pmatrix}1\\0\end{pmatrix}s(r)e^{-iE_{\mathrm{CK}}t}, \\ \phi_{s\downarrow}(t,\boldsymbol{x}) &= -\sqrt{\frac{1}{4\pi}}\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\begin{pmatrix}0\\-1\end{pmatrix}s(r)e^{-iE_{\mathrm{CK}}t}, \end{aligned} \quad (28)$$

where the p-wave nature is in the combination $\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}$, $s(r)$ the corresponding radial function and $E_{\rm CK}$ the energy. Once again the minus sign in the lower component of the second line of (28) reflects properly the spin transformation rule. The functions $k(r)$ and $s(r)$ are obtained by solving the Klein-Gordon-like eigenvalue equations [12,13,22,23]. Their normalization needs to be treated properly to reflect the structure of the Klein-Gordon-like equations, as shown in Appendix B.

C. Baryon states

The isosinglet state of Λ(1405) is formed by the two isospin 1/2 states of the nucleon and the kaon:

$$|\Lambda(1405)\rangle = \sqrt{\frac{1}{2}}|pK^-\rangle - \sqrt{\frac{1}{2}}|n\bar{K}^0\rangle = \sqrt{\frac{1}{2}}\psi^N_{p\uparrow}(A)a^\dagger_{K^-}|0\rangle - \sqrt{\frac{1}{2}}\psi^N_{n\uparrow}(A)a^\dagger_{\bar{K}^0}|0\rangle. \quad (29)$$

The proton (p) and neutron (n) wave functions with spin up and down $\psi_{pn,\uparrow\downarrow}(A)$ are given by the collective coordinate A:

$$A = a_0 + i\boldsymbol{\tau}\cdot\boldsymbol{a} = i\pi\begin{pmatrix} -\psi^N_{n\uparrow} & -\psi^N_{n\downarrow} \\ \psi^N_{p\uparrow} & \psi^N_{p\downarrow}. \end{pmatrix}. \quad (30)$$

The Σ state is given by a combination of diquarklike wave functions of spin and isospin 1 and of the strange quark. For neutral spin up Σ,

$$|\Sigma^0(J_3 = 1/2)\rangle = \sqrt{\frac{2}{3}}|d(J_3 = 1)s_\downarrow\rangle - \sqrt{\frac{1}{3}}|d(J_3 = 0)s_\uparrow\rangle = \sqrt{\frac{2}{3}}\psi^d_{10}(A)a^\dagger_{s\downarrow}|0\rangle - \sqrt{\frac{1}{3}}\psi^d_{00}(A)a^\dagger_{s\uparrow}|0\rangle, \quad (31)$$

where the vector-isovector diquark wave functions are labeled by its spin J_3 and isospin I_3, $\psi^d_{J_3I_3}$, and the relevant ones here are given as

$$\psi^d_{10}(A) = \frac{\sqrt{3}}{\pi}(a_1 + ia_2)(a_0 + ia_3), \quad (32)$$

$$\psi^d_{00}(A) = \sqrt{\frac{3}{2}}\frac{i}{\pi}({a_0}^2 - {a_1}^2 - {a_2}^2 + {a_3}^2). \quad (33)$$

IV. CALCULATION OF THE MATRIX ELEMENT

After establishing the axial current and the wave function, we demonstrate how the matrix element (18) is computed. The procedure is rather straightforward, though actual computation is quite long and tedious. Therefore, we will show the outline briefly. Let us consider the transition to the neutral Σ ($a = 3$). Replacing the kaon fields as in (24) and the time derivatives by the eigenenergies of the relevant terms of (26) and (28), we find

$$J_5^{0,3}(x,A) = -(E_{\rm EH} + E_{\rm CK})\frac{1}{4}{\rm tr}(\xi^\dagger\tau^3\xi - \xi\tau^3\xi^\dagger)AK_{\rm CK}K^\dagger_{\rm EH}, \quad (34)$$

where we have indicated that the current is a function of x and the collective coordinate A. Using the rotating hedgehog configuration $\xi = A\xi_H A^\dagger$ with

$$\xi_H = \cos\frac{F}{2} + i\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\sin\frac{F}{2}, \quad (35)$$

we obtain

$$J_5^{0,3}(x,A) = -(E_{\rm EH} + E_{\rm CK})\frac{\sin(F/2)}{4}{\rm tr}(\tau^3\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}' - \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}'\tau^3) \times AK_{\rm CK}K^\dagger_{\rm EH}, \quad (36)$$

where $\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}' = A\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}A^\dagger$.

For the transition amplitude, we need to take the matrix element of the interaction Lagrangian (16) with the initial Λ(1405) and the final Σπ, with a finite pion momentum $q^\mu = (E_\pi, \boldsymbol{q})$, $E_\pi = \sqrt{m_\pi^2 + \boldsymbol{q}^2}$. Performing necessary trace algebra for the relevant 2 × 2 matrices, we integrate over the space-time d^4x and collective coordinates $d\mu(A)$, where

$$\int d\mu(A) = \int_0^\pi d\theta_1 d\theta_2 \int_0^{2\pi} d\theta_3 \sin^2\theta_1 \sin\theta_2, \quad (37)$$

and the relation between the three angles and the SU(2) rotation matrix is given by

$$a_0 = \cos\theta_1,$$
$$a_1 = \sin\theta_1\sin\theta_2\cos\theta_3,$$
$$a_2 = \sin\theta_1\sin\theta_2\sin\theta_3,$$
$$a_3 = \sin\theta_1\cos\theta_2.$$

The time integral leads to the δ function for energy conservation. After these manipulations, we arrive at a rather compact expression:

$$\langle\pi^0(\boldsymbol{q})\Sigma^0|\mathcal{L}_{\rm int}|\Lambda(1405)\rangle = \frac{2}{F_\pi}\int d^4x d\mu(A)\langle\pi|\partial^0\pi^3(x)|0\rangle\langle\Sigma|J_5^{0,3}(x,A)|\Lambda(1405)\rangle = i\delta(E_\pi + E_{\rm CK} - E_{\rm EH})\frac{2}{F_\pi}\int_0^\infty drr^2 j_0(qr) \times \frac{E_\pi(E_{\rm EH} + E_{\rm CK})}{9}\sin F s(r)k^*(r). \quad (38)$$

The presence of the spherical Bessel function $j_0(qr) = \sin(qr)/(qr)$ indicates that the decaying pion is in the s wave as it should be.

So far we have shown the result for the second-order derivative term. The computation goes similarly for the Skyrme and WZW terms. The results are summarized as follows:

$$\begin{aligned}
&\langle \pi^0(\boldsymbol{q})\Sigma^0|\mathcal{L}_{\rm int}|\Lambda(1405)\rangle \\
&= \frac{2}{F_\pi}\int d^4x d\mu(A)\langle \pi|\partial^0\pi^3(x)|0\rangle\langle\Sigma|J_5^{0,3}(x,A)^{(2+4+{\rm WZW})}|\Lambda(1405)\rangle \\
&= i\delta(E_\pi + E_{\rm CK} - E_{\rm EH})\frac{2}{F_\pi}\int_0^\infty drr^2 j_0(qr)(\mathcal{I}_2 + \mathcal{I}_2 + \mathcal{I}_{\rm WZW}),
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
\mathcal{I}_2 &= \frac{E_\pi(E_{\rm EH}+E_{\rm CK})}{9}\sin F s(r)k^*(r), \\
\mathcal{I}_4 &= -\frac{(E_s + E_{\bar K})}{3}s(r)k^*(r)\left[\frac{2}{3}\sin F\left\{(F')^2 + \frac{\sin^2 F}{r^2}\right\}\right] \\
&\quad + \frac{E_{\bar K}}{3}s(r)k^*(r)\left[\frac{4}{3}\frac{c^2\sin F}{r^2}(5c^2 - s^2)\right] + \frac{E_s}{3}s(r)k^*(r)\left[\frac{4}{3}\frac{s^2\sin F}{r^2}(-c^2 + 5s^2)\right] \\
&\quad - \frac{E_{\bar K}}{3}s'(r)k^*(r)\left[2F'\left(-\frac{5c^2}{3} + \frac{7s^2}{3}\right)\right] + \frac{E_s}{3}s(r)k^{*\prime}(r)\left[2F'\left(\frac{7c^2}{3} - \frac{5s^2}{3}\right)\right], \\
\mathcal{I}_{\rm WZW} &= \left(s(r)k^*(r)\frac{4\sin F F'}{r^2} + s'(r)k^*(r)\frac{2\sin^2 F}{r^2} - s(r)k^{*\prime}(r)\frac{2\sin^2 F}{r^2}\right).
\end{aligned} \tag{40}$$

Here we have introduced the notation $c = \cos(F/2)$ and $s = \sin(F/2)$.

V. RESULTS AND DISCUSSIONS

In this section, we present and discuss our numerical results for the decay of $\Lambda(1405) \to \pi\Sigma$. The formulas that are derived in the previous sections determine the coupling constant $g_{\Lambda(1405)\pi\Sigma}$ as defined in the effective Lagrangian (18). The decay width is then computed by the formula

$$\Gamma_{\Lambda(1405)\to\pi\Sigma} = g^2_{\Lambda(1405)\pi\Sigma}\frac{|\boldsymbol{q}|}{\pi}\frac{E_\Sigma + m_\Sigma}{4(E_\Sigma + E_\pi)} \times 3. \tag{41}$$

The factor 3 is for isospin sum. For kinematic parameters we employ the physical values that are fixed by the experiment as summarized in Table I. Here we take the mass of $\Lambda(1405)$ slightly higher than the nominal value, that is, 1420 MeV, considering the recent discussions of the two-pole structure of $\Lambda(1405)$, and the $\bar{K}N$ quasibound state is considered to locate at around the higher-mass region [29].

TABLE I. Kinematical inputs for the decay of $\Lambda(1405)$ in units of MeV.

m_π	m_K	m_Σ	$m_{\Lambda(1405)}$	$\lvert\boldsymbol{q}\rvert$	E_π	E_Σ
138	495	1193	1420	166	216	1204

Our main results in this paper are shown in Table II, where various contributions to the coupling constants and the resulting decay widths are given for three sets of the Skyrme model parameters, A, B and C. In set A, the decay constant F_π is taken at an average of the pion and kaon decay constants, considering the difference in the two constants, while in set B it is set at the pion decay constant. Set C is from Ref. [20]. In all cases, the Skyrme parameter e is determined such that the $N\Delta$ mass splitting is reproduced.

As seen from Table II, the present model predictions of the decay width Γ are small as compared to the experimental data and scatter in a range from the minimum value to the maximum value that is about 3 times larger than the minimum value. The experimental data is taken from PDG where they quote the average number 50.5 ± 2.0 MeV [29]. There are, however, discussions about the two-pole structure of $\Lambda(1405)$ having the $\bar{K}N$ and $\pi\Sigma$ origin. The $\bar{K}N$ originated one locates relatively higher in mass at around 1420 MeV and has a narrower width, while the $\pi\Sigma$ originated one locates lower with a wider width. Our present result is to be compared with the former $\bar{K}N$ dominant one, whose width is expected to be around 20 MeV [29]. Thus the corresponding coupling constants are shown in parentheses.

TABLE II. Results for the three sets of Skyrme model parameters. Contributions of the coupling constant from the second-order, fourth-order and WZW terms are shown separately as g_2, g_4 and g_{WZW}, respectively. The experimental data for $\Lambda(1405)$ are taken from the averaged value from PDG and the corresponding coupling constant $g_{\Lambda(1405)\pi\Sigma}$ is evaluated by them. The numbers in the parentheses are those expected for the $\bar{K}N$ dominant pole. The data for e are also shown when identifying them with the coupling of $\rho \to \pi\pi$ decay [30].

	F_π (MeV)	e	B.E. (MeV)	g_2	g_4	g_{WZW}	$g_{\Lambda(1405)\pi\Sigma}$	Γ (MeV)
Set A	205	4.67	20.6	0.0545	0.0385	0.0938	0.187	2.3
Set B	186	4.82	32.2	0.0609	0.0439	0.1180	0.223	3.3
Set C	129	5.45	81.3	0.0437	0.0520	0.2371	0.333	7.4
Data	186	5.75	30				0.87(∼0.55)	50.5(∼20)

The reason that the model predictions scatter in a rather wide range is that the amplitude is proportional to $1/F_\pi$ and that the overlap integral in the matrix element is sensitive to the structure of the kaon wave functions of $\Lambda(1405)$ and of Σ. It is not difficult to see that these factors may change the coupling constant by a few times. Then a possible reason for small values may be explained by the overlap integral; in the present approach the two limits are employed for the construction of the wave functions of $\Lambda(1405)$ and Σ, the weak coupling and strong coupling limits. The matrix elements for the transition amplitudes computed by the integral of the two wave functions are therefore suppressed. In a realistic situation, both wave functions are between the two limits and therefore the overlap integral would gain some strength. We also consider that the suppression is related to the bound state approach where the kaon is regarded as a heavy meson and is, as well as hyperons, not treated as flavor SU(3) multiplets. Physically, the transition from $\bar{K}N$ to $\pi\Sigma$ requires an exchange of a (heavy) strange quark from $\bar{K}$ to Σ. It is natural to consider that such a heavy particle exchange is suppressed.

Aside from the quantitative aspect, it is worth emphasizing as the main conclusion of the present study that the resulting decay width turns out to be narrow. This enables the $\bar{K}N$ bound state to remain as a Feshbach resonance, seemingly a natural consequence that the Skyrme model supports.

The final remark is the relevance of the present analysis with the two-pole structure of $\Lambda(1405)$ in the chiral unitary approach [9–11]. The Skyrme model is a realization of chiral symmetry with baryon structure described by solitons. Therefore, it shares common features in meson-baryon dynamics with the chiral unitary approach. To show this explicitly in the present approach we need the $\pi\Sigma - \pi\Sigma$ interaction, the derivation of which requires terms of meson fluctuations of fourth order, $\pi\pi KK$ soliton; kaons here are used to generate the Σ baryon as a $\bar{K}N$ bound state. Although the inclusion of such higher-order terms seems rather formidable, we expect to have an energy-dependent $\pi\Sigma$ potential, generating a $\pi\Sigma$ resonance as obtained in the chiral unitary approach. Hence what we can conclude at this point is that the $\Lambda(1405)$ resonance is dominated by the Feshbach molecule of $\bar{K}N$ complemented by a broad structure due to the $\pi\Sigma$ dynamics, the feature of the chiral approach with the inclusion of the internal structure of the nucleon.

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APPENDIX A: EXPLICIT EXPRESSIONS OF THE AXIAL CURRENT

In this Appendix, we show the explicit form of the axial current:

$$J_5^{\mu,a} = \frac{iF_\pi^2}{16}\mathrm{tr}[\lambda^a(R^\mu - L^\mu)] + \frac{i}{16e^2}\mathrm{tr}[\lambda^a\{[R^\nu,[R_\nu,R^\mu]] - [L^\nu,[L_\nu,L^\mu]]\}] - \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{\lambda^a}{2}(L_\nu L_\alpha L_\beta + R_\nu R_\alpha R_\beta)\right]. \quad (A1)$$

The term from the second derivative term has been already given in (23) for $\mu = 0$:

$$J_5^{0,a}(\mathrm{2nd}) = \frac{i}{4}\mathrm{tr}(\xi^\dagger\tau^a\xi - \xi\tau^a\xi^\dagger)(K\dot{K}^\dagger - \dot{K}K^\dagger). \quad (A2)$$

For the term from the fourth derivative (Skyrme) term, we find

$$J_5^{0,a}(\mathrm{4th}) = -\frac{i}{4e^2F_\pi^2}\mathrm{tr}(\lambda^a[\alpha^{i,(0)},[\alpha_i^{(0)},\alpha^{0,(2)}]] + 2\lambda^a[\alpha^{i,(0)},[\alpha_i^{(1)},\alpha^{0,(1)}]] + 2\lambda^a[\alpha^{i,(1)},[\alpha_i^{(0)},\alpha^{0,(1)}]]) - (\xi \leftrightarrow \xi^\dagger), \quad (A3)$$

where

$$\alpha_i^{(0)} = \begin{pmatrix} \tilde{U}_H\partial_i\tilde{U}_H^\dagger & 0 \\ 0 & 0 \end{pmatrix}, \quad (A4)$$

$$\alpha_i^{(1)} = \begin{pmatrix} 0 & -\tilde{U}_H \partial_i (\tilde{\xi}^\dagger A K_{\rm CK}) \\ K_{\rm EH}^\dagger \tilde{\xi} \partial_i \tilde{U}_H^\dagger - \partial_i (K_{\rm EH}^\dagger \tilde{\xi}^\dagger) & 0 \end{pmatrix}, \tag{A5}$$

$$\alpha_0^{(1)} = \begin{pmatrix} 0 & -\tilde{\xi} A \partial_0 (K_{\rm CK}) \\ -\partial_0 K_{\rm EH}^\dagger \tilde{\xi}^\dagger & 0 \end{pmatrix}, \tag{A6}$$

$$\alpha_0^{(2)} = \begin{pmatrix} \tilde{\xi} A \dot{K}_{\rm CK} K_{\rm EH}^\dagger \tilde{\xi}^\dagger - \tilde{\xi} A K_{\rm CK} \dot{K}_{\rm EH}^\dagger \tilde{\xi}^\dagger & 0 \\ 0 & -K_{\rm EH}^\dagger A \dot{K}_{\rm CK} + \dot{K}_{\rm EH}^\dagger A K_{\rm CK} \end{pmatrix}. \tag{A7}$$

For the term from the WZW term, we find

$$J_5^{0,a}({\rm WZW}) = \frac{N_c \epsilon^{ijk}}{24\pi^2 F_\pi^2} [\lambda^a (\beta_i^{(0)} \beta_j^{(0)} \beta_k^{(2)} + \beta_i^{(0)} \beta_j^{(2)} \beta_k^{(0)} + \beta_i^{(2)} \beta_j^{(0)} \beta_k^{(0)}) \\ + 2\lambda^a (\beta_i^{(0)} \beta_j^{(1)} \beta_k^{(1)} + \beta_i^{(1)} \beta_j^{(0)} \beta_k^{(1)} + \beta_i^{(1)} \beta_j^{(1)} \beta_k^{(0)})] + \{\xi \leftrightarrow \xi^\dagger\}, \tag{A8}$$

where

$$\beta_\mu^{(0)} = U_\pi^\dagger \partial_\mu U_\pi = \begin{pmatrix} L_\mu & 0 \\ 0 & 0 \end{pmatrix}, \tag{A9}$$

$$\beta_\mu^{(1)} = \begin{pmatrix} 0 & -\xi^{\dagger 2} \partial_\mu (\xi K) \\ K^\dagger \xi^\dagger \partial_\mu \xi^2 - \partial_\mu (K^\dagger \xi) & 0 \end{pmatrix}, \tag{A10}$$

$$\beta_\mu^{(2)} = \begin{pmatrix} \xi^{\dagger 2} \partial_\mu (\xi K) K^\dagger \xi - \xi^\dagger K \partial_\mu (K^\dagger \xi^\dagger) \xi^2 & 0 \\ 0 & -2K^\dagger \xi^\dagger \partial_\mu (\xi K) + \partial_\mu (K^\dagger K) \end{pmatrix}. \tag{A11}$$

APPENDIX B: NORMALIZATION CONDITIONS

In this Appendix, we show the normalization conditions for the kaon and s-quark wave functions which are consistent with the solutions of the Klein-Gordon equation and with the canonical commutation relations. First, in the CK approach, the normalization is given by [22,23]

$$4\pi \int drr^2 s_n^*(\boldsymbol{r}) s_m(\boldsymbol{r}) [f(r)(\omega_n + \omega_m) + 2\lambda(r)] = \delta_{nm}, \\ 4\pi \int drr^2 \tilde{s}_n^*(\boldsymbol{r}) \tilde{s}_m(\boldsymbol{r}) [f(r)(\tilde{\omega}_n + \tilde{\omega}_m) - 2\lambda(r)] = \delta_{nm}, \\ 4\pi \int drr^2 s_n^*(\boldsymbol{r}) \tilde{s}_m(\boldsymbol{r}) [f(r)(\omega_n - \tilde{\omega}_m) + 2\lambda(r)] = 0, \tag{B1}$$

where $s_m(\boldsymbol{r})$ and $\tilde{s}_m(\boldsymbol{r})$ are the wave functions of the s and $\bar{s}$ quark in the m mode, respectively, and ω_m and $\tilde{\omega}_m$ the corresponding eigenenergies. The radial-dependent functions $f(r)$ and $\lambda(r)$ are given, respectively, by

$$f(r) = 1 + \frac{1}{(eF_\pi)^2} \left[2 \frac{\sin^2 F}{r^2} + F'^2 \right], \tag{B2}$$

$$\lambda(r) = -\frac{N_c E}{2\pi^2 F_\pi{}^2} \frac{\sin^2 F}{r^2} F'. \tag{B3}$$

In the EH approach, the normalization conditions are given by

$$4\pi \int drr^2 k_n^*(\boldsymbol{r}) k_m(\boldsymbol{r}) \left[f(\omega_n + \omega_m) + 2\{\rho_1 + \lambda_1\} - \frac{1}{r^2} \frac{d}{dr} (r^2 \rho_2) \right] = \delta_{nm}, \tag{B4}$$

$$4\pi \int drr^2 \tilde{k}_n^*(\boldsymbol{r}) \tilde{k}_m(\boldsymbol{r}) \left[f(\tilde{\omega}_n + \tilde{\omega}_m) - 2\{\rho_1 + \lambda_1\} + \frac{1}{r^2} \frac{d}{dr} (r^2 \rho_2) \right] = \delta_{nm}, \\ 4\pi \int drr^2 k_n^*(\boldsymbol{r}) \tilde{k}_m(\boldsymbol{r}) \left[f(\omega_n - \tilde{\omega}_m) + 2\{\rho_1 + \lambda_1\} - \frac{1}{r^2} \frac{d}{dr} (r^2 \rho_2) \right] = 0, \tag{B5}$$

where

$$\rho_1(r) = -\frac{4\sin^2(F/2)}{3\Lambda}\boldsymbol{I}_K \cdot \boldsymbol{I}_N\left[1+\frac{1}{(eF_\pi)^2}\left(\frac{4}{r^2}\sin^2 F + F'^2\right)\right] - \frac{\sin^2(F/2)}{\Lambda}\left[1+\frac{1}{(eF_\pi)^2}\left(\frac{5}{r^2}\sin^2 F + F'^2\right)\right], \quad \text{(B6)}$$

$$\rho_2(r) = \frac{1}{(eF_\pi)^2}\left[\frac{\sin F}{\Lambda}F'(4\boldsymbol{I}_K\cdot\boldsymbol{I}_N + 3)\right], \quad \text{(B7)}$$

$$\lambda_1(r) = \frac{N_c}{{F_\pi}^2}B^0, \qquad B^0 = -\frac{1}{2\pi^2}\frac{\sin^2 F}{r^2}F', \quad \text{(B8)}$$

where $k_m(\boldsymbol{r})$ and ω_m are the wave functions and the corresponding eigenenergies, respectively, and the tilded variables are for the kaon.

These normalization conditions Eqs. (B2) and (B6) are obtained in order to satisfy the canonical quantization condition

$$[k_n(\boldsymbol{r},t), \pi_m(\boldsymbol{r}',t)] = i\delta_{nm}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}'), \quad \text{(B9)}$$

where $\pi_m(\boldsymbol{r}',t)$ is the canonical momentum conjugate to $k_m(\boldsymbol{r},t)$.

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