$SO(2)$ gauged Skyrmions in $4+1$ dimensions

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We study the simplest $SO(2)$ gauged $O(5)$ Skyrme model in $4+1$ (flat) dimensions. In the gauge decoupled limit, the model supports topologically stable solitons (Skyrmions) and after gauging, the static energy of the solutions is bounded from below by a "baryon number." The studied model features both Maxwell and Maxwell–Chern-Simons dynamics. The considered configurations are subject to biazimuthal symmetry in the $\mathbb{R}^4$ subspace resulting in a two dimensional subsystem, as well as subject to an enhanced symmetry relating the two planes in the $\mathbb{R}^4$ subspace, which results in a one dimensional subsystem. Numerical solutions are constructed in both cases. In the purely magnetic case, fully biazimuthal solutions were given, while electrically charged and spinning solutions were constructed only in the radial (enhanced symmetric) case, both in the presence of a Chern-Simons term, and in its absence. We find that, in contrast with the analogous models in $2+1$ dimensions, the presence of the Chern-Simons term in the model under study here results only in quantitative effects.

I. INTRODUCTION

The gauging of the Skyrmion, namely of the soliton of the $O(4)$ sigma model on $\mathbb{R}^3$, is recognized to be of physical relevance in the study of the electrically charged nucleon. This was considered by Callan and Witten [1] in the context of baryon number violation. Gauging a Skyrme (sigma model) scalar results in the deformation of the lower bound on the energy, which prior to gauging is the topological charge, namely the winding number. The most prominent such example is the $U(1)$, or $SO(2)$, gauged Skyrme system in $3+1$ dimensions, the earliest work being [1], where the emphasis was on baryon number violation, and subsequently [2], where the dependence of the mass of the proton on the electric charge was studied, and [3], where the spin of the proton was considered. The gauging prescription used in [2,3] coincides with that used in [1].

These studies, [1–3], apply to the $SO(2)$ gauged $O(4)$ Skyrme system on $\mathbb{R}^3$. However, these models possess

generalizations for other dimensions of the background geometry, solitons of the $SO(2)$ gauged $O(3)$ Skyrme system on $\mathbb{R}^2$ being constructed by Schoers [4]. The simpler problem of gauging the planar Skyrmions [4] is much more transparent, and has led to a proposal for the $SO(D)$ gauging of $O(D+1)$ Skyrme system on $\mathbb{R}^D$, in Ref. [5].

A lower bound on the energy of a gauged Skyrmion in D dimensions persists also for $SO(N)$ gauge groups with $2 \le N \le D$, as e.g., in [1–3]. The prescription for constructing such lower bounds is systematically explained in Appendix A of [6], where, in the specific case of the $O(5)$ Skyrme model on $\mathbb{R}^4$ of interest here, only one pair of the components of the 5-component Skyrme scalar are gauged. This is unsatisfactory in the context of the problem at hand, where it is desirable to gauge two pairs of the Skyrme scalar with $SO(2)$, with the aim of imposing biazimuthal symmetry in $\mathbb{R}^4$. Such a gauging prescription together with the corresponding topological charge density is constructed in Appendix A of the present paper. There we start with the density pertaining to the system gauged with the full $SO(4)$ group, which acts on four of the components of the 5-component Skyrme scalar, and then perform a group contraction to $SO(2)$. Here, and in Refs. [5,6] the integral of this lower bound is loosely described as a "topological charge" in analogy with its Higgs analogue.

Systematic and quantitative studies of $SO(2)$ gauged Skyrme systems in $2 + 1$ dimensions were recently carried out in [7–9]. There, the emphasis was on the effect of the Chern-Simons dynamics, which is possible to define in all odd spacetime dimensions.[1] It was found that the presence of the Chern-Simons term resulted in a *nonstandard* relation between mass and charge/spin, seen in [7], and in the dissipation of the baryon number, seen in [8,9]. Both these effects are striking new results, and it is not unreasonable to expect that they are not exclusive to this low dimension only.

To carry out such a study in the physically most interesting case of the Abelian gauged Skyrmion in $3 + 1$ dimensions is technically a very substantial problem since the definition of the relevant Chern-Simons density, proposed in Ref. [6], involves a supplementary $O(6)$ Skyrme scalar in addition to the $O(4)$ Skyrme scalar that describes the nucleon. It is therefore reasonable to defer that investigation and proceed instead with the study of an Abelian gauged $O(5)$ Skyrmion in $4 + 1$ (odd) dimensions, where a (usual) Chern-Simons density is defined. This is the remit of the present work, which is a preliminary attempt at inquiring whether the phenomena exposed in Refs. [7–9] persist.

In $3 + 1$ dimensions, irrespective of the absence of the (usual) Chern-Simons density there is the additional technical complication that the largest symmetry that can be applied to the Abelian field is axial symmetry, resulting in a two-dimensional residual subsystem [2,3,8]. This is of course the case in all higher even dimensional spacetimes, which results in the necessity of tackling multidimensional partial differential equations (PDEs) of the reduced systems. An advantage of $4 + 1$ dimensions, i.e., static $\mathbb{R}^4$, is that it is possible to impose an enhanced symmetry on the system that renders the residual system one-dimensional, depending only on the radial variable. The enhanced symmetry in question is that which is imposed on the bipolar (biazimuthal) symmetry in $\mathbb{R}^4$, and leads to a simplified form of the equations which are solved in a numerical approach.

In the present work we study solutions in both the purely magnetic sector where the electric component of the Abelian field vanishes, $A_0 = 0$, and also when $A_0 \neq 0$, where electric charge and angular momentum are present. In both sectors, we consider the symmetry enhanced systems resulting in one dimensional ODEs. In the purely magnetic sector, the fully biazimuthal solutions to two dimensional PDEs are also constructed. In the $A_0 \equiv V \neq 0$ sector, where only radial solutions were considered, the electric charge Q_e was given by the asymptotic solutions for $V = V_0 + \frac{Q_e}{4\pi^2 r^2} + \cdots$, in agreement with the solution of

the Laplace equation on $\mathbb{R}^4$. In this sense, our electrically (and magnetically) charged solutions are analogues of the Julia-Zee dyons [10] in $3 + 1$ dimensions. This definition of electric charge contrasts with that given by Paul and Khare [11] for the Abelian gauged Maxwell–Chern-Simons system interacting with a scalar in $2 + 1$ dimensions. While in the latter [11] the existence of electric charge and spin depends on a nonvanishing Chern-Simons (CS) density, here these are present independently of CS dynamics as in the case of JZ dyons [10]. This is because Q_e in [11] is proportional to the *first* Pontryagin charge (and the spin to the square of the Pontryagin charge) on $\mathbb{R}^2$, while here the corresponding quantity is the *second* Pontryagin charge on $\mathbb{R}^4$, for which not to vanish the gauge group must be $SU(2)$ and with the Abelian gauge field at hand it vanishes. We plan to return to this question elsewhere.

The paper is structured as follows. In Sec. II, we present the model and subject the system to the symmetries described above, and in Sec. III, we present the results of the numerical analysis. In Sec. IV, we summarize our results and point out to future developments. In addition, we have supplied two Appendices. Appendix A defines the "topological charge" supplying the lower bound of the energy. Such a charge density is provided in [6], which is not adequate for the present application since only two of the five components of the Skyrme scalar are gauged in that case. Here, we need to gauge two pairs of Skyrme scalars to enable the imposition of the enhanced symmetry rendering the biazimuthal system a radial one. (Appendix A in fact stands on its own as a supplement to the corresponding result in [6].) In Appendix B, we have established the Belavin inequalities that give the Bogomol'nyi lower bounds, a task which is appreciably more involved than the corresponding one for the ungauged $O(5)$ model, studied in [12].

A. Conventions

Throughout the paper, mid alphabet Greek indices, $\mu, \nu, \ldots$, label spacetime coordinates running from 1 to 5 (with $x^5 = t$). When referring to spacelike coordinates only we will use mid alphabet Latin letters, $i, j, \ldots$. Early Latin letters, $a, b, \ldots$ label the internal indices of the scalar field multiplet, running from 1 to 5, when primed $a', b', \ldots$ they just refer to the first four internal indices 1, 2, 3, 4. Since we will gauge the Skyrme scalar field by pairs of components, we will indicate the pair (1,2) by early Greek letters, $\alpha, \beta, \ldots$, while for the pair (3,4) we will employ early capital Latin indices, $A, B, \ldots$. As standard, we use Einstein's summation convention, but to alleviate notation, no distinction is made between covariant and contravariant *internal* indices.

The background of the theory is Minkowski spacetime, where the spatial $\mathbb{R}^4$ is written in terms of bipolar spherical coordinates,

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[1]The prescription of constructing Chern-Simons densities for gauged Skyrme systems in even spacetime dimensions is given in [6], but to date its effect has not been quantitatively studied.

$$ds^2 = dr^2 + r^2 d\Omega_3^2 - dt^2, \qquad (1)$$

where r, t are the radial and time coordinates respectively, while $d\Omega_3^2$ is the metric of the three sphere, with

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta \, d\varphi_1^2 + \cos^2\theta \, d\varphi_2^2, \qquad (2)$$

where $0 \le \theta \le \pi/2$ and $0 \le \varphi_{1,2} < 2\pi$.

In addition to using the coordinates (r, θ), we will find it convenient to employ instead

$$\rho = r\sin\theta, \quad \sigma = r\cos\theta, \qquad (3)$$

(with $0 \le \sigma, \rho < \infty$) in some expressions, such that (1) becomes

$$ds^2 = d\rho^2 + \rho^2 d\varphi_1^2 + d\sigma^2 + \sigma^2 d\varphi_2^2 - dt^2. \qquad (4)$$

II. THE MODEL

A. Gauging prescription and the action

In $4+1$ spacetime dimensions, the Skyrme model is described by the real scalar field $\phi^a = (\phi^\alpha, \phi^A, \phi^5)$, $\alpha = 1, 2$; $A = 3, 4$, subject to the sigma model constraint

$$|\phi^a|^2 = (\phi^\alpha)^2 + (\phi^A)^2 + (\phi^5)^2 = 1. \qquad (5)$$

The gauging prescription [5,6] for a $O(D+1)$ Skyrme scalar in $D+1$ spacetime involves gauging only the first D components, ϕ^a, $a = 1, \ldots, D$. Thus in the present case in $4+1$ dimensions, only the first four components ϕ^a, $a = 1, 2, 3, 4$ are gauged. The gauging prescription in the present case is stated by the definition of the covariant derivatives

$$\phi_\mu^\alpha = D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A_\mu(\varepsilon\phi)^\alpha, \qquad (6)$$

$$\phi_\mu^A = D_\mu \phi^A = \partial_\mu \phi^A + A_\mu(\varepsilon\phi)^A, \qquad (7)$$

$$\phi_\mu^5 = D_\mu \phi^5 = \partial_\mu \phi^5, \qquad (8)$$

with $\alpha = 1, 2$; $A = 3, 4$. Here ε denotes the Levi-Civita symbol in each of the two-dimensional subsets of internal indices, (1,2) and (3,4), respectively. More specifically, $(\varepsilon\phi)^1 = \phi^2$, $(\varepsilon\phi)^2 = -\phi^1$ and similar for indices (3,4).

In what follows, we shall use the abbreviated notation

$$\phi_{\mu_1\mu_2\ldots\mu_p}^{a_1a_2\ldots a_p}(p) = \phi_{\mu_1}^{a_1} \wedge \phi_{\mu_2}^{a_2} \ldots \wedge \phi_{\mu_p}^{a_p},$$

for the p-fold antisymmetrized products of the 1-form ϕ_μ^a defined by (6)–(8). The squares of these quantities describe the Skyrme kinetic terms, which in this case are allowed for $p = 1, 2, 3, 4$. Here, we restrict our attention to the

quadratic, quartic and sextic terms with $p = 1, 2, 3$, and eschew the octic term with $p = 4$.

We will use the further abbreviated notation

$$\Phi^2 \equiv \phi_{\mu_1}^a \phi_{\mu_2}^a g^{\mu_1\mu_2},$$

$$\Phi^4 \equiv \phi_{\mu_1\nu_1}^{ab} \phi_{\mu_2\nu_2}^{ab} g^{\mu_1\mu_2} g^{\nu_1\nu_2},$$

$$\Phi^6 \equiv \phi_{\mu_1\nu_1\lambda_1}^{abc} \phi_{\mu_2\nu_2\lambda_2}^{abc} g^{\mu_1\mu_2} g^{\nu_1\nu_2} g^{\lambda_1\lambda_2}, \qquad (9)$$

where $g^{\mu\nu}$ are contravariant to the metric tensor $g_{\mu\nu}$ of the five dimensional background geometry. Thus, $\phi_a^\mu = \phi_\nu^a g^{\mu\nu}$, $\phi_{ab}^{\mu\nu} = \phi_{\rho\sigma}^{ab} g^{\mu\rho} g^{\nu\sigma}$, *etc.*. Note that $a, b, \ldots$ are the global $O(5)$ indices for which we do not distinguish *upper* and *lower*, for typographical convenience.

We consider the following action

$$S = \int d^5x \sqrt{-g} \left[\frac{\lambda_1}{2} \Phi^2 + \frac{\lambda_2}{4} \Phi^4 + \frac{\lambda_3}{36} \Phi^6 + \lambda_0 U(\phi^5) \right.$$
$$\left. + \frac{1}{4} \lambda_M F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma\lambda} A_\lambda F_{\mu\nu} F_{\rho\sigma} \right], \qquad (10)$$

which apart from the above quantities features the (standard) Maxwell and Chern-Simons terms, and a Skyrme potential U. Thus $F_{\mu\nu}$ is the Maxwell field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ being the gauge connection. Our choice for the Skyrme potential is

$$U = 1 - \phi^5, \qquad (11)$$

which is the analogue of the "pion mass potential," often used in the three dimensional and planar Skyrme models. $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_M) \ge 0$ are coupling constants. We shall also define

$$\lambda_M = \frac{1}{g^2}, \qquad (12)$$

with g the gauge coupling constant, such that the gauge decoupling limit in [12] is approached for $g = 0$.

Again, to accommodate the eventual formulation in curved coordinates, we replace all partial derivatives ∂_μ in (6)–(8) formally by ∇_μ. Varying the Lagrangian (10) with respect to the scalars ϕ^a leads to the Euler-Lagrange equations

$$(\delta^{da} - \phi^d\phi^a)\left\{ 2\lambda_1 D^\mu \phi_\mu^a + 8\lambda_2 \phi_b^\nu D^\mu \phi_{\mu\nu}^{ab} + 9\lambda_3 \phi_{bc}^{\nu\lambda} D^\mu \phi_{\mu\nu\lambda}^{abc} \right.$$
$$\left. + \lambda_0 \frac{\partial U}{\partial \phi^a} \right\} = 0, \qquad (13)$$

while the corresponding equations for the Maxwell field are

$$\lambda_M \nabla_\nu F^{\nu\mu} = J^\mu + \kappa \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\nu\rho} F_{\sigma\lambda}, \qquad (14)$$

where $J^\mu = J^\mu[\phi(p)]$ is the Skyrme current arising from the variation with respect to the Maxwell potential.

Variation of (10) with respect to the metric tensor $g_{\mu\nu}$ leads to the energy-momentum tensor of the model

$$T_{\mu\nu} = \lambda_M T^{(M)}_{\mu\nu} + \lambda_0 T^{(0)}_{\mu\nu} + \lambda_1 T^{(1)}_{\mu\nu} + \lambda_2 T^{(2)}_{\mu\nu} + \lambda_3 T^{(3)}_{\mu\nu}, \quad (15)$$

in terms of the contributions of the distinct terms in (10), which read

$$T^{(M)}_{\mu\nu} = F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma},$$

$$T^{(0)}_{\mu\nu} = -g_{\mu\nu} U(\phi^5),$$

$$T^{(1)}_{\mu\nu} = \phi^a_\mu \phi^a_\nu - \frac{1}{2} g_{\mu\nu} \Phi^2,$$

$$T^{(2)}_{\mu\nu} = \phi^{ab}_{\mu\rho} \phi^{ab}_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\rho\sigma} \Phi^4,$$

$$T^{(3)}_{\mu\nu} = \frac{1}{6} \left(\phi^{abc}_{\mu\rho\tau} \phi^{abc}_{\nu\sigma\lambda} g^{\rho\sigma} g^{\tau\lambda} - \frac{1}{6} g_{\mu\nu} \Phi^6 \right). \quad (16)$$

As usual, the tt component of the mixed energy-momentum tensor, $T^t{}_t$, (taken with minus sign) corresponds to the local mass-energy density, while the angular momentum densities in the (x_1, x_2) and (x_3, x_4) planes are given by the $t\varphi_1$ and $t\varphi_2$ components, $T^t{}_{\varphi_1}$ and $T^t{}_{\varphi_2}$, respectively.

B. The Ansatz and boundary conditions

1. The general case

We consider a U(1) Ansatz in terms of three potentials, two magnetic $a_{1,2}(r,\theta)$, and one electric, $V(r,\theta)$, with

$$A = a_1(r,\theta) d\varphi_1 + a_2(r,\theta) d\varphi_2 + V(r,\theta) dt, \quad (17)$$

and the following expression of the scalars

$$\phi^1 = \Psi_1(r,\theta) \cos(n_1\varphi_1 - \omega t), \qquad \phi^2 = \Psi_1(r,\theta) \sin(n_1\varphi_1 - \omega t),$$
$$\phi^3 = \Psi_2(r,\theta) \cos(n_2\varphi_2 - \omega t), \qquad \phi^4 = \Psi_2(r,\theta) \sin(n_2\varphi_2 - \omega t),$$
$$\phi^5 = \Psi_3(r,\theta), \qquad (18)$$

with n_1, n_2 two positive integers (the winding numbers) and $\omega \geq 0$ the field frequency. Also, the functions Ψ_1, Ψ_2, Ψ_3 are subject to the constraint (5)

$$\Psi_1^2 + \Psi_2^2 + \Psi_3^2 = 1. \qquad (19)$$

In this approach, the problem reduces to solving a set of six PDEs with dependence on only two coordinates. As usual, these equations result by varying (10) with respect to the functions Ψ_1, Ψ_2, Ψ_3 and a_1, a_2, V, respectively. Before stating the boundary conditions, we display the expression of several terms which enter the action S (10), subject to the general Ansätze (17)–(18).

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{r^2} \left[\frac{1}{\sin^2\theta} \left(a_{1,r}^2 + \frac{1}{r^2} a_{1,\theta}^2 \right) + \frac{1}{\cos^2\theta} \left(a_{2,r}^2 + \frac{1}{r^2} a_{2,\theta}^2 \right) - \left(V_{,r}^2 + \frac{1}{r^2} V_{,\theta}^2 \right) \right], \qquad (20)$$

$$\varepsilon^{\mu\nu\rho\sigma\lambda} A_\lambda F_{\mu\nu} F_{\rho\sigma} = \frac{8}{\sin\theta \cos\theta} [(a_1 a_{2,\theta} - a_2 a_{1,\theta}) V_{,r} + (a_2 a_{1,r} - a_1 a_{2,r}) V_{,\theta} + (a_{2,r} a_{1,\theta} - a_{1,r} a_{2,\theta}) V], \qquad (21)$$

and

$$\Phi^2 \equiv \phi^a_{i_1} \phi^a_{i_2} g^{i_1 i_2} = \Psi_{1,r}^2 + \Psi_{2,r}^2 + \Psi_{3,r}^2 + \frac{1}{r^2} (\Psi_{1,\theta}^2 + \Psi_{2,\theta}^2 + \Psi_{3,\theta}^2)$$
$$+ \frac{1}{r^2} \left[\left(\frac{(n_1 - a_1)^2}{\sin^2\theta} + \frac{a_2^2}{\cos^2\theta} \right) \Psi_1^2 + \left(\frac{(n_2 - a_2)^2}{\cos^2\theta} + \frac{a_1^2}{\sin^2\theta} \right) \Psi_2^2 \right] - (\Psi_1^2 + \Psi_2^2)(\omega + V)^2, \qquad (22)$$

while the expressions of Φ^4 and Φ^6 are too complicated to include here.

The boundary conditions satisfied by the functions Ψ_1, Ψ_2, Ψ_3 are

$$\Psi_1|_{r=0} = -1, \quad \Psi_2|_{r=0} = 0, \quad \Psi_3|_{r=0} = -1, \quad \Psi_1|_{r=\infty} = 0, \quad \Psi_2|_{r=\infty} = 0, \quad \Psi_3|_{r=\infty} = 1,$$
$$\Psi_1|_{\theta=0} = \partial_\theta \Psi_2|_{\theta=0} = \partial_\theta \Psi_3|_{\theta=0} = 0, \quad \partial_\theta \Psi_1|_{\theta=\pi/2} = \Psi_2|_{\theta=\pi/2} = \partial_\theta \Psi_3|_{\theta=\pi/2} = 0, \qquad (23)$$

while for the gauge potentials we impose

$$a_1|_{r=0} = a_2|_{r=0} = 0, \quad V_{,r}|_{r=0} = 0, \quad a_1|_{r=\infty} = a_2|_{r=\infty} = 0, \quad V|_{r=\infty} = V_0,$$

$$a_1|_{\theta=0} = \partial_\theta a_2|_{\theta=0} = 0, \qquad \partial_\theta a_1|_{\theta=\pi/2} = a_2|_{\theta=\pi/2} = 0, \qquad \partial_\theta V|_{\theta=0,\pi/2} = 0. \tag{24}$$

These boundary conditions are compatible with an approximate form of the solutions on the boundaries of the domain of integration, together with some physical requirements (e.g., regularity and finiteness of global charges). Another criteria here (and an important guideline in selecting among possible sets of boundary conditions) is the compatibility with the spherically symmetric ungauged limit in [12], together with the radially enhanced limit in Section II 2 2. For example, Eqs. (29) and (30) in Section II 2 2 imply that the functions Ψ_1 and a_1 vanish at $\theta = 0$, while Ψ_2, Ψ_3, a_2 and V should satisfy Newman boundary conditions. Then one assumes the existence of a generic small θ-expansion of the form $\mathcal{U} = \sum_{k\geq 0} u_k(r)\theta^k$ (with $\mathcal{U} = \{\Psi_1, \Psi_2, \Psi_3; a_1, a_2, V\}$) which is plugged into the field equations. (Note that the coefficients $u_0(r)$ vanish for Ψ_1 and a_1, while $u_1(r)$ is zero for the remaining functions). A similar approach is implemented for $\theta = \pi/2$ and at the limits of the r-interval.

As usual the total mass-energy, M, and angular momenta, $J_{1,2}$, of a solution are defined as

$$M = -\int d^4x \sqrt{-g} T^t{}_t, \qquad J_1 = \int d^4x \sqrt{-g} T^t{}_{\varphi_1},$$

$$J_2 = \int d^4x \sqrt{-g} T^t{}_{\varphi_2}, \tag{25}$$

while the electric charge Q_e is computed from the electric flux at infinity,

$$Q_e = \oint_\infty dS_{rt} F^{rt}, \tag{26}$$

and thus can also be evaluated from the asymptotics of the electric potential

$$V = V_0 + \frac{Q_e}{4\pi^2 r^2} + \dots, \tag{27}$$

with V_0 a constant. However, by using the field equations, the volume integral in the expression (25) of $J_{1,2}$ can be converted into surface integrals at infinity in terms of Maxwell potentials, and one finds[2]

$$J_1 = \frac{1}{2}\lambda_M n_1 Q_e, \qquad J_2 = \frac{1}{2}\lambda_M n_2 Q_e. \tag{28}$$

2. $n_1 = n_2 = 1$: a symmetry enhanced Ansatz

Remarkably, it turns out that the choice

$$\Psi_1 = \sin\psi(r)\sin\theta, \qquad \Psi_2 = \sin\psi(r)\cos\theta,$$

$$\Psi_3 = \cos\psi(r), \tag{29}$$

$$a_1(r,\theta) = a_\varphi(r)\sin^2\theta, \qquad a_2(r,\theta) = a_\varphi(r)\cos^2\theta,$$

$$V(r,\theta) = V(r), \tag{30}$$

provides a consistent factorization of the angular dependence for the general model, provided that[3]

$$n_1 = n_2 = 1. \tag{31}$$

This restrictive Ansatz greatly reduces the complexity of the system and simplifies the numerical construction of the lowest topological charge solutions, which are found in this case by solving a set of three ordinary differential equations (ODEs). For example, with the above Ansatz, the effective action of the model reads

$$L_{\text{eff}} = r^3 \left\{ \frac{\lambda_M}{2r^2} \left(a_\varphi'^2 + \frac{4a_\varphi^2}{r^2} - r^2 V'^2 \right) + \frac{16\kappa}{r^3}(V a_\varphi' - a_\varphi V')a_\varphi \right.$$

$$+ \frac{\lambda_1}{2}\left[\psi'^2 + \frac{\sin^2\psi}{r^2}(2 + (1-a_\varphi)^2 - r^2(\omega + V)^2) \right]$$

$$+ \lambda_2 \frac{\sin^2\psi}{r^2}\left[\psi'^2(2 + (1-a_\varphi)^2 - r^2(\omega + V)^2) + \frac{\sin^2\psi}{r^2}(1 + 2((1-a_\varphi)^2 - r^2(\omega + V)^2)) \right]$$

$$\left. + \lambda_3 \frac{\sin^4\psi}{r^4}\left[\psi'^2(1 + 2(1-a_\varphi)^2 - 2r^2(\omega + V)^2) + \frac{\sin^2\psi}{r^2}((1-a_\varphi)^2 - r^2(\omega + V)^2) \right] + \lambda_0(1-\cos\psi) \right\}, \tag{32}$$

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[2]Note, however, that the corresponding densities are not equal.
[3]This is similar to the factorization of the θ-dependence on the S^3-sphere employed in the scalar field Ansätze in [13,14].

the contribution of various terms being transparent. The boundary conditions satisfied by the functions $\{\psi(r), a_\varphi(r), V(r)\}$ results directly from (23)–(24). Also, in this case it is possible to compute an approximate form of the solutions at the limits for the domain of integration. For example, one finds the following small-r expression

$$a_\varphi(r) = m_2 r^2 + m_4 r^4 + O(r^6), \qquad \psi(r) = \pi + f_1 r + O(r^3), \qquad V(r) = v_0 + v_4 r^4 + O(r^5), \qquad (33)$$

which contains three essential parameters f_1, m_2 and v_0, while

$$m_4 = \frac{96\kappa^2 m_2^3}{\lambda_M^2} - \frac{f_1^2(\lambda_1 + 6f_1^2(\lambda_2 + f_1^2\lambda_3))}{12\lambda_M},$$

$$v_4 = \frac{f_1^2(v_0 + \omega)}{24\lambda_M}(\lambda_1 + 6f_1^2(\lambda_2 + f_1^2\lambda_3)) + \frac{\kappa m_2}{\lambda_M^3}[-1152\kappa^2 m_2^3 + f_1^2\lambda_M(\lambda_1 + 6f_1^2(\lambda_2 + f_1^2\lambda_3))]. \qquad (34)$$

The leading order terms in the large-r expansion of the solutions are

$$a_\varphi(r) = \frac{\bar{m}_2}{r^2} + \dots, \quad \psi(r) = c\sqrt{\frac{\pi}{2}}\frac{e^{-r\sqrt{\frac{\lambda_0}{\lambda_1}}}}{r^{3/2}} + \dots, \quad V = V_0 + \frac{Q_e}{4\pi^2 r^2} + \dots, \qquad (35)$$

with $\bar{m}_2$ Q_e, V_0 and c some constants which are determined by numerics.

The corresponding expressions of the mass-energy and angular momenta densities are also of interest, with[4]

$$-T^t_{\ t} = \frac{\lambda_M}{2r^2}\left(a'^2_\varphi + \frac{4a_\varphi^2}{r^2} + r^2 V'^2\right) + \frac{\lambda_1}{2}\left[\psi'^2 + \frac{\sin^2\psi}{r^2}(2 + (1 - a_\varphi)^2 + r^2 V^2)\right]$$

$$+ \lambda_2 \frac{\sin^2\psi}{r^2}\left[\psi'^2(2 + (1 - a_\varphi)^2 + r^2 V^2) + \frac{\sin^2\psi}{r^2}(1 + 2((1 - a_\varphi)^2 + r^2 V^2))\right]$$

$$+ \lambda_3 \frac{\sin^4\psi}{r^4}\left[\psi'^2(1 + 2(1 - a_\varphi)^2 + 2r^2 V^2) + \frac{\sin^2\psi}{r^2}((1 - a_\varphi)^2 + r^2 V^2)\right] + \lambda_0(1 - \cos\psi), \qquad (36)$$

$$\frac{T^t_{\ \varphi_1}}{\sin^2\theta} = \frac{T^t_{\ \varphi_2}}{\cos^2\theta} = -\lambda_M a'_\varphi V' + \sin^2\psi(1 - a_\varphi)V\left[\lambda_1 + 2\lambda_2\left(\psi'^2 + \frac{2\sin^2\psi}{r^2}\right) + 4\lambda_3 \frac{\sin^2\psi}{r^2}\left(\psi'^2 + \frac{\sin^2\psi}{2r^2}\right)\right]. \qquad (37)$$

Then, by using the Maxwell equations, one can easily show that

$$\sqrt{-g}T^t_{\ \varphi_1} = \sin^3\theta\cos\theta\,\mathcal{S}', \qquad \sqrt{-g}T^t_{\ \varphi_2} = \cos^3\theta\sin\theta\,\mathcal{S}', \quad \text{with} \quad \mathcal{S} = \lambda_M(1 - a_\varphi)r^3 V' + 8\kappa a_\varphi^2(3 - 2a_\varphi), \quad (38)$$

which makes manifest the total derivative structure of $T^t_{\ \varphi_i}$. Also, one observes that despite entering the angular momenta density, the integral contribution of the CS term vanishes, since $a_\varphi \to 0$ as $r \to 0$ and as $r \to \infty$.

C. Scaling symmetry and numerical approach

The model (10) contains four input parameters λ_i together with the gauge coupling constant g (we recall $\lambda_M = 1/g^2$). However, the constant multiplying the quadratic term can be taken as an overall factor for the Skyrme action. Also, the equations are invariant under the transformation

$$r \to \tau r, \qquad \lambda_0/\lambda_1 \to \tau^2\lambda_0/\lambda_1, \qquad \lambda_2/\lambda_1 \to \lambda_2/(\tau^2\lambda_1),$$
$$\lambda_3/\lambda_1 \to \lambda_3/(\tau^4\lambda_1), \qquad (39)$$

(with τ some arbitrary positive parameter), which was used to set $\lambda_3 = 1$.

Then the problem still contains three free constants λ_0, λ_2 and λ_M. In this work, in order to simplify the picture, we have chosen to solve a model without the quartic term, i.e., we set

$$\lambda_1 = \lambda_3 = 1, \qquad \lambda_2 = 0, \qquad (40)$$

such that the only input parameters are λ_0 and the gauge coupling constant g (i.e., the coefficient λ_M of the Maxwell term in the action (10)). The choice (40) will used for all the numerical solutions presented in this paper.

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[4]Note that these expressions are given in gauge with $\omega = 0$, which was employed in numerics.

Starting with the case of solutions within the general Ansatz (17)–(18), the constraint (19) is imposed by using the Lagrange multiplier method, as explained e.g., in [15,16]. The numerical calculations were performed by using the professional software based on the Newton-Raphson method CADSOL [17]. The field equations are first discretized on a nonequidistant grid and the resulting system is solved iteratively until convergence is achieved. In this scheme, a new radial variable $x = r/(1 + r)$ is introduced which maps the semi-infinite region $[0, \infty)$ to the closed region $[0, 1]$. Also, this software package provides error estimates for each unknown function, which

allows judging the quality of the computed solution. The numerical error for the solutions reported in this work is estimated to be typically $< 10^{-4}$.

The solutions within the symmetry enhanced Ansatz (29)–(30) were found by using the professional software package COLSYS [18] (although some of them were also computed by using the same approach as in the general case). This solver employs a collocation method for boundary-value ODEs and a damped Newton method of quasi-linearization. At each iteration step a linearized problem is solved by using a spline collocation at Gaussian points. In this approach, the linearized problem
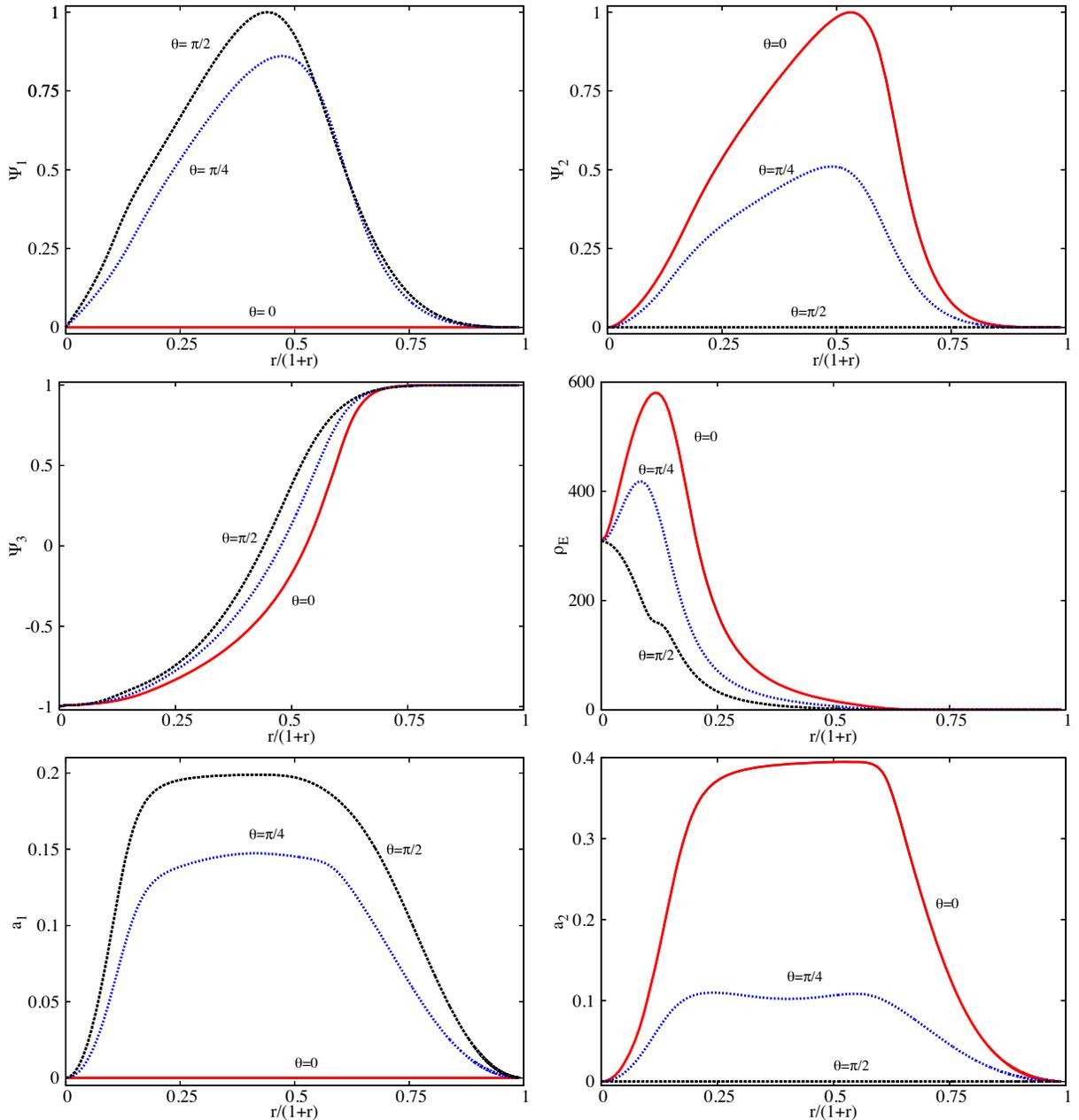

FIG. 1. The scalar functions Ψ_1, Ψ_2, Ψ_3 and the gauge potentials a_1, a_2 which enter the Ansatz (17)–(18) are shown together with the mass-energy density $\rho_E = -T^t{}_t$ for a solution with $n_1 = 1$, $n_2 = 2$.
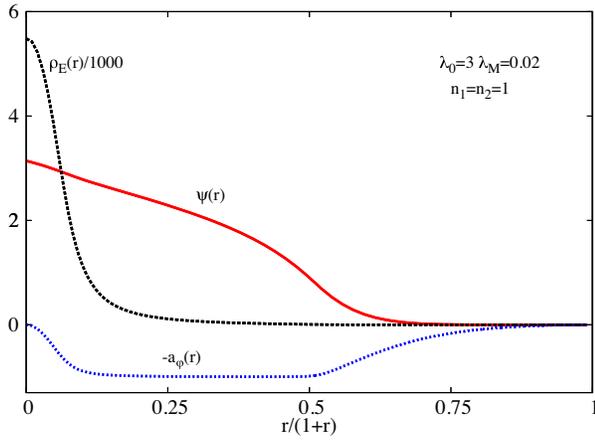
FIG. 2. The scalar function ψ and the gauge potential a_φ which enter the simplified Ansatz (29)-(30) are shown together with the mass-energy density $\rho_E = -T^t{}_t$ for a solution with $n_1 = n_2 = 1$.
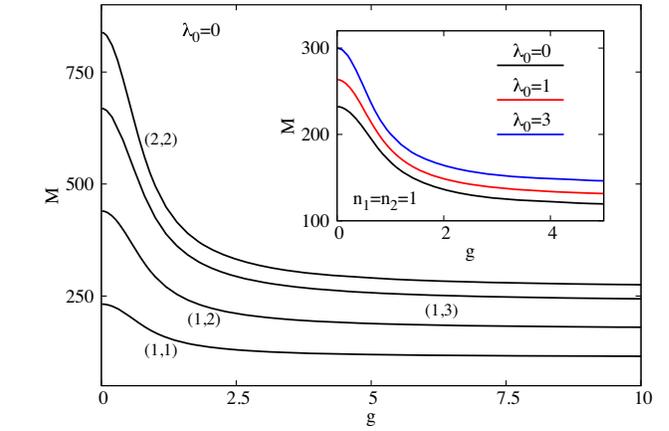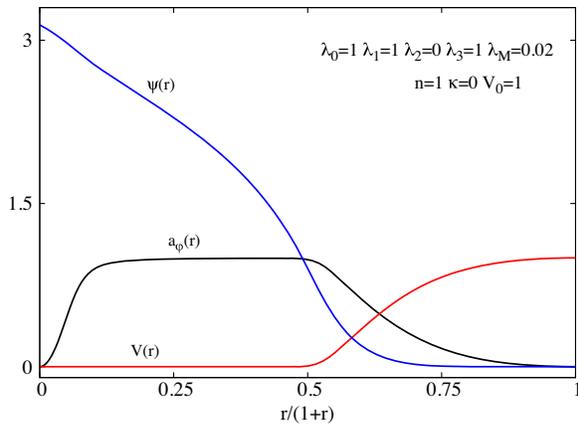

FIG. 3. The mass of the $SO(2)$ gauged Skyrmions is shown as a function of gauge coupling constant g (with $\lambda_M = 1/g^2$) for several values of the winding numbers n_1, n_2.

is solved on a sequence of meshes until the successful stopping criterion is reached, a compactified radial variable $x = r/(1 + r)$ being again employed.

III. NUMERICAL RESULTS

A. Purely magnetic, static solutions

These configurations (characterized by $\kappa = 0$) have $V = 0$, and their angular momenta and electric charge density vanish identically. As an illustrative example for the general case, we show in Fig. 1 the profiles of a typical solution with $n_1 = 1$, $n_2 = 2$ and $\lambda_M = 1/25$, $\lambda_0 = 0$. One can see that the gauge and scalar functions (except Ψ_3) as well as the energy density depend strongly on θ.

For completeness, in Fig. 2 we give a similar plot for the special case $n_1 = n_2 = 1$ [note however that, here we show the functions ψ and a_φ which enter the simplified Ansatz (29)–(30)].

The dependence of the solutions on the gauge coupling constant g is shown in Fig. 3 for the lowest values of the pair (n_1, n_2). One can notice the existence of some universal behaviour there, the total mass of the solutions decreasing with $\lambda_M = 1/g^2$. The mass is maximized by the $g = 0$ configurations, a limit which corresponds to the ungauged O(5) sigma model, whose solutions were discussed in [12] (albeit for the special case $n_1 = n_2 = 1$ only). A curious result there is that, for the same other input parameters, the $n_1 = n_2 = 2$ solutions have a larger mass than the $n_1 = 1$, $n_2 = 3$ configurations.

Although the results there were found for solutions without a potential, the same behavior is found for $\lambda_0 \neq 0$, as shown in the inset of Fig. 3. Moreover, as expected, the mass of the solutions always increases with the parameter λ_0. Also, one remarks that the generic properties of the static solutions appear to be the same for any choice of the integers n_1, n_2.
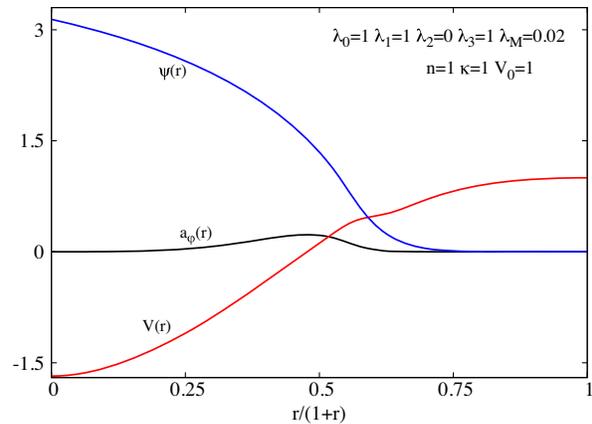

FIG. 4. The scalar function ψ is shown together with the magnetic and electric gauge potentials, a_φ and V, for a spinning solution with $n_1 = n_2 = n = 1$ for $\kappa = 0$ (left panel) and $\kappa = 1$ (right panel).
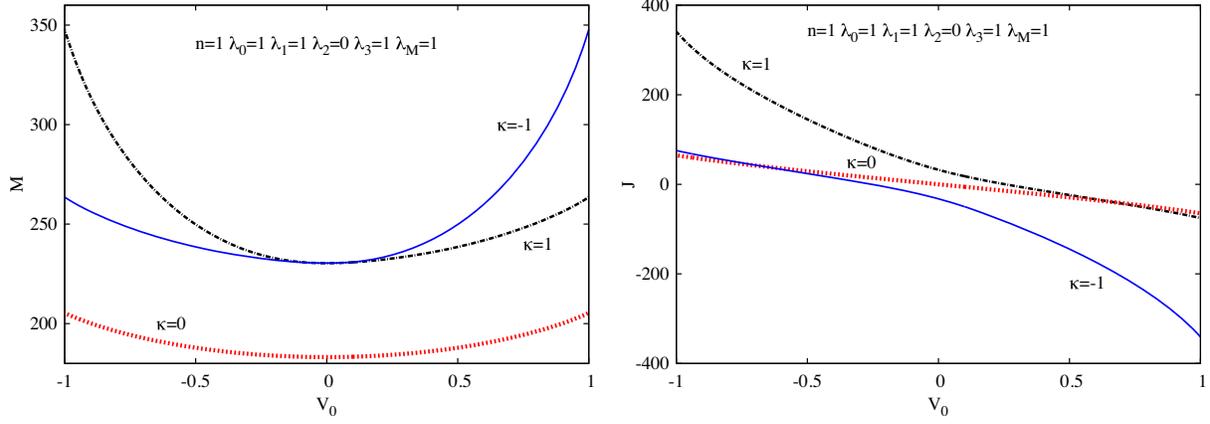
FIG. 5. The total mass-energy M (left panel) and the angular momenta $J_1 = J_2 = J$ (right panel) are shown as functions of the asymptotic value of the electric potential V_0 for several values of the Chern-Simons coupling constant κ for solutions with $n_1 = n_2 = n = 1$.

B. Electrically charged, spinning solutions

The only rotating solutions considered in this work were found for the enhanced symmetry Ansatz[5] (29)–(30), which means they have

$$J_1 = J_2 = J = \frac{1}{2}\lambda_M Q_e. \qquad (41)$$

Also, we use the residual gauge symmetry of the model $V \to V - \omega$ to set $\omega = 0$ in the numerical approach. Then, for spinning solutions the constant V_0, which fixes the asymptotic value of the electric potential, is the only extra-input parameter as compared to the purely static case.

The profile of two typical solutions without a CS term ($\kappa = 0$, left panel) and with a CS term ($\kappa = 1$, right panel) are shown in Fig. 4. Note that while all other input parameters are kept constant there (in particular the asymptotic value of the electric potential V_0), the presence of a CS term leads to a rather different shape of the gauge potentials $a_\varphi(r)$ and $V(r)$. Also, in the $\kappa = 0$ case (no CS term), the profiles of $a_\varphi(r)$, $\psi(r)$ are not very different as compared to the static limit, Fig. 2.

Our numerical results indicate that any static configuration appears to possess rotating generalizations. As we increase V_0 from zero while keeping fixed other input parameters, a branch of solutions forms. Along this branch, the total mass-energy M increases monotonically with V_0. The dependence of mass-energy M and angular momentum J on the value of the electric potential at infinity V_0 is shown in Fig. 5 for several values of the CS coupling constant κ. As one can see, the symmetry

$$V_0 \to -V_0, \quad M \to M, \quad J \to -J, \qquad (42)$$

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[5]Note that the results obtained in the static limit, Sec. III A, strongly suggests that this limit contains already all basic features of the general case.

exists for solutions without a Chern-Simons term, $\kappa = 0$, only. Also, no upper bound appears to exist for the value of $|\kappa|$, both the mass-energy and angular momenta increasing proportionally with $|\kappa|$. Moreover, for $\kappa = 0$, the solutions with $V_0 = 0$ have no electric field and correspond to static configurations discussed in Sec. III A. However, as expected, the angular momentum does not vanish for solutions with a CS term which have an electric potential which vanishes asymptotically, $V_0 = 0$. However, in all cases the minimal value of M is approached for $V_0 = 0$. Finally, let us remark on the existence of a special set of solutions with $J = Q_e = 0$, which, for $\kappa \neq 0$ still possess a nonvanishing angular momentum/electric charge density.

IV. SUMMARY AND OUTLOOK

In this work, we have formulated the $SO(2)$ gauged $O(5)$ Skyrme model in $4 + 1$ dimensions. This is the gauged analogue of a previous work in Ref. [12], where the gravitating system was studied, while here we consider the Maxwell dynamics instead of gravity. The present work is a preliminary step toward a comprehensive investigation of the properties of solitons of $U(1)$ gauged Skyrmions, in principle in all dimensions but most importantly in $3 + 1$. Here we start with $4 + 1$ dimensions since this is the simplest next case to $2 + 1$ dimensions, like which $4 + 1$ is an odd dimension. Our ultimate aim is (a) studying the charge-mass and spin-mass dependences, and (b) tracking the evolution of values of the effective baryon charge. The "effective baryon charge" in question is given by the lower bound on the energy of the gauged static soliton. The solutions we seek include those supporting the global charges: electric charge and angular momentum.

We are motivated by a number of unexpected results that we have obtained in the study of an analogous model in $2 + 1$ dimensions, namely the $U(1)$ gauged planar Skyrme system, in which case the presence of electric charge and spin is contingent on the presence of the Chern-Simons

term in the Lagrangian. Specifically, we found that in $2 + 1$ dimensions (a) charge-mass and spin-mass curves are not monotonically increasing *as usual* but rather can also decrease in some areas of the parameter space (see Sec. 4 of Ref. [7]), and (b) that in the given model solitons characterized with continuous values of the baryon charge can exist (see Refs. [8,9]).

One may expect that some of the properties observed in [7–9] can be reproduced in the case of charged Skyrmions in $3 + 1$ dimensions. But in the absence of a Chern-Simons term in $3 + 1$ dimensions, this does not turn out to be the case, as shown in our work of Refs. [2,3]. While it is possible to employ a "new" $U(1)$ Chern-Simons density proposed in [6], where it is called a Skyrme–Chern-Simons density, this would involve the interaction with an $O(6)$ Skyrme scalar in addition to the $O(4)$ Skyrme scalar that supports the soliton. This complication, along with the necessity to tackle a two-dimensional PDEs problem, is one reason we defer that $3 + 1$ dimensional problem and proceed in the present work to the study of the $U(1)$ gauged $O(5)$ model in $4 + 1$ dimensions. Moreover, it is convenient that in $4 + 1$ dimensions the "usual" Chern-Simons term is available.

The main results of this work can be summarized as follows. First, we have established the existence of $SO(2)$ gauged generalizations of $O(5)$ Skyrme model introduced in Ref. [12]. (Note that here the solutions were found for a fixed Minkowski spacetime, the gravity effects being ignored.) Both static and spinning configurations were studied, subject to a specific Ansatz which reduces the problem to solving a set of PDEs. Moreover, the Ansatz allows for an "enhanced symmetry" limit that renders the residual system one-dimensional, depending on the radial variable only. The static purely magnetic solutions were studied for both the general Ansatz and the enhanced symmetry Ansatz. It turns out that the numerical results show that the basic qualitative features are rather similar in both cases, as shown, e.g., in Fig. 3; for example, the mass of the solutions always decreases monotonically as the gauge coupling constant increases. In the spinning case, only enhanced symmetry configurations were studied, solutions with and without a Chern-Simons (CS) term being considered. Our numerical results indicate that, different from the case of charged Skyrmions in $2 + 1$ dimensions, the presence of a CS term in the Lagrangian of the present model leads only to some quantitative features, the unusual features unveiled in Refs. [7–9], being not recovered in our study. For example, while in $2 + 1$ dimensions the electric charge and spin were not supported by the Skyrmion when the CS term was absent, here the situation is more like as for gauged Skyrmions in $3 + 1$ dimensions [2,3] or like in the case of JZ-dyons [10] where electric charge (but not spin) is present despite the absence of a CS term. Also, the mass of the solutions and angular momentum always possess a monotonic dependence on the asymptotic value of the electric potential, see Fig. 5.

To summarize, we conclude that the presence of the CS term has only a quantitative effect on the $4 + 1$ dimensional model proposed in this paper. This strongly contrasts with the $2 + 1$ dimensional case in [7–9], where it had the qualitative effect of featuring "nonstandard" mass-charge and mass-spin dependences, and, moreover, changing "baryon number" inside a given theory. The salient difference between the Julia-Zee (JZ) type of dyons [2,3,10] studied here, and the Paul-Khare (PK) type dyons [11] is that the former exist with or without the presence of Chern-Simons dynamics, while the existence of the latter (PK) type is predicated on the presence of Chern-Simons dynamics in which case the Chern-Pontryagin (CP) index (in the spatial subspace) determines the electric charge (and spin) quantitatively. Given that in the model studied here the only gauge field is Abelian, the corresponding CP charge vanishes. Technically, in the case of PK dyons the presence of the CS term in the Lagrangian leads to solutions where the electric function $V(r)$ can take on a continue range of values, in contrast with the JZ type here. It is this property of the solutions which gives rise to the unusual mass-electric charge/spin relation and the variable "baryon charge" seen in [7–9].

To recover these properties the model at hand needs to be extended, and for this there are two distinct possibilities available:

(i) To extend the model to feature an $SU(2) \sim SO_{\pm}(4)$ field, such that the volume integral of the second Chern-Pontryagin (CP) term in the Gauss law equation [resulting from the Maxwell equation (14)], which yields the electric charge Q_e, does not vanish, as it does in the solely $U(1)$ gauged model here. This would cause the value of Q_e to depend on the CS coupling κ, unlike here. This done, the electric charge would still get contributions even when $\kappa = 0$. To change this, namely to cause Q_e to be entirely dependent on κ for its support, it would be necessary to replace $F_{\mu\nu}^2$, the kinetic term of the gauge field, with $F_{\mu\nu\rho\sigma}^2$. Such a model is under active consideration now.

(ii) An alternative extension of the model is motivated by our study of the $SU(3)$ and $SO(5)$ gauged Higgs (YMH) model with algebra-valued Higgs field [19,20] in $3 + 1$ dimensions, augmented with a Higgs–Chern-Simons (HCS) term [21,22]. Those models, where a new Chern-Simons density (the HCS) is present, both electric charge and spin are supported. Most importantly, they feature "nonstandard" mass-charge and mass-spin dependencies in [7]. In those models, this effect has been enabled by the larger [than $SU(2)$] gauge group. This may signal the possibility that in the present case where the scalar matter is a Skyrme rather than a Higgs scalar, incorporating a larger target space sigma model may be useful. In this direction, it is natural

to augment the Abelian gauged $O(5)$ sigma model with a Skyrme–Chern-Simons [6] density which is defined by the supplementary $O(7)$ Skyrme scalar. Compared to the above described possibility, this alternative is a technically more challenging problem.

APPENDIX A: "TOPOLOGICAL CHARGE" OF $SO(2)$ GAUGED $O(5)$ MODEL ON $\mathbb{R}^4$

What we refer to as the "topological charge" density of a gauged Skyrme (sigma model) system is the density that results from the deformation of the topological charge density of the sigma model, *prior to gauging*. While the topological charge of the latter presents a lower bound on the energy of the ungauged sigma model, the latter presents the energy lower bound for the gauged system. As such, it is not strictly speaking a topological charge, as is the case for gauged Higgs systems.

The lower bound on the energy of a given sigma model in the appropriate dimensions is given by a topological [23] charge density. Such charge densities are not explicitly *total divergence*, in contrast to the case of Higgs models [21], but when the sigma model scalar is expressed in a parametrization that is compliant with the sigma model constraint, they become explicitly *total divergence*. We refer to the charge densities in the generic parametrization as *essentially total divergence*. In the context of the present work, the sigma models in question are the $O(D + 1)$ Skyrme models on $\mathbb{R}^D$.

When it comes to gauging Skyrme models, the situation strictly differs from the (gauged[6]) Higgs models [21], where the charge densities supplying the lower bounds

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[6]The gauge decoupled Higgs models, referred to as Goldstone models in [21], do support topologically stable solitons.

on the energy densities are all descended from Chern-Pontryagin densities in some higher dimension, and are topological densities. The gauge group of a Higgs model is fixed by the representation of the Higgs scalar, in which the topological charge is encoded [24,25]. Gauging a Skyrme model with the requirement that a charge density giving a lower bound on the energy density be defined, contrasts starkly with the definition of the corresponding density of a Higgs model. The physical requirements that the charge density of a gauged Skyrmion must satisfy (a) that it be *(essentially) total divergence* to enable the evaluation of the charge integral as a surface integral in terms of the asymptotic fields, and, (b) that it be gauge invariant. A prescription for achieving such a definition was given in [5].

The definition of the topological charge density of an $SO(N)$ gauged Skyrme scalar $\phi^a = (\phi^a, \phi^{D+1})$, $a = 1, 2, …, D$ on $\mathbb{R}^D$ given in [5,6] relies on the relation between the winding number density ϱ_0, prior to gauging, which is *effectively total divergence* but is *gauge variant*, and the density ϱ_G defined by replacing all the partial derivatives in ϱ_0 by covariant derivatives. While ϱ_0 is *effectively total divergence* and is *gauge variant*, ϱ_G is *gauge invariant* but is not *total divergence*. The physical charge density must be both *gauge invariant* and *total divergence*. The definition of ϱ_G follows from that of ϱ_0, by formally replacing all partial derivatives by the $SO(N)$ covariant derivatives.

The generic expression

$$\varrho = \varrho_0 + \partial_i \Omega_i[\mathcal{A}, \phi] \qquad (A1)$$

$$= \varrho_G + W[\mathcal{F}, \mathcal{D}\phi], \qquad (A2)$$

defines the topological charge density where $\Omega_i[\mathcal{A}, \phi]$, like ϱ_0, is *gauge variant* while $W[\mathcal{F}, \mathcal{D}\phi]$, like ϱ_G is *gauge-invariant*. (A1) and (A2) are equivalent definitions for ϱ, which as required is both *gauge invariant* and *total divergence*.

Clearly, the definition (A1)–(A2) depends on the gauging prescription employed and here, in contrast with Higgs models, the gauge group can be chosen to be $SO(N)$, for all N in the range $D \geq N \geq 2$. Thus, at most D components of the $D + 1$ component Skyrme scalar are gauged with $SO(D)$, down to only two of the components with (Abelian) $SO(2)$. For example in [2,3], the $O(4)$ Skyrme system on $\mathbb{R}^3$ is gauged with $SO(2)$, while in [26] this system is gauged with $SO(3)$. Numerous such examples are presented in [6], for models on $\mathbb{R}^D$, $D = 2, 3, 4, 5$.

In the present work, we are concerned with the $O(5)$ model on $\mathbb{R}^4$, with the Abelian gauging prescription (6), (7) and (8), i.e., gauging two pairs of Skyrme scalars, in contrast to the example given in [6] where only one pair is gauged. Our choice for gauging two pairs here is driven by our desire of having a gauging prescription that is symmetric in $a = 1, 2$ and $a = 3, 4$ that enables the enhanced

radial symmetry of the biazimuthal symmetry. For this reason it is convenient to start with the maximal $SO(4)$ gauging and proceeding to the desired subgauging prescriptions by group contraction. We describe these two steps in the next two subsections.

1. Topological charge of $SO(4)$ gauged $O(5)$ Skyrme system

We denote the $O(5)$ Skyrme scalar ϕ^a, $a = a', 5$, $a' = 1, 2, 3, 4$ and the densities ϱ_0 and ϱ_G appearing in (A1)–(A2) are

$$\varrho_0 = \varepsilon_{ijkl}\varepsilon^{abcde}\partial_i\phi^a \partial_j\phi^b \partial_k\phi^c \partial_l\phi^d \phi^e, \quad (A3)$$

$$\varrho_G = \varepsilon_{ijkl}\varepsilon^{abcde}\mathcal{D}_i\phi^a \mathcal{D}_j\phi^b \mathcal{D}_k\phi^c \mathcal{D}_l\phi^d \phi^e, \quad (A4)$$

where calligraphic $\mathcal{D}$ and $\mathcal{A}$ are used to denote the $SO(4)$ gauged covariant derivative and connections. This is to distinguish these quantities from D and A of the $SO(2)$ gauging in the main body of the work. Thus the $SO(4)$ connection and curvature are expressed by $(\mathcal{A}_i^{a'b'}, \mathcal{F}_{ij}^{a'b'})$, and the gauging prescription by the covariant derivative

$$\mathcal{D}_i\phi^{a'} = \partial_i\phi^{a'} + \mathcal{A}_i\phi^{a'}, \qquad \mathcal{D}_i\phi^5 = \partial_i\phi^5,$$

where $\mathcal{A}_i\phi^{a'} = \mathcal{A}_i^{a'b'}\phi^{b'}$ and $i = 1, 2, 3, 4$.

The quantities $\Omega_i[\mathcal{A}, \phi]$, $W[\mathcal{F}, \mathcal{D}\phi]$ in (A1)–(A2) are given in [5,6] to be

$$\Omega_i = 3!\varepsilon_{ijkl}\varepsilon^{a'b'c'd'}\phi^5\left\{\partial_j\left[\mathcal{A}_l^{a'b'}\phi^{c'}\left(\partial_k\phi^{d'} + \frac{1}{2}\mathcal{A}_k\phi^{d'}\right)\right]\right.$$
$$\left. + \frac{1}{4}\left(1 - \frac{1}{3}(\phi^5)^2\right)\mathcal{A}_l^{a'b'}\left[\partial_j\mathcal{A}_k^{c'd'} + \frac{2}{3}(\mathcal{A}_j\mathcal{A}_k)^{c'd'}\right]\right\}, \quad (A5)$$

$$W = 3!\varepsilon_{ijkl}\varepsilon^{a'b'c'd'}\left\{\frac{1}{16}\phi^5\left(1 - \frac{1}{3}(\phi^5)^2\right)\mathcal{F}_{ij}^{a'b'}\mathcal{F}_{kl}^{c'd'}\right.$$
$$\left. + \frac{1}{2}\mathcal{F}_{ij}^{a'b'}\phi^{c'}\mathcal{D}_k\phi^{d'}\partial_l\phi^5\right\}, \quad (A6)$$

in which $\Omega_i[\mathcal{A}, \phi]$ is manifestly gauge variant, displaying the (Euler)–Chern-Simons density which is typical in all even dimensions, and $W[\mathcal{F}, \mathcal{D}\phi]$ which is manifestly gauge invariant. Clearly, the Euler density can be added to the definitions of both (A5) and (A6), but this is unnecessary here, especially since we do not anticipate the introduction of a potential term in the Lagrangian.

What is important to realize here is that the gauge variant density (A5), consists *exclusively* of gauge variant elements. This feature, which occurred in the 2 and 3 dimensional cases seen in [5,6], is not a general feature in all dimensions. In dimensions $D \geq 4$, there is the freedom to remove the total divergence part of the second

term in (A6) and placing it in (A5). The result is again a gauge invariant definition of the topological charge. $D \geq 4$, the definitions can be altered such that the gauge variant density consists of *both* gauge variant and gauge invariant terms. In this redefinition, Ω_i remains gauge variant, while W remains gauge invariant as required.

By removing a total divergence term in (A6) and placing it in (A5), we have the more aesthetic definitions for the densities

$$\hat{\Omega}_i = 3!\varepsilon_{ijkl}\varepsilon^{a'b'c'd'}\phi^5\left\{\frac{1}{2}\mathcal{F}_{kl}^{c'd'}\phi^{a'}\mathcal{D}_j\phi^{b'}\right.$$
$$+ \partial_j\left[\mathcal{A}_l^{a'b'}\phi^{c'}\left(\partial_k\phi^{d'} + \frac{1}{2}\mathcal{A}_k\phi^{d'}\right)\right]$$
$$\left. + \frac{1}{4}\left(1 - \frac{1}{3}(\phi^5)^2\right)\mathcal{A}_l^{a'b'}\left[\partial_j\mathcal{A}_k^{c'd'} + \frac{2}{3}(\mathcal{A}_j\mathcal{A}_k)^{c'd'}\right]\right\}, \quad (A7)$$

$$\hat{W} = 3!\varepsilon_{ijkl}\varepsilon^{a'b'c'd'}\phi^5$$
$$\times \left\{\frac{1}{24}(\phi^5)^2\mathcal{F}_{ij}^{a'b'}\mathcal{F}_{kl}^{c'd'} + \frac{1}{2}\mathcal{F}_{ij}^{a'b'}\mathcal{D}_{[k}\phi^{c'}\mathcal{D}_{l]}\phi^{d'}\right\}. \quad (A8)$$

Apart from its aesthetic appearance, the density $\hat{W}$, (A8), is necessary for the statement of the relevant Belavin inequalities in Appendix B below.

2. Group contraction

In (A5)–(A6) and (A7)–(A8), $\mathcal{A}_i^{a'b'}$ is the $SO(4)$ connection, $\mathcal{F}_{ij}^{a'b'}$ is the curvature, and $\mathcal{D}_i\phi^{a'} = \partial_i\phi^{a'} + \mathcal{A}_i\phi^{a'}$ is the covariant derivative, with $a' = 1, 2, 3, 4$. In the notation of (6)–(7), $a' = \alpha, A$; $\alpha = 1, 2$; $A = 3, 4$.

We now contract the gauge group $SO(4)$ by setting the components of the connection $\mathcal{A}_i^{a'b'} = (\mathcal{A}_i^{\alpha\beta}, \mathcal{A}_i^{AB}, \mathcal{A}_i^{\alpha A})$ to $\mathcal{A}_i^{\alpha\beta} = A_i\varepsilon^{\alpha\beta}, \mathcal{A}_i^{AB} = B_i\varepsilon^{AB}$, and $\mathcal{A}_i^{\alpha A} = 0$, where $A_i^{\alpha\beta}$ and B_i^{AB} are now two $SO(2)$ connections inside $SO(4)$. The corresponding curvatures are $F_{ij} = \partial_{[i}A_{j]}, G_{ij} = \partial_{[i}B_{j]}$ and $F_{ij}^{\alpha A} = 0$. The covariant derivative $\mathcal{D}_i\phi^{a'} = (\mathcal{D}_i\phi^\alpha, \mathcal{D}_i\phi^A)$ now splits up into

$$\mathcal{D}_i\phi^\alpha = \partial_i\phi^\alpha + A_i(\varepsilon\phi)^\alpha, \quad (A9)$$

$$\mathcal{D}_i\phi^A = \partial_i\phi^A + B_i(\varepsilon\phi)^A. \quad (A10)$$

Finally, identifying[7] the two Abelian fields $A_i = B_i$, (A9)–(A10) we have the desired gauging prescription (6)–(7).

As a result of this group contraction, the topological charge densities following from (A7)–(A8) are expressed by

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[7]Alternatively, setting $B_i = 0$, (A7)–(A8) leads to the corresponding topological charge density displayed in [6], where only one pair of Skyrme scalars is gauged with $SO(2)$.

$$\hat{\Omega}_i = 3! \varepsilon_{ijkl} \phi^5 A_l \left\{ -\frac{1}{3} (\phi^5)^2 F_{jk} \right.$$

$$\left. + 2(\varepsilon^{AB} \partial_j \phi^A \partial_k \phi^B + \varepsilon^{\alpha\beta} \partial_j \phi^\alpha \partial_k \phi^\beta) \right\}, \qquad \text{(A11)}$$

$$\hat{W} = 3! \varepsilon_{ijkl} \phi^5 F_{ij} \left\{ \left(1 - \frac{1}{3} (\phi^5)^2 \right) F_{kl} \right.$$

$$\left. + 2(\varepsilon^{\alpha\beta} D_k \phi^\alpha D_l \phi^\beta + \varepsilon^{AB} D_k \phi^A D_l \phi^B) \right\}, \qquad \text{(A12)}$$

where $D_i \phi^\alpha$ and $D_i \phi^A$ are now given by (6) and (7) respectively, and $F_{ij} = \partial_i A_j - \partial_j A_i$.

Inserting $\hat{\Omega}_i$ and $\hat{W}$ given by (A11)–(A12) into (A1)–(A2) defines the topological charge density for the $SO(2)$ gauged $O(5)$ model studied here.

3. Charge integrals

We adopt the definition of the topological charge density (A1) with Ω_i being given by (A11), and we denote the second term by $\varrho_1 = \partial_i \hat{\Omega}_i$. This term is manifestly total divergence, while the first term in (A1), namely ϱ_0 defined by (A3) is not manifestly total divergence but becomes such when a constraint compliant parametrization of the scalar ϕ^a satisfying (19) is employed. For this purpose, we adopt the parametrization

$$\Psi_1 = \sin f \sin g, \quad \Psi_2 = \sin f \cos g, \quad \Psi_3 = \cos f.$$
$$\text{(A13)}$$

In terms of the functions $f(\rho, \sigma)$ and $g(\rho, \sigma)$ (with $\rho = r \sin \theta$, $\sigma = r \cos \theta$), ϱ_0 reduces to the antisymmetric product

$$\varrho_0 = 2 \cdot 3! \frac{n_1 n_2}{\rho \sigma} \partial_{[\rho} F \partial_{\sigma]} G, \qquad \text{(A14)}$$

where $F(\rho, \sigma)$ and $G(\rho, \sigma)$ are the functions

$$F = \cos f + \frac{2}{3} \cos^3 f - \frac{3}{5} \cos^5 f, \quad G = \sin^2 g. \qquad \text{(A15)}$$

Denoting $(\rho, \sigma) = y_i, i = 1, 2$, the volume integral of ϱ_0 can be cast in the form

$$\int \varrho_0 d^4 x = (2\pi)^2 n_1 n_2 \int \varepsilon_{ij} \partial_i F \partial_j G d^2 y$$

$$= \frac{1}{2} (2\pi)^2 n_1 n_2 \int \varepsilon_{ij} (F \overset{\leftrightarrow}{\partial}_j G) ds_i. \qquad \text{(A16)}$$

It is interesting to point out here that in evaluating the Stokes integral (A16), instead of taking $0 \le \theta \le \frac{\pi}{2}$ one can take the limits $0 \le \theta \le \frac{m\pi}{2}$, with m an integer. For even m, the solutions should be Skyrme-antiSkyrme as in Yang-Mills.

The corresponding integral of the term ϱ_1 can also be evaluated using Stokes theorem, since in that case this density is manifestly total divergence in terms of the functions (f, g)

$$\int \varrho_1 d^4 x$$

$$= (2\pi)^2 \int \varepsilon_{ij} \left\{ \frac{2}{3} \Psi_3^3 [(a_{(1)} - n_1) \partial_j a_2 - (a_{(2)} - n_2) \partial_j a_1] \right.$$

$$\left. + 2\Psi_3 [n_1 (a_{(2)} - n_2) \partial_j \Psi_1^2 - n_2 (a_{(1)} - n_1) \partial_j \Psi_2^2] \right\} ds_i,$$
$$\text{(A17)}$$

where the volume integral is evaluated by applying Stokes' Theorem.

APPENDIX B: THE BELAVIN INEQUALITIES AND THE MODELS

We establish the Belavin inequalities for the $SO(4)$ gauged system, from which follow the corresponding inequalities pertaining to the gauge-contracted systems, in particular those giving the lower bound on the static Hamiltonian of the Lagrangian (10) of the $SO(2)$ gauged model studied here.

The Belavin inequalities are most conveniently derived from definition (A2) of the topological charge, given by $\hat{W}$

Consider now the inequalities

$$\left| \phi^5 \mathcal{F}_{ij}^{a'b'} - \frac{1}{2!^2} \varepsilon_{ijkl} \varepsilon^{a'b'c'd'} \mathcal{F}_{kl}^{c'd'} \right|^2 \ge 0, \qquad \text{(B1)}$$

$$\left| \phi^5 \mathcal{F}_{ij}^{a'b'} - \frac{1}{2!^2} \varepsilon_{ijkl} \varepsilon^{a'b'c'd'} D_{[k} \phi^{c'} D_{l]} \phi^{d'} \right|^2 \ge 0, \qquad \text{(B2)}$$

$$\left| D_{[i} \phi^a D_{j]} \phi^b - \frac{1}{2!^2} \varepsilon_{ijkl} \varepsilon^{abcde} D_{[k} \phi^c D_{l]} \phi^d \phi^e \right|^2 \ge 0,$$
$$a = a', 5 \quad \text{(B3)}$$

The inequalities (B1)–(B3) yield

$$(\phi^5)^2 (1 + (\phi^5)^2) |\mathcal{F}_{ij}^{a'b'}|^2 \ge \frac{1}{4} \varepsilon_{ijkl} \varepsilon^{a'b'c'd'} (\phi^5)^3 \mathcal{F}_{ij}^{a'b'} \mathcal{F}_{kl}^{c'd'},$$
$$\text{(B4)}$$

$$(\phi^5)^2 |\mathcal{F}_{ij}^{a'b'}|^2 + |D_{[i} \phi^{a'} D_{j]} \phi^{b'}|^2$$

$$\ge \frac{1}{2} \varepsilon_{ijkl} \varepsilon^{a'b'c'd'} \phi^5 \mathcal{F}_{ij}^{a'b'} D_{[k} \phi^{c'} D_{l]} \phi^{d'}, \qquad \text{(B5)}$$

$$|D_{[i} \phi^a D_{j]} \phi^b|^2 \ge \varrho_G. \qquad \text{(B6)}$$

Adding $\frac{1}{6}$ times (B4) to (B5) and (B6), the right-hand sides yield the "topological charge" density

$$\varrho = \varrho_G + \hat{W}[\mathcal{F}, D\phi],$$

defined by (A2), with W there replaced by $\hat{W}$ in (A8).

Concerning the left-hand side of that inequality, this can be cast into the form

$$c_1 |\mathcal{F}_{ij}^{a'b'}|^2 + c_2 |\varphi_{ij}^{a'b'}|^2,$$

(c_1, $c_2 > 0$) by simply adding positive definite quantities, recognizing also that in (A8), the quantity $\frac{1}{2}\phi^5(1 - \frac{1}{3}(\phi^5)^2)$ is always positive.

Thus, after the group contraction described in Appendix A.1 with $\mathcal{F}_{ij}^{a'b'} \rightarrow F_{ij}^{a'b'}$ and $\varphi_i^a \rightarrow \phi_i^a$, the static energy density functional pertaining to (10) is the bounded from below by $\varrho_G[\phi^a]$ plus $\hat{W}$ given by (A12). Clearly, the positive definite quadratic and sextic kinetic Skyrme terms in (10) can be added without invalidating the bound.

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