Gravitational and electromagnetic memory

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We present a unified investigation of memory effect in Einstein-Maxwell theory. We specify two types of memory effect, a velocity kick and a position displacement, by examining the motion of a single free-falling charged test particle. Our result recovers the two known gravitational memory effect formulas and the two known electromagnetic memory effect formulas.

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I. INTRODUCTION

In the last few years, there has been renewed interest on gravitational [1–7] and electromagnetic [8] memory effects. Although both of them have been investigated for a long time (see also Refs. [9–16] for the realization in experimental detections), the new enthusiasm comes from a purely theoretical side. In 2014, Strominger and Zhiboedov discovered a fundamental connection between the gravitational memory effect and Weinberg's soft graviton theorem [17]. They are mathematically equivalent. This equivalence was shortly extended to gauge theories [18–20]. Inspired by this fascinating equivalence, new gravitational [21] and new electromagnetic [22] memory effects were reported.

The investigation in the literature on memory effect are performed independently for different theories, either gravitational memory in Einstein theory or electromagnetic memory in Maxwell theory.[1] A unified treatment of different types of memory effects in a coupled theory is still missing. Though gravitational memory effect and electromagnetic memory effect seem to be present at an order in which there is no coupling between the gravitational term and electromagnetic term, the main gap of connecting memory in different theories is encoded in the different types of observation. In Einstein or Maxwell theory, memory effect is interpreted as a change in the waveform of gravitational or electromagnetic wave burst. The memory effect is completely determined by the solution of Einstein equation or Maxwell's equation. The gravitational memory [7] and the new gravitational memory [21] are characterized by the change of the asymptotic shear of the outgoing null surfaces $\Delta\sigma^0$ and its u integral $\int\sigma^0 du$. The electromagnetic memory [8] and the new electromagnetic memory [22] are characterized by the change of the asymptotic data of the gauge field ΔA_z^0 and its u integral $\int A_z^0 du$. In general relativity, it is important to focus upon the coordinate invariant observable. The gravitational memory effect [17] is a relative displacement of nearby observers, while the new gravitational memory effect [21] is a relative time delay between different orbiting light rays. When we turn to the electromagnetic memory, a single charged test particle is utilized. The electromagnetic memory effect [8] is a change of the velocity (a "kick") of the charged particle, while the new electromagnetic memory effect [22] is a position displacement of the charged particle. Hence, one has to implement completely different detections to explore gravitational and electromagnetic memory effects. The aim of the present work is to provide a unified treatment for gravitational and electromagnetic memory effects in Einstein-Maxwell theory. To achieve this, we will give up the requirement of coordinate invariant observable, e.g., the proper separation between two test particles or the proper time of a single test particle. Alternatively, we will study the motion of charged particles.

Free-falling observers receive a velocity kick when gravitational waves with memory pass by [24–30] (see also Refs. [31–34] for earlier but less relevant investigations). This is the observational effect we will adopt from the gravitational side to connect with the electromagnetic memory effect. In this work, we examine the memory effect via studying the motion of a charged free-falling particle.[2] By solving the equations of motion, we find that the charged particle, which is initially static, is forced to orbit over some tiny angle about the "center" of the spacetime by the gravitational and electromagnetic radiation. The velocity change of the charged particle induced by gravitational and electromagnetic radiation is determined

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[1]Memory effect was investigated in Ref. [23] in Einstein-Maxwell theory. But only gravitational effect was involved.

[2]These are test particles. We do not consider them as a local source of the Einstein-Maxwell theory.

by $\Delta\sigma^0$ and ΔA_z^0. Hence, they recover the gravitational and electromagnetic memory formulas, respectively. The position displacement of the charged particle involves the u integral of σ^0 and A_z^0. The gravitational and electromagnetic contributions reproduce the spin memory formula in Ref. [21] and the new electromagnetic memory formula in Ref. [22], respectively.[3] The charged particle receives a time delay. The contributions to the time delay are from the massive objects with or without electric charge in the spacetime [35–37], gravitational radiation [29,30], and electromagnetic radiation. The gravitational and electromagnetic memory effects happen at the same order, while the contribution of electromagnetic radiation to the time delay of the charged particle shows up at one order higher than gravitational radiation.

Our plan is as follows. In the next section, we study the Einstein-Maxwell theory in the Newman-Penrose (NP) formalism [38]. We work in the NP formalism because it makes the geometrical property of the spacetime more transparent. Hence, we can easily find the connection between the memory formula and the geometrical property of the spacetime. The NP formalism also has a natural connection with the spinor formalism, which is the most satisfactory way of investigating fermion coupled theories. We obtain the most general asymptotic solutions of Einstein-Maxwell theory that asymptotically approach flatness. The solution space generalizes the result of Refs. [39,40] by relaxing the unit 2-sphere boundary to the case of an arbitrary 2-surface boundary, although such relaxation is not really needed for deriving the memory formulas in the present work. The solution space of Einstein-Maxwell theory allows us to derive the memory formulas and to compute the time delay of the charged particle in Sec. III. Finally, the two known gravitational memory effects and the two known electromagnetic memory effects are recovered. We then conclude with a discussion. The NP equations are listed in the Appendix.

II. EINSTEIN-MAXWELL THEORY IN THE NP FORMALISM

The NP formalism is a tetrad formalism where two real null vectors $e_1 = l, e_2 = n$, one complex null vector $e_3 = m$ and its complex conjugate vector $e_4 = \bar{m}$ are chosen as the basis vectors. The metric is constructed from the basis vectors as

$$g_{\mu\nu} = n_\mu l_\nu + l_\mu n_\nu - m_\mu \bar{m}_\nu - m_\nu \bar{m}_\mu. \tag{2.1}$$

In a hyperbolic Riemannian manifold [38], it is always possible to introduce a coordinate system (u, r, x^A) where $(A = z, \bar{z})$ and $z = e^{i\phi}\cot\frac{\theta}{2}, \bar{z} = e^{-i\phi}\cot\frac{\theta}{2}$ are the standard stereographic coordinates, such that the basis vectors and the cotetrad have the form

$$\begin{aligned}
n^\mu\partial_\mu &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A\frac{\partial}{\partial x^A}, \qquad l^\mu\partial_\mu = \frac{\partial}{\partial r},\\
m^\mu\partial_\mu &= \omega\frac{\partial}{\partial r} + L^A\frac{\partial}{\partial x^A},\\
n_\mu dx^\mu &= [-U - X^A(\bar{\omega}L_A + \omega\bar{L}_A)]du + dr\\
&\quad + (\omega\bar{L}_A + \bar{\omega}L_A)dx^A,\\
l_\mu dx^\mu &= du, \qquad m_\mu dx^\mu = -X^A L_A du + L_A dx^A,
\end{aligned} \tag{2.2}$$

where $L_A L^A = 0$, $L_A\bar{L}^A = -1$. The connection coefficients are called spin coefficients in the NP formalism with special greek symbols (we will follow the convention of Ref. [41]),

$$\begin{aligned}
\kappa &= \Gamma_{311} = l^\nu m^\mu \nabla_\nu l_\mu, \qquad \pi = -\Gamma_{421} = -l^\nu \bar{m}^\mu \nabla_\nu n_\mu,\\
\epsilon &= \frac{1}{2}(\Gamma_{211} - \Gamma_{431}) = \frac{1}{2}(l^\nu n^\mu \nabla_\nu l_\mu - l^\nu \bar{m}^\mu \nabla_\nu m_\mu),\\
\tau &= \Gamma_{312} = n^\nu m^\mu \nabla_\nu l_\mu, \qquad \nu = -\Gamma_{422} = -n^\nu \bar{m}^\mu \nabla_\nu n_\mu,\\
\gamma &= \frac{1}{2}(\Gamma_{212} - \Gamma_{432}) = \frac{1}{2}(n^\nu n^\mu \nabla_\nu l_\mu - n^\nu \bar{m}^\mu \nabla_\nu m_\mu),\\
\sigma &= \Gamma_{313} = m^\nu m^\mu \nabla_\nu l_\mu, \qquad \mu = -\Gamma_{423} = -m^\nu \bar{m}^\mu \nabla_\nu n_\mu,\\
\beta &= \frac{1}{2}(\Gamma_{213} - \Gamma_{433}) = \frac{1}{2}(m^\nu n^\mu \nabla_\nu l_\mu - m^\nu \bar{m}^\mu \nabla_\nu m_\mu),\\
\rho &= \Gamma_{314} = \bar{m}^\nu m^\mu \nabla_\nu l_\mu, \qquad \lambda = -\Gamma_{424} = -\bar{m}^\nu \bar{m}^\mu \nabla_\nu n_\mu,\\
\alpha &= \frac{1}{2}(\Gamma_{214} - \Gamma_{434}) = \frac{1}{2}(\bar{m}^\nu n^\mu \nabla_\nu l_\mu - \bar{m}^\nu \bar{m}^\mu \nabla_\nu m_\mu).
\end{aligned}$$

The freedom of the rotations of the basis vectors allows one to set

$$\pi = \kappa = \epsilon = 0, \qquad \rho = \bar{\rho}, \qquad \tau = \bar{\alpha} + \beta. \tag{2.3}$$

Ten independent components of the Weyl tensors are represented by five complex scalars

$$\begin{aligned}
\Psi_0 &= -C_{1313}, \qquad \Psi_1 = -C_{1213},\\
\Psi_2 &= -C_{1342}, \qquad \Psi_3 = -C_{1242},\\
\Psi_4 &= -C_{2324}.
\end{aligned}$$

Ricci tensors are defined in terms of four real and three complex scalars:

[3]The displacement effect is from a single test particle, while the displacement discovered in Refs. [1–3] is a relative displacement of nearby observers. So, they are different types of memory effect.

$$\Phi_{00}=-\frac{1}{2}R_{11},\qquad \Phi_{22}=-\frac{1}{2}R_{22},\qquad \Phi_{02}=-\frac{1}{2}R_{33},\qquad \Phi_{20}=-\frac{1}{2}R_{44},$$
$$\Phi_{11}=-\frac{1}{4}(R_{12}+R_{34}),\qquad \Phi_{01}=-\frac{1}{2}R_{13},\qquad \Phi_{12}=-\frac{1}{2}R_{23},$$
$$\frac{1}{24}R=\frac{1}{12}(R_{12}-R_{34}),\qquad \Phi_{10}=-\frac{1}{2}R_{14},\qquad \Phi_{21}=-\frac{1}{2}R_{24}.$$

The Maxwell tensor is replaced by three complex scalars:

$$\phi_0=F_{\mu\nu}l^\mu m^\nu,\qquad \phi_1=\frac{1}{2}F_{\mu\nu}(l^\mu n^\nu+\bar{m}^\mu m^\nu),\qquad \phi_2=F_{\mu\nu}\bar{m}^\mu n^\nu.$$

The Lagrangian of four-dimensional Einstein-Maxwell theory is

$$\mathcal{L}=\sqrt{-g}\left[R-\frac{1}{2}F^2\right],\qquad F=dA. \tag{2.4}$$

For the coupled theory, $R=0$ and Φ_{ab} should be replaced by $\phi_a\bar{\phi}_b$. As directional derivatives, the basis vectors are designated with special symbols

$$D=l^\mu\partial_\mu,\qquad \Delta=n^\mu\partial_\mu,\qquad \delta=m^\mu\partial_\mu. \tag{2.5}$$

The Newman-Penrose equations that we will deal with are listed in the Appendix.

The main conditions of approaching flatness at infinity are $\Psi_0=\frac{\Psi_0^0}{r^5}+\mathcal{O}(r^{-6})$ and $\phi_0=\frac{\phi_0^0}{r^3}+\mathcal{O}(r^{-4})$. The solutions of the NP equations in asymptotic expansions were first obtained in Refs. [39,40]. However, a special choice of the boundary topology S^2 was adopted in Ref. [40]. We remove this restriction, and a more general solution space with arbitrary 2-surface boundary topology is given by[4]

$$\begin{aligned}
\Psi_0&=\frac{\Psi_0^0(u,z,\bar{z})}{r^5}+\frac{\Psi_0^1(u,z,\bar{z})}{r^6}+\mathcal{O}(r^{-7}),\qquad \phi_0=\frac{\phi_0^0(u,z,\bar{z})}{r^3}+\frac{\phi_0^1(u,z,\bar{z})}{r^4}+\mathcal{O}(r^{-5}),\\
\Psi_1&=\frac{\Psi_1^0(u,z,\bar{z})}{r^4}+\frac{3\phi_0^0\bar{\phi}_1^0-\bar{\eth}\Psi_0^0}{r^5}+\mathcal{O}(r^{-6}),\qquad \phi_1=\frac{\phi_1^0(u,z,\bar{z})}{r^2}-\frac{\bar{\eth}\phi_0^0}{r^3}+\mathcal{O}(r^{-4}),\\
\Psi_2&=\frac{\Psi_2^0(u,z,\bar{z})}{r^3}+\frac{\phi_1^0\bar{\phi}_1^0-\bar{\eth}\Psi_1^0}{r^4}+\frac{1}{2r^5}[\lambda^0\Psi_0^0+\bar{\eth}^2\Psi_0^0+3\sigma^0\bar{\sigma}^0\Psi_2^0+4\Psi_1^0\eth\bar{\sigma}^0+\bar{\sigma}^0\eth\Psi_1^0\\
&\quad-2\phi_1^0\eth\bar{\phi}_0^0-6\bar{\phi}_1^0\bar{\eth}\phi_0^0-3\phi_0^0\bar{\eth}\bar{\phi}_1^0+(\gamma^0+3\bar{\gamma}^0)\phi_0^0\bar{\phi}_0^0+\bar{\phi}_0^0\partial_u\phi_0^0)]+\mathcal{O}(r^{-6}),\\
\phi_2&=\frac{\phi_2^0(u,z,\bar{z})}{r}-\frac{\bar{\eth}\phi_1^0}{r^2}+\frac{\lambda^0\phi_0^0+\sigma^0\bar{\sigma}^0\phi_2^0+2\phi_1^0\eth\bar{\sigma}^0+\bar{\sigma}^0\eth\phi_1^0+\bar{\eth}^2\phi_0^0}{r^3}+\mathcal{O}(r^{-4})\\
\Psi_3&=\frac{\Psi_3^0}{r^2}+\frac{\phi_2^0\bar{\phi}_1^0-\bar{\eth}\Psi_2^0}{r^3}+\mathcal{O}(r^{-4}),\qquad \Psi_4=\frac{\Psi_4^0}{r}-\frac{\bar{\eth}\Psi_3^0}{r^2}+\mathcal{O}(r^{-3}),\\
\rho&=-\frac{1}{r}-\frac{\sigma^0\bar{\sigma}^0}{r^3}+\frac{\sigma^0\bar{\Psi}_0^0+\bar{\sigma}^0\Psi_0^0-6(\sigma^0\bar{\sigma}^0)^2-2\phi_0^0\bar{\phi}_0^0}{6r^5}+\mathcal{O}(r^{-6}),\\
\sigma&=\frac{\sigma^0(u,z,\bar{z})}{r^2}+\frac{\sigma^0\sigma^0\bar{\sigma}^0-\frac{1}{2}\Psi_0^0}{r^4}-\frac{\Psi_0^1}{3r^5}+O(r^{-6}),
\end{aligned} \tag{2.6}$$

[4]The relaxation is encoded in the leading order of $L^{\bar{z}}$. We have an arbitrary function $P(u,z,\bar{z})$ rather than a particular choice $\frac{1+z\bar{z}}{\sqrt{2}}$ for a unit 2-sphere. The relaxation in the solution space is mainly shown in the integration constant (2.8) and the evolution equations (2.9)–(2.14).

$$
\begin{aligned}
\alpha &= \frac{\alpha^0}{r} + \frac{\bar\sigma^0\bar\alpha^0}{r^2} + \frac{\sigma^0\bar\sigma^0\alpha^0}{r^3} + \frac{6\bar\alpha^0\sigma^0(\bar\sigma^0)^2 - \bar\alpha^0\bar\Psi_0^0 + \bar\sigma^0\Psi_1^0 - 2\phi_1^0\bar\phi_0^0}{6r^4} + \mathcal{O}(r^{-5}),\\
\beta &= -\frac{\bar\alpha^0}{r} - \frac{\sigma^0\alpha^0}{r^2} - \frac{\sigma^0\bar\sigma^0\bar\alpha^0 + \frac{1}{2}\Psi_1^0}{r^3} + \frac{\bar\eth\Psi_0^0 + \frac{1}{2}\alpha^0\Psi_0^0 - 3\alpha^0(\sigma^0)^2\bar\sigma^0 - 3\phi_0^0\bar\phi_1^0}{3r^4} + \mathcal{O}(r^{-5}),\\
\tau &= -\frac{\Psi_1^0}{2r^3} + \frac{\bar\eth\Psi_0^0 + \frac{1}{2}\sigma^0\bar\Psi_1^0 - 4\phi_0^0\bar\phi_1^0}{3r^4} + \mathcal{O}(r^{-5}),\\
\mu &= \frac{\mu^0}{r} - \frac{\sigma^0\lambda^0 + \Psi_2^0}{r^2} + \frac{\sigma^0\bar\sigma^0\mu^0 + \frac{1}{2}\bar\eth\Psi_1^0 - \phi_1^0\bar\phi_1^0}{r^3} + \mathcal{O}(r^{-4}),\\
\lambda &= \frac{\lambda^0}{r} - \frac{\bar\sigma^0\mu^0}{r^2} + \frac{\sigma^0\bar\sigma^0\lambda^0 + \frac{1}{2}\bar\sigma^0\Psi_2^0 - \frac{1}{2}\phi_2^0\bar\phi_0^0}{r^3} + \mathcal{O}(r^{-4}),\\
\gamma &= \gamma^0 - \frac{\Psi_2^0}{2r^2} + \frac{2\bar\eth\Psi_1^0 + \alpha^0\Psi_1^0 - \bar\alpha^0\bar\Psi_1^0 - 6\phi_1^0\bar\phi_1^0}{6r^3}\\
&\quad + \frac{1}{24r^4}[-3\lambda^0\Psi_0^0 - 3\bar\eth^2\Psi_0^0 - 3\bar\sigma^0\eth\Psi_1^0 - 9\sigma^0\bar\sigma^0\Psi_2^0 - 12\Psi_1^0\eth\bar\sigma^0\\
&\quad + 4(\bar\alpha^0\bar\sigma^0\Psi_1^0 - \alpha^0\sigma^0\bar\Psi_1^0) + 2(\bar\alpha^0\eth\bar\Psi_0^0 - \alpha^0\bar\eth\Psi_0^0) + 8(\alpha^0\phi_0^0\bar\phi_1^0 - \bar\alpha^0\bar\phi_0^0\phi_1^0)\\
&\quad - 3(\gamma^0 + 3\bar\gamma^0)\phi_0^0\bar\phi_0^0 + 12\phi_1^0\eth\bar\phi_0^0 + 24\bar\phi_1^0\bar\eth\phi_0^0 + 9\phi_0^0\bar\eth\bar\phi_1^0 - 3\bar\phi_0^0\partial_u\phi_0^0] + \mathcal{O}(r^{-5}),\\
\nu &= \nu^0 - \frac{\Psi_3^0}{r} + \frac{\bar\eth\Psi_2^0 - 2\phi_2^0\bar\phi_1^0}{2r^2} + \mathcal{O}(r^{-3}),\\
X^z &= \frac{\bar P\Psi_1^0}{6r^3} + \frac{\bar P}{12r^4}(-\bar\eth\Psi_0^0 - 2\sigma^0\bar\Psi_1^0 + 4\phi_0^0\bar\phi_1^0) + \mathcal{O}(r^{-5}),\\
\omega &= \frac{\bar\eth\sigma^0}{r} - \frac{\sigma^0\eth\bar\sigma^0 + \frac{1}{2}\Psi_1^0}{r^2} + \frac{\bar\eth\Psi_0^0 + 6\sigma^0\bar\sigma^0\bar\eth\sigma^0 + 2\sigma^0\bar\Psi_1^0 - 4\phi_0^0\bar\phi_1^0}{6r^3} + \mathcal{O}(r^{-4}),\\
U &= -r(\gamma^0 + \bar\gamma^0) + \mu^0 - \frac{\Psi_2^0 + \bar\Psi_2^0}{2r} + \frac{\bar\eth\Psi_1^0 + \eth\bar\Psi_1^0 - 6\phi_1^0\bar\phi_1^0}{6r^2} - \frac{1}{24r^3}[\lambda^0\Psi_0^0 + \bar\lambda^0\bar\Psi_0^0\\
&\quad + \bar\eth^2\Psi_0^0 + \eth^2\bar\Psi_0^0 + \bar\sigma^0\eth\Psi_1^0 + \sigma^0\bar\eth\bar\Psi_1^0 + 3\sigma^0\bar\sigma^0(\Psi_2^0 + \bar\Psi_2^0)\partial_u(\phi_0^0\bar\phi_0^0)\\
&\quad + 4(\gamma^0 + \bar\gamma^0)\phi_0^0\bar\phi_0^0 - 12\phi_1^0\eth\bar\phi_0^0 - 12\bar\phi_1^0\bar\eth\phi_0^0 - 3\bar\phi_0^0\eth\phi_1^0 - 3\phi_0^0\bar\eth\bar\phi_1^0] + \mathcal{O}(r^{-4}),\\
L^z &= -\frac{\sigma^0\bar P(u,z,\bar z)}{r^2} - \frac{\bar P}{r^4}\left((\sigma^0)^2\bar\sigma^0 - \frac{1}{6}\Psi_0^0\right) + \frac{\bar P\Psi_0^1}{12r^5} + \mathcal{O}(r^{-6}),\\
L^{\bar z} &= \frac{P(u,z,\bar z)}{r} + \frac{\sigma^0\bar\sigma^0 P}{r^3} + \frac{P}{12r^5}(12(\sigma^0\bar\sigma^0)^2 + \phi_0^0\bar\phi_0^0 - 2\bar\sigma^0\Psi_0^0 - \sigma^0\bar\Psi_0^0) + \mathcal{O}(r^{-6}),\\
L_z &= -\frac{r}{\bar P} + \frac{\bar\sigma^0\Psi_0^0 + \phi_0^0\bar\phi_0^0}{12\bar P r^3} + \mathcal{O}(r^{-4}), L_{\bar z} = -\frac{\sigma^0}{P} + \frac{\Psi_0^0}{6Pr^2} + \frac{\Psi_0^1}{12Pr^3} + \mathcal{O}(r^{-4}),
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
\alpha^0 &= \frac{1}{2}\bar P\partial_z \ln P, \qquad \mu^0 = -\frac{1}{2}P\bar P\partial_z\partial_{\bar z}\ln P\bar P,\\
\lambda^0 &= \partial_u\bar\sigma^0 + \bar\sigma^0(3\gamma^0 - \bar\gamma^0),\\
\gamma^0 &= -\frac{1}{2}\partial_u \ln\bar P, \qquad \nu^0 = \bar\eth(\gamma^0 + \bar\gamma^0),
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
\Psi_2^0 - \bar\Psi_2^0 &= \bar\eth^2\sigma^0 - \eth^2\bar\sigma^0 + \bar\sigma^0\bar\lambda^0 - \sigma^0\lambda^0,\\
\Psi_3^0 &= \bar\eth\mu^0 - \eth\lambda^0, \qquad \Psi_4^0 = \bar\eth\nu^0 - \partial_u\lambda^0 - 4\gamma^0\lambda^0,\\
\partial_u\phi_0^0 + (\gamma^0 + 3\bar\gamma^0)\phi_0^0 &= \eth\phi_1^0 + \sigma^0\phi_2^0,
\end{aligned}
\tag{2.9}
$$

$$
\partial_u\phi_1^0 + 2(\gamma^0 + \bar\gamma^0)\phi_1^0 = \eth\phi_2^0,
\tag{2.10}
$$

TABLE I. Spin weights.

	ð	∂_u	γ^0	ν^0	μ^0	σ^0	λ^0	Ψ_4^0	Ψ_3^0	Ψ_2^0	Ψ_1^0	Ψ_0^0	ϕ_2^0	ϕ_1^0	ϕ_0^0
s	1	0	0	−1	0	2	−2	−2	−1	0	1	2	−1	0	1

$$\partial_u\Psi_0^0 + (\gamma^0 + 5\bar{\gamma}^0)\Psi_0^0 = ð\Psi_1^0 + 3\sigma^0\Psi_2^0 + 3\phi_0^0\bar{\phi}_2^0, \quad (2.11)$$

$$\partial_u\Psi_1^0 + 2(\gamma^0 + 2\bar{\gamma}^0)\Psi_1^0 = ð\Psi_2^0 + 2\sigma^0\Psi_3^0 + 2\phi_1^0\bar{\phi}_2^0, \quad (2.12)$$

$$\partial_u\Psi_2^0 + 3(\gamma^0 + \bar{\gamma}^0)\Psi_2^0 = ð\Psi_3^0 + \sigma^0\Psi_4^0 + \phi_2^0\bar{\phi}_2^0, \quad (2.13)$$

$$\partial_u\Psi_3^0 + 2(2\gamma^0 + \bar{\gamma}^0)\Psi_3^0 = ð\Psi_4^0. \quad (2.14)$$

The "ð" operator is defined as

$$ð\eta^s = P\bar{P}^{-s}\partial_{\bar{z}}(\bar{P}^s\eta^s) = P\partial_{\bar{z}}\eta^s + 2s\bar{\alpha}^0\eta^s,$$
$$\bar{ð}\eta^s = \bar{P}P^s\partial_z(P^{-s}\eta^s) = \bar{P}\partial_z\eta^s - 2s\alpha^0\eta^s, \quad (2.15)$$

where s is the spin weight of the field η. The spin weights of relevant fields are listed in Table I.

We will work in retarded radial gauge $A_r = 0$. In terms of the gauge fields A_μ, the solution of the electromagnetic fields is

$$A_u^0 = -(\phi_1^0 + \bar{\phi}_1^0), \qquad \partial_u A_z^0 = -\frac{\phi_2^0}{\bar{P}},$$
$$A_z^1 = -\frac{\bar{\phi}_0^0}{\bar{P}}, \qquad (\partial_z A_{\bar{z}}^0 - \partial_{\bar{z}}A_z^0) = \frac{\phi_1^0 - \bar{\phi}_1^0}{P\bar{P}}, \quad (2.16)$$

$$\partial_u\left(\frac{A_u^0}{P\bar{P}}\right) = \partial_u(\partial_z A_{\bar{z}}^0 + \partial_{\bar{z}}A_z^0), \quad (2.17)$$

where

$$A_u = \frac{A_u^0(u,z,\bar{z})}{r} + \mathcal{O}(r^{-2}),$$
$$A_z = A_z^0(u,z,\bar{z}) + \frac{A_z^1(u,z,\bar{z})}{r} + \mathcal{O}(r^{-2}). \quad (2.18)$$

III. MEMORY EFFECTS

The memory effects are all encoded in the solution space derived in the previous section. To specify the observational effects, we will examine the motion of a massive charged particle. The charged particle will be constrained to a fixed radial distance r_0 that is very far from the gravitational and electromagnetic source, for instance, constrained on the Earth. The $r = r_0$ hypersurface is timelike; its induced metric can be derived easily by inserting the solution space in the previous section into (2.1) and (2.2). The induced metric in series expansions is given by

$$ds^2 = \left[1 + \frac{\Psi_2^0 + \bar{\Psi}_2^0}{r_0} - \frac{\bar{ð}\Psi_1^0 + ð\bar{\Psi}_1^0 - 6\phi_1^0\bar{\phi}_1^0}{3r_0^2} + O(r^{-3})\right]du^2 - 2\left[\frac{ð\bar{\sigma}^0}{P_s} - \frac{2\bar{\Psi}_1^0}{3P_s r_0} + O(r_0^{-2})\right]dudz$$
$$-2\left[\frac{\bar{ð}\sigma^0}{P_s} - \frac{2\Psi_1^0}{3P_s r_0} + O(r_0^{-2})\right]dud\bar{z} - \left[2\frac{\bar{\sigma}^0 r}{P_s^2} - \frac{\bar{\Psi}_0^0}{3P_s^2 r_0} + O(r_0^{-2})\right]dz^2 - \left[2\frac{\sigma^0 r_0}{P_s^2} - \frac{\Psi_0^0}{3P_s^2 r_0} + O(r_0^{-2})\right]d\bar{z}^2$$
$$-2\left[\frac{r_0^2}{P_s^2} + \frac{\sigma^0\bar{\sigma}^0}{P_s^2} + O(r_0^{-2})\right]dzd\bar{z}, \quad (3.1)$$

where $P_s = \frac{1+z\bar{z}}{\sqrt{2}}$. We now work in the unit 2-sphere case by setting $P = \bar{P} = P_s$. The induced Maxwell field on the $r = r_0$ hypersurface is

$$F_{uz} = -\frac{\phi_2^0}{P_s} + \frac{\bar{ð}\phi_1^0 - \bar{\sigma}^0\bar{\phi}_2^0}{P_s r_0} + \mathcal{O}(r_0^{-2}), \quad F_{u\bar{z}} = -\frac{\bar{\phi}_2^0}{P_s} + \frac{ð\bar{\phi}_1^0 - \sigma^0\phi_2^0}{P_s r_0} + \mathcal{O}(r_0^{-2}), \quad F_{z\bar{z}} = \frac{\phi_1^0 - \bar{\phi}_1^0}{P_s^2} + \frac{ð\bar{\phi}_0^0 - \bar{ð}\phi_0^0}{P_s^2 r_0} + \mathcal{O}(r_0^{-2}). \quad (3.2)$$

A free-falling charged particle with a net charge q on this hypersurface will of course not travel along the geodesic. The tangent vector V of the particle worldline satisfies

$$V^\nu(\bar{\nabla}_\nu V^\mu + q\bar{F}_\nu{}^\mu) = 0, \quad (3.3)$$

where $\bar{\nabla}$ is the covariant derivative on this three-dimensional hypersurface. Following Ref. [29], we impose that V is given in series expansion as

$$V^u = 1 + \sum_{a=1}^{\infty} \frac{V^u_a}{r^a}, \qquad V^z = \sum_{a=2}^{\infty} \frac{V^z_a}{r^a}. \tag{3.4}$$

Then, we can solve (3.3) order by order. The solution up to relevant order is

$$V^u_1 = -\frac{\Psi^0_2 + \bar{\Psi}^0_2}{2}, \tag{3.5}$$

$$V^u_2 = \frac{1}{6}(\bar{\eth}\Psi^0_1 + \eth\bar{\Psi}^0_1) - \eth\bar{\sigma}^0\bar{\eth}\sigma^0 + \frac{3}{8}(\Psi^0_2 + \bar{\Psi}^0_2)^2 - \phi^0_1\bar{\phi}^0_1 + q^2P_s^2A^0_zA^0_{\bar{z}}, \tag{3.6}$$

$$V^z_2 = -P_s\bar{\eth}\sigma^0 + qP_s^2A^0_{\bar{z}}, \tag{3.7}$$

$$V^z_3 = P_s\left[2\eth\bar{\sigma}^0\sigma^0 + \frac{2}{3}\Psi^0_1 + \frac{1}{2}\bar{\eth}\sigma^0(\Psi^0_2 + \bar{\Psi}^0_2)\right] - P_s\int dv \frac{\eth(\Psi^0_2 + \bar{\Psi}^0_2 + 2qA^0_u)}{2} - 2qP_s^2\sigma^0A^0_z + qP_s^2A^1_{\bar{z}}. \tag{3.8}$$

We have set all integration constants of u to zero as we require that the charged particle is initially static.

At r_0^{-2} order, V has angular components due to the presence of gravitational waves characterized by σ^0 and electromagnetic waves characterized by $A^0_{\bar{z}}$. In other words, the radiation forces the charged particle to rotate over some tiny angle about the center of the spacetime $r = 0$. The memory effect is the velocity kick of the charged particle

$$\Delta V^z = -\frac{1}{r_0^2}(P_s\bar{\eth}\Delta\sigma^0 - qP_s^2\Delta A^0_{\bar{z}}) + \mathcal{O}(r_0^{-3}). \tag{3.9}$$

It includes two parts: namely, the gravitational contribution $-P_s\bar{\eth}\Delta\sigma^0$ and electromagnetic contribution $qP_s^2\Delta A^0_{\bar{z}}$. They precisely recover the gravitational memory formula in Ref. [7] and the electromagnetic memory formula in Ref. [8].

Both gravitational and electromagnetic radiation have a decomposition into the E mode and B modes [42]. The decomposition into electric and magnetic parts is achieved by relating σ^0 or ϕ^0_2 to spin-weight-0 fields

$$\sigma^0 = \eth^2[A(u,z,\bar{z}) + iB(u,z,\bar{z})],$$
$$\phi^0_2 = \partial_u\bar{\eth}[C(u,z,\bar{z}) + iD(u,z,\bar{z})],$$

where the second relation is equivalent to

$$A^0_{\bar{z}} = -\partial_{\bar{z}}(C + iD).$$

Inserting those decomposition into (2.10) and (2.13), one obtains

$$\eth\bar{\eth}\Delta C = \frac{1}{2}\Delta(\phi^0_1 + \bar{\phi}^0_1),$$
$$\eth^2\bar{\eth}^2\Delta A = -\frac{1}{2}\Delta(\Psi^0_2 + \bar{\Psi}^0_2 + \sigma^0\partial_u\bar{\sigma}^0 + \bar{\sigma}^0\partial_u\sigma^0) + \int du(\partial_u\sigma^0\partial_u\bar{\sigma}^0 + \phi^0_2\bar{\phi}^0_2), \tag{3.10}$$

and

$$i\eth\bar{\eth}\Delta D = \frac{1}{2}\Delta(\phi^0_1 - \bar{\phi}^0_1),$$
$$i\eth^2\bar{\eth}^2\Delta B = \frac{1}{2}\Delta(\Psi^0_2 - \bar{\Psi}^0_2 + \sigma^0\partial_u\bar{\sigma}^0 - \bar{\sigma}^0\partial_u\sigma^0). \tag{3.11}$$

Note that we now work in the unit 2-sphere case. The E-mode electromagnetic memory in (3.10) only has the ordinary part $\frac{1}{2}\Delta(\phi^0_1 + \bar{\phi}^0_1)$ following the classification of Ref. [8], because there is no charged matter coupled to the theory. Hence, no charged radiation reaches null infinity. The E-mode gravitational memory has both [43,44] the ordinary part

$$-\frac{1}{2}\Delta(\Psi^0_2 + \bar{\Psi}^0_2 + \sigma^0\partial_u\bar{\sigma}^0 + \bar{\sigma}^0\partial_u\sigma^0)$$

and the null part

$$\int du(\partial_u\sigma^0\partial_u\bar{\sigma}^0 + \phi^0_2\bar{\phi}^0_2).$$

The B-mode memory (3.11) cannot be studied from a purely asymptotic argument [42]. The B-mode memory just recovers the relations

$$(\partial_z A^0_{\bar{z}} - \partial_{\bar{z}}A^0_z) = \frac{\phi^0_1 - \bar{\phi}^0_1}{P\bar{P}},$$
$$\bar{\eth}^2\sigma^0 - \eth^2\bar{\sigma}^0 = \Psi^0_2 - \bar{\Psi}^0_2 + \sigma^0\lambda^0 - \bar{\sigma}^0\bar{\lambda}^0$$

in (2.16) and (2.8). However, the B-mode memory can been seen in the position displacement as we will show below.

Following the treatment in electromagnetism [22], one can define a second memory effect by a position displacement of the charged particle

$$\Delta z = \int V^z du = -\frac{1}{r_0^2}\int du(P_s\bar{\eth}\sigma^0 - qP_s^2A^0_{\bar{z}}) + \mathcal{O}(r_0^{-3}), \tag{3.12}$$

where we have used the fact that $du = d\chi + \mathcal{O}(r_0^{-1})$ and χ is the proper time. It also includes two parts, namely the gravitational contribution $-\int(P_s\bar{\eth}\sigma^0)du$ and electromagnetic contribution $\int(qP_s^2A^0_{\bar{z}})du$. They precisely recover the spin memory formula in Ref. [21] and the displacement memory formula in Ref. [22].

Inserting the B-mode decomposition of electromagnetic and gravitational radiation into (2.9) and (2.12), we obtain

$$i\bar{\eth}\eth\bar{\eth}\eth\int Ddu = \frac{1}{2}\Delta(\bar{\eth}\phi_0^0 - \eth\bar{\phi}_0^0) + \frac{1}{2}\int du[\eth(\bar{\sigma}^0\bar{\phi}_0^0) - \bar{\eth}(\sigma^0\phi_2^0)], \qquad (3.13)$$

and

$$\begin{aligned} i\bar{\eth}\eth\bar{\eth}^2\eth^2\int Bdu &= \frac{1}{2}\Delta(\bar{\eth}\Psi_1^0 - \eth\bar{\Psi}_1^0) + \frac{1}{2}\int du(\sigma^0\partial_u\bar{\sigma}^0 - \bar{\sigma}^0\partial_u\sigma^0) \\ &+ \int du[\sigma^0\bar{\eth}\eth\bar{\sigma}^0 - \bar{\sigma}^0\eth\bar{\eth}\sigma^0 + \eth(\bar{\phi}_1^0\phi_2^0) - \bar{\eth}(\phi_1^0\bar{\phi}_2^0)]. \end{aligned} \qquad (3.14)$$

Interestingly, the B-mode electromagnetic memory (3.13) now has a null part. The mixed term $\sigma^0\phi_2^0$ in (2.9) is the "magnetic" source that reaches null infinity.

Another observational memory effect is a time delay of the free-falling particle [29,30]. The electromagnetic radiation can also contribute to the time delay of a charged particle. Since V is timelike, the infinitesimal change of the proper time can be derived from the covector[5]

$$\begin{aligned} d\chi = \Bigg[1 &+ \frac{1}{2r_0}(\Psi_2^0 + \bar{\Psi}_2^0) \\ &- \frac{1}{r_0^2}\left(\frac{1}{8}(\Psi_2^0 + \bar{\Psi}_2^0)^2 + \frac{1}{6}(\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0) - \bar{\eth}\sigma^0\eth\bar{\sigma}^0 \right. \\ &\left. - \phi_1^0\bar{\phi}_1^0 + q^2P_s^2A_z^0A_{\bar{z}}^0\right)\Bigg]du + \mathcal{O}(r_0^{-3}). \end{aligned} \qquad (3.15)$$

Clearly, the electromagnetic contribution $(\phi_1^0\bar{\phi}_1^0 - q^2P_s^2A_z^0A_{\bar{z}}^0)$ comes one order higher than the gravitational contribution $\frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0)$ in the $\frac{1}{r_0}$ expansion.

IV. CONCLUSIONS

In this work, the gravitational memory effect and the electromagnetic memory effect are investigated in a unified fashion by examining the motion of a charged test particle. Some interesting applications and open questions may cross the reader's mind. We have only concerned ourselves with the memory effects that are related to soft theorems in the present work. However, as reported in Ref. [45], the memory effect can be defined as infinite towers at every order. We believe that the unified method we proposed here is also applicable for the higher-order memory effect. One just needs to check more orders in (3.4). Since our motivation is to provide a unified treatment of memory effect in coupled theories. It would be of interest to test our treatment in more generic theories with more matter fields coupled in various ways or even string theory [46]. And the equivalence between soft theorems and memory effects could be investigated in a systematical way with our treatment. In the present work, we applied the Newman-Unti gauge [47], which is the most convenient one to derive the solution space and hence the memory effect. However, the universality of the leading soft theorems implies a gauge-independent deviation of the memory effect, e.g., symmetry or conformal structure [48]. It is of interest to study this issue elsewhere. Another interesting point is about the double soft theorem (see, e.g., Refs. [49,50]). Hopefully, our treatment can shine light on the understanding of the memory effect which is connected to the double soft theorem.

[5]We have used the fact that $dz = \frac{V_0^z}{r_0^2}du + \mathcal{O}(r_0^{-3})$.

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APPENDIX: NP EQUATIONS

Radial equations are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \phi_0\bar{\phi}_0, \qquad (A1)$$

$$D\sigma = 2\rho\sigma + \Psi_0, \qquad (A2)$$

$$D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1 + \phi_0\bar{\phi}_1, \qquad (A3)$$

$$D\alpha = \rho\alpha + \beta\bar{\sigma} + \phi_1\bar{\phi}_0, \qquad (A4)$$

$$D\beta = \alpha\sigma + \rho\beta + \Psi_1, \qquad (A5)$$

$$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 + \phi_1\bar{\phi}_1, \qquad (A6)$$

$$D\lambda = \rho\lambda + \bar{\sigma}\mu + \phi_2\bar{\phi}_0, \qquad (A7)$$

$$D\mu = \rho\mu + \sigma\lambda + \Psi_2, \qquad (A8)$$

$$D\nu = \bar{\tau}\mu + \tau\lambda + \Psi_3 + \phi_2\bar{\phi}_1, \qquad (A9)$$

$$DU = \bar{\tau}\omega + \tau\bar{\omega} - (\gamma + \bar{\gamma}), \qquad (A10)$$

$$DX^A = \bar{\tau}L^A + \tau\bar{L}^A, \qquad (A11)$$

$$D\omega = \rho\omega + \sigma\bar{\omega} - \tau, \qquad (A12)$$

$$DL^A = \rho L^A + \sigma \bar{L}^A, \tag{A13}$$

$$D\Psi_1 - \bar{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0 + \bar{\phi}_1 D\phi_0 - \bar{\phi}_0\delta\phi_0 - 2\sigma\phi_1\bar{\phi}_0 + 2\beta\phi_0\bar{\phi}_0, \tag{A14}$$

$$D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0 + \bar{\phi}_1\bar{\delta}\phi_0 - \bar{\phi}_0\Delta\phi_0 - 2\alpha\phi_0\bar{\phi}_1 + 2\rho\phi_1\bar{\phi}_1 + 2\gamma\phi_0\bar{\phi}_0 - 2\tau\phi_1\bar{\phi}_0, \tag{A15}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1 + \bar{\phi}_1 D\phi_2 - \bar{\phi}_0\delta\phi_2 + 2\mu\phi_1\bar{\phi}_0 - 2\beta\phi_2\bar{\phi}_0, \tag{A16}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2 - \bar{\phi}_0\Delta\phi_2 + \bar{\phi}_1\bar{\delta}\phi_2 + 2\alpha\phi_2\bar{\phi}_1 + 2\nu\phi_1\bar{\phi}_0 - 2\gamma\phi_2\bar{\phi}_0 - 2\lambda\phi_1\bar{\phi}_1, \tag{A17}$$

$$D\phi_1 - \bar{\delta}\phi_0 = 2\rho\phi_1 - 2\alpha\phi_0, \tag{A18}$$

$$D\phi_2 - \bar{\delta}\phi_1 = \rho\phi_2 - \lambda\phi_0. \tag{A19}$$

Nonradial equations are

$$\Delta\lambda = \bar{\delta}\nu - (\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + 2\alpha\nu - \Psi_4, \tag{A20}$$

$$\Delta\rho = \bar{\delta}\tau - \rho\bar{\mu} - \sigma\lambda - 2\alpha\tau + (\gamma + \bar{\gamma})\rho - \Psi_2, \tag{A21}$$

$$\Delta\alpha = \bar{\delta}\gamma + \rho\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3, \tag{A22}$$

$$\Delta\mu = \delta\nu - \mu^2 - \lambda\bar{\lambda} - (\gamma + \bar{\gamma})\mu + 2\beta\nu - \phi_2\bar{\phi}_2, \tag{A23}$$

$$\Delta\beta = \delta\gamma - \mu\tau + \sigma\nu + \beta(\gamma - \bar{\gamma} - \mu) - \alpha\bar{\lambda} - \phi_1\bar{\phi}_2, \tag{A24}$$

$$\Delta\sigma = \delta\tau - \sigma\mu - \rho\bar{\lambda} - 2\beta\tau + (3\gamma - \bar{\gamma})\sigma - \phi_0\bar{\phi}_2, \tag{A25}$$

$$\Delta\omega = \delta U + \bar{\nabla}u - \bar{\lambda}\,\bar{\omega} + (\gamma - \bar{\gamma} - \mu)\omega, \tag{A26}$$

$$\Delta L^A = \delta X^A - \bar{\lambda}\bar{L}^A + (\gamma - \bar{\gamma} - \mu)L^A, \tag{A27}$$

$$\delta\rho - \bar{\delta}\sigma = \rho\tau - \sigma(3\alpha - \bar{\beta}) - \Psi_1 + \phi_0\bar{\phi}_1, \tag{A28}$$

$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \Psi_2 + \phi_1\bar{\phi}_1, \tag{A29}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\bar{\tau} + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 + \phi_2\bar{\phi}_1, \tag{A30}$$

$$\delta\bar{\omega} - \bar{\delta}\omega = \mu - \bar{\mu} - (\alpha - \bar{\beta})\omega + (\bar{\alpha} - \beta)\bar{\omega}, \tag{A31}$$

$$\delta\bar{L}^A - \bar{\delta}L^A = (\bar{\alpha} - \beta)\bar{L}^A - (\alpha - \bar{\beta})L^A, \tag{A32}$$

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2 - \bar{\phi}_2 D\phi_0 + \bar{\phi}_1\delta\phi_0 - 2\beta\phi_0\bar{\phi}_1 + 2\sigma\phi_1\bar{\phi}_1, \tag{A33}$$

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + (2\gamma - 2\mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \bar{\phi}_1\Delta\phi_0 - \bar{\phi}_2\bar{\delta}\phi_0 - 2\rho\phi_1\bar{\phi}_2 - 2\gamma\phi_0\bar{\phi}_1 + 2\tau\phi_1\bar{\phi}_1 + 2\alpha\phi_0\bar{\phi}_2, \tag{A34}$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + (2\beta - 2\tau)\Psi_3 + \sigma\Psi_4 - \bar{\phi}_2 D\phi_2 + \bar{\phi}_1\delta\phi_2 - 2\mu\phi_1\bar{\phi}_1 + 2\beta\phi_2\bar{\phi}_1, \tag{A35}$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - (2\gamma + 4\mu)\Psi_3 + (4\beta - \tau)\Psi_4 + \bar{\phi}_1\Delta\phi_2 - \bar{\phi}_2\bar{\delta}\phi_2 - 2\alpha\phi_2\bar{\phi}_2 - 2\nu\phi_1\bar{\phi}_1 + 2\gamma\phi_2\bar{\phi}_1 + 2\lambda\phi_1\bar{\phi}_2, \tag{A36}$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2, \tag{A37}$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 - (\bar{\alpha} - \beta)\phi_2. \tag{A38}$$

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