Spectrum of the hidden-bottom and the hidden-charm-strange exotics in the dynamical diquark model

Jesse F. Giron [*] and Richard F. Lebed [†]

Department of Physics, Arizona State University, Tempe, Arizona 85287, USA

The lightest hidden-bottom tetraquarks in the dynamical diquark model fill an S-wave multiplet consisting of 12 isomultiplets. We predict their masses and dominant bottomonium decay channels using a simple three-parameter Hamiltonian that captures the core fine-structure features of the model, including isospin dependence. The only experimental inputs needed are the corresponding observables for $Z_b(10610)$ and $Z_b(10650)$. The mass of X_b, the bottom analogue to $X(3872)$, is highly constrained in this scheme. In addition, using lattice-calculated potentials we predict the location of the center of mass of the P-wave multiplet and find that $Y(10860)$ fits well but the newly discovered $Y(10750)$ does not, more plausibly being a D-wave bottomonium state. Using similar methods, we also examine the lowest S-wave multiplet of six $c\bar{c}s\bar{s}$ states, assuming as in earlier work that $X(3915)$ and $Y(4140)$ are members, and predict the masses and dominant charmonium decay modes of the other states. We again use lattice potentials to compute the centers of mass of higher multiplets and find them to be compatible with the masses of $Y(4626)$ $(1P)$ and $X(4700)$ $(2S)$, respectively.

I. INTRODUCTION

The modern study of hadrons that manifest exotic valence-quark content has produced numerous surprises in both experiment and theory, as reviewed in Refs. [1–11]. As of this writing, more than 40 candidates have been observed in the heavy-quark sector. However, the fundamental organizing principle underlying their spectroscopy has proved elusive, unlike the clear structure derived from quark-potential models in the conventional $c\bar{c}$ and $b\bar{b}$ sectors [12].

For instance, one may attempt to model multiquark exotics using the original molecular picture of two conventional hadrons bound via light-meson (e.g., π) exchange [13,14]. This approach can provide some guidance regarding which thresholds might be expected to support a molecule [15,16]. However, hadronic molecules lack a regularly spaced spectrum because the pattern of mass splittings among the light and heavy-light hadrons acting as their constituents is itself nontrivial, being obscured by the specifics of strong-interaction dynamics. In addition, a composite state of a given width cannot form if either constituent hadron has a larger width, and it remains unclear

whether molecular formation is limited to the case in which the constituents are in a relative S wave. Indeed, calculating the detailed properties of hadronic molecules appears to require the careful consideration of a variety of near-threshold effects such as cusps and rescattering diagrams [5].

The $J^{PC} = 1^{++}$ $X(3872)$, the first heavy-quark exotic state discovered [17], is touted as the *ne plus ultra* of hadronic molecules, and no reasonable researcher can deny that the absurdly small splitting $m_{X(3872)} - m_{D^0} - m_{D^{*0}} = +0.01 \pm 0.18$ MeV indicates the controlling influence of the state $D^0\bar{D}^{*0}$ (charge conjugates understood) over the nature of the resonance. And yet, the very proximity of $X(3872)$ to threshold indicates that it is almost certainly not a "traditional" molecule of the type described above, but rather its precise mass eigenvalue relies in an intrinsic way upon threshold effects. Several observed features of $X(3872)$ point to a complicated structure; for example, its substantial collider prompt production rate suggests that $X(3872)$ possesses a tightly bound component, but suppression of this rate with increasing charged-particle multiplicity in pp collisions as compared to that of $\psi(2S)$ [18] suggests that $X(3872)$ may more easily dissociate in a dense particle environment, as one expects for a molecule. A typical resolution of this conundrum is to suppose that the 1^{++} conventional charmonium state $\chi_{c1}(2P)$, predicted by potential models to lie around 3925 MeV [19] but conspicuously absent from the data, mixes to a significant degree with a $D^0\bar{D}^{*0}$ state to form the physical $X(3872)$.

The $\chi_{c1}(2P)$ is not the only example of a tightly bound state that can mix with $X(3872)$. Diquark models also

[*] jfgiron@asu.edu
[†] Richard.Lebed@asu.edu

produce a single isoscalar 1^{++} tetraquark state as one of their lowest hidden-charm excitations, appearing in the color-attractive arrangement $(cq)_{\bar{3}}(\bar{c}\,\bar{q})_3$ [20]. Typical diquark $\delta \equiv (cq)_{\bar{3}}$ masses of ~ 1.9 GeV naturally produce such a 1^{++} $\delta - \bar{\delta}$ state in the vicinity of 3.9 GeV [21]. $X(3872)$ might actually, in the end, prove to be a perfect storm of a $D^0 \bar{D}^{*0}$ molecular state enhanced by threshold effects, mixing with the otherwise isolated conventional charmonium $\chi_{c1}(2P)$ state and the lowest-lying isoscalar 1^{++} δ-$\bar{\delta}$ state.

The variant diquark model used in this work is the so-called "dynamical" diquark model, which was developed [22] to address the issue of how δ-$\bar{\delta}$ states persist long enough to be observed, rather than their quarks instantly recombining through the more strongly attractive $3 \otimes \bar{3} \to 1$ color coupling into meson pairs. The physical picture has two components: First, a heavy quark Q must be created in closer proximity to a quark q than to an antiquark $\bar{q}$, and form a somewhat compact diquark quasiparticle $\delta \equiv (Qq)_{\bar{3}}$, and vice versa for $\bar{\delta}$; and second, the large energy release of the production process drives apart the δ-$\bar{\delta}$ pair before recombination into a meson pair can occur, creating an observable resonance.[1] A similar mechanism using color-triplet *triquarks* extends the picture to pentaquark formation [23].

This physical picture was developed into a predictive model [24] by describing the color flux tube that connects the separating δ-$\bar{\delta}$ pair using the language of potentials in the Born-Oppenheimer (BO) approximation. These potentials are the same ones appearing in QCD lattice gauge-theory simulations of heavy-quark hybrid mesons [25–29], so they may be applied directly to obtain numerical results for the δ-$\bar{\delta}$ spectrum [21], since both systems involve glue connecting heavy color 3 and $\bar{3}$ sources. The lowest BO multiplets are all found numerically to lie in the Σ_g^+ potential,[2] and in order of increasing mass are $1S$, $1P$, $2S$, $1D$, and $2P$. The parity of all states in each Σ_g^+ multiplet is simply given by $(-1)^L$.

As first proposed in Ref. [30], the dominant spin-spin couplings in the δ-$\bar{\delta}$ states (as supported by comparison to observation) appear to be the ones *within* each of δ and $\bar{\delta}$. The strength of this coupling is denoted by κ_{qQ}, where Q refers to the heavy quark and q the light quark in δ. The

dominance of these particular spin couplings arises naturally if δ and $\bar{\delta}$ are more compact than the full exotic state in which they appear. Furthermore, the near-universal prediction that spin-singlet couplings within diquarks are more attractive than spin-triplet couplings leads to the expectation that $\kappa_{qQ} > 0$. A detailed numerical examination of the effect of including a finite diquark size is one of the primary thrusts of Ref. [31]; there it is found that the calculated state masses are remarkably stable as long as the diquark wave functions no longer significantly overlap when the distance R between their centers exceeds the critical value of 0.8 fm. In other words, the diquarks may have radii as large as $R/2 \simeq 0.4$ fm and still be considered compact for the purpose of the model. Indeed, Ref. [31] also found that observation [specifically, the experimental absence of a charged partner to $X(3872)$] does not support the dominant isospin dependence in the δ-$\bar{\delta}$ state being one that couples to diquarks as truly pointlike objects, but the model works quite well when the dominant isospin dependence is instead taken to couple only to the light quarks within δ and $\bar{\delta}$. (And, of course, isospin exchange is irrelevant for $c\bar{c}s\bar{s}$ states.)

The mass spectrum and preferred heavy-quark spin-eigenstate decay modes of the 12 isomultiplets (6 isosinglets and 6 isotriplets) comprising the $c\bar{c}q\bar{q}'$ $\Sigma_g^+(1S)$ multiplet $(q, q' \in \{u, d\})$ was studied in Ref. [31]. This was the first work to differentiate $I = 0$ and $I = 1$ states in a diquark model. The model naturally produces scenarios in which $X(3872)$ is the lightest member; and of the two $I = 1$, $J^{PC} = 1^{+-}$ states, the $Z_c(3900)$ naturally decays to J/ψ and the $Z_c(4020)$ to h_c, as is observed. The simplest model uses a three-parameter Hamiltonian: a common multiplet mass, an internal diquark-spin coupling, and a long-distance isospin-dependent coupling between the light quark q in δ and light antiquark $\bar{q}'$ in $\bar{\delta}$. The corresponding analysis of the 28 states of the negative-parity $c\bar{c}q\bar{q}'$ $\Sigma_g^+(1P)$ multiplet, which includes precisely 4 Y states ($J^{PC} = 1^{--}$), was performed in Ref. [32]. In this case, the simplest model has five parameters, including now spin-orbit and tensor terms. An earlier diquark analysis using a similar Hamiltonian but not including isospin appears in Ref. [33].

In this paper we extend the study of the dynamical diquark model to the $\Sigma_g^+(1S)$ multiplet in the hidden-bottom ($b\bar{b}q\bar{q}'$) sector (again, 12 isomultiplets) and the hidden-charm, hidden-strange ($c\bar{c}s\bar{s}$) sector (six states). Remarkably, using only the well-known $Z_b(10610)$ and $Z_b(10650)$ states—often themselves identified as $B\bar{B}^*$ and $B^*\bar{B}^*$ molecules, respectively—and rough information from their Υ and h_b branching ratios, one can predict masses of the remaining 10 states and their preferred heavy-quark decay channels with surprising accuracy. The analysis of the $c\bar{c}s\bar{s}$ states builds upon that of Ref. [34] [which assumes that $X(3915)$ and $Y(4140)$ are $c\bar{c}s\bar{s}$ states] to reflect the current state of data and to develop a better understanding of the underpinnings of the model. The negative-parity $\Sigma_g^+(1P)$ multiplet consists of

---

[1]Equivalently, the full four-quark wave function has a large overlap with two-meson states when its $\delta, \bar{\delta}$ components have a small relative momentum, and a large overlap with an idealized δ-$\bar{\delta}$ state (and a suppressed overlap with two-meson states) when this relative momentum is large.

[2]A full definition of the standard BO potential notation is presented in Ref. [24]. The specific case Σ_g^+ means that the projection of angular momentum along the axis connecting the heavy sources has eigenvalue 0, and that the light degrees of freedom are symmetric under two reflections: through a plane perpendicular to and bisecting this axis, and under the CP inversion of the light degrees of freedom (using the midpoint of the heavy sources as the origin).

TABLE I.   All bottomoniumlike exotic-meson candidates catalogued by the Particle Data Group (PDG) [12]. Also included is the recently observed $Y(10750)$ [39]. Both the particle name most commonly used in the literature and its label as given in the PDG are listed. Only bottomonium decays are listed, and branching ratios are given where available.

Particle	PDG label	$I^G J^{PC}$	Mass [MeV]	Width [MeV]	Production and decay	
$Z_b(10610)^\pm$	$Z_b(10610)^\pm$	1^+1^{+-}	10607.2 ± 2.0	18.4 ± 2.4	$e^+e^- \to Z; \quad Z \to$	$\begin{cases} \Upsilon(1S)\pi^+\pi^- & (5.4^{+1.9}_{-1.5}) \times 10^{-3} \\ \Upsilon(2S)\pi^+\pi^- & (3.6^{+1.1}_{-0.8})\% \\ \Upsilon(3S)\pi^+\pi^- & (2.1^{+0.8}_{-0.6})\% \\ h_b(1P)\pi^+\pi^- & (3.5^{+1.2}_{-0.9})\% \\ h_b(2P)\pi^+\pi^- & (4.7^{+1.7}_{-1.3})\% \end{cases}$
$Z_b(10610)^0$	$Z_b(10610)^0$	1^+1^{+-}	10609 ± 6	18.4 ± 2.4	$e^+e^- \to Z; \quad Z \to$	$\begin{cases} \Upsilon(2S)\pi^0 \\ \Upsilon(3S)\pi^0 \end{cases}$
$Z_b(10650)^\pm$	$Z_b(10650)^\pm$	1^+1^{+-}	10652.2 ± 1.5	11.5 ± 2.2	$e^+e^- \to Z; \quad Z \to$	$\begin{cases} \Upsilon(1S)\pi^+\pi^- & (1.7^{+0.8}_{-0.6}) \times 10^{-3} \\ \Upsilon(2S)\pi^+\pi^- & (1.4^{+0.6}_{-0.4})\% \\ \Upsilon(3S)\pi^+\pi^- & (1.6^{+0.7}_{-0.5})\% \\ h_b(1P)\pi^+\pi^- & (8.4^{+2.9}_{-2.4})\% \\ h_b(2P)\pi^+\pi^- & (15 \pm 4)\% \end{cases}$
$Y(10750)$	$\Upsilon(10750)$	0^-1^{--}	$10752.7^{+5.9}_{-6.0}$	$35.5^{+18.0}_{-11.8}$	$e^+e^- \to \gamma Y; \quad Y \to$	$\begin{cases} \Upsilon(1S)\pi^+\pi^- \\ \Upsilon(2S)\pi^+\pi^- \\ \Upsilon(3S)\pi^+\pi^- \end{cases}$
$Y(10860)$	$\Upsilon(10860)$	0^-1^{--}	$10889.9^{+3.2}_{-2.6}$	51^{+6}_{-7}	$e^+e^- \to \gamma Y; \quad Y \to$	$\begin{cases} \Upsilon(1S)\pi^+\pi^- & (5.3 \pm 0.6) \times 10^{-3} \\ \Upsilon(2S)\pi^+\pi^- & (7.8 \pm 1.3) \times 10^{-3} \\ \Upsilon(3S)\pi^+\pi^- & (4.8^{+1.9}_{-1.7}) \times 10^{-3} \\ \Upsilon(1S)K^+K^- & (6.1 \pm 1.8) \times 10^{-4} \\ h_b(1P)\pi^+\pi^- & (3.5^{+1.0}_{-1.3}) \times 10^{-3} \\ h_b(2P)\pi^+\pi^- & (5.7^{+1.7}_{-2.1}) \times 10^{-3} \\ \eta\Upsilon_J(1D) & (4.8 \pm 1.1) \times 10^{-3} \\ \chi_{b1}(1P)\pi^+\pi^-\pi^0 & (1.85 \pm 0.33) \times 10^{-3} \\ \chi_{b2}(1P)\pi^+\pi^-\pi^0 & (1.17 \pm 0.30) \times 10^{-3} \end{cases}$

28 states for $b\bar{b}q\bar{q}'$ and 14 states for $c\bar{c}s\bar{s}$, but only a very small number of candidates have been observed for each type; nevertheless, we use the approach of Ref. [21] to predict the centers of mass of the $\Sigma_g^+(1P)$ and $\Sigma_g^+(2S)$ multiplets, and we find that most of the candidates lie in the anticipated mass regions [the exception being $Y(10750)$, which we argue to be a conventional bottomonium state].

This paper is organized as follows: In Sec. II we review the current data on $b\bar{b}q\bar{q}'$ and $c\bar{c}s\bar{s}$ candidates. Section III reprises the analysis of Ref. [31], as applied to these sectors. The naming scheme for levels comprising the $\Sigma_g^+(1S)$ multiplets is defined in Sec. IV, and we present explicit expressions for their masses in terms of the model parameters. Numerical analysis of states in the $b\bar{b}q\bar{q}'$ and the $c\bar{c}s\bar{s}$ sectors appears in Sec. V, where both mass eigenvalues and mixing parameters relevant to heavy-quark decay modes are predicted. We conclude in Sec. VI.

II. EXPERIMENTAL REVIEW

A. The $b\bar{b}q\bar{q}'$ sector

Of all hidden-bottom states thus far observed, only a handful are exotic candidates, which are summarized in Table I. The most familiar examples are the $I = 1$,

$J^{PC} = 1^{+-}$ states $Z_b(10610)$ and $Z_b(10650)$. Their proximity to the thresholds for $B\bar{B}^*$ (10604.2 ± 0.3 MeV) and $B^*\bar{B}^*$ (10649.4 ± 0.4 MeV), respectively, suggests a natural identification as molecular states. These states also possess hidden-charm analogues $Z_c(3900)$ and $Z_c(4020)$ that carry the same quantum numbers, which indeed lie near the $D\bar{D}^*$ and $D^*\bar{D}^*$ thresholds, respectively.[3] Nevertheless, $Z_c(3900)$ and $Z_c(4020)$ were found in Ref. [31] to serve naturally as the $I = 1$, $J^{PC} = 1^{+-}$ members of the ground-state $\Sigma_g^+(1S)$ multiplet of the dynamical diquark model, and so in this work we interpret the two Z_b states analogously. Furthermore, both $Z_b(10610)$ and $Z_b(10650)$, as the Z_c states, have been observed to decay to both closed [37] [$\Upsilon(nS)$, $h_b(nP)$] and open [38] [$B^{(*)}\bar{B}^*$] heavy-flavor states. However, the Z_b and Z_c states differ in one important regard: The observed charmonium decays of $Z_c(3900)$ to date all have total charm-quark spin $s_{c\bar{c}} = 1$ (i.e., that of J/ψ), while those of $Z_c(4020)$ have $s_{c\bar{c}} = 0$ (i.e., that of h_c), and obtaining this

---

[3]Although we identify the $Z_c(3900)$ as a δ-$\bar{\delta}$ state, its nature is still fiercely debated in the literature (among many references, note Ref. [35] for its discussion in amplitude analyses and Ref. [36] for a recent lattice simulation).

TABLE II. All candidate hidden-charm-strange states catalogued by the Particle Data Group [12]. Also included is $Y(4626)$ [63,64]. Both the particle name most commonly used in the literature and its label as given in the PDG are listed.

Particle	PDG label	$I^G J^{PC}$	Mass [MeV]	Width [MeV]	Production and decay
$X(3915)$	$\chi_{c0}(3915)$	$0^+(0 \text{ or } 2)^{++}$	3918.4 ± 1.9	20 ± 5	$e^+e^- \to X; X \to \begin{cases} \omega J/\psi \\ \gamma\gamma \end{cases}$
$Y(4140)$	$\chi_{c1}(4140)$	0^+1^{++}	4146.8 ± 2.4	22^{+8}_{-7}	$\left. \begin{array}{c} B \to KY \\ p\bar{p} \to Y + \text{anything} \end{array} \right\}; Y \to \phi J/\psi$
$Y(4274)$	$\chi_{c1}(4274)$	0^+1^{++}	4274^{+8}_{-6}	49 ± 12	$B \to KY; Y \to \phi J/\psi$
$X(4350)$	$X(4350)$	$0^+?^{?+}$	4351 ± 5	13^{+18}_{-10}	$\gamma\gamma \to X; X \to \phi J/\psi$
$X(4500)$	$\chi_{c0}(4500)$	0^+0^{++}	4506^{+16}_{-19}	92 ± 29	$p\bar{p} \to X; X \to \phi J/\psi$
$Y(4626)$	$\psi(4626)$	0^-1^{--}	4624 ± 5	49 ± 13	$e^+e^- \to \gamma Y; Y \to D_s^+ D_{s1}(2536)^-, D_s^+ D_{s2}^*(2573)^-$
$X(4700)$	$\chi_{c0}(4700)$	0^+0^{++}	4704^{+17}_{-26}	120 ± 50	$p\bar{p} \to X; X \to \phi J/\psi$

idealized mixing in a natural way is one of the central results of Ref. [31]. However, a glance at Table I shows that the Z_b system is rather different: *Both* Z_b states decay to states with $s_{b\bar{b}} = 0$ and 1 (Υ and h_b) with fairly comparable branching ratios.

The current experimental situation for the hidden-bottom sector also differs from the hidden-charm sector in one obvious respect: In the latter, the most obvious and best-studied state is the neutral 1^{++} $X(3872)$. However, the hidden-bottom analogue X_b ($I^G = 0^+$, $J^{PC} = 1^{++}$) has not yet been observed, despite a number of searches [40–42]. Partly, this absence reflects the relative difficulty of probing the hidden-bottom sector with limited energy [e.g., at the original Belle experiment operating at a center-of-momentum energy equal to the $\Upsilon(4S)$ mass [40]] or limited only to the decay channel $\Upsilon(1S)\pi^+\pi^-$ (at the LHC [41,42]), which has opposite G-parity to that expected for X_b. It would be truly surprising in both molecular models and diquark models (as well as in coupled-channel and QCD sum-rule approaches) were the X_b state to fail to exist; as a result, a great deal of theoretical effort has been invested in studying the conjectured X_b [14,16,40–58]. Bounding the possible range for the X_b mass and determining whether any hidden-bottom exotics can be even lighter constitute a major goal of this work.

Table I presents two further observed exotic candidates, both with $J^{PC} = 1^{--}$. The $Y(10750)$ was recently observed at Belle [39] and has already been studied as a diquark state [59] and within QCD sum rules [60]. Additionally, $Y(10750)$ and the remaining exotic candidate $Y(10860)$ have been argued to be conventional bottomonium states [61,62]. One should note, however, that $Y(10860)$ (as the Z_b states) has both $s_{b\bar{b}} = 0$ and $s_{b\bar{b}} = 1$ decay modes (thus violating heavy-quark spin symmetry in its decays if it is conventional bottomonium). In contrast, $Y(10750)$ has thus far been observed to decay only to $\Upsilon(nS)$. In addition, $Y(10750)$ lies only 100 MeV above the Z_b states, which would indicate a much smaller $1P$-$1S$ splitting (assuming they share a related structure) than between corresponding bottomonium states (e.g., $m_{\chi_{bJ}}(1P) - m_{\Upsilon(1S)} > 400$ MeV).

We argue in Sec. V that $Y(10860)$ is well suited to being a $\Sigma_g^+(1P)$ excitation of $\Sigma_g^+(1S)$ states such as $Z_b(10610)$ and $Z_b(10610)$, but $Y(10750)$ is not.

1. The $c\bar{c}s\bar{s}$ sector

The most likely hidden-charm-strange ($c\bar{c}s\bar{s}$) exotic candidates are listed in Table II. Almost all have been seen exclusively in the decay channel $\phi J/\psi$, which indicates that each either has a valence $c\bar{c}s\bar{s}$ quark content or is a pure $c\bar{c}$ state decaying through an Okubo-Zweig-Iizuka (OZI)-suppressed channel. Similar statements apply to the newly observed $Y(4626)$ [63,64], which has been observed to decay thus far only to channels of open charm and strangeness.

$X(3915)$ has been included in Table II as the lightest $c\bar{c}s\bar{s}$ candidate despite having no observed decays to states of hidden or open strangeness.[4] Upon its discovery, $X(3915)$ was immediately assigned by the PDG as the first radial excitation $\chi_{c0}(2P)$ of the conventional charmonium state $\chi_{c0}(1P)$. However, this identification was found to be problematic for several reasons [66–69]: First, the mass splitting between $\chi_{c2}(2P)$ and $X(3915)$ (only about 10 MeV [12]) is smaller than the $\chi_{c2}(2P)$-$\chi_{c0}(2P)$ splitting expected from quark potential models; furthermore, one would expect $\chi_{c0}(2P)$ (or a $c\bar{c}q\bar{q}$ exotic) to decay prominently into $D\bar{D}$, but the dominant observed $X(3915)$ decay channel is actually the OZI-suppressed mode $\omega J/\psi$. These features led Ref. [34] to suppose that $X(3915)$ is actually a $c\bar{c}s\bar{s}$ state, its $\omega J/\psi$ decay possibly proceeding by means of a small $s\bar{s}$ component in ω. Indeed, the subsequent Belle discovery of $\chi_{c0}(3860)$ [70] as a candidate with the expected properties of the missing $\chi_{c0}(2P)$ sharpens the case for arguing that $X(3915)$ is exotic [71].

S-wave hidden-charm-strange exotics have been discussed by multiple authors [72–93], using methods as

---

[4]$X(3915)$ lies below both the $\phi J/\psi$ and the $D_s\bar{D}_s$ thresholds. The mode $\eta\eta_c$ is possible, but here only an upper bound is known [65].

varied as ordinary (tetra)quark models, diquark models, molecular/rescattering models, and QCD sum rules (as well as combinations of these). Following on the observation of the negative-parity $Y(4626)$, P-wave $c\bar{c}s\bar{s}$ states have also recently been considered [94,95].

The precise nature of the two 1^{++} states $Y(4140)$ and $Y(4274)$ in Table II is particularly interesting. On the one hand, they both appear in the mass range predicted for the conventional 1^{++} charmonium state $\chi_{c1}(3P)$. One might naively think that since the (missing) $\chi_{c1}(2P)$ state is expected to be quite wide (> 100 MeV), its radial excitation $\chi_{c1}(3P)$ should be even wider. However, it has been known for some time that the more complicated wave-function nodal structure of $\chi_{c1}(3P)$ actually suppresses its width [19] to the same order of magnitude as that of both $Y(4140)$ and $Y(4274)$. So then which one, if either, is the $\chi_{c1}(3P)$? Studies in which the $Y(4140)$-$Y(4274)$ sector is described in terms of conventional charmonium appear in Refs. [96–101]. Moreover, as seen in these papers and in Refs. [67,79], no true consensus has emerged on the assignment of either one. Additionally, in the simplest diquark models such as the one used in this work, the ground-state $\Sigma_g^+(1S)$ multiplet contains only one 1^{++} $c\bar{c}s\bar{s}$ state [see Eq. (3)]. In this paper, we show that the most natural assignment identifies $Y(4140)$ as the unique $J^{PC} = 1^{++}$ $c\bar{c}s\bar{s}$ state and $Y(4274)$ as $\chi_{c1}(3P)$.

III. MASS HAMILTONIAN

In the most minimal model variant associated with the dynamical diquark picture, $b\bar{b}q\bar{q}'$ exotics ($q, q' \in \{u, d\}$) connected by a color flux tube in its ground state (the $1S$ multiplet of the Σ_g^+ BO potential) can be described using a very simple three-parameter Hamiltonian:

$$H = M_0 + \Delta M_{\kappa_{qb}} + \Delta M_{V_0}$$
$$= M_0 + 2\kappa_{qb}(\mathbf{s}_q \cdot \mathbf{s}_b + \mathbf{s}_{\bar{q}'} \cdot \mathbf{s}_{\bar{b}}) + V_0(\boldsymbol{\tau}_q \cdot \boldsymbol{\tau}_{\bar{q}'})(\boldsymbol{\sigma}_q \cdot \boldsymbol{\sigma}_{\bar{q}'}). \quad (1)$$

Here, M_0 is the common $\Sigma_g^+(1S)$ multiplet mass, which depends only upon the chosen diquark [$\delta \equiv (bq)_{\bar{\mathbf{3}}}$ or $\bar{\delta} \equiv (\bar{b}\,\bar{q})_{\mathbf{3}}$] mass and a central potential $V(r)$ computed numerically on the lattice from pure glue configurations that connect $\mathbf{3}$ and $\bar{\mathbf{3}}$ sources, as done in Ref. [21]. M_0 is the lowest eigenvalue of the Schrödinger equation using the Σ_g^+ BO potential $V_{\Sigma_g^+}(r)$; higher eigenvalues have also been computed for this potential [e.g., for $\Sigma_g^+(1P)$, $\Sigma_g^+(2S)$], as well as eigenvalues for lattice-computed excited-glue configurations (e.g., for BO potentials Π_u^+, Σ_u^-).

The second term in Eq. (1) represents the spin-spin interaction within diquarks, assumed to couple only $q \leftrightarrow b$ and $\bar{q}' \leftrightarrow \bar{b}$, and κ_{qb} indicates the strength of this interaction. These couplings are singled out as having a greater physical effect upon the nature of the state by assuming that δ and $\bar{\delta}$ are at least somewhat separated quasiparticles

within the full exotic state, so that their internal spin couplings are expected to be stronger than the ones between δ and $\bar{\delta}$. This ansatz originates with Ref. [30] and is incorporated into the motivation behind the dynamical diquark picture, as described in the Introduction and discussed in further detail in Refs. [22,31].

The final term in Eq. (1) is an isospin-spin-dependent interaction between the light-quark spins, where V_0 is the strength of the coupling. The exotic candidates, appearing in distinct $I = 0$ and $I = 1$ multiplets, undisputedly exhibit nontrivial isospin dependence, thus requiring a term such as this to be included in the Hamiltonian. Its precise form as given in Eq. (1) is, of course, motivated by that of chiral pion exchanges in hadronic physics, and one plausible interpretation of this operator [31] is to represent the effect of exchanging a Goldstone-boson-like mode across the flux tube connecting the light quarks q and $\bar{q}'$ in δ and $\bar{\delta}$, respectively. Nevertheless, one could argue for alternate forms that still carry isospin dependence. For example, Refs. [31,32] consider the possibility that the final operator in Eq. (1) couples not to light-quark spins $\mathbf{s}_{q,\bar{q}}$, but to the full diquark spins $\mathbf{s}_{\delta,\bar{\delta}}$, which would be an appropriate scheme were the diquarks truly pointlike. However, as seen in Refs. [31,32] for the hidden-charm sector, this alternate formulation leads to results inconsistent with experiment, such as degeneracy between $X(3872)$ and its (unobserved) $I = 1$ partners.

The form of Eq. (1) has been presented for use in the $b\bar{b}q\bar{q}'$ $\Sigma_g^+(1S)$ sector, but as indicated above, it was originally used for $c\bar{c}q\bar{q}'$ [31]. It can be generalized to B_c-like exotics $b\bar{c}q\bar{q}'$ by using the reduced mass obtained from unequal m_δ and $m_{\bar{\delta}}$ in the Schrödinger equation and introducing unequal κ_{qb} and κ_{qc} coefficients into the relevant Hamiltonian terms. Equation (1) has also been generalized to the $\Sigma_g^+(1P)$ sector [32] by the addition of spin-orbit and (isospin-dependent) tensor couplings.

The ground-state [$\Sigma_g^+(1S)$] hidden-charm-strange ($c\bar{c}s\bar{s}$) exotics can be described using an even simpler Hamiltonian, since the states lack isospin dependence:

$$H = M_0 + \Delta M_{\kappa_{sc}}$$
$$= M_0 + 2\kappa_{sc}(\mathbf{s}_s \cdot \mathbf{s}_c + \mathbf{s}_{\bar{s}} \cdot \mathbf{s}_{\bar{c}}), \quad (2)$$

where M_0 and κ_{sc} are defined analogously to the parameters above. This Hamiltonian actually first appeared in Ref. [34] and also included orbital and spin-orbit terms to allow comparison between S- and P-wave states; in the current model, the S-P splitting (as well as the $2S$-$1S$ splitting) can be computed directly using the techniques of Ref. [21], as seen in Sec. V. Moreover, subsequent experimental findings that confirm the existence and J^{PC} quantum numbers of relevant states, as well as the

discovery of $X(4500)$ and $X(4700)$ (Table II) and their assignment to the $2S$ multiplet in a diquark model [80], make a fresh analysis of the $c\bar{c}s\bar{s}$ sector quite relevant.

IV. MASS FORMULA

The fully general notation for all states in the dynamical diquark model appears in Ref. [24]. Since the current work focuses solely on states in the lowest BO potential Σ_g^+, and most often those in its lowest multiplet $1S$, we can reduce to a much more compact notation. For diquark-antidiquark (δ-$\bar{\delta}$) states of good total J^{PC} in the S-wave band (i.e., zero orbital angular momentum), the defining notation is

$$J^{PC} = 0^{++}: \ X_0 = |0_\delta, 0_{\bar{\delta}}\rangle_0, \quad X_0' = |1_\delta, 1_{\bar{\delta}}\rangle_0,$$
$$J^{PC} = 1^{++}: \ X_1 = \frac{1}{\sqrt{2}}(|1_\delta, 0_{\bar{\delta}}\rangle_1 + |0_\delta, 1_{\bar{\delta}}\rangle_1),$$
$$J^{PC} = 1^{+-}: \ Z = \frac{1}{\sqrt{2}}(|1_\delta, 0_{\bar{\delta}}\rangle_1 - |0_\delta, 1_{\bar{\delta}}\rangle_1),$$
$$Z' = |1_\delta, 1_{\bar{\delta}}\rangle_1,$$
$$J^{PC} = 2^{++}: \ X_2 = |1_\delta, 1_{\bar{\delta}}\rangle_2, \tag{3}$$

where outer subscripts indicate total quark spin $S = J$ in the absence of orbital angular momentum. The same states may be expressed in any other spin-coupling basis by using angular momentum recoupling coefficients, specifically $9j$ symbols. For both the simplest evaluation of the final operator in Eq. (1) and for convenient physical interpretation, the most useful alternate basis is that of definite heavy-quark (and light-quark) spin eigenvalues, $(Q\bar{Q}) + (q\bar{q})$:

$$\langle (s_q s_{\bar{q}}) s_{q\bar{q}}, (s_Q s_{\bar{Q}}) s_{Q\bar{Q}}, S | (s_q s_Q) s_\delta, (s_{\bar{q}} s_{\bar{Q}}) s_{\bar{\delta}}, S \rangle$$
$$= ([s_{q\bar{q}}][s_{Q\bar{Q}}][s_\delta][s_{\bar{\delta}}])^{1/2} \begin{Bmatrix} s_q & s_{\bar{q}} & s_{q\bar{q}} \\ s_Q & s_{\bar{Q}} & s_{Q\bar{Q}} \\ s_\delta & s_{\bar{\delta}} & S \end{Bmatrix}, \tag{4}$$

where $[s] \equiv 2s + 1$ denotes the multiplicity of a spin-s state. Using Eqs. (3) and (4), one then obtains

$$J^{PC} = 0^{++}: \ X_0 = \frac{1}{2}|0_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_0 + \frac{\sqrt{3}}{2}|1_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_0,$$
$$X_0' = \frac{\sqrt{3}}{2}|0_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_0 - \frac{1}{2}|1_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_0,$$
$$J^{PC} = 1^{++}: \ X_1 = |1_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_1,$$
$$J^{PC} = 1^{+-}: \ Z = \frac{1}{\sqrt{2}}(|1_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_1 - |0_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_1),$$
$$Z' = \frac{1}{\sqrt{2}}(|1_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_1 + |0_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_1),$$
$$J^{PC} = 2^{++}: \ X_2 = |1_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_2. \tag{5}$$

A similar recoupling can be used to express these states in terms of equivalent heavy-light meson spins, $(q\bar{Q}) + (\bar{q}Q)$.

The pairs of states X_0, X_0' and Z, Z' carry the same values of J^{PC} and can therefore mix. One may define the equivalent heavy-quark spin eigenstates, which are X_1, X_2, and

$$\tilde{X}_0 \equiv |0_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_0 = +\frac{1}{2}X_0 + \frac{\sqrt{3}}{2}X_0',$$
$$\tilde{X}_0' \equiv |1_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_0 = +\frac{\sqrt{3}}{2}X_0 - \frac{1}{2}X_0',$$
$$\tilde{Z} \equiv |1_{q\bar{q}}, 0_{Q\bar{Q}}\rangle_1 = \frac{1}{\sqrt{2}}(Z' + Z),$$
$$\tilde{Z}' \equiv |0_{q\bar{q}}, 1_{Q\bar{Q}}\rangle_1 = \frac{1}{\sqrt{2}}(Z' - Z). \tag{6}$$

Assuming $q, q' \in \{u, d\}$, the $\Sigma_g^+(1S)$ multiplet for either $c\bar{c}q\bar{q}'$ or $b\bar{b}q\bar{q}'$ then consists of precisely 12 isomultiplets: an isosinglet and an isotriplet corresponding to each of the six states in Eqs. (3) or (5) [or as reorganized in Eq. (6)]. The current PDG nomenclature [12] adopted for the $b\bar{b}q\bar{q}'$ states is $X_J^{(\prime)I=0} \to \chi_{bJ}$, $X_J^{(\prime)I=1} \to W_{bJ}$, $Z^{(\prime)I=0} \to h_b$, $Z^{(\prime)I=1} \to Z_b$. The corresponding multiplet for B_c-like exotics would also contain 12 isomultiplets, but which are no longer C-parity eigenstates. If the light quarks are replaced by an $s\bar{s}$ pair, then only six distinct states remain; in PDG notation, the $c\bar{c}s\bar{s}$ states are labeled $X_J^{(\prime)} \to \chi_{cJ}$, $Z^{(\prime)} \to h_c$.

A. Bottomoniumlike exotics

Using the Hamiltonian of Eq. (1) and working (for definiteness) in the heavy-quark spin basis of Eq. (6), one obtains mass matrices for all 12 isomultiplets of the $b\bar{b}q\bar{q}'$ $\Sigma_g^+(1S)$ multiplet. The cases with nonvanishing off-diagonal elements, for which the entries are arranged in the order $s_{b\bar{b}} = 0, 1$, read

$$\tilde{M}_{0^{++}}^{I=0} = M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \kappa_{qb} \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} - 3V_0 \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\tilde{M}_{0^{++}}^{I=1} = M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \kappa_{qb} \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix} + V_0 \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\tilde{M}_{1^{+-}}^{I=0} = M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \kappa_{qb} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - 3V_0 \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix},$$
$$\tilde{M}_{1^{+-}}^{I=1} = M_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \kappa_{qb} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + V_0 \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}. \tag{7}$$

Diagonalizing these expressions, and appending the expressions for the other states (whose mass matrices are already diagonal), one obtains the mass eigenvalues for all 12 isomultiplets of the $b\bar{b}q\bar{q}'$ $\Sigma_g^+(1S)$ multiplet:

$$M_{0^{++}}^{I=0} = (M_0 - \kappa_{qb} + 3V_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2V_1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{0^{++}}^{I=1} = (M_0 - \kappa_{qb} - V_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2V_2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{1^{+-}}^{I=0} = (M_0 + 3V_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + V_3\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{1^{+-}}^{I=1} = (M_0 - V_0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + V_4\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{1^{++}}^{I=0} = M_0 - \kappa_{qb} - 3V_0,$$

$$M_{1^{++}}^{I=1} = M_0 - \kappa_{qb} + V_0,$$

$$M_{2^{++}}^{I=0} = M_0 + \kappa_{qb} - 3V_0,$$

$$M_{2^{++}}^{I=1} = M_0 + \kappa_{qb} + V_0, \tag{8}$$

where we abbreviate

$$V_1 \equiv \sqrt{\kappa_{qb}{}^2 + 3\kappa_{qb}V_0 + 9V_0^2},$$

$$V_2 \equiv \sqrt{\kappa_{qb}{}^2 - \kappa_{qb}V_0 + V_0^2},$$

$$V_3 \equiv \sqrt{\kappa_{qb}{}^2 + 36V_0^2},$$

$$V_4 \equiv \sqrt{\kappa_{qb}{}^2 + 4V_0^2}. \tag{9}$$

The pairs of states in Eq. (8) degenerate in J^{PC} are arranged in order of increasing mass.

To obtain the mixing angles, one must first derive the corresponding normalized eigenvectors for the four mixed pairs of states with $J^{PC} = 0^{++}, 1^{+-}$. Further denoting

$$\epsilon_{qQ} \equiv \text{sgn}(\kappa_{qQ}), \tag{10}$$

the normalized eigenvectors collected into columns of unitary matrices R read

$$R_{0^{++}}^{I=0} = \frac{1}{2\sqrt{V_1}}$$
$$\times \begin{pmatrix} \sqrt{2V_1-(\kappa_{qb}+6V_0)} & \epsilon_{qb}\sqrt{2V_1+(\kappa_{qb}+6V_0)} \\ \epsilon_{qb}\sqrt{2V_1+(\kappa_{qb}+6V_0)} & -\sqrt{2V_1-(\kappa_{qb}+6V_0)} \end{pmatrix},$$

$$R_{0^{++}}^{I=1} = \frac{1}{2\sqrt{V_2}}$$
$$\times \begin{pmatrix} \sqrt{2V_2-(\kappa_{qb}-2V_0)} & \epsilon_{qb}\sqrt{2V_2+(\kappa_{qb}-2V_0)} \\ \epsilon_{qb}\sqrt{2V_2+(\kappa_{qb}-2V_0)} & -\sqrt{2V_2-(\kappa_{qb}-2V_0)} \end{pmatrix},$$

$$R_{1^{+-}}^{I=0} = \frac{1}{\sqrt{2V_3}}\begin{pmatrix} \epsilon_{qb}\sqrt{V_3+6V_0} & \sqrt{V_3-6V_0} \\ -\sqrt{V_3-6V_0} & \epsilon_{qb}\sqrt{V_3+6V_0} \end{pmatrix},$$

$$R_{1^{+-}}^{I=1} = \frac{1}{\sqrt{2V_4}}\begin{pmatrix} \epsilon_{qb}\sqrt{V_4-2V_0} & \sqrt{V_4+2V_0} \\ -\sqrt{V_4+2V_0} & \epsilon_{qb}\sqrt{V_4-2V_0} \end{pmatrix}. \tag{11}$$

The probability P of the lighter mass eigenstate in each mixed case to be measured to have heavy-quark spin eigenvalue $s_{b\bar{b}} = 1$, which is simply obtained by squaring the 1,2 element in each matrix of Eq. (11), is given by

$$P_{0^{++},s_{b\bar{b}}=1}^{I=0} = \frac{1}{2} + \frac{\kappa_{qb}+6V_0}{4\sqrt{\kappa_{qb}{}^2+3\kappa_{qb}V_0+9V_0^2}},$$

$$P_{0^{++},s_{b\bar{b}}=1}^{I=1} = \frac{1}{2} + \frac{\kappa_{qb}-2V_0}{4\sqrt{\kappa_{qb}{}^2-\kappa_{qb}V_0+V_0^2}},$$

$$P_{1^{+-},s_{b\bar{b}}=1}^{I=0} = \frac{1}{2} - \frac{3V_0}{\sqrt{\kappa_{qb}{}^2+36V_0^2}},$$

$$P_{1^{+-},s_{b\bar{b}}=1}^{I=1} = \frac{1}{2} + \frac{V_0}{\sqrt{\kappa_{qb}{}^2+4V_0^2}}. \tag{12}$$

Assuming that heavy-quark symmetry is unbroken in the decays of these states, the P values give the relative branching ratios for the lighter mass eigenstate in each case to decay into a bottomonium state with $s_{b\bar{b}} = 1$ (Υ, χ_b) vs $s_{b\bar{b}} = 0$ (η_b, h_b).

B. Hidden-charm-strange exotics

Using the Hamiltonian of Eq. (2) and working (for definiteness) in the heavy-quark spin basis of Eq. (6), one obtains mass matrices for all six states of the $c\bar{c}s\bar{s}$ $\Sigma_g^+(1S)$ multiplet. The cases with nonvanishing off-diagonal elements, for which the entries are arranged in the order $s_{c\bar{c}} = 0, 1$, read

$$\tilde{M}_{0^{++}} = M_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \kappa_{sc}\begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix},$$

$$\tilde{M}_{1^{+-}} = M_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \kappa_{sc}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

Diagonalizing these expressions, and appending the expressions for the other states (whose mass matrices are already diagonal), one obtains the mass eigenvalues for all six states of the $c\bar{c}s\bar{s}$ $\Sigma_g^+(1S)$ multiplet:

$$M_{0^{++}} = (M_0 - \kappa_{sc})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2|\kappa_{sc}|\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{1^{+-}} = M_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + |\kappa_{sc}|\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$M_{1^{++}} = M_0 - \kappa_{sc},$$

$$M_{2^{++}} = M_0 + \kappa_{sc}. \tag{14}$$

The pairs of states in Eq. (14) degenerate in J^{PC} are arranged in order of increasing mass. Note at this point we have not constrained the spin-spin coupling κ_{sc} to assume a positive value.

To obtain the mixing angles, one must first derive the corresponding normalized eigenvectors for the two mixed pairs of states with $J^{PC} = 0^{++}, 1^{+-}$. Collected into columns of unitary matrices R, the eigenvectors read

$$R_{0^{++}} = \frac{1}{2} \begin{pmatrix} \sqrt{2 - \epsilon_{sc}} & \epsilon_{sc}\sqrt{2 + \epsilon_{sc}} \\ \epsilon_{sc}\sqrt{2 + \epsilon_{sc}} & -\sqrt{2 - \epsilon_{sc}} \end{pmatrix},$$

$$R_{1^{+-}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_{sc} & 1 \\ -1 & \epsilon_{sc} \end{pmatrix}. \qquad (15)$$

The probability P of the lighter mass eigenstate in each mixed case to be measured to have heavy-quark spin eigenvalue $s_{c\bar{c}} = 1$, which is simply obtained by squaring the 1,2 element in each matrix of Eq. (15), is given by

$$P_{0^{++},s_{c\bar{c}}=1} = \frac{1}{2} + \frac{1}{4}\epsilon_{sc},$$

$$P_{1^{+-},s_{c\bar{c}}=1} = \frac{1}{2}. \qquad (16)$$

Assuming that heavy-quark symmetry is unbroken in the decays of these states, the P values give the relative branching ratios for the lighter mass eigenstate in each case to decay into a charmonium state with $s_{c\bar{c}} = 1$ (ψ, χ_c) vs $s_{c\bar{c}} = 0$ (η_c, h_c).

V. ANALYSIS AND RESULTS

A. $c\bar{c}q\bar{q}'$ Exotics redux

The masses of the 12 isomultiplets in the $b\bar{b}q\bar{q}'$ $\Sigma_g^+(1S)$ multiplet depend upon only three Hamiltonian parameters: M_0, κ_{qb}, and V_0, as seen in Eqs. (8) and (9). A similar, but not identical, analysis of the 12 $c\bar{c}q\bar{q}'$ $\Sigma_g^+(1S)$ isomultiplets appears in Ref. [31] (with, of course, $\kappa_{qb} \to \kappa_{qc}$, and different M_0 and V_0 numerical values for the $c\bar{c}q\bar{q}'$ and $b\bar{b}q\bar{q}'$ systems). There, the masses of the three states $X(3872)$, $Z_c(3900)$, and $Z_c(4020)$ [12] are used as inputs, and the mixing angles of 0^{++} and 1^{+-} states are allowed to vary under the reasoning that any additional operators omitted from the minimal three-parameter form have small numerical coefficients and would leave the mass spectrum stable, but could nevertheless substantially change the precise values of the mixing angles. Using the additional phenomenological observation that $X_1^{I=0}$ [corresponding to $X(3872)$] appears to be the lightest $c\bar{c}q\bar{q}'$ state, Ref. [31] obtained

$$M_0 = 3988.75 \text{ MeV}, \quad \kappa_{qc} = 17.76 \text{ MeV}, \quad V_0 = 33.10 \text{ MeV}. \qquad (17)$$

From these values, Ref. [31] found that $Z_c(3900)$ decays almost exclusively to J/ψ and $Z_c(4020)$ to h_c, in full accord with current observations.

However, one may just as easily adopt the strict three-parameter form of Eq. (1) for the $c\bar{c}q\bar{q}'$ sector, and use the three measured mass eigenvalues for $M_{1^{++}}^{I=0}$ and $M_{1^{+-}}^{I=1}$ in

Eqs. (8) and (9) to obtain values for the parameters M_0, κ_{qc}, and V_0, as well as for the mixing parameters P of Eq. (12). A double-valued set of equations then arises; one solution gives nearly identical values to Eq. (17),

$$M_0 = 3988.69 \text{ MeV}, \quad \kappa_{qc} = 17.89 \text{ MeV}, \quad V_0 = 33.04 \text{ MeV}, \qquad (18)$$

and the very satisfactory value $P_{s_{c\bar{c}}=1}[Z_c(3900)] = 0.983$. The other solution gives rather different values,

$$M_0 = 3964.59 \text{ MeV}, \quad \kappa_{qc} = 66.07 \text{ MeV}, \quad V_0 = 8.94 \text{ MeV}, \qquad (19)$$

and the phenomenologically unacceptably small value $P_{s_{c\bar{c}}=1}[Z_c(3900)] = 0.631$. One learns from this exercise that the value of P, even if not precisely measured, serves as a decisive input to the model.

But one also finds, using the fit values of Eq. (18) in the minimal three-parameter model, that $X_1^{I=0}$ is no longer the lightest $c\bar{c}q\bar{q}'$ state; $X_0^{I=0}$ (the 0^{++} isosinglet) assumes that status, with

$$M_{X_0^{I=0}} = 3851.6 \text{ MeV}. \qquad (20)$$

This prediction is remarkable in that it overlaps with the observed mass 3862^{+50}_{-35} MeV of the conventional charmonium $\chi_{c0}(2P)$ candidate [70], which shares the same quantum numbers. The large observed width 201^{+180}_{-110} MeV indicates unimpeded S-wave decays into $D\bar{D}$ pairs (threshold ≈ 3740 MeV) for either $\chi_{c0}(2P)$ or $X_0^{I=0}$, and indeed, the observed $\chi_{c0}(3860)$ could be a mixture of the two.

B. Bottomoniumlike exotics

Table I shows that only 2 out of 12 $b\bar{b}q\bar{q}'$ candidates in the positive-parity $\Sigma_g^+(1S)$ multiplet have been observed to date, both with $(I^G)J^{PC} = (1^+)1^{+-}$: $Z_b(10610)$ and $Z_b(10650)$. Two known masses for a model with three Hamiltonian parameters hardly seems sufficient input to draw many conclusions, but the results of the previous subsection indicate that using the $s_{b\bar{b}} = 1$ content $P_{1^{+-},s_{b\bar{b}}=1}^{I=1}$ of $Z_b(10610)$ can be helpful. Indeed, we define

$$\bar{M}_0 \equiv M_0 - V_0$$
$$= \frac{1}{2}[m_{Z_b(10650)} + m_{Z_b(10610)}] = 10629.7 \text{ MeV},$$

$$V_4 \equiv \sqrt{\kappa_{qb}^2 + 4V_0^2}$$
$$= \frac{1}{2}[m_{Z_b(10650)} - m_{Z_b(10610)}] = 22.5 \text{ MeV},$$

$$P \equiv P_{1^{+-},s_{b\bar{b}}=1}^{I=1}$$
$$= \frac{1}{2} + \frac{V_0}{\sqrt{\kappa_{qb}^2 + 4V_0^2}} = \frac{1}{2} + \frac{V_0}{V_4}, \qquad (21)$$

where the definitions of V_4 and $P^{I=1}_{1^{+-},s_{b\bar{b}}=1}$ are the same as in Eqs. (9) and (12), respectively, and for definiteness our numerical analysis uses the mass of the charged $Z_b(10650)$. Using these definitions, one may express the original parameters in Eq. (1) as

$$M_0 = \bar{M}_0 + V_4\left(P - \frac{1}{2}\right),$$

$$|\kappa_{qb}| = 2V_4\sqrt{P(1-P)},$$

$$V_0 = V_4\left(P - \frac{1}{2}\right). \tag{22}$$

Given a particular numerical value for P, the only remaining ambiguity in predicting the entire $\Sigma^+_g(1S)$ mass spectrum is the sign of κ_{qb}. With reference to Eq. (1), $\kappa_{qb} > 0$ indicates a scenario in which the spin-singlet diquark $\delta \equiv (qb)$ is lighter than the spin-triplet, which is the expectation of virtually every model. Thus making the mild assumption that $\kappa_{qb} > 0$, the formulas of Eqs. (8) and (9) for the mass eigenstates (indicated henceforth by overlines, with primes for the heavier of states that are degenerate in J^{PC}) then read

$$M^{I=0}_{\bar{X}_0} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} - 2(2P-1) + C_1(P)],$$

$$M^{I=0}_{\bar{X}_{0'}} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} - 2(2P-1) - C_1(P)],$$

$$M^{I=1}_{\bar{X}_0} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} + C_2(P)],$$

$$M^{I=1}_{\bar{X}_{0'}} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} - C_2(P)],$$

$$M^{I=0}_{\bar{Z}} = \bar{M}_0 + V_4[2(2P-1) - \sqrt{9 - 32P(1-P)}],$$

$$M^{I=0}_{\bar{Z}'} = \bar{M}_0 + V_4[2(2P-1) + \sqrt{9 - 32P(1-P)}],$$

$$M^{I=1}_{\bar{Z}} = \bar{M}_0 - V_4 = M_{Z_b(10610)},$$

$$M^{I=1}_{\bar{Z}'} = \bar{M}_0 + V_4 = M_{Z_b(10650)},$$

$$M^{I=0}_{\bar{X}_1} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} + (2P-1)],$$

$$M^{I=1}_{\bar{X}_1} = \bar{M}_0 - V_4[2\sqrt{P(1-P)} - (2P-1)],$$

$$M^{I=0}_{\bar{X}_2} = \bar{M}_0 + V_4[2\sqrt{P(1-P)} - (2P-1)],$$

$$M^{I=1}_{\bar{X}_2} = \bar{M}_0 + V_4[2\sqrt{P(1-P)} + (2P-1)], \tag{23}$$

where we abbreviate

$$C_1(P) \equiv \sqrt{9 - 20P(1-P) + 12(2P-1)\sqrt{P(1-P)}},$$

$$C_2(P) \equiv \sqrt{1 + 12P(1-P) - 4(2P-1)\sqrt{P(1-P)}}. \tag{24}$$

The expressions in Eq. (12) for the heavy-quark spin content of the remaining mixed states then assume the forms

$$P^{I=0}_{\bar{X}_0,s_{b\bar{b}}=1} = \frac{1}{2}\left[1 + \frac{2\sqrt{P(1-P)} + 3(2P-1)}{C_1(P)}\right],$$

$$P^{I=1}_{\bar{X}_0,s_{b\bar{b}}=1} = \frac{1}{2}\left[1 + \frac{2\sqrt{P(1-P)} - (2P-1)}{C_2(P)}\right],$$

$$P^{I=0}_{\bar{Z},s_{b\bar{b}}=1} = \frac{1}{2}\left[1 - \frac{3(2P-1)}{\sqrt{9 - 32P(1-P)}}\right], \tag{25}$$

and all observables for the entire $\Sigma^+_g(1S)$ multiplet are now expressed as functions of the single parameter $P \equiv P^{I=0}_{\bar{Z},s_{b\bar{b}}=1}$, which varies between 0 and 1; the only numerical inputs are the $Z_b(10610)$ and $Z_b(10650)$ masses.

In fact, sufficient data exist to go even further: An examination of the exclusive Υ- and h_b-channel branching ratios in Table I reveals some interesting effects. First, the branching ratios to $\Upsilon(1S)$ are the smallest among bottomonium decays for both $Z_b(10610)$ and $Z_b(10650)$, and the branching ratios to $h_b(2S)$ are the largest. Noting from simple quark-potential models that $\Upsilon(1S)$ has by far the most spatially compact bottomonium wave function while $h_b(2P)$ has the largest of those kinematically allowed in the Z_b decays, one is led to the qualitative conclusion that the Z_b states are not spatially compact. Moreover, $h_b(2P)$ has a complicated wave function with not only angular but radial nodes, suggesting initial Z_b wave functions that are similarly nonuniform in their spatial density. For our immediate purposes, however, the most interesting feature arises in a direct comparison of the branching ratios for individual Υ and h_b channels, noting that the phase space factors for exclusive $Z_b(10610)$ and $Z_b(10650)$ decay modes are almost identical. With the possible exception of the $\Upsilon(3S)$, the Υ branching ratios of the $Z_b(10610)$ appear to be a factor of about 3 times larger than those of the $Z_b(10650)$, and the h_b branching ratios of the $Z_b(10610)$ appear to be a factor of about 3 times smaller than those of the $Z_b(10650)$. One is therefore led to the natural estimate $P \approx 3/4$.

In addition, the last of Eq. (22) shows that the sign of $P - \frac{1}{2}$ directly gives the sign of V_0. Since as mentioned below its definition in Eq. (1), the V_0 term is motivated [31] by its similarity in form to the attractive pion interaction in hadronic physics, the value of $P \approx 3/4$ obtained above gives $V_0 > 0$ and suggests a similar interaction in the $\Sigma^+_g(1S)$ multiplet for both charmonium-like and bottomoniumlike states. Based upon these considerations, we expect $1/2 < P < 1$. In fact, independently of V_0, the values of M_0 and κ_{qb} obtained solely from the Z_b masses are numerically very stable over the entire range of P, and demonstrate this fact by exhibiting results at $P = 1/4$ in addition to $P = 1/2$ and $3/4$:

TABLE III. Predictions for the 12 isomultiplet masses (in MeV) of the $\Sigma_g^+(1S)$ $b\bar{b}q\bar{q}'$ multiplet, using the Hamiltonian of Eq. (1) as evaluated using Eqs. (21), (23), and (24). Boldface indicates the measured Z_b mass inputs.

| J^{PC} | $P = P_{\bar{Z},s_{b\bar{b}}=1}^{I=1} = 1/4$ | | | | $P = P_{\bar{Z},s_{b\bar{b}}=1}^{I=1} = 1/2$ | | | | $P = P_{\bar{Z},s_{b\bar{b}}=1}^{I=1} = 3/4$ | | | |
	$I = 0$		$I = 1$		$I = 0$		$I = 1$		$I = 0$		$I = 1$	
0^{++}	10551.1	10624.4	10564.6	10655.9	10562.2	10652.2	10562.2	10652.2	10569.7	10695.7	10575.4	10644.9
1^{++}	10621.5		10599.0		10607.2		10607.2		10598.9		10621.4	
1^{+-}	10568.3	10646.2	**10607.2**	**10652.2**	10607.2	10652.2	**10607.2**	**10652.2**	10613.2	10691.1	**10607.2**	**10652.2**
2^{++}	10660.5		10638.0		10652.2		10652.2		10637.9		10660.4	

$$P = \frac{1}{4} \rightarrow M_0 = 10624.08 \text{ MeV},$$

$$\kappa_{qb} = 19.49 \text{ MeV},$$

$$V_0 = -5.63 \text{ MeV}, \qquad (26)$$

$$P = \frac{1}{2} \rightarrow M_0 = 10629.70 \text{ MeV},$$

$$\kappa_{qb} = 22.50 \text{ MeV},$$

$$V_0 = 0.00 \text{ MeV}, \qquad (27)$$

$$P = \frac{3}{4} \rightarrow M_0 = 10635.33 \text{ MeV},$$

$$\kappa_{qb} = 19.49 \text{ MeV},$$

$$V_0 = +5.63 \text{ MeV}. \qquad (28)$$

Of special note, the allowed values of κ_{qb} in this range of P are numerically very close to those obtained in Eq. (18) for κ_{qc}, indicating a common physical origin for both. In contrast, the allowed values of V_0 are several times smaller in the hidden-bottom sector, reflecting the expectation that V_0 contains a coefficient scaling inversely with a power of the heavy-quark mass and therefore being smaller for bottomoniumlike than for charmoniumlike states.

Inserting the values of parameters determined in Eq. (28) from these three choices of P, we obtain predictions for masses of all 12 isomultiplets of the $b\bar{b}q\bar{q}'$ $\Sigma_g^+(1S)$ multiplet in Table III. The most notable feature of these results is the remarkably small numerical variation of individual state mass predictions over the whole range $1/4 \leq P \leq 3/4$, keeping in mind that the expected sign of V_0 and the decay pattern disfavor $P < 1/2$.

Another way to visualize these results is to impose our expectation that $P \geq 1/2$ and consider the entire range $1/2 \leq P \leq 1$. We then plot the results from combining Eqs. (21), (23), and (24) for all 12 $\Sigma_g^+(1S)$ isomultiplet masses in Fig. 1. The ordering of the states in this range of P is remarkably stable. Of particular note: Over most of the allowed range for P, the isosinglet $J^{PC} = 0^{++}$ state $\bar{X}_0^{I=0}$ is lightest, and its isotriplet partner $\bar{X}_0^{I=1}$ is second lightest. Both lie above the threshold (≈ 10560 MeV) of their expected dominant $B\bar{B}$ decay channel but not excessively

so, suggesting that reasonably narrow 0^{++} $b\bar{b}q\bar{q}'$ states will be discovered in future experiments. Meanwhile, $\bar{X}_b \equiv X_1^{I=0}$, the $b\bar{b}q\bar{q}'$ analogue to the $X(3872)$, only becomes the second-lightest $b\bar{b}q\bar{q}'$ state for values of P very close to 1 (which is what occurs in the $c\bar{c}q\bar{q}'$ system). More interestingly, $\bar{X}_1^{I=0}$ lies at most only a few MeV below the $B\bar{B}^*$ threshold (≈ 10605 MeV) over almost the whole range $1/2 \leq P \leq 1$, and thus analogously to $X(3872)$ in its relation to $D\bar{D}^*$, $\bar{X}_1^{I=0}$ will need to be analyzed by considering the impact of $B\bar{B}^*$ threshold effects. Explicitly, we predict

$$10598 \text{ MeV} \leq m_{X_b} \equiv m_{\bar{X}_1^{I=0}} \leq 10607 \text{ MeV}. \qquad (29)$$
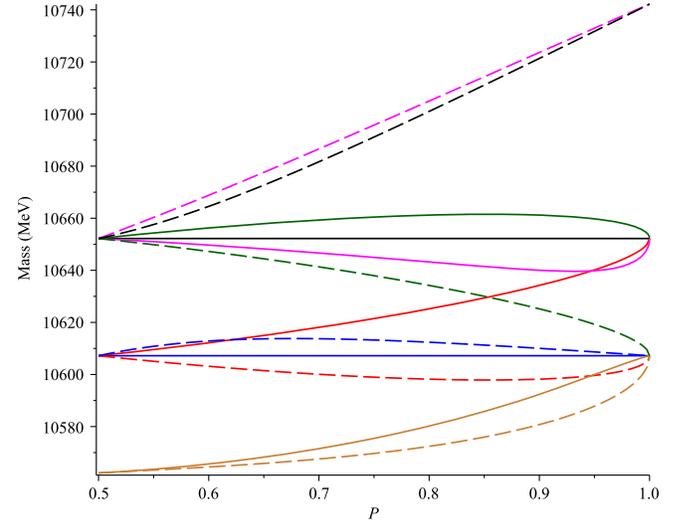
FIG. 1. Prediction of the 12 isomultiplet masses (in MeV) of the $\Sigma_g^+(1S)$ multiplet as functions of the heavy-quark $s_{b\bar{b}} = 1$ spin-content parameter P of $Z_b(10610)$ defined in Eq. (21). Solid (dashed) lines indicate $I = 1$ ($I = 0$) states. Using the naming scheme of Eq. (3) with isospin superscripts, an overline for mass eigenstates, and a prime for the heavier of mixed eigenstates, the levels from top to bottom at $P = 3/4$ are as follows: $\bar{X}_0^{\prime I=0}$ (dashed magenta); $\bar{Z}^{\prime I=0}$ (dashed black); $X_2^{I=1}$ (solid green); $\bar{Z}^{\prime I=1}$ (solid black); $\bar{X}_0^{I=1}$ (solid magenta); $X_2^{I=0}$ (dashed green); $X_1^{I=1}$ (solid red); $\bar{Z}^{I=0}$ (dashed blue); $\bar{Z}^{I=1}$ (solid blue); $X_1^{I=0}$ (dashed red); $\bar{X}_0^{I=1}$ (solid gold); $\bar{X}_0^{I=0}$ (dashed gold).

The heavy-quark spin structure of the mixed eigenstates can also be computed solely as functions of P, according to Eqs. (24) and (25). The results are presented in Fig. 2. We find in the range $1/2 \leq P \leq 1$ that $\bar{X}_0^{I=0}$ decays preferentially to Υ or χ_b, $\bar{Z}^{I=0}$ decays to h_b or η_b, and the proportion for $\bar{X}_0^{I=1}$ depends sensitively upon the precise value of P.

Having completed the analysis of the $\Sigma_g^+(1S)$ multiplet, we now use the techniques of Ref. [21] to compute the center of mass M_0 for any other multiplet. The results of Eq. (28) indicate that $M_0(1S) = 10630$ MeV, with an uncertainty of no more than 5 MeV. Using this mass eigenvalue in a Schrödinger equation with the lattice-computed potentials of Refs. [25–29], one finds the diquark $\delta \equiv (bq)_{\bar{3}}$ and its antiparticle $\bar{\delta} \equiv (\bar{b}\,\bar{q})_3$ to have mass

$$m_\delta = m_{\bar{\delta}} = 5383.1\text{–}5406.2 \text{ MeV}, \tag{30}$$

where the range indicates the effect of varying over potentials taken from the different lattice simulations. In turn, these $m_{\delta,\bar{\delta}}$ values serve as inputs used to compute other multiplet mass eigenvalues, and we predict

$$M_0(1P) = 10960.9\text{–}10966.3 \text{ MeV},$$
$$M_0(2S) = 11087.7\text{–}11093.2 \text{ MeV}. \tag{31}$$

One immediately notes that of the two remaining $b\bar{b}q\bar{q}'$ candidates in Table I (both with negative parity), $Y(10860)$ lies about 70 MeV below $M_0(1P)$ and thus uncontroversially fits into the $1P$ multiplet.[5] On the other hand, $Y(10750)$ does not fit well into this scheme; indeed, it is only about 100 MeV heavier than the $1S$ state $Z_b(10650)$. The nP-nS average mass splitting for conventional bottomonium is about 450 MeV for $n = 1$ and 250 MeV for $n = 2$, suggesting that $Y(10750)$ is not sufficiently heavy to be a $\Sigma_g^+(1P)$ $b\bar{b}q\bar{q}'$ state. However, it was noted even in the discovery paper [39] that $Y(10750)$ is a natural candidate for a higher conventional Υ state, likely identifying with a missing $\Upsilon(nD)$ state, and possibly mixing with $\Upsilon(nS)$ states, although the exact composition remains a matter of debate [61,62]. In support of this view, note from Table I that only $Y(10750) \rightarrow \Upsilon$ (but not h_b) decay modes have been observed to date, thus promoting the hypothesis of a pure $s_{b\bar{b}} = 1$ state, as expected for conventional bottomonium.

C. Hidden-charm-strange exotics

As noted in the Introduction, the $c\bar{c}s\bar{s}$ sector was first considered using a model with separated (cs) and $(\bar{c}\,\bar{s})$ diquarks in Ref. [34]. The possibility that the lightest $c\bar{c}s\bar{s}$

---

[5]In comparison, the lowest 1^{--} state $Y(4230)$ in the $c\bar{c}q\bar{q}'$ $\Sigma_g^+(1P)$ multiplet lies about 140 MeV below the multiplet center of mass and yet fits well in the multiplet [32].
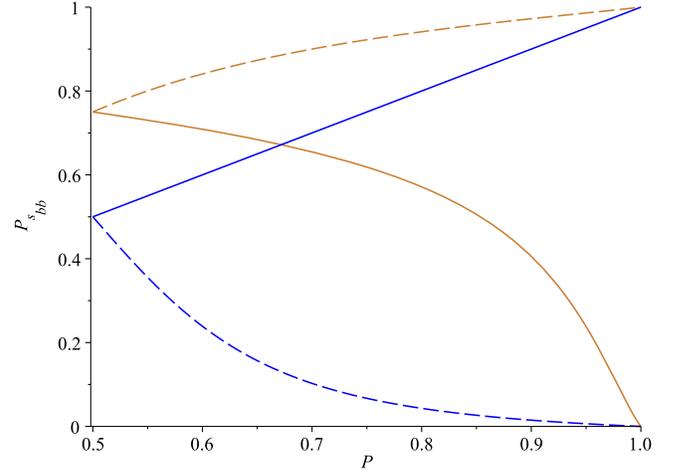
FIG. 2. Prediction of the heavy-quark spin-content parameters $P_{s_{b\bar{b}}=1}$ of Eq. (25) for the lighter of mass eigenstates that are degenerate in J^{PC}, as functions of the parameter P defined in Eq. (21). Solid (dashed) lines indicate $I = 1$ ($I = 0$) states. These levels from top to bottom at $P = 3/4$ are as follows: $\bar{X}_0^{I=0}$ (dashed gold); $\bar{Z}^{I=1}$ (solid blue, which is P itself); $\bar{X}_0^{I=1}$ (solid gold); $\bar{Z}^{I=0}$ (dashed blue).

state is $X(3915)$ was introduced in that work, a reprise of the arguments in favor of this assignment appearing in Sec. II A 1. We also noted that two strong candidates for the sole 1^{++} state in the $c\bar{c}s\bar{s}$ $\Sigma_g^+(1S)$ multiplet, $Y(4140)$ and $Y(4274)$, have been experimentally confirmed (Table II), but also that the conventional charmonium state $\chi_{c1}(3P)$ is predicted to have a mass and a width comparable to those observed for the two candidates. Indeed, the early calculation of Ref. [19] predicts

$$m_{\chi_{c1}(3P)} = 4271 \text{ MeV}, \qquad \Gamma_{\chi_{c1}(3P)} = 39 \text{ MeV}. \tag{32}$$

The Hamiltonian introduced in Ref. [34] restricted to the $\Sigma_g^+(1S)$ multiplet is actually identical to the one given in Eq. (2). In Ref. [34] it was introduced as a purely phenomenological construct, but in this work it is seen to be the direct expression of the dynamical diquark model, and mass splittings between different BO multiplets can be computed using lattice-calculated potentials, as in Ref. [21].

A nagging difficulty with the $X(3915)$ has been an ambiguity in its measured J^{PC} quantum numbers. As suggested in Table II and discussed by the PDG [12], the original 0^{++} assignment relies on the assumption of dominance by a particular $\gamma\gamma$ helicity component in $X(3915)$ production, and if this assumption is relaxed, then the assignment 2^{++} is also possible.

Using the measured masses in Table II, we therefore obtain fits to the Hamiltonian of Eq. (2) under two alternate assumptions: that the $X(3915)$ is the lighter of the two 0^{++} states in $\Sigma_g^+(1S)$, or that it is the sole 2^{++} state. For the moment we also assign $Y(4140)$ to be the sole 1^{++} state,

TABLE IV. Prediction of the six state masses (in MeV) of the $\Sigma_g^+(1S)$ $c\bar{c}s\bar{s}$ multiplet, using the Hamiltonian of Eq. (2). Boldface indicates the measured $X(3915)$ and $Y(4140)$ masses used as inputs for the fit.

J^{PC}	$J^{PC}_{X(3915)} = 0^{++}$		$J^{PC}_{X(3915)} = 2^{++}$	
0^{++}	**3918.4**	4375.2	4375.2	3918.4
1^{+-}	4146.8	4375.2	4375.2	3918.4
1^{++}	**4146.8**		**4146.8**	
2^{++}	4375.2		**3918.4**	

supposing by default that $Y(4274)$ is $\chi_{c1}(3P)$. The results of fits with both $X(3915)$ assignments are presented in Table IV. In either case, the spectrum is quite simple, consisting of only three distinct (and equally spaced) mass eigenvalues for the six states.

A stunning feature of Table IV is that both assignments predict a 0^{++} state at the mass of the $X(3915)$, which suggests one remarkable scenario in which the observed $X(3915)$ is actually a mixture of 0^{++} and 2^{++} states. Furthermore, the third distinct mass in either case, 4375.2 MeV, lies quite close to that of the $X(4350)$, another $c\bar{c}s\bar{s}$ candidate in Table IV. Confirmation of this state and a precise measurement of its mass and J^P quantum numbers ($C = +$ is known) at Belle II will be quite incisive.

These two fits, however, have a major difference that selects one as more relevant to the spirit of the dynamical diquark model. If $X(3915)$ is 0^{++}, then one obtains

$$M_0 = 4261.0 \text{ MeV}, \qquad \kappa_{sc} = +114.2 \text{ MeV}, \qquad (33)$$

while taking $X(3915)$ to be 2^{++} gives

$$M_0 = 4032.6 \text{ MeV}, \qquad \kappa_{sc} = -114.2 \text{ MeV}. \qquad (34)$$

We have already noted in Sec. V B that the diquark spin-spin coupling κ_{qQ} is positive in virtually every model, so the scenario of Eq. (34) leading to a large, negative value of κ_{sc} and the $X(3915)$ being a degenerate 0^{++}-2^{++} combination seems phenomenologically less appealing.

The large value of κ_{sc} obtained in Eq. (33) as compared to κ_{cq} in Eq. (18) or κ_{qb} in Eq. (28) (a factor of 5-6) suggests that the lighter constituent of the diquark δ has a significantly greater influence on the size of the spin-spin coupling within δ than does the flavor of the heavy quark. One may argue that the s quark, being much heavier than u or d, has less Fermi motion and allows δ to be substantially more compact, thus enhancing the effects of spin couplings within δ. In the language of quark models, the equivalent spin-spin operator would have an expectation value scaling as some inverse power of the δ size.

Turning now to the identity of the sole 1^{++} state, we consider the alternate possibility that $Y(4274)$ is a $c\bar{c}s\bar{s}$ state and $Y(4140)$ is $\chi_{c1}(3P)$. Then the third distinct mass

eigenvalue in the fits of Table IV becomes 4629.6 MeV, a much higher value than in the previous fit, and completely unsuitable for the $X(4350)$.

Using the methods of Ref. [21] and the inputs of Eq. (33) [taking $X(3915)$ as the unique lightest state and $Y(4140)$ as the sole 1^{++} state in the $c\bar{c}s\bar{s}$ $\Sigma_g^+(1S)$ multiplet], we obtain

$$m_\delta = m_{\bar{\delta}} = 2063.7\text{–}2085.5 \text{ MeV} \qquad (35)$$

and predict

$$M_0(1P) = 4625.3\text{–}4628.8 \text{ MeV},$$
$$M_0(2S) = 4814.9\text{–}4818.1 \text{ MeV}. \qquad (36)$$

In comparison with the remaining states of Table II, the $\Sigma_g^+(1P)$ multiplet center of mass lies extraordinarily close to that of $Y(4626)$, while $X(4500)$ is somewhat light to serve as a $\Sigma_g^+(2S)$ state [plausibly, it could even be the heavier $\Sigma_g^+(1S)0^{++}$ state], but $X(4700)$ works well as the lighter 0^{++} state in the $\Sigma_g^+(2S)$ multiplet. Had $Y(4274)$ instead been used for these fits, the results would have been hundreds of MeV higher, reinforcing the conclusion that $Y(4140)$ works much better as a $c\bar{c}s\bar{s}$ state and $Y(4274)$ as $\chi_{c1}(3P)$.

VI. CONCLUSIONS

This paper expands upon the work of Refs. [21,31,32] to incorporate the hidden-bottom ($b\bar{b}q\bar{q}'$) and hidden-charm-strange ($c\bar{c}s\bar{s}$) sectors into the dynamical diquark model, primarily (but not exclusively) for the states that lie in their respective ground-state [$\Sigma_g^+(1S)$] multiplets.

Starting from a Hamiltonian with only three parameters (for $b\bar{b}q\bar{q}'$) or two parameters (for $c\bar{c}s\bar{s}$) that describe the fine structure within each multiplet of the model, we obtain explicit, closed-form expressions for all 12 $b\bar{b}q\bar{q}'$ isomultiplet masses and all six $c\bar{c}s\bar{s}$ masses.

In the $b\bar{b}q\bar{q}'$ sector, the masses of the $Z_b(10610)$ and $Z_b(10650)$ combined with their relative preferences to decay to Υ or h_b states are sufficient to highly constrain all other masses and heavy-quark-spin decay-mode preferences in the $\Sigma_g^+(1S)$ multiplet. In particular, the lightest states carry $J^{PC} = 0^{++}$ and lie only a few 10's of MeV above the $B\bar{B}$ threshold, and thus may have observably small widths. The 1^{++} analogue of the $X(3872)$ is predicted to lie in an especially constrained range (10598–10607 MeV), near the $B\bar{B}^*$ threshold.

In a redux of the $c\bar{c}q\bar{q}'$ sector (following on Ref. [31]), we find that the three-parameter Hamiltonian also predicts an isoscalar 0^{++} state that is lighter than $X(3872)$, but with exactly the right mass to merge with the conventional charmonium $\chi_{c0}(2P)$ candidate at 3860 MeV. Moreover, the fit values of the diquark internal spin coupling κ_{qb} in the $b\bar{b}q\bar{q}'$ sector and κ_{qc} in the $c\bar{c}q\bar{q}'$ sector are numerically

equal, but both are much smaller than κ_{sc} in the $c\bar{c}s\bar{s}$. The isospin-dependent couplings V_0 in the $b\bar{b}q\bar{q}'$ and $c\bar{c}q\bar{q}'$ sectors are both positive, having the same sign as the corresponding pion-exchange operator in hadronic physics.

Once the center of mass for the $\Sigma_g^+(1S)$ multiplet is determined from this analysis, we use potentials calculated in lattice simulations to compute the corresponding centers for higher multiplets, such as $\Sigma_g^+(1P)$ and $\Sigma_g^+(2S)$. We find that $Y(10860)$ works well as a $b\bar{b}q\bar{q}'$ $1P$ state but $Y(10750)$ is too light, very likely being primarily a D-wave conventional bottomonium state.

In the $c\bar{c}s\bar{s}$ sector, we find it possible to identify $X(3915)$ as a 2^{++} state, but only if the diquark spin coupling κ_{sc} has opposite sign to the positive one nearly universally accepted. Thus the assignment $J^{PC} = 0^{++}$ is much more natural in the dynamical diquark model. Additionally, $X(4350)$ emerges directly as a $c\bar{c}s\bar{s}$ state. We also find that $Y(4140)$ is much more likely the sole 1^{++} $\Sigma_g^+(1S)$ $c\bar{c}s\bar{s}$

state and $Y(4274)$ is the conventional charmonium state $\chi_{c1}(3P)$. Computing higher center-of-multiplet masses, we find that $Y(4626)$ fits the $\Sigma_g^+(1P)$ multiplet well and $X(4700)$ [but not $X(4500)$] fits the $\Sigma_g^+(2S)$ multiplet well.

To summarize, the dynamical diquark model produces a large number of remarkable results, both in the fine structure of individual multiplets by employing an extremely simple model and in the calculated splittings between multiplets, by using potentials calculated from first principles on the lattice. It further produces interesting physical insights in multiple sectors of exotic states, thus far including $c\bar{c}q\bar{q}'$, $c\bar{c}s\bar{s}$, and $b\bar{b}q\bar{q}'$. One could similarly analyze hidden-charm–open-strange states, B_c-like exotics, pentaquarks, and other possibilities.

ACKNOWLEDGMENTS

[1] R. Lebed, R. Mitchell, and E. Swanson, Prog. Part. Nucl. Phys. **93**, 143 (2017).

[2] H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Phys. Rep. **639**, 1 (2016).

[3] A. Hosaka, T. Iijima, K. Miyabayashi, Y. Sakai, and S. Yasui, Prog. Theor. Exp. Phys. **2016**, 062C01 (2016).

[4] A. Esposito, A. Pilloni, and A. Polosa, Phys. Rep. **668**, 1 (2017).

[5] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Rev. Mod. Phys. **90**, 015004 (2018).

[6] A. Ali, J. Lange, and S. Stone, Prog. Part. Nucl. Phys. **97**, 123 (2017).

[7] S. Olsen, T. Skwarnicki, and D. Zieminska, Rev. Mod. Phys. **90**, 015003 (2018).

[8] M. Karliner, J. Rosner, and T. Skwarnicki, Annu. Rev. Nucl. Part. Sci. **68**, 17 (2018).

[9] C.-Z. Yuan, Int. J. Mod. Phys. A **33**, 1830018 (2018).

[10] Y.-R. Liu, H.-X. Chen, W. Chen, X. Liu, and S.-L. Zhu, Prog. Part. Nucl. Phys. **107**, 237 (2019).

[11] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.-P. Shen, C. Thomas, A. Vairo, and C.-Z. Yuan, arXiv:1907 .07583.

[12] M. Tanabashi et al. (Particle Data Group), Phys. Rev. D **98**, 030001 (2018).

[13] M. Voloshin and L. Okun, Pisma Zh. Eksp. Teor. Fiz. **23**, 369 (1976) [, JETP Lett. **23**, 333 (1976)].

[14] N. Törnqvist, Z. Phys. C **61**, 525 (1994).

[15] M. Cleven, F.-K. Guo, C. Hanhart, Q. Wang, and Q. Zhao, Phys. Rev. D **92**, 014005 (2015).

[16] M. Karliner and J. Rosner, Phys. Rev. Lett. **115**, 122001 (2015).

[17] S. Choi et al. (Belle Collaboration), Phys. Rev. Lett. **91**, 262001 (2003).

[18] J. Durham (LHCb Collaboration), 28th International Conference on Ultrarelativistic Nucleus-Nucleus Collisions (Quark Matter 2019), Wuhan, China, 2019, arXiv:2002 .01551.

[19] T. Barnes, S. Godfrey, and E. Swanson, Phys. Rev. D **72**, 054026 (2005).

[20] L. Maiani, F. Piccinini, A. Polosa, and V. Riquer, Phys. Rev. D **71**, 014028 (2005).

[21] J. Giron, R. Lebed, and C. Peterson, J. High Energy Phys. 05 (2019) 061.

[22] S. Brodsky, D. Hwang, and R. Lebed, Phys. Rev. Lett. **113**, 112001 (2014).

[23] R. Lebed, Phys. Lett. B **749**, 454 (2015).

[24] R. Lebed, Phys. Rev. D **96**, 116003 (2017).

[25] K. Juge, J. Kuti, and C. Morningstar, Nucl. Phys. B, Proc. Suppl. **63**, 326 (1998).

[26] K. Juge, J. Kuti, and C. Morningstar, Phys. Rev. Lett. **82**, 4400 (1999).

[27] K. Juge, J. Kuti, and C. Morningstar, Phys. Rev. Lett. **90**, 161601 (2003).

[28] http://www.andrew.cmu.edu/user/cmorning/static_potentials/ SU3_4D/greet.html.

[29] S. Capitani, O. Philipsen, C. Reisinger, C. Riehl, and M. Wagner, Phys. Rev. D **99**, 034502 (2019).

[30] L. Maiani, F. Piccinini, A. Polosa, and V. Riquer, Phys. Rev. D **89**, 114010 (2014).

[31] J. Giron, R. Lebed, and C. Peterson, J. High Energy Phys. 01 (2020) 124.

[32] J. Giron and R. Lebed, Phys. Rev. D **101**, 074032 (2020).

[33] A. Ali, L. Maiani, A. Borisov, I. Ahmed, M. Jamil Aslam, A. Parkhomenko, A. Polosa, and A. Rehman, Eur. Phys. J. C **78,** 29 (2018).

[34] R. Lebed and A. Polosa, Phys. Rev. D **93,** 094024 (2016).

[35] A. Pilloni, C. Fernandez-Ramirez, A. Jackura, V. Mathieu, M. Mikhasenko, J. Nys, and A. Szczepaniak (JPAC Collaboration), Phys. Lett. B **772,** 200 (2017).

[36] Y. Ikeda, S. Aoki, T. Doi, S. Gongyo, T. Hatsuda, T. Inoue, T. Iritani, N. Ishii, K. Murano, and K. Sasaki (HAL QCD Collaboration), Phys. Rev. Lett. **117,** 242001 (2016).

[37] A. Bondar *et al.* (Belle Collaboration), Phys. Rev. Lett. **108,** 122001 (2012).

[38] A. Garmash *et al.* (Belle Collaboration), Phys. Rev. Lett. **116,** 212001 (2016).

[39] R. Mizuk *et al.* (Belle Collaboration), J. High Energy Phys. 10 (2019) 220.

[40] X.-H. He *et al.* (Belle Collaboration), Phys. Rev. Lett. **113,** 142001 (2014).

[41] S. Chatrchyan *et al.* (CMS Collaboration), Phys. Lett. B **727,** 57 (2013).

[42] G. Aad *et al.* (ATLAS Collaboration), Phys. Lett. B **740,** 199 (2015).

[43] N. Törnqvist, Phys. Rev. Lett. **67,** 556 (1991).

[44] N. Törnqvist, Phys. Lett. B **590,** 209 (2004).

[45] E. Swanson, Phys. Rep. **429,** 243 (2006).

[46] W.-S. Hou, Phys. Rev. D **74,** 017504 (2006).

[47] D. Ebert, R. Faustov, and V. Galkin, Mod. Phys. Lett. A **24,** 567 (2009).

[48] A. Ali, C. Hambrock, I. Ahmed, and M. Aslam, Phys. Lett. B **684,** 28 (2010).

[49] F.-K. Guo, C. Hidalgo-Duque, J. Nieves, and M. Valderrama, Phys. Rev. D **88,** 054007 (2013).

[50] M. Karliner and S. Nussinov, J. High Energy Phys. 07 (2013) 153.

[51] F.-K. Guo, U.-G. Meißner, W. Wang, and Z. Yang, Eur. Phys. J. C **74,** 3063 (2014).

[52] G. Li and W. Wang, Phys. Lett. B **733,** 100 (2014).

[53] M. Karliner and J. Rosner, Phys. Rev. D **91,** 014014 (2015).

[54] G. Li and Z. Zhou, Phys. Rev. D **91,** 034020 (2015).

[55] S. Patel and P. Vinodkumar, Eur. Phys. J. C **76,** 356 (2016).

[56] Q. Wu, G. Li, F. Shao, Q. Wang, R. Wang, Y. Zhang, and Y. Zheng, Adv. High Energy Phys. **2016,** 3729050 (2016).

[57] Z.-Y. Zhou, D.-Y. Chen, and Z. Xiao, Phys. Rev. D **99,** 034005 (2019).

[58] Z.-G. Wang, Eur. Phys. J. C **79,** 489 (2019).

[59] A. Ali, L. Maiani, A. Parkhomenko, and W. Wang, Phys. Lett. B **802,** 135217 (2020).

[60] Z.-G. Wang, Chin. Phys. C **43,** 123102 (2019).

[61] Q. Li, M.-S. Liu, Q.-F. Lü, L.-C. Gui, and X.-H. Zhong, Eur. Phys. J. C **80,** 59 (2020).

[62] B. Chen, A. Zhang, and J. He, Phys. Rev. D **101,** 014020 (2020).

[63] S. Jia *et al.* (Belle Collaboration), Phys. Rev. D **100,** 111103 (2019).

[64] S. Jia *et al.* (Belle Collaboration), Phys. Rev. D **101,** 091101 (2020).

[65] A. Vinokurova *et al.* (Belle Collaboration), J. High Energy Phys. 06 (2015) 132; 02 (2017) 088(E).

[66] F.-K. Guo and U.-G. Meißner, Phys. Rev. D **86,** 091501 (2012).

[67] H. Wang, Y. Yang, and J. Ping, Eur. Phys. J. A **50,** 76 (2014).

[68] S. Olsen, Phys. Rev. D **91,** 057501 (2015).

[69] S. Olsen, EPJ Web Conf. **202,** 01003 (2019).

[70] K. Chilikin *et al.* (Belle Collaboration), Phys. Rev. D **95,** 112003 (2017).

[71] S. Olsen, EPJ Web Conf. **212,** 02009 (2019).

[72] F. Stancu, J. Phys. G **37,** 075017 (2010); **46,** 019501(E) (2019).

[73] Z.-G. Wang and Y.-F. Tian, Int. J. Mod. Phys. A **30,** 1550004 (2015).

[74] X. Li and M. Voloshin, Phys. Rev. D **91,** 114014 (2015).

[75] H.-X. Chen, E.-L. Cui, W. Chen, X. Liu, and S.-L. Zhu, Eur. Phys. J. C **77,** 160 (2017).

[76] Z.-G. Wang, Eur. Phys. J. C **77,** 78 (2017).

[77] A. Martinez Torres, K. Khemchandani, J. Dias, F. Navarra, and M. Nielsen, Nucl. Phys. **A966,** 135 (2017).

[78] Z.-G. Wang, Eur. Phys. J. C **76,** 657 (2016).

[79] X.-H. Liu, Phys. Lett. B **766,** 117 (2017).

[80] L. Maiani, A. Polosa, and V. Riquer, Phys. Rev. D **94,** 054026 (2016).

[81] Z.-G. Wang, Eur. Phys. J. A **53,** 19 (2017).

[82] Q.-F. Lü and Y.-B. Dong, Phys. Rev. D **94,** 074007 (2016).

[83] J. Wu, Y.-R. Liu, K. Chen, X. Liu, and S.-L. Zhu, Phys. Rev. D **94,** 094031 (2016).

[84] Z.-G. Wang, Eur. Phys. J. C **77,** 174 (2017).

[85] S. Agaev, K. Azizi, and H. Sundu, Phys. Rev. D **95,** 114003 (2017).

[86] A. Turkan and H. Dag, Nucl. Phys. **A985,** 38 (2019).

[87] W. Chen, H.-X. Chen, X. Liu, T. Steele, and S.-L. Zhu, Phys. Rev. D **96,** 114017 (2017).

[88] M. Anwar, J. Ferretti, and E. Santopinto, Phys. Rev. D **98,** 094015 (2018).

[89] Z.-G. Wang and Z.-Y. Di, Eur. Phys. J. C **79,** 72 (2019).

[90] Z.-G. Wang, Acta Phys. Pol. B **51,** 435 (2020).

[91] R. Albuquerque, J. Dias, K. Khemchandani, A. Martínez Torres, F. Navarra, M. Nielsen, and C. Zanetti, J. Phys. G **46,** 093002 (2019).

[92] Y. Yang and J. Ping, Phys. Rev. D **99,** 094032 (2019).

[93] S. Agaev, K. Azizi, and H. Sundu, Turk. J. Phys. **44,** 95 (2020).

[94] C. Deng, H. Chen, and J. Ping, Phys. Rev. D **101,** 054039 (2020).

[95] J.-R. Zhang, arXiv:2004.10985.

[96] P. Ortega, J. Segovia, D. Entem, and F. Fernández, Phys. Rev. D **94,** 114018 (2016).

[97] D.-Y. Chen, Eur. Phys. J. C **76,** 671 (2016).

[98] A. Badalian and B. Bakker, Phys. Rev. D **100,** 054036 (2019).

[99] W. Hao, G.-Y. Wang, E. Wang, G.-N. Li, and D.-M. Li, arXiv:1909.13099.

[100] R. Chaturvedi and A. Rai, arXiv:1910.06025.

[101] J. Ferretti, E. Santopinto, M. Anwar, and Y. Lu, Eur. Phys. J. C **80,** 464 (2020).