

Comment on “Linear superposition of regular black hole solutions of Einstein nonlinear electrodynamics”

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It is argued that in the paper by A. A. Garcia-Diaz and G. Gutierrez-Cano [Phys. Rev. D **100**, 064068 (2019)] on nonlinear electrodynamics coupled to general relativity, along with some interesting results and useful observations, many statements are either inaccurate or incomplete. In particular, the authors only consider solutions with an electric charge, whereas their magnetic counterparts have features of equal interest, both similar to and different from those of electric ones. Moreover, it is not mentioned that in electric solutions with a regular center the Lagrangian function $L(f)$ ($f = F_{\mu\nu}F^{\mu\nu}$) cannot have a Maxwell weak-field limit. The observation on superpositions of regular solutions suffers some inaccuracies. The present Comment tries to fill these gaps and to provide necessary corrections.

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In their paper [1], A. A. Garcia-Diaz and G. Gutierrez-Cano focused on the properties of static, spherically, planarly and pseudospherically symmetric metrics of general relativity (GR) coupled to nonlinear electrodynamics (NED) that, in their opinion, had been previously unnoticed. In particular:

- (1) Extension of the Birkhoff theorem to Einstein-NED space-times.
- (2) Determination of the algebraic types of the NED stress-energy tensor (SET) T_{μ}^{ν} , hence of the Einstein tensor G_{μ}^{ν} .
- (3) A formulation of the inverse integration method.
- (4) The existence of linear superpositions of Einstein-NED solutions with given metric functions.
- (5) A generating technique for obtaining multiparametric and asymptotically Reissner-Nordström (RN) solutions.
- (6) A description of a class of regular black hole solutions.

Also, the authors consider only solutions with an electric field, which is important since, in general, there is no electric-magnetic duality in NED.

Let us briefly discuss all these points. For better transparency, let us restrict ourselves to static, spherically symmetric metrics (an extension to the planar and pseudospherical symmetries is simple and evident) and apply more usual notations than those in [1]. On the other hand, for completeness, systems with both electric and magnetic charges will be considered.

It makes sense, for further discussion, to begin with reproducing some well-known facts on the static, spherically symmetric Einstein-NED system according to [2–5] (and many others) in the presence of both electric and magnetic charges. We consider the action

$$S = \frac{1}{2} \int \sqrt{-g} d^4x [R - L(f)], \quad f = F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where $F_{\mu\nu}$ is the electromagnetic tensor, $L(f)$ is an arbitrary function, and we use units in which $c = 8\pi G = 1$. In the general static, spherically symmetric metric

$$ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

the only possible nonzero components of $F_{\mu\nu}$ are $F_{tr} = -F_{rt}$ (a radial electric field) and $F_{\theta\phi} = -F_{\phi\theta}$ (a radial magnetic field). The electromagnetic field equations $\nabla_{\mu}(L_f F^{\mu\nu}) = 0$ and $\nabla_{\mu}{}^*F^{\mu\nu} = 0$ (the asterisk denotes duality) imply

$$r^2 L_f F^{tr} = q_e, \quad F_{\theta\phi} = q_m \sin\theta, \quad (3)$$

where $L_f \equiv dL/df$, and the constants q_e and q_m are the electric and magnetic charges, respectively. The only nonzero SET components are

$$T_t^t = T_r^r = \frac{L}{2} + f_e L_f, \quad T_{\theta}^{\theta} = T_{\phi}^{\phi} = \frac{L}{2} - f_m L_f, \quad (4)$$

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$$f_e = 2F_{tr}F^{rt} = \frac{2q_e^2}{L_f^2 r^4}, \quad f_m = 2F_{\theta\phi}F^{\theta\phi} = \frac{2q_m^2}{r^4}, \quad (5)$$

so that $f = f_m - f_e$. The equality $T_t^t = T_r^r$, through the Einstein equations, leads without loss of generality to $A(r) \equiv B(r)$, so that

$$ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6)$$

with only one unknown metric function $A(r)$. From the Einstein equation $G_t^t = -T_t^t$ it follows

$$A(r) = 1 - \frac{2M(r)}{r}, \quad M(r) = \frac{1}{2} \int T_t^t(r)r^2 dr, \quad (7)$$

where $M(r)$ is called the mass function. This relation involves both electric and magnetic charges according to (3) and (4) [3,4]. A possible inclusion of the cosmological constant Λ adds the term $-\Lambda r^2/3$ to the expression (7) for $A(r)$, the corresponding solutions are discussed, e.g., in [6–9].

Now we can pass on to discussing items 1–6.

(1) The Birkhoff theorem. There was no need to prove this theorem anew for the system in question because it is a special case of the extended Birkhoff theorem proved in [10,11], where sufficient conditions for its validity were formulated as requirements to the SET components, and these conditions ($T_t^t = 0$ and that some combination $T_r^r + \text{const} \cdot T_t^t$ should not depend on A if, in the metric (2), both A and B are allowed to depend on both r and t) are manifestly fulfilled for the tensor (4). Particularly, the Birkhoff theorem for systems with $T_r^r = T_t^t$ (“Dymnikova’s vacuum”) was discussed in [12].

Another, more geometric formulation of the extended Birkhoff theorem was presented in [13,14], and it includes, in particular, systems with SETs of Segre types [(111,1)] and [(11)(1,1)] that correspond to a cosmological constant and NED, respectively.

One should note that the meaning of the (extended) Birkhoff theorem is not that a certain kind of matter in space-times of given (here, spherical) symmetry necessarily creates a static metric, but that this metric, due to the field equations, necessarily contains an *additional* symmetry, not initially postulated. The corresponding Killing vector may be timelike (then the metric is static), spacelike (as happens in time-dependent black hole interiors) or null (see examples, e.g., in [15]). This important circumstance was not mentioned in [1].

(2) Algebraic types of T_μ^ν and G_μ^ν . As already said, having the structure (4), the SET of NED with spherical symmetry belongs to the Segre type [(11)(1,1)] that corresponds to two different pairs

of eigenvalues. This is certainly well known and cannot be regarded a result. However, of certain interest is the observation that in the Einstein-NED system the traceless Ricci tensor $S_\mu^\nu := R_\mu^\nu - \frac{1}{4}\delta_\mu^\nu R$ has the form $S_\mu^\nu = S \text{diag}(1, -1, 1, -1)$.

What the authors of [1] call a *theorem* (top of the right column on page 4), sounds really strange: “Besides the vacuum with Λ solutions, static Schwarzschild-like metrics only allow electromagnetic solutions to the Einstein (linear or nonlinear) electrodynamics equations.” First, the authors repeatedly use the words “Schwarzschild-like metrics” but *nowhere* define what they mean by them. Even if this term is used somewhere else, it is not widely known and must be clearly defined. Second, very probably “Schwarzschild-like” means the metric (6) with any $A(r)$. But anyway, the algebraic type of the SET certainly does not uniquely prescribe the kind of matter it belongs to. For example, the Segre type [(11)(1,1)] of SET pertains not only to NED but also to non-Abelian Yang-Mills fields.

(3) The inverse integration method. For electric solutions ($q_e \neq 0$, $q_m = 0$), the inverse integration method is formulated in [1], presenting F_{tr} and $L(f)$ in terms of the “structure function” $Q(r) = A(r)/r^2$ in Eqs. (18) and (19). In our notations, in terms of the metric function $A(r)$, we have equivalently

$$2q_e F_{tr}(r) = -1 + A - \frac{1}{2}r^2 A'', \quad (8)$$

$$L(r) = -\Lambda - A'' - 2\frac{A'}{r} \quad (9)$$

(the prime stands for d/dr). This gives the quantities F_{tr} (hence $f = -f_e = -2F_{tr}^2$) and L as functions of r . It is, however, important but ignored in [1], that with chosen $A(r)$ [or $Q(r)$], the function $f(r)$ will not always be monotonic, and thus it is not always possible to obtain an unambiguous Lagrangian function $L(f)$.

Unlike that, for systems with pure magnetic charge ($q_e = 0$, $q_m \neq 0$), the function $f(r) = f_m = 2q_m^2/r^4$ is monotonic, hence for given $A(r)$ we always obtain a well-defined Lagrangian function $L(f)$ [3]: as follows from (7) with possible $\Lambda \neq 0$,

$$L(r) = \frac{4M'}{r^2} = -2\Lambda + \frac{2}{r^2}(1 - A - rA'). \quad (10)$$

In the dyonic case with both nonzero q_e and q_m , the situation is more complicated [5,16,17]: given $A(r)$, there is no direct expression for $L(r)$; instead of (10), we obtain from (7)

$$\frac{4M'}{r^2} = -2\Lambda + \frac{2}{r^2}(1 - A - rA') = L + \frac{4q_e^2}{L_f r^4}. \quad (11)$$

- (4) Linear superpositions of solutions. As noticed in [1], Eqs. (18) and (19) (equivalent to (8) and (9) in the present Comment) are linear in the structure function $Q(y)$ or, equivalently, in the metric function $A(r)$ in the present notations. It then directly follows that if each of the functions $Q_i(y)$ (or $A_i(r)$) describes a solution to the NED-Einstein equations with electric charges q_{ei} and the quantities $q_{ei}F_{ir}(r)$, $L_i(r)$ and Λ_i , then their linear combinations (with constant coefficients c_i) also describe electric solutions, in which the quantities $q_e F_{ir}(r)$, $L(r)$ and Λ are linear combinations of the constituent quantities with the same coefficients.

The authors have formulated this result as a theorem: “For static Schwarzschild metrics coupled to electrodynamics (linear and nonlinear) and a Λ term (if any), any linear superposition of structural functions leads to linear superpositions of Lagrangian functions and the corresponding electromagnetic field functions.”

This formulation, as well as the unnumbered equation after it, are not precise, even forgetting that the term “Schwarzschild metric” is used here in an unusual manner. The following points are missing: (i) For $\sum_i c_i A_i(r)$ to satisfy (8), it is required $\sum_i c_i = 1$; (ii) not F_{ir} but $q_e F_{ir}$ is a subject of superposition (the statement on F_{ir} in [1] is correct only if the charge q_e is the same in all constituent solutions), and (iii) the resulting Λ is $\sum_i c_i \Lambda_i$.

One should add that, as before, in all thus obtained electric solutions one should take care of the monotonicity of $f(r) = -2F_{ir}^2$, otherwise $L(f)$ is ill defined.

In the magnetic case, Eq. (10) is also linear with respect to $A(r)$, therefore emerges a similar superposition method of constructing new solutions from known ones, but the magnetic charge values are not directly related to (10), and this issue should be analyzed separately.

- (5) Multiparametric and asymptotically RN solutions. The above-described superpositions are characterized in [1] as a generating technique for obtaining multiparametric solutions. Indeed, it allows for obtaining new solutions from known ones. Though, concerning the number of parameters, let us recall that actually, if $L(f)$ is not specified, we have an arbitrary function $A(r)$ (or $Q(y)$ in [1]), which can be endowed with *any* number of parameters. As to solutions with RN asymptotic behavior, it is clear that if $A(r)$ is specified with a proper large r behavior, the whole solution will also behave properly at large r . The example of a multiparametric

electric solution discussed in [1] confirms that. Similar examples of magnetic solutions can also be constructed.

- (6) Regular black holes and solitons. It is correctly said [1] that regularity at the center $r = 0$ (as at any other location) requires finiteness of all curvature invariants. However, there is no need to calculate them for each particular solution since it is well known that the metric (6) is regular at $r = 0$ if and only if $A(r) = 1 + \text{const} \cdot r^2 + o(r^2)$ as $r \rightarrow 0$ (see, e.g., [18]). Any such function provides a regular center, both in the electric and magnetic cases. Moreover, a superposition of solutions regular at $r = 0$ is also regular at $r = 0$ under the evident condition $\sum_i c_i = 1$.

However, a well-known important property of electric solutions, not mentioned in [1], is that a Lagrangian function $L(f)$ providing a solution with a regular center cannot have a Maxwell weak field limit ($L \sim f$ as $f \rightarrow 0$) [19]. At such a center the electric field should be zero, so that $f \rightarrow 0$, but it then follows from the field equations that $L_f \rightarrow \infty$ as $r \rightarrow 0$, which means a strongly non-Maxwell behavior at small f . In dyonic solutions ($q_e \neq 0$, $q_m \neq 0$), a regular center also requires a non-Maxwell weak field limit of $L(f)$ [3]. Only pure magnetic solutions are compatible with a correct weak-field limit of $L(f)$: in this case, at a regular center, $f = 2q_m^2/r^4 \rightarrow \infty$ but $L(f) \rightarrow \text{const} < \infty$, providing finite limits of both the SET components and the curvature invariants.

An electric solution can have a regular center (it describes a black hole if there are zeros of $A(r)$ at $r > 0$ or a solitonic particlelike object if $A(r) > 0$ at all r), where $L_f \rightarrow \infty$ as $f \rightarrow 0$, and a RN asymptotic behavior at large r , where $L \propto f$ at small f , which, however, means that these are *different* functions $L(f)$. In other words, different NED theories are acting in different parts of space. It is an example of what happens if, in a particular solution, $f(r)$ is not monotonic. Pure magnetic solutions are free from this shortcoming. A more detailed discussion of such situations can be found in [3,5,20].

To conclude, the paper [1], containing some interesting results and observations, is not free from significant gaps and inaccuracies which I tried to fill or correct in this Comment.

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