Two $\theta_{\mu\nu}$-deformed covariant relativistic quantum phase spaces as Poincaré-Hopf algebroids

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We consider two quantum phase spaces which can be described by two Hopf algebroids linked with the well-known $\theta_{\mu\nu}$-deformed $D = 4$ Poincaré-Hopf algebra $\mathbb{H}$. The first algebroid describes $\theta_{\mu\nu}$-deformed relativistic phase space with canonical NC space-time (constant $\theta_{\mu\nu}$ parameters) and the second one incorporates dual to $\mathbb{H}$ quantum $\theta_{\mu\nu}$-deformed Poincaré-Hopf group algebra $\mathbb{G}$, which contains non-commutative space-time translations given by Λ-dependent $\Theta_{\mu\nu}$ parameters ($\Lambda \equiv \Lambda_{\mu\nu}$ parametrize classical Lorentz group). The canonical $\theta_{\mu\nu}$-deformed space-time algebra and its quantum phase space extension is covariant under the quantum Poincaré transformations described by $\mathbb{G}$. We will also comment on the use of Hopf algebroids for the description of multiparticle structures in quantum phase spaces.

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I. INTRODUCTION

There have been proposed in recent years various models of noncommutative (NC) space-times which characterizes space-time geometry if the quantum gravity (QG) effects are included (see e.g., [1–8]). In this paper we shall study the canonical case of quantum space-times, with NC counterparts $\hat{x}_\mu$ ($\mu = 0, 1, 2, 3$) of space-time coordinates satisfying the well-known formula

$$[\hat{x}_\mu, \hat{x}_\nu] = i\lambda^2\theta_{\mu\nu}, \tag{1}$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ is a numerical 4×4 matrix[1] and $[\lambda] = [l]$ describes an elementary length, which for QG-generated noncommutativity is linked with Planck length

$$\lambda_P = \frac{\hbar}{m_P c} = \sqrt{\frac{\hbar G}{c^3}}, \tag{2}$$

where m_P is the Planck mass, and G describes the Newton constant characterizing the gravitational interactions. If λ is proportional to λ_P, from formula (2) due to the presence of Planck constant $\hbar$, one can deduce the quantum-mechanical and QG origin of relation (1) ($\lambda_P = 0$ if $\hbar \to 0$ or $G \to 0$). Further, for simplicity we shall use the choice of units $\hbar = c = 1$.

[1] In this paper we shall consider the case $D = 3 + 1$.

The well-known $\theta_{\mu\nu}$-deformed quantum space-times [see (1)] and associated quantum phase spaces are generated by the following Abelian twist [9,10]:

$$\mathcal{F} \equiv \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)} = \exp\left(\frac{i}{2}\lambda^2\theta^{\mu\nu} p_\mu \otimes p_\nu\right). \tag{3}$$

It defines $\theta_{\mu\nu}$-deformed quantum Poincaré-Hopf algebra $\mathbb{H}$, with nondeformed Poincaré algebra sectors, however, with modified coproducts and antipodes [11]

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\circ\Delta_0(h)\circ\mathcal{F}^{-1}, \qquad h = \{p_\mu, M_{\mu\nu}\}, \tag{4}$$

$$S_{\mathcal{F}}(h) = US_0(h)U^{-1}, \qquad U = \mathcal{F}_{(1)}S_0(\mathcal{F}_{(2)}). \tag{5}$$

The twist (3) can be employed in two ways:

(i) We introduce the classical Minkowski space-time coordinates $x_\mu \in \mathbf{X}$ as vectorial representation of relativistic symmetries described by classical Poincaré algebra $\mathcal{P}$, with the algebraic structure of $\mathcal{P} \otimes \mathbf{X}$ given by the semidirect product $\mathcal{P} \rtimes \mathbf{X}$. In quantum-deformed theory one introduces the NC space-times ($\hat{x}_\mu \in \hat{\mathbf{X}}$) as the module algebra (NC algebraic representation) of quantum Poincaré-Hopf algebra $\mathbb{H}$. In the case of $\theta_{\mu\nu}$-deformation the quantum space-time coordinates $\hat{x}_\mu \in \hat{\mathbf{X}}$ are obtained from Drinfeld twisting procedure [11] by star product technique [12,13].

Let us consider the algebra $\hat{A}$ of functions on $\hat{\mathbf{X}}$ ($f(\hat{x}) \in \hat{A}$). In the scheme of twist quantization one can represent the algebra $(\hat{A}, \cdot)$ by the algebra

$(A, \star_{\mathcal{F}})$ of classical functions, with their multiplication defined by the nonlocal star product $\star_{\mathcal{F}}$[2]

$$f(\hat{x})\cdot g(\hat{x}) \simeq f(x)\star_{\mathcal{F}} g(x) = m[\mathcal{F}^{-1}\circ(f\otimes g)] = (\mathcal{F}^{-1}_{(1)}\triangleright f)(\mathcal{F}^{-1}_{(2)}\triangleright g). \tag{6}$$

Putting in (6) $f(\hat{x}) = \hat{x}_\mu, g(\hat{x}) \equiv 1$ one gets

$$\hat{x}_\mu = m[\mathcal{F}^{-1}(\triangleright\otimes 1)(x_\mu\otimes 1)] = (\mathcal{F}^{-1}_{(1)}\triangleright x_\mu)\mathcal{F}^{-1}_{(2)}. \tag{7}$$

Formula (7) provides the quantum map expressing the deformed NC space-time coordinates by a nonlocal map in undeformed relativistic phase space $(x_\mu, p_\mu = \frac{1}{i}\partial_\mu)$. If we put $f(x) = x_\mu, g(x) \equiv x_\nu$ one can calculate from (6) the commutator given by the formula (1).

$$[\hat{x}_\mu,\hat{x}_\nu]\simeq[x_\mu,x_\nu]_{\star_{\mathcal{F}}} \equiv x_\mu\star_{\mathcal{F}} x_\nu - x_\nu\star_{\mathcal{F}} x_\mu = i\lambda^2\theta_{\mu\nu}. \tag{8}$$

(ii) By Hopf-algebraic duality one can define the $\theta_{\mu\nu}$-deformed quantum Poincaré group $\mathbb{G}$, with generators $\hat{g} = \{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\}$, describing generalized NC coordinates on algebraic $\theta_{\mu\nu}$-deformed Poincaré group manifold [14] with quantum Lorentz group parameters $\hat{\Lambda}_{\mu\nu}$ and the NC quantum Poincaré group translations $\hat{\xi}_\mu$ satisfying the following algebra:

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = i\lambda^2\theta^{\rho\sigma}(\eta_{\mu\rho}\eta_{\nu\sigma} - \hat{\Lambda}_{\mu\rho}\hat{\Lambda}_{\nu\sigma}) \coloneqq i\lambda^2\Theta_{\mu\nu}(\hat{\Lambda}). \tag{9}$$

Using the Heisenberg double construction [5,15] given by the particular choice of semidirect product $\mathbb{G}\rtimes\mathbb{H}$ called smash product, one obtains $(10+10)$-dimensional generalized $\theta_{\mu\nu}$-deformed quantum phase space $\mathcal{H}^{(10+10)} = (\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}, p_\mu, M_{\mu\nu})$. Such phase space can be employed in physical applications for the description of NC dynamics on algebraic $\theta_{\mu\nu}$-deformed quantum Poincaré group manifold.[3]

It appears that both $\theta_{\mu\nu}$-deformed structures presented above [see (1) and (9)] are necessary in order to describe in a complete way the NC space-time (8) as describing quantum Poincaré-covariant $\mathbb{H}$-module, which transforms under quantum Poincaré group $\mathbb{G}$ in the following standard way:

$$\hat{x}'_\mu = \hat{\Lambda}_{\mu\nu}\hat{x}^\nu + \hat{\xi}_\mu, \tag{10}$$

where NC translations $\hat{\xi}_\mu \in \mathbf{T}$ satisfy the relation (9). We shall show that to describe the quantum Poincaré transformations (10) using a star-product formula one should extend the star multiplication (6) to the functions of NC variables $\hat{x}_\mu$ (NC Minkowski space) and $\hat{\xi}_\mu$ (NC Poincaré translations) (see [14]).[4]

The plan of our paper is the following. In Sec. II we recall the $\theta_{\mu\nu}$-deformed Poincaré-Hopf algebra $\mathbb{H}$ extended by NC space-time coordinates $\hat{x}_\mu \in \hat{\mathbf{X}}$, which describe the Lorentz group extension of the relativistic phase space $(\hat{x}_\mu, p_\mu) \in \hat{\mathbf{X}}\rtimes\mathcal{T}^4$

$$\hat{\mathbf{X}}\rtimes\mathbb{H} = \hat{\mathbf{X}}\rtimes(\mathcal{T}^4\rtimes O(3,1)). \tag{11}$$

In Sec. III we describe the $\theta_{\mu\nu}$-deformed Poincaré quantum group $\mathbb{G}$ and calculate the $\theta_{\mu\nu}$-deformed Heisenberg double $\mathcal{H}^{(10+10)} = \mathbb{H}\rtimes\mathbb{G}$, with generalized NC Poincaré coordinates $\{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\} \in \mathbb{G}$ and generalized momenta $\{p_\mu, M_{\mu\nu}\} \in \mathbb{H}$. In Sec. IV we shall derive the covariance under the quantum Poincaré group transformations given by formula (10). In Sec. V we specify the data which define two Hopf algebroids, first providing the quantum Poincaré-covariant $\theta_{\mu\nu}$-deformed phase space $(\hat{x}_\mu, p_\mu) \in \hat{\mathbf{X}}\rtimes\mathcal{T}^4$ with positions (coordinates) described by algebra $\hat{\mathbf{X}}$ $(\hat{x}_\mu \in \hat{\mathbf{X}})$ supplemented with Lorentz transformations (Lorentz parameters $\Lambda_{\mu\nu}$) and the second Hopf algebroid describing quantum $\theta_{\mu\nu}$-deformed Poincaré symmetry transformations [see (10)], with the coordinate sector described by quantum $\theta_{\mu\nu}$-deformed Poincaré group $\mathbb{G}$. In particular in Sec. V D. by following earlier applications of Hopf algebroids to the description of κ-deformed quantum phase spaces [19,20] we shall consider the applicability of the coalgebra sector of $\theta_{\mu\nu}$-deformed Hopf algebroids to the phase space description of the multiparticle states.

II. $\theta_{\mu\nu}$-DEFORMED QUANTUM POINCARÉ ALGEBRA $\mathbb{H}$ AND NC SPACE-TIME AS $\mathbb{H}$-MODULE

A. Twist deformed quantum Poincaré algebra $\mathbb{H}$

The classical $D=4$ Poincaré-Hopf algebra looks as follows:

[2]Because the twist $\mathcal{F}$ is the function of classical Poincaré algebra generators $\hat{g} = (p_\mu, M_{\mu\nu})$, the action $\hat{g}\triangleright f(x)$ in formula (6) is described by the differential realization of classical Poincaré algebra on functions of standard Minkowski coordinates x_μ.

[3]The idea of phase space description of the dynamics on the classical group and coset group manifolds is due to Souriau [16] and Kostant [17].

[4]See [18] for the use of star product to represent the quantum group transformations.

$$\begin{aligned}[p_\mu, p_\nu] &= 0 \\ [M_{\mu\nu}, p_\rho] &= i(\eta_{\nu\rho} p_\mu - \eta_{\mu\rho} p_\nu) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}),\end{aligned} \tag{12}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and is supplemented by primitive costructure maps

$$\Delta_0(h) = h \otimes 1 + 1 \otimes h, \qquad S_0(h) = -h, \qquad \epsilon_0(h) = 0. \tag{13}$$

The twist $\mathcal{F}$ is an element of $\mathbb{H} \otimes \mathbb{H}$ ($\mathbb{H} = \mathcal{U}(\mathcal{P})$) which has an inverse, satisfies the cocycle condition $\mathcal{F}_{12}(\Delta_0 \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta_0)\mathcal{F}$ and the normalization condition $(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1$, where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$.

The twist $\mathcal{F}$ does not modify the algebraic part and the counit, but changes the coproducts $\Delta: \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ and the antipodes $S: \mathbb{H} \to \mathbb{H}$ according to formulas (4)–(5). The quantum θ-deformation is generated by the twist (3). From the formula (4) one gets the coproducts

$$\Delta_{\mathcal{F}}(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu \tag{14}$$

$$\begin{aligned}\Delta_{\mathcal{F}}(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ - \frac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu} p_\nu - \eta_{\alpha\nu} p_\mu) \otimes p_\beta \\ + p_\alpha \otimes (\eta_{\beta\mu} p_\nu - \eta_{\beta\nu} p_\mu)].\end{aligned} \tag{15}$$

From (3) and (5), it follows that in the considered case of $\theta_{\mu\nu}$-deformations $U = 1$ and the antipodes remain unchanged, i.e., $S_{\mathcal{F}}(h) = S_0(h) = -h$.

B. Algebra of generalized coordinates $\hat{\mathbb{X}}$ as twisted $\mathbb{H}$-module

For the twist-deformed case we can introduce the deformed coordinates algebra $\hat{\mathbb{X}} \ni \hat{X}_A = \{\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}\}$ with the multiplication given by the star product formula

$$\begin{aligned}\hat{X}_A \cdot \hat{X}_B \simeq X_A \star_{\mathcal{F}} X_B &= m[\mathcal{F}^{-1} \circ (X_A \otimes X_B)] \\ &= (\mathcal{F}^{-1}_{(1)} \triangleright X_A)(\mathcal{F}^{-1}_{(2)} \triangleright X_B),\end{aligned} \tag{16}$$

where

$$h \triangleright X_A = [h, X_A], \qquad h = \{p_\mu, M_{\mu\nu}\}, \qquad X_A = \{x_\mu, \Lambda_{\mu\nu}\}, \tag{17}$$

and in the undeformed case we obtain

$$[p_\mu, x_\nu] = i\eta_{\mu\nu}, \qquad [M_{\mu\nu}, x_\rho] = i(\eta_{\rho\nu} x_\mu - \eta_{\rho\mu} x_\nu) \tag{18}$$

$$[p_\mu, \Lambda_{\rho\sigma}] = 0, \qquad [M_{\mu\nu}, \Lambda_{\rho\sigma}] = \eta_{\rho\nu}\Lambda_{\mu\sigma} - \eta_{\rho\mu}\Lambda_{\nu\sigma}. \tag{19}$$

If we choose $X_A = f(X), X_B = g(X)$ the formula (16) can be also written as follows:

$$f(X) \star_{\mathcal{F}} g(X') = \widehat{f(X)} \triangleright g(X'), \tag{20}$$

where $\widehat{f(X)}$ denotes the noncommutative star representation of $f(\hat{X})$ defined by the formula [see also (7)]

$$f(\hat{X}) \simeq \widehat{f(X)} = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(f(X) \otimes 1)]. \tag{21}$$

For the twist (3) we get from (21) the following explicit formulas describing generalized coordinates $\hat{X}_A = \{\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}\}$ in terms of undeformed relativistic quantum phase space variables (x_μ, p_μ) and $\Lambda_{\mu\nu}$:

$$\hat{x}^\mu = x^\mu + \frac{1}{2}\theta^{\mu\alpha} p_\alpha, \qquad \hat{\Lambda}_{\rho\sigma} = \Lambda_{\rho\sigma}, \tag{22}$$

i.e., Lorentz group parameters, remain classical. Due to (18) and (22) one gets the expected algebraic relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \tag{23}$$

$$[\hat{x}_\mu, \hat{\Lambda}_{\rho\sigma}] = [\hat{\Lambda}_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = 0. \tag{24}$$

Using the relation

$$h \triangleright (\hat{X}_A \hat{X}_B) = (h_{(1)} \triangleright \hat{X}_A)(h_{(2)} \triangleright \hat{X}_B) \tag{25}$$

following [10] one can check easily that the commutators (23)–(24) are covariant under the action (25) of $\theta_{\mu\nu}$-deformed Poincaré-Hopf algebra.

C. $\theta_{\mu\nu}$-deformed quantum phase space $\mathcal{H}_\theta^{(10+10)} = (\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$

Using (22) one can check the following set of cross commutators:

$$[p_\mu, \hat{x}_\nu] = i\eta_{\mu\nu} \tag{26}$$

$$[p_\mu, \hat{\Lambda}_{\rho\sigma}] = 0 \tag{27}$$

$$[M_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = -i(\eta_{\rho\mu}\hat{\Lambda}_{\nu\sigma} - \eta_{\rho\nu}\hat{\Lambda}_{\mu\sigma}) \tag{28}$$

$$\begin{aligned}[M_{\mu\nu}, \hat{x}_\rho] = i\eta_{\rho\nu}\left(\hat{x}_\mu - \frac{1}{2}\theta_\mu{}^\alpha p_\alpha\right) - i\eta_{\rho\mu}\left(\hat{x}_\nu - \frac{1}{2}\theta_\nu{}^\alpha p_\alpha\right) \\ - \frac{i}{2}(\theta_{\rho\mu} p_\nu - \theta_{\rho\nu} p_\mu).\end{aligned} \tag{29}$$

Together with commutators (23)–(24) the set of relations (26)–(29) satisfies the Jacobi identities and defines the algebra of $\theta_{\mu\nu}$-deformed quantum phase space $\mathcal{H}_\theta^{(10+10)}$.

III. $\theta_{\mu\nu}$-DEFORMED QUANTUM POINCARÉ MATRIX GROUP $\mathbb{G}$ AND CORRESPONDING HEISENBERG DOUBLE $\mathbb{G}\rtimes\mathbb{H}$

A. *RTT* quantization method and $\theta_{\mu\nu}$-deformed quantum Poincaré group algebra $\mathbb{G}$

The universal $\mathcal{R}$-matrix $(a\wedge b = a\otimes b - b\otimes a)$

$$\mathcal{R} = \mathcal{F}^T\mathcal{F}^{-1} = \exp[-i\theta^{\mu\nu}p_\mu\otimes p_\nu] \qquad (a\otimes b)^T = b\otimes a \tag{30}$$

can be used for the description of the 10-generator deformed $D=4$ Poincaré group. Using the 5×5-matrix realization of the Poincaré generators,

$$(M_{\mu\nu})^A{}_B = \delta^A{}_\mu\eta_{\nu B} - \delta^A{}_\nu\eta_{\mu B} \qquad (p_\mu)^A{}_B = \delta^A{}_\mu\delta^4{}_B, \tag{31}$$

we can show that in (30) only the linear term is non-vanishing, i.e.,

$$\mathcal{R} = 1\otimes 1 - i\theta^{\mu\nu}p_\mu\otimes p_\nu. \tag{32}$$

To find the matrix quantum group, which provides the Hopf algebra dual to $\mathbb{H}$ in the matrix realization (31), we introduce the following 5×5-matrices:

$$\hat{\mathcal{T}}_{AB} = \begin{pmatrix} \hat{\Lambda}_{\mu\nu} & \hat{\xi}_\mu \\ 0 & 1 \end{pmatrix}, \tag{33}$$

where $\hat{\Lambda}_{\mu\nu}$ parametrizes the quantum Lorentz rotation and $\hat{\xi}_\mu$ denotes quantum translations. In the framework of the $\mathcal{FRT}$ procedure, the algebraic relations defining such a quantum group $\mathbb{G}$ are described by the following relation:

$$\mathcal{R}\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_1\mathcal{R}, \tag{34}$$

while the composition law for the coproduct remains classical $\Delta(\hat{\mathcal{T}}_{AB}) = \hat{\mathcal{T}}_{AC}\otimes\hat{\mathcal{T}}^C_B$ with $\hat{\mathcal{T}}_1 = \hat{\mathcal{T}}\otimes 1$, $\hat{\mathcal{T}}_2 = 1\otimes\hat{\mathcal{T}}$ and quantum $\mathcal{R}$-matrix (32) given in the representation (31).

In terms of the basis $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})$ of $\mathbb{G}$ the algebraic relations (34), describing the quantum group algebra, can be written as follows:

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = i\theta^{\rho\sigma}(\eta_{\mu\rho}\eta_{\nu\sigma} - \hat{\Lambda}_{\mu\rho}\hat{\Lambda}_{\nu\sigma}) \coloneqq i\Theta_{\mu\nu}(\hat{\Lambda}), \tag{35}$$

$$[\hat{\xi}_\mu, \hat{\Lambda}_{\rho\sigma}] = 0, \qquad [\hat{\Lambda}_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = 0, \tag{36}$$

while the coproduct takes the well known classical form

$$\Delta(\hat{\Lambda}_{\mu\nu}) = \hat{\Lambda}_{\mu\rho}\otimes\hat{\Lambda}^\rho{}_\nu \qquad \Delta(\hat{\xi}_\mu) = \hat{\Lambda}_{\mu\nu}\otimes\hat{\xi}^\nu + \hat{\xi}_\mu\otimes 1. \tag{37}$$

One can check that coproducts (37) are homomorphic to the algebra (35)–(36) defining the $\theta_{\mu\nu}$-deformed quantum Poincaré group.

B. Duality between quantum Hopf algebras $\mathbb{H}$ and $\mathbb{G}$ and Heisenberg double $\mathcal{H}=\mathbb{H}\rtimes\mathbb{G}$

Two Hopf algebras $\mathbb{H}$, $\mathbb{G}$ are said to be dual if there exists a nondegenerate bilinear form $\langle,\rangle: \mathbb{H}\times\mathbb{G}\to\mathbb{C}, (h,\hat{g})\to\langle h,\hat{g}\rangle$ such that the duality relations

$$\langle h, \hat{g}\hat{g}'\rangle = \langle\Delta(h), \hat{g}\otimes\hat{g}'\rangle \tag{38}$$

$$\langle hh', \hat{g}\rangle = \langle h\otimes h', \Delta(\hat{g})\rangle \tag{39}$$

are satisfied. In our considerations the following pairing relations:

$$\begin{aligned} \langle p_\mu, \hat{\xi}_\nu\rangle &= i\eta_{\mu\nu} \\ \langle M_{\mu\nu}, \hat{\Lambda}_{\alpha\beta}\rangle &= -i(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\nu\alpha}\eta_{\mu\beta}) \\ \langle 1, \hat{\Lambda}_{\mu\nu}\rangle &= \eta_{\mu\nu} \end{aligned} \tag{40}$$

are appropriate. The basic action of $\mathbb{H}$ on $\mathbb{G}$ promoting $\mathbb{G}$ to the $\mathbb{H}$-module is given by the following relation:

$$h\blacktriangleright\hat{g} = \hat{g}_{(1)}\langle h, \hat{g}_{(2)}\rangle. \tag{41}$$

After using (38) one gets the relation

$$\begin{aligned} h\blacktriangleright(\hat{g}\hat{g}') &= \hat{g}\hat{g}'_{(1)}\langle\Delta h, \hat{g}_{(2)}\otimes\hat{g}'_{(2)}\rangle \\ &= \hat{g}\hat{g}'_{(1)}\langle h_{(1)}, \hat{g}_{(2)}\rangle\langle h_{(1)}, \hat{g}'_{(2)}\rangle \\ &= (h_{(1)}\blacktriangleright\hat{g})(h_{(2)}\blacktriangleright\hat{g}') \end{aligned} \tag{42}$$

which establishes that algebra $\mathbb{G}$ is indeed the $\mathbb{H}$-module.

In the Heisenberg double framework we can obtain cross commutators between the algebra $\mathbb{H}$ and group $\mathbb{G}$ by the following relation:

$$\begin{aligned} &[h, \hat{g}] = \hat{g}_{(2)}\langle h_{(1)}, \hat{g}_{(1)}\rangle h_{(2)} - \hat{g}h \\ &h = \{p_\mu, M_{\mu\nu}\}; \qquad \hat{g} = \{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\}. \end{aligned} \tag{43}$$

In such a way we obtain the quantum phase space algebra.[5] Using pairing (40), coproducts (14), (15), and formula (37) we get

$$[p_\mu, \hat{\xi}_\nu] = i\eta_{\mu\nu} \tag{44}$$

$$[p_\mu, \hat{\Lambda}_{\rho\sigma}] = 0 \tag{45}$$

$$[M_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = -i(\eta_{\rho\mu}\hat{\Lambda}_{\nu\sigma} - \eta_{\rho\nu}\hat{\Lambda}_{\mu\sigma}) \tag{46}$$

[5]For the case of κ-deformed quantum phase space see [15].

$$[M_{\mu\nu}, \hat{\xi}_\rho] = i\eta_{\rho\nu}\left(\hat{\xi}_\mu - \frac{1}{2}\theta_\mu{}^\alpha p_\alpha\right) - i\eta_{\rho\mu}\left(\hat{\xi}_\nu - \frac{1}{2}\theta_\nu{}^\alpha p_\alpha\right) - \frac{i}{2}(\theta_{\rho\mu}p_\nu - \theta_{\rho\nu}p_\mu). \quad (47)$$

The Hopf algebroid $\tilde{\mathcal{H}}^{(10+10)} = (\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$ introduces an alternative model of $\theta_{\mu\nu}$-deformed quantum phase space. It describes the quantum phase space characterizing the dynamical system with the coordinates $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})$ which are specified by the NC quantum Poincaré group manifold $\mathbb{G}$.

IV. THE COVARIANCE OF $\hat{\mathbb{X}}$ UNDER QUANTUM POINCARÉ GROUP $\mathbb{G}$ AND THE GENERALIZED STAR PRODUCT

A. The covariance of $\hat{\mathbb{X}}$ under quantum Poincaré group $\mathbb{G}$

We recall that $\hat{\mathbb{X}}$ is the algebra of generalized coordinates $\hat{X}_A = \{\hat{x}_\mu, \hat{\Lambda}_{\mu\nu}\}$, and $\mathbb{G}$ is the algebra of Poincaré symmetry parameters $\hat{g} = \{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\}$. One performs the quantum Poincaré transformations of $\hat{X}_A$ in the following way:

$$\hat{x}_\mu \to \hat{x}'_\mu = \hat{\Lambda}_{\mu\nu}\hat{x}^\nu + \hat{\xi}_\mu \quad (48)$$

$$\hat{\Lambda}_{\mu\nu} \to \hat{\Lambda}'_{\mu\nu} = \hat{\Lambda}_\mu{}^\alpha\hat{\Lambda}_\nu{}^\beta\hat{\Lambda}_{\alpha\beta} = \hat{\Lambda}_{\mu\nu}. \quad (49)$$

The commutators of algebra $\hat{\mathbb{X}}$ [see (23)–(24)] are invariant under a such transformation provided that the quantum Poincaré symmetry parameters $\mathbb{G} \ni \hat{g} = \{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\}$ satisfy the relations defining the algebra $\mathbb{G}$ [see (35)–(36)] and $[\hat{X}_A, \hat{g}] = 0$. In particular, we have

$$[\hat{x}_\mu, \hat{x}_v] = i\theta_{\mu\nu} \overset{\hat{x}_\mu \to \hat{x}'_\mu = \hat{\Lambda}_{\mu\nu}\hat{x}^\nu + \hat{\xi}_\mu}{\longrightarrow} [\hat{x}'_\mu, \hat{x}'_\nu] = i\theta_{\mu\nu}. \quad (50)$$

We see that the generalized coordinates algebra $\hat{\mathbb{X}}$ in different $\mathbb{G}$-frames specified by (48)–(49) transform covariantly under the transformations of quantum Poincaré group $\mathbb{G}$.

In order to describe effectively the quantum Poincaré transformations (48) of NC functions $f(\hat{x}_\mu)$ it is convenient to introduce the generalized star products (see also [18]) representing the algebra of functions $F(\hat{x}_\mu, \hat{\xi}_\mu)$ depending as well on NC translations $\hat{\xi}_\mu$.

B. Star product on the product $\mathbf{X} \otimes \mathbf{T}$ with noncommutative translations of coordinates

Let us consider firstly the star product $\star'_{\mathcal{F}}$ describing the NC product of algebra of functions $F(\hat{\xi}_\mu)$ which depend on NC translations $\hat{\xi}_\mu \in \mathbb{G}$ satisfying the relation (9):

$$F(\hat{\xi}_\mu) \cdot G(\hat{\xi}_\nu) \simeq F(\xi_\mu)\star'_{\mathcal{F}}G(\xi_\nu) = m\circ \exp\left[\frac{i}{2}\Theta^{\alpha\beta}(\hat{\Lambda})\frac{\partial}{\partial\xi^\alpha}\otimes\frac{\partial}{\partial\xi^\beta}\right]F(\mu)\otimes G(\nu), \quad (51)$$

where in formula (51) one treats the Lorentz group parameters $\Lambda \equiv \Lambda_{\mu\nu}$ as the numerical ones. Discussing the quantum Poincaré transformations (10) in field theory we should deal with NC algebra of functions on $\mathbf{X}$ ($x_\mu \in \mathbf{X}$) as well as on $\mathbf{T}$ ($\xi_\mu \in \mathbf{T}$), and subsequently use the composite star product $\tilde{\star}_{\mathcal{F}} = \star_{\mathcal{F}} \cdot \star'_{\mathcal{F}}$ (see also [21]):

$$F(\hat{x}_\mu, \hat{\xi}_\mu) \cdot G(\hat{x}_\nu, \hat{\xi}_\nu) \simeq F(x_\mu, \xi_\mu)\tilde{\star}_{\mathcal{F}}G(x_\nu, \xi_\nu) = m\circ \exp\left[\frac{i}{2}\left(\theta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\otimes\frac{\partial}{\partial x^\beta} + \Theta^{\alpha\beta}(\hat{\Lambda})\frac{\partial}{\partial\xi^\alpha}\otimes\frac{\partial}{\partial\xi^\beta}\right)\right] \times (F(x_\mu, \xi_\mu)\otimes G(x_\nu, \xi_\nu)). \quad (52)$$

In formula (52) both relations (1), (9) are taken into account and we can represent the NC quantum Poincaré transformations given by $\mathbb{G}$ by using such a star product language. In particular one can show that using the relation $\Lambda_{\mu\alpha}\Lambda_{\nu\beta}\theta^{\alpha\beta} + \Theta_{\mu\nu}(\Lambda) = \theta_{\mu\nu}$ [see (9)] we get

$$\begin{aligned} F(\Lambda_{\mu\alpha}x^\alpha + \xi_\mu)\tilde{\star}_{\mathcal{F}}G(\Lambda_{\nu\beta}x^\beta + \xi_\nu) &= F(x'_\mu)\star_{\mathcal{F}}G(x'_\nu)|_{x_\mu\to x'_\mu=\Lambda_{\mu\alpha}x^\alpha+\xi_\mu; x_\nu\to x'_\nu=\Lambda_{\nu\alpha}x^\alpha+\xi_\nu} \\ &= m\circ\exp\left[\frac{i}{2}\theta^{\alpha\beta}\frac{\partial}{\partial x'^\alpha}\otimes\frac{\partial}{\partial x'^\beta}\right](F(x'_\mu)\otimes G(x'_\nu)) \\ &\equiv F(x'_\mu)\star_{\mathcal{F}}G(x'_\nu). \end{aligned} \quad (53)$$

In such a way we expressed in star product language the NC Poincaré group transformations by classical Poincaré group transformations. We see that the NC structure of quantum Poincaré group translations is encoded in the replacement of the star product $\star_{\mathcal{F}} \to \tilde{\star}_{\mathcal{F}}$. It can be shown by explicit calculation that three star products $\star_{\mathcal{F}}, \star'_{\mathcal{F}}$, and $\tilde{\star}_{\mathcal{F}}$ are associative.

V. TWO $\theta_{\mu\nu}$-DEFORMED HOPF ALGEBROIDS

A. Briefly on Hopf algebroids

The Hopf algebroids, introduced in [22] (see also [23]), are described by bialgebroids with supplemented antipodes. It has been argued (see e.g., [24–28]) that the Hopf algebroids are well adjusted to the description of physically important quantum (canonical and noncanonical) phase space.

The bialgebroid $\mathcal{B}$ is specified by the set of the data $(H, A; s, t; m, \tilde{\Delta}, \epsilon)$ where H is the total algebra with product m and its subalgebra $A \subset H$ is called the base algebra. The source map $s(a): A \to H$ is a homomorphism and the target map $t(a){:}A\rightarrow H$ an antihomomorphism, with their images commuting

$$[s(a),t(b)]=0\qquad a,b\in A\qquad s(a),t(b)\in H. \tag{54}$$

The canonical choice of the source map is $s(a)=a$. One can introduce natural (A,A)-bimodule structure on H using source and target maps in the basic (A,A)-bimodule formula, namely, $ahb=ht(a)s(b)$. The coproducts $\tilde{\Delta}$ are described by the maps $H\rightarrow H\otimes_A H$ from H into (A,A) bimodules $H\otimes_A H$, satisfying the coassociativity condition

$$(\tilde{\Delta}\otimes_A id_H)\tilde{\Delta}=(id_H\otimes_A\tilde{\Delta})\tilde{\Delta}. \tag{55}$$

Because $H\otimes_A H$ as the codomain of coproducts $\tilde{\Delta}$ does not inherit the algebra structure from $H\otimes H$, in order to have well defined multiplication one introduces the submodule $H\times_A H\subset H\otimes H$ defined by the Takeuchi coproduct [29].

The algebra H with the product m and the coalgebra with Takeuchi coproduct $\tilde{\Delta}$ are compatible, i.e.,

$$\tilde{\Delta}(hh')=\tilde{\Delta}(h)\tilde{\Delta}(h'). \tag{56}$$

The coproduct $\tilde{\Delta}$ in $H\times_A H$ can be expressed in terms of standard tensor product $H\otimes H$ by the equivalence classes satisfying the condition

$$\tilde{\Delta}(h)\mathcal{I}_L(a)=0,\qquad \mathcal{I}_L(a)=t(a)\otimes 1-1\otimes s(a), \tag{57}$$

where $\mathcal{I}_L$ defines the left ideal in $H\otimes H$. The equivalence classes defined by (57) are parametrized by so-called coproduct gauge (see e.g., [20]).

The counit map $\epsilon{:}H\rightarrow A$ is satisfying $\epsilon(1_H)=1_A$, and

$$(\epsilon\otimes_A id_H)\tilde{\Delta}=(id_H\otimes_A\epsilon)\tilde{\Delta}=id_H. \tag{58}$$

We get Hopf algebroids if we are able to introduce an antipode (bijective map) $S{:}H\rightarrow H$, which is an algebra antihomomorphism and satisfies the following properties:

$$S(t)=s \tag{59}$$

$$m[(1\otimes S)\circ\gamma\tilde{\Delta}]=s\epsilon=\epsilon \tag{60}$$

$$m[(S\otimes 1)\circ\tilde{\Delta}]=t\epsilon S, \tag{61}$$

where in the general case we need an additional linear map $\gamma{:}H\otimes_A H\rightarrow H\otimes H$, the so-called anchor map.

B. First choice: The coordinates $\hat{\mathbb{X}}$ as the base algebra (following [30])

Let us choose the bialgebroid group coproducts for base algebra generators $\hat{X}_A=\{\hat{x}_\mu,\hat{\Lambda}_{\mu\nu}\}$:

$$\tilde{\Delta}(\hat{X}_A)=\hat{X}_A\otimes 1 \tag{62}$$

satisfies the group and generalized quantum phase space algebra (26)–(29).

The source $s(\hat{X}_A)$ and the target $t(\hat{X}_A)$ maps are the following:

$$s(\hat{X}_A)=m[\mathcal{F}^{-1}(\triangleright\otimes 1)(s_0(X_A)\otimes 1)]=\hat{X}_A \tag{63}$$

$$t(\hat{x}_\mu)=m[(\mathcal{F}^{-1})^\tau(\triangleright\otimes 1)(t_0(x_\mu)\otimes 1)]$$
$$=x_\mu-\frac{1}{2}\theta_\mu^{\ \alpha}p_\alpha=\hat{x}_\mu-\theta_\mu^{\ \alpha}p_\alpha \tag{64}$$

$$t(\hat{\Lambda}_{\mu\nu})=m[(\mathcal{F}^{-1})^\tau(\triangleright\otimes 1)(t_0(\Lambda_{\mu\nu})\otimes 1)]=\Lambda_{\mu\nu}=\hat{\Lambda}_{\mu\nu}, \tag{65}$$

where $s_0(X_A)=X_A$, $t_0(X_A)=X_A$. The maps (63)–(65) satisfy the following relations:

$$[s(\hat{X}_A),t(\hat{X}_B)]=0 \tag{66}$$

$$[s(\hat{x}_\mu),s(\hat{x}_\nu)]=i\theta_{\mu\nu},\qquad [t(\hat{x}_\mu),t(\hat{x}_\nu)]=-i\theta_{\mu\nu} \tag{67}$$

$$[s(\cdot),s(\cdot)]=[t(\cdot),t(\cdot)]=0\quad\text{(for the other choices of }\hat{X}_A). \tag{68}$$

The counit has the canonical form

$$\epsilon(\hat{X}_A)=m[\mathcal{F}^{-1}(\triangleright\otimes 1)(\epsilon_0(X_A)\otimes 1)]=\hat{X}_A. \tag{69}$$

Using (59) one gets explicit formulas for the antipodes

$$S(\hat{X}_A)=t(\hat{X}_A). \tag{70}$$

In our case we have $S^2=1$. One can check that for ideal $\mathcal{I}=t\otimes 1-1\otimes s$, it is true that

$$m[(1\otimes S)]\mathcal{I}=m[(S\otimes 1)]\mathcal{I}=0, \tag{71}$$

and it follows that we do not need the anchor map [see (60)].

In order to determine the ideal (57) let us start form the nondeformed ideal (for $\theta_{\mu\nu}=0$)

$$\mathcal{I}_0(X_A)=X_A\otimes 1-1\otimes X_A. \tag{72}$$

We can obtain $\hat{X}_A$ from X_A by using twisting formula (21). One gets for our twisted algebroid the following twist-deformed ideal [see (63)–(65)]:

$$\mathcal{I}_{\mathcal{L}}(\hat{X}_A) = \mathcal{F}\mathcal{I}_0(X_A)\mathcal{F}^{-1} = t(\hat{X}_A) \otimes 1 - 1 \otimes s(\hat{X}_A). \quad (73)$$

In particular one can check the following relations:

$$[\tilde{\Delta}(\hat{X}_A), \mathcal{I}_{\mathcal{L}}(\hat{X}_A)] = [\mathcal{I}_{\mathcal{L}}(\hat{X}_A), \mathcal{I}_{\mathcal{L}}(\hat{X}_B)] = 0. \quad (74)$$

If we change the bialgebroid coproduct (62) by introducing the following coproduct gauge transformation (see also [19,24,26]):

$$\tilde{\Delta}(\hat{X}_A) \to \tilde{\Delta}_\lambda(\hat{X}_A) = \tilde{\Delta}(\hat{X}_A) + \lambda\mathcal{I}_{\mathcal{L}}(\hat{X}_A), \quad (75)$$

it follows from (74) that the new bialgebroid coproduct (75) describes the homomorphism of the commutation relations (23)–(24) extended by the phase space commutators (26)–(29).

The formula (75) after substituting the formulas (63)–(65) for source as well as target map and using the exponential parametrization,

$$\hat{\Lambda}_{\mu\nu} \equiv (e^{\hat{\omega}})_{\mu\nu} = \eta_{\mu\nu} + \hat{\omega}_{\mu\nu} + \frac{1}{2}\hat{\omega}_\mu{}^\rho\hat{\omega}_{\rho\nu} + \mathcal{O}(\hat{\omega}^3), \quad (76)$$

takes for example for $\lambda = -\frac{1}{2}$ the following explicit form:

$$\tilde{\Delta}_{-\frac{1}{2}}(\hat{x}_\mu) = \frac{1}{2}(\hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu) - \frac{1}{2}\theta_\mu{}^\nu p_\nu \otimes 1, \quad (77)$$

$$\tilde{\Delta}_{-\frac{1}{2}}(\hat{\Lambda}_{\mu\nu}) = \frac{1}{2}(\hat{\Lambda}_{\mu\nu} \otimes 1 + 1 \otimes \hat{\Lambda}_{\mu\nu}) \equiv \eta_{\mu\nu} 1 \otimes 1 + \frac{1}{2}(\hat{\omega}_{\mu\nu} \otimes 1 + 1 \otimes \hat{\omega}_{\mu\nu}) + \mathcal{O}(\hat{\omega}^2), \quad (78)$$

where (77) describes the $\theta_{\mu\nu}$-deformation of the symmetric formula $\tilde{\Delta}_{-\frac{1}{2}}(x_\mu) = \frac{1}{2}(x_\mu \otimes 1 + 1 \otimes x_\mu)$.[6]

C. Second choice: Quantum group $\mathbb{G}$ as the base algebra (following [22])

The half-primitive bialgebroid coproducts for $\hat{g} = \{\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}\}$

$$\tilde{\Delta}(\hat{g}) = \hat{g} \otimes 1 \quad (79)$$

together with the coproducts (14)–(15) satisfy the Heisenberg double commutators [see (35)–(36) and (44)–(47)].

The source $s(\hat{g})$ and the target $t(\hat{g})$ maps should be consistent with base algebra in the following sense:

$$[s(\hat{g}), t(\hat{g}')] = 0 \quad (80)$$

[6]For a pair of free nonrelativistic two-particles with the same masses, the coproduct $\tilde{\Delta}_{-\frac{1}{2}}(x_\mu)$ describes the global coordinates describing the center of mass (see [19], Sec. 4).

$$[s(\hat{\xi}_\mu), s(\hat{\xi}_\nu)] = i\Theta_{\mu\nu}(s(\hat{\Lambda})), \qquad [t(\hat{\xi}_\mu), t(\hat{\xi}_\nu)] = -i\Theta_{\mu\nu}(t(\hat{\Lambda})) \quad (81)$$

$$[s(\cdot), s(\cdot)] = [t(\cdot), t(\cdot)] = 0 \quad \text{(for the other choices of } \hat{g}), \quad (82)$$

where due to (35) the relations (81) describe quadratic algebras. Subsequently, it can be easily shown that analogously to (63)–(65) one gets

$$s(\hat{g}) = \hat{g} \quad (83)$$

$$t(\hat{\Lambda}_{\mu\nu}) = \hat{\Lambda}_{\mu\nu}, \qquad t(\hat{\xi}_\mu) = \hat{\xi}_\mu - \Theta_\mu{}^\alpha(\hat{\Lambda})p_\alpha. \quad (84)$$

The counit is

$$\epsilon(\hat{g}) = \hat{g}, \quad (85)$$

and the antipodes which are given by

$$S(\hat{g}) = t(\hat{g}) \quad (86)$$

satisfy the required relations (59)–(61). Similarly as in Sec. V B, $S^2 = 1$ and we do not need the anchor map.

If we consider the ideal

$$\mathcal{I}_{\mathcal{L}}(\hat{g}) = t(\hat{g}) \otimes 1 - 1 \otimes s(\hat{g}) \quad (87)$$

and use the formulas (83)–(84) one can introduce as well the counterpart of the coproduct gauge transformations (75) and (77)–(78).

VI. OUTLOOK

In this paper we considered the most popular in the literature Moyal quantum deformation of space-time coordinates and the corresponding quantum-deformed noncanonical phase-spaces, described by $\theta_{\mu\nu}$-deformation of relativistic Heisenberg algebra. Our aim was to present the pair of $\theta_{\mu\nu}$-deformed phase spaces in the language of Hopf algebroid, with the extension of translational sectors $(\hat{x}_\mu, p_\mu)$ or $(\hat{\xi}_\mu, p_\mu)$ by the rotational Lorentz phase space coordinates $(\hat{\Lambda}_{\mu\nu}, M_{\mu\nu})$, describing in relativistic particle models the spin degrees of freedom.

There were introduced two different relativistic quantum phase spaces with Hopf-algebroid structure described in two ways:

[–] by twist deformation of classical canonical Heisenberg bialgebroid, describing $\theta_{\mu\nu}$-deformed relativistic quantum phase space with NC Minkowski space-time coordinates and dual commuting fourmomenta (see e.g., [23–26,30]),

[–] by considering dual pairs of quantum-deformed Poincaré-Hopf algebras (see [20]) which define the Poincaré-Heisenberg double as a semidirect (smash) product of the quantum $\theta_{\mu\nu}$-deformed Poincaré group describing generalized coordinate sector and dual quantum Poincaré algebra which provides the generalized momenta sector (see e.g., [20,27]).

These methods were applied in order to obtain two versions of $\theta_{\mu\nu}$-deformed quantum phase spaces. We considered the $(10+10)$-dimensional generalized phase spaces, with generalized coordinates described, respectively, by $(\hat{x}_\mu, \hat{\Lambda}_{\mu\nu})$ and $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})$ and different noncommutativity for $\hat{x}_\mu$ and $\hat{\xi}_\mu$: the first one characterized by constant $\theta_{\mu\nu}$ [see (1)], and the second one with Λ-dependent $\Theta_{\mu\nu}(\hat{\Lambda})$, with commutation relations quadratic in $\hat{\Lambda}$ [see (35)].

The Hopf algebroids introduce a new class of quantum spaces H endowed with bialgebroid structure, suitable for the description of quantum manifolds with symplectic structure. The bialgebroidal coproducts $\tilde{\Delta}$ can be introduced in the framework of standard tensor products $H \otimes H$ as algebras defined by modulo so-called coproduct gauges [19,20]. The coproduct gauge freedom can parametrize various classes of dynamical particle models, with different values of physical parameters, but with the same symplectic algebraic structure in H, in $H \otimes H$, and higher multiparticle sectors. A simple example has been provided in [19] where it was shown that the coproduct gauges in the model describing pairs of free NR particles with masses m_1, m_2 depend on the mass ratio $\frac{m_1}{m_2}$, which is a physical parameter. It should be pointed out however that at present the role of coproduct gauges, e.g., in phase space description of physically important interacting relativistic particles still is not well understood.

The second question which could be studied is the description and classification of infinitesimal quantum deformations of Hopf algebroids. In particular it would be interesting to introduce for Hopf algebroids the notion analogous to classical r-matrices for Hopf algebras, as well as the bialgebroidal counterpart of Yang-Baxter and pentagon equations. The answer to the last problem is linked with the previous problem, i.e., the understanding of the physical content of the notion of coproduct gauges.

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[1] M. P. Bronstein, Zh. Èksp. Teor. Fiz. **6**, 195 (1936) [Quantum theory of week gravitational fields, Gen. Relativ. Gravit. **44**, 267 (2012)].
[2] H. S. Snyder, Quantized space-time, Phys. Rev. D **71**, 38 (1947).
[3] S. Majid, Hopf algebras for physics at the Planck scale, Classical Quantum Gravity **5**, 1587 (1988).
[4] J. Lukierski, A. Nowicki, H. Ruegg, and V. Tolstoy, q-deformation of Poincaré algebra, Phys. Lett. B **264**, 331 (1991).
[5] S. Majid, *Foundations of Quantum Groups* (Cambridge University Press, Cambridge, England, 1995).
[6] A. Connes, *Non-Commutative Geometry* (Academic Press, New York, 1944).
[7] S. Doplicher, K. Fredenhagen, and J. E. Roberts, The quantum structure of space time at the Planck scale and quantum fields, Commun. Math. Phys. **172**, 187 (1995).
[8] E. J. Beggs and S. Majid, in *Quantum Riemannian Geometry*, Comprehensive Studies in Mathematics, Vol. 355 (Springer, 2020).
[9] R. Oeckl, Untwisting noncommutative R^d and the equivalence of quantum field theories, Nucl. Phys. **B581**, 559 (2000).
[10] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT, Phys. Lett. B **604**, 98 (2004).
[11] V. G. Drinfeld, *Proceedings of XXth Math. Congress, Berkeley, 1985* (Berkeley Press, Berkeley, 1986, Vol. 1, p. 798.
[12] C. Blohmann, Covariant realization of quantum spaces as star products by Drinfeld twists, J. Math. Phys. (N.Y.) **44**, 4736 (2003).
[13] P. Aschieri, A. Borowiec, and A. Pachol, Observables and dispersion relations in κ-Minkowski spacetime, J. High Energy Phys. 10 (2017) 152.
[14] C. Gonera, P. Kosinski, P. Maslanka, and S. Giller, Spacetime symmetry of noncommutative field theory, Phys. Lett. B **622**, 192 (2005); Global symmetries of noncommutative space-time, Phys. Rev. D **72**, 067702 (2005).
[15] J. Lukierski and A. Nowicki, Heisenberg double description of κ-Poincare algebra and κ-deformed phase space, *Proc. XXI Int. Coll. Group. Theor. Methods in Physics*, edited by V. K. Dobrev and H. D. Doebner (Heron Press, Sofia, 1997), p. 186.
[16] J. M. Souriau, *Structure des Systemes Dynamiques* (Editions Dunod, Paris, 1970).
[17] B. Kostant, Quantization and representation theory, Springer Lect. Notes Math. **170**, 87 (1970).
[18] P. Kosinski, J. Lukierski, and P. Maslanka, Local field theory on kappa Minkowski space, star products and noncommutative translations, Czech. J. Phys. **50**, 1283 (2000).
[19] J. Lukierski, Z. Skoda, and M. Woronowicz, κ-deformed covariant quantum phase spaces as Hopf algebroids, Phys. Lett. B **750**, 401 (2015).

[20] J. Lukierski, Z. Skoda, and M. Woronowicz, On Hopf algebroid structure of κ-deformed Heisenberg algebra, Phys. At. Nucl. **80**, 576 (2017).
[21] C. Gonera, P. Kosinski, P. Maslanka, and S. Giller, Global symmetries of noncommutative field theory, J. Phys. Conf. Ser. **53**, 865 (2006).
[22] J-H. Lu, Hopf algebroids and quantum grupoids, Int. J. Math. **7**, 47 (1996).
[23] T. Brzezinski and G. Militaru, Bialgebroids, $\times_A$-bialgebras and duality, J. Algebra **251**, 279 (2002).
[24] T. Jurić, D. Kovačević, and S. Meljanac, κ-deformed phase space, Hopf algebroid and twisting, SIGMA **10**, 106 (2014).
[25] S. Meljanac, Z. Skoda, and M. Stojić, Lie algebra type noncommutative phase spaces as Hopf algebroid, Lett. Math. Phys. **107**, 475 (2017).
[26] A. Borowiec and A. Pachoł, Twisted bialgebroids versus bialgebroids from a Drinfeld twist, J. Phys. A **50**, 055205 (2017).
[27] J. Lukierski, D. Meljanac, S. Meljanac, D. Pikutic, and M. Woronowicz, Lie-deformed quantum Minkowski spaces from twists: Hopf-algebraic versus Hopf-algebroid approach, Phys. Lett. B **777**, 1 (2018).
[28] J. Lukierski, S. Meljanac, and M. Woronowicz, Quantum twist-deformed $D = 4$ phase spaces with spin sector and Hopf algebroid structures, Phys. Lett. B **789**, 82 (2019).
[29] M. Takeuchi, Groups of algebras over $A \otimes \bar{A}$, J. Math. Soc. Jpn. **29**, 459 (1977).
[30] P. Xu, Quantum grupoids, Commun. Math. Phys. **216**, 539 (2001).