Perturbations and linearization stability of closed Friedmann universes

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We consider perturbations of closed Friedmann universes. Perturbation modes of two lowest wave numbers ($L = 0$ and 1) are generally known to be fictitious, but here we show that both are physical. The issue is more subtle in Einstein static universes where closed background space has a timelike Killing vector with the consequent occurrence of linearization instability. Proper solutions of the linearized equation need to satisfy the Taub constraint on a quadratic combination of first-order variables. We evaluate the Taub constraint in the two available fundamental gauge conditions, and show that in both gauges the $L \geq 1$ modes should accompany the $L = 0$ (homogeneous) mode for vanishing sound speed, c_s. For $c_s^2 > 1/5$ (a scalar field supported Einstein static model belongs to this case with $c_s^2 = 1$), the $L \geq 2$ modes are known to be stable. In order to have a stable Einstein static evolutionary stage in the early universe, before inflation and without singularity, although the Taub constraint does not forbid it, we need to find a mechanism to suppress the unstable $L = 0$ and $L = 1$ modes.

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I. INTRODUCTION

The study of scalar, vector and tensor perturbations of the Friedmann universes of general relativity began with the famous paper of Lifshitz in 1946 [1,2]. Curiously, just as the isotropic and homogeneous Newtonian cosmologies were found later by Milne and McCrea [3], in 1934, then their general relativistic counterparts by Friedmann [4,5] in 1922, so the Newtonian treatment of their scalar perturbations, by Bonnor [6] in 1957, also followed the general relativistic treatment of Lifshitz. Recently, the studies of the stability of the Einstein static universe by Barrow *et al.* [7] and Losic and Unruh [8] have drawn attention to a subtle feature of the homogeneous and isotropic background cosmological model that can cause perturbation theory to fail due to the phenomenon of linearization instability. This is the motivation for our study.

Linearization instability arises when the sum of the two leading terms in perturbation around an exact solution cannot be completed to a convergent expansion. That is, if the metric is expanded as

$$g_{ab} = g^{(0)}_{ab} + \epsilon g^{(1)}_{ab} + \epsilon^2 g^{(2)}_{ab} + \cdots, \tag{1}$$

where g_{ab} and $g^{(0)}_{ab}$ are solutions of the full Einstein equations, and $g^{(1)}_{ab}$ is a solution of the linearized Einstein equations, then the series expansion is said to be *linearization stable* if the series (1) can be completed to form a convergent series. If not, it is said to be *linearization unstable*. In general relativity, Fischer and Marsden, Arms, and Moncrief [9–11] showed that compact spaces in vacuum with Killing vectors are linearization unstable: $g^{(1)}_{ab}$ is linearization stable if and only if $g^{(0)}_{ab}$ has no Killing fields. In Ref. [12], this feature was discussed in relation to series expansions about the Mixmaster universe, which has compact space sections and Killing symmetries, and Brill provides several examples [13]. A comprehensive overview is also given in the thesis of Altas [14].

Heuristically, the geometry of the solution space of cosmologies with compact Cauchy surfaces is conical at the points with Killing symmetries and so the perturbation expansion is like trying to draw a tangent through the apex of a cone: there are an infinite number of possible tangents and the ones that form the leading order of an expansion that converges to a true solution corresponds to the tangents that run down the side of the cone. This reminds us that there are two ways to obtain a perturbed version of an exact solution. The first (definitive but unrealistic) method is to find the general solution *of the equations* and linearize about the exact solution in question. The other method (used in practice) is to linearize the equations about the exact solution and *solve the linearized equations*. This does not necessarily lead to the same result unless some extra constraints are *imposed* (which we shall discuss below in the general relativistic context).

A typical example is provided by the equation

$$f(x,y) = x(x^2+y^2) = 0, \tag{2}$$

with the set of solutions $(x,y) = (0,y)$, *where* y *is arbitrary*. Now linearize Eq. (2) about the particular solution (0,0). This yields

$$(3x^2+y^2)\delta x + 2xy\delta y = 0. \tag{3}$$

We see that for $(x,y) = (0,0)$ there is no restriction on the linearized solutions and $(\delta x, \delta y)$ are completely arbitrary. However, from the exact solution, we know that although there are linearized solutions to the linearized Eq. (3) with $\delta x \neq 0$, they cannot arise from *the linearizations of any exact solution of* Eq. (2) [14].

Fischer, Marsden and Moncrief [9] showed that the $g^{(1)}_{ab}$ is not a spurious solution if and only if it satisfies a second-order constraint, involving integrals of the Taub conserved quantity which therefore vanish [15]. In this paper, we will evaluate the Taub constraint in different gauges and determine the status of the first-order neutral stability results for the Einstein static universe, which is a prime candidate for the phenomenon of linearization instability as it has compact space sections and many Killing symmetries.

In the course of this analysis we will also identify some features of gauge invariant perturbation claims *in the literature* that appear to be discrepant in ways that do appear to have been noticed in the past. Specifically, we will address two issues in the cosmological scalar perturbations of the homogeneous and isotropic Friedmann universes. In *some of* the literature, the perturbations with the two lowest wave numbers ($L = 0$ and 1) are claimed to be fictitious. Here, we show that both are physical.

In a closed background space with Killing vectors, *in order to be linearization stable* the solution of a linearized equation should satisfy a constraint on a quadratic combination of first-order variables; we call it the Taub constraint. When the Taub constraint is evaluated under two gauge conditions for a timelike Killing vector in an Einstein static background, it implies that $L \geq 1$ modes should accompany the $L = 0$ (homogeneous) mode, but this is true only for vanishing sound speed.

In Sec. II, we review the equations and solutions for linear perturbations of scalar type in the presence of background curvature and we consider a complete set of exact solutions with zero pressure and cosmological constant (see Appendix A).

In Sec. III we investigate the physical nature of the two lowest wave number modes ($L = 0$ and 1) in the positive curvature background. In Sec. IV we analyze the stability in the Einstein static background in the presence of pressure or a scalar field. In Sec. V we evaluate the Taub constraint for a timelike Killing vector in the Einstein static model; the Taub constraint is derived in Appendix B. Section VI is a discussion of our results and their consequences. In Secs. II–IV, we consider scalar-type linear perturbation in the Friedmann background with spatial curvature, while Sec. V considers second-order perturbations. Sections II and III consider general background curvature K, while Secs. IV and V are concerned with the positive curvature background. In the case of the scalar field we set $c \equiv 1 \equiv \hbar$.

II. LINEAR PERTURBATIONS WITH GENERAL CURVATURE

All results in this section are known in the literature, but we pay special attention to three simple cases of perturbed Friedmann universes. These are (i) the Einstein static background with $H = 0$, (ii) the homogeneous perturbation with $\Delta = 0$, and (iii) the case with $\Delta + 3K = 0$, where K is the curvature parameter in the Friedmann equation, equal to 0 or ± 1; the latter two cases are considered in the spherical geometry; H is the Hubble-Lemaître parameter and Δ is a Laplacian operator of the comoving three-space of the Friedmann metric. In these simple cases some terms in the perturbation equations automatically vanish, thus the analysis and final results are often invalid; in such cases a simple cure is to go back to the original perturbation equations and check each case. We will study these simple cases in more detail in later sections.

A. Basic equations

We consider perturbations of scalar-type in the Friedmann background. Our metric convention follows Bardeen's in [16]

$$\tilde{g}_{00} = -a^2(1+2\alpha), \qquad \tilde{g}_{0i} = -a^2\beta_{,i},$$
$$\tilde{g}_{ij} \equiv a^2[(1+2\varphi)\gamma_{ij} + 2\gamma_{,i|j}], \tag{4}$$

with $x^0 = \eta$ the conformal time and $a(t)$ is the expansion scale factor. We introduce $\chi \equiv a(\beta + \frac{a}{c}\dot{\gamma})$ where the time derivative is with respect to t where $cdt \equiv ad\eta$. The energy-momentum tensor is decomposed into fluid quantities based on a timelike fluid four-vector, $\tilde{u}_a$, normalized with $\tilde{u}^a\tilde{u}_a \equiv -1$, so

$$\tilde{T}_{ab} = \tilde{\mu}\tilde{u}_a\tilde{u}_b + \tilde{p}(\tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b) + \tilde{\pi}_{ab}, \tag{5}$$

$$\tilde{\mu} = \tilde{\varrho}c^2, \qquad \tilde{\varrho} \equiv \varrho + \delta\varrho, \qquad \tilde{p} = p + \delta p,$$
$$\tilde{u}_i \equiv -\frac{a}{c}v_{,i}, \qquad \tilde{\pi}_{ij} \equiv \frac{1}{a^2}\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta\right)\Pi. \tag{6}$$

We may set $\delta p \equiv c_s^2c^2\delta\varrho + e$ with $c_s^2 \equiv \dot{p}/(\dot{\varrho}c^2)$, and $w \equiv p/(\varrho c^2)$; e is the entropic perturbation and Π is the anisotropic stress. The spatial indices are raised and lowered by γ_{ij} and its inverse, and a vertical bar indicates

the covariant derivative based on the metric tensor γ_{ij}. One representation of γ_{ij} is

$$d\ell^2 \equiv \gamma_{ij}dx^i dx^j = \begin{cases} d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) & (K=+1), \\ d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\phi^2) & (K=0), \\ d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) & (K=-1), \end{cases} \tag{7}$$

with a normalized background curvature K.

The Friedmann equations are [4,5]

$$H^2 = \frac{8\pi G}{3}\varrho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3},$$
$$\dot{H} + H^2 = -\frac{4\pi G}{3}\left(\varrho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3},$$
$$\dot{H} = -4\pi G\left(\varrho + \frac{p}{c^2}\right) + \frac{Kc^2}{a^2},$$
$$\dot{\varrho} + 3H\left(\varrho + \frac{p}{c^2}\right) = 0, \tag{8}$$

with $H \equiv \dot{a}/a$ the Hubble-Lemaître parameter. The pressure term was first considered by Lemaître [17–19]. The general perturbation in the Friedmann background was studied first by Lifshitz [1,2]. Lifshitz studied the scalar-, vector- and tensor-type perturbations to the linear order in the synchronous gauge ($\alpha \equiv 0 \equiv \beta$). Here we consider the scalar-type perturbation in all fundamental gauges. Unless mentioned otherwise, our study in Secs. II and III is valid for all types of K, whereas Secs. IV and V concern the positive curvature model.

To linear order in perturbation, the basic equations for the scalar-type perturbation, without imposing the gauge conditions, are [16]

$$\kappa = 3H\alpha - 3\dot{\varphi} - c\frac{\Delta}{a^2}\chi, \tag{9}$$

$$4\pi G\delta\varrho + H\kappa + c^2\frac{\Delta + 3K}{a^2}\varphi = 0, \tag{10}$$

$$\kappa + c\frac{\Delta + 3K}{a^2}\chi - \frac{12\pi G}{c^2}a\left(\varrho + \frac{p}{c^2}\right)v = 0, \tag{11}$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\varrho + \frac{3\delta p}{c^2}\right), \tag{12}$$

$$\varphi + \alpha - \frac{1}{c}(\dot{\chi} + H\chi) = -\frac{8\pi G}{c^4}\Pi, \tag{13}$$

$$\delta\dot{\varrho} + 3H\left(\delta\varrho + \frac{\delta p}{c^2}\right) + \left(\varrho + \frac{p}{c^2}\right)\left(3H\alpha - \kappa - \frac{\Delta}{a}v\right) = 0, \tag{14}$$

$$\frac{1}{a^4}\left[a^4\left(\varrho + \frac{p}{c^2}\right)v\right]^{\cdot} = \frac{1}{a}\left[\delta p + (\varrho c^2 + p)\alpha + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right]. \tag{15}$$

We consider a gauge transformation, $\hat{x}^c = x^c + \tilde{\xi}^c(x^e)$ with $\tilde{\xi}^0 = \xi^0 \equiv \frac{1}{a}\xi^t$ and $\tilde{\xi}^i = \xi^i \equiv \frac{1}{a}\xi^{|i}$; index of ξ_i is raised and lowered using γ_{ij} as the metric. To the linear order we have [16]

$$\hat{\alpha} = \alpha - \frac{1}{c}\dot{\xi}^t, \qquad \hat{\beta} = \beta - \frac{1}{a}\xi^t + \frac{a}{c}\left(\frac{1}{a}\xi\right)^{\cdot},$$
$$\hat{\gamma} = \gamma - \frac{1}{a}\xi, \quad \hat{\chi} = \chi - \xi^t, \quad \hat{\kappa} = \kappa + \frac{1}{c}\left(3\dot{H} + c^2\frac{\Delta}{a^2}\right)\xi^t,$$
$$\hat{\varphi} = \varphi - \frac{1}{c}H\xi^t, \qquad \delta\hat{\varrho} = \delta\varrho - \frac{1}{c}\dot{\varrho}\xi^t,$$
$$\delta\hat{p} = \delta p - \frac{1}{c}\dot{p}\xi^t, \qquad \hat{v} = v - \frac{c}{a}\xi^t, \qquad \hat{e} = e,$$
$$\hat{\Pi} = \Pi, \qquad \delta\hat{\phi} = \delta\phi - \frac{1}{c}\dot{\phi}\xi^t. \tag{16}$$

By using χ instead of β and γ, all the perturbation variables are spatially gauge invariant. We have the following possible fundamental gauge conditions: the uniform-curvature gauge (UCG, $\varphi \equiv 0$), the uniform-density gauge (UDG, $\delta\varrho \equiv 0$), the uniform-expansion gauge (UEG, $\kappa \equiv 0$), the comoving gauge (CG, $v \equiv 0$), the zero-shear gauge (ZSG, $\chi \equiv 0$), and the synchronous gauge (SG, $\alpha \equiv 0$). We introduce gauge-invariant notations, like $v_\chi \equiv v - (c/a)\chi \equiv -(c/a)\chi_v$, where v_χ is gauge invariant the *same* as v in the ZSG. One exception is the SG; after imposing the gauge condition we still have nonvanishing $\xi^t(\mathbf{x})$ which is the remnant gauge mode in the SG. Thus, $\chi_\alpha \equiv \chi - c\int^t \alpha dt$ is not gauge invariant; the lower bound of integration gives the remnant gauge mode with $\chi \propto \xi^t(\mathbf{x})$. Concerning the spatial gauge transformation, our definitions of χ and v are spatially gauge-invariant combinations; χ is the same as (equivalent to) $a\beta$ under the spatial gauge condition $\gamma \equiv 0$.

We note that in a static background with $H = 0$, both φ and $\delta\varrho$ become gauge invariant. In addition, for $\Delta = 0$ (thus, a homogeneous) mode, κ becomes gauge invariant as well.

B. Exact equations and asymptotic solutions

A powerful large-scale conserved behavior of a combination of variables in the presence of K is known already. The following analysis is valid for $H \neq 0$; for $H = 0$, φ

and δ are gauge invariant, and we can show $\Phi = 0$, (see Sec. IV). We define

$$\Phi \equiv \varphi_v - \frac{\frac{Kc^2}{a^2}}{4\pi G(\varrho + \frac{p}{c^2})}\varphi_\chi = \varphi_v + \frac{K}{\Delta + 3K}\frac{\delta_v}{1+w}, \tag{17}$$

where we used

$$c^2\frac{\Delta + 3K}{a^2}\varphi_\chi = -4\pi G\delta\varrho_v, \tag{18}$$

which follows from Eqs. (10) and (11); from Eq. (16) we have

$$\varphi_\chi \equiv \varphi - \frac{H}{c}\chi \quad \text{and} \quad \delta\varrho_v \equiv \delta\varrho + \frac{1}{c^2}3aH\left(\varrho + \frac{p}{c^2}\right)v.$$

Note that for $\Delta = -3K$, Eq. (18) gives $\delta\varrho_v = 0$, and the second expression in Eq. (17) does not apply; $\delta\varrho_v = 0$ follows from Eqs. (10) and (11) evaluated in the CG with $\Delta = -3K$.

In Eq. (17), from the first relation, using Eqs. (9), (11) and (13), and from the second relation, using Eqs. (9)–(11), (14) and (15), respectively, we can derive

$$\Phi = \frac{H^2}{4\pi G(\varrho + \frac{p}{c^2})a}\left[\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + \frac{8\pi G}{c^4}a\Pi\right], \tag{19}$$

$$\dot{\Phi} = \frac{Hc_s^2c^2}{4\pi G(\varrho + \frac{p}{c^2})}\frac{\Delta}{a^2}\varphi_\chi - \frac{H}{\varrho c^2 + p}\left(e + \frac{2}{3}\frac{\Delta}{a^2}\Pi\right). \tag{20}$$

Although we used Eq. (18) in deriving Eq. (20), we can check by using the original Eqs. (9)–(15) that the result is valid even for $\Delta = -3K$. Ignoring the imperfect fluid contribution, thus setting $e \equiv 0 \equiv \Pi$, we have

$$\frac{H^2c_s^2}{(\varrho + \frac{p}{c^2})a^3}\left[\frac{(\varrho + \frac{p}{c^2})a^3}{H^2c_s^2}\dot{\Phi}\right]^{\cdot} - c_s^2c^2\frac{\Delta}{a^2}\Phi = 0. \tag{21}$$

Using

$$v \equiv z\Phi, \qquad z \equiv \frac{a\sqrt{\varrho + \frac{p}{c^2}}}{Hc_s}, \tag{22}$$

we have

$$\frac{1}{c^2az^2}(az^2\dot{\Phi})^{\cdot} - c_s^2\frac{\Delta}{a^2}\Phi = \frac{1}{a^2z}\left[v'' - \left(\frac{z''}{z} + c_s^2\Delta\right)v\right] = 0, \tag{23}$$

where a prime is the time derivative with respect to the conformal time, η. In the large-scale (super-sound-horizon scale) limit, $z''/z \gg c_s^2\Delta$, we have a general solution:

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}) + d(\mathbf{x})\int^t \frac{H^2c_s^2}{4\pi G(\varrho + \frac{p}{c^2})a^3}dt. \tag{24}$$

Thus, the relatively growing solution of Φ remains constant in the super-sound-horizon scale. The sound-horizon vanishes for zero-pressure fluid, in which case we have $\dot{\Phi} = 0$, and so $\Phi = C(\mathbf{x})$ exactly.

The well-known equation in terms of v and z in Eq. (23) first appeared in Eq. (44) of Field and Shepley's 1968 paper [20] in the context with general K (see also [21], Sec. V of [22] and Sec. III of [23]; in the absence of K, see [24–26]). Using Eqs. (10) and (14), Eq. (17) can be arranged as

$$c^2\frac{\Delta + 3K}{a^2H^2}\Phi = -\left(\frac{\delta\varrho_\alpha}{(\varrho + \frac{p}{c^2})H}\right)^{\cdot} - \frac{3e}{\varrho c^2 + p}, \tag{25}$$

which is related to Eqs. (31) and (43) in [20]. Here, we used $\alpha_v \equiv \alpha - \frac{1}{c^2}(av)^{\cdot}$, $v_\alpha \equiv v - \frac{c^2}{a}\int^t \alpha dt$ and $\delta\varrho_\alpha \equiv \delta\varrho - \dot{\varrho}\int^t \alpha dt$ which follow from Eq. (16); notice that the remnant gauge degree of freedom in the SG imbedded in the lower bound of integration of $\delta\varrho_\alpha$ in Eq. (25) disappears because of the time derivative. For $\Delta = -3K$ Eq. (25) is identically satisfied as we have $\delta\varrho_\alpha = \delta\varrho_v - \dot{\varrho}\int^t \alpha_v dt$ with $\delta\varrho_v = 0$ and $\alpha_v = -e/(\varrho c^2 + p)$ which follows from Eq. (15).

C. Exact solutions for zero-pressure fluid

In the zero-pressure situation, with $p = 0 = \delta p$ and $\Pi = 0$, but with general K and Λ, we have [see Eqs. (20) and (24)]

$$\Phi = C(\mathbf{x}). \tag{26}$$

Again, the following analysis is valid for $H \neq 0$; for $H = 0$, φ and δ are gauge invariant, and we have $\Phi = 0$; the static case will be studied in Sec. IV. In the CG, Eq. (15) gives $\alpha_v = 0$. Using Eqs. (10) and (14), the second relation in Eq. (17) gives

$$\left(\frac{\delta_v}{H}\right)^{\cdot} = -c^2\frac{\Delta + 3K}{a^2H^2}\Phi, \tag{27}$$

with an exact solution:

$$\delta_v = -c^2(\Delta + 3K)CH\int^t \frac{dt}{\dot{a}^2}. \tag{28}$$

The relatively decaying solution is absorbed in the lower bound of the integration. From this one solution we can derive all the other solutions in the same gauge and, using the complete solutions in one gauge, we can derive all solutions with all other gauge conditions. The complete solutions are presented in Table 1 of [27], and are reproduced in Appendix A in this paper; in our notation,

paying particular attention to the $k^2 = 0$ and $3K$ modes in the spherical geometry; we introduced the comoving wave number with $\Delta = -k^2$.

For $\Delta = -3K$, we have $\delta_v = 0$ and we cannot begin with the above two equations which become trivial. We need to start from a nonvanishing solution. In the ZSG, from Eqs. (9), (11), (13) and (15), we have

$$\frac{1}{a}(a\varphi_\chi)^{\cdot} = -\frac{4\pi G\varrho}{c^2} a v_\chi,$$
$$\frac{1}{a}(a v_\chi)^{\cdot} = -\frac{c^2}{a}\varphi_\chi, \quad \text{thus}$$
$$\frac{1}{a^3}[a^2(a\varphi_\chi)^{\cdot}]^{\cdot} = 4\pi G\varrho\varphi_\chi. \tag{29}$$

This can be written as

$$\frac{1}{a^3 H}\left[a^2 H^2\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} = 0, \tag{30}$$

with the solution

$$\varphi_\chi = 4\pi G\varrho a^2 HC\int^t \frac{dt}{\dot{a}^2}. \tag{31}$$

The normalization is made using Eq. (19). This solution coincides with the one in Appendix A. From this we obtain solutions of every variable in all gauge conditions. The results are naturally (because φ_χ coincides) the same as the ones derived from δ_v in Eq. (28) presented in Appendix A.

D. Scalar fields

For a minimally coupled scalar field, the equations for the fluid, Eq. (8) for the background, and Eqs. (9)–(15) for perturbations, remain valid with the fluid quantities replaced by the ones for the scalar field. Additionally, we have the scalar field equation of motion which also follows from the conservation equations, the last one in Eq. (8) and Eq. (14). For the background, we have

$$\varrho = \frac{1}{2}\dot{\phi}^2 + V, \qquad p = \frac{1}{2}\dot{\phi}^2 - V, \tag{32}$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{33}$$

For the perturbation, we have

$$\delta\varrho = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi,$$
$$\delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi,$$
$$(\varrho + p)v = \frac{1}{a}\dot{\phi}\delta\phi, \qquad \Pi = 0, \tag{34}$$

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(V_{,\phi\phi} - \frac{\Delta}{a^2}\right)\delta\phi = \dot{\phi}(\kappa + \dot{\alpha}) + (2\ddot{\phi} + 3H\dot{\phi})\alpha. \tag{35}$$

The gauge transformation property of the scalar field is presented in Eq. (16).

The scalar field can be treated as a fluid, as identified in Eqs. (32) and (34). The CG ($v \equiv 0$) coincides with the uniform-field gauge (UFG, $\delta\phi \equiv 0$). In this gauge we have

$$\delta\phi_v = \delta\mu_v, \quad \text{thus} \quad e = (1 - c_s^2)\delta\mu_v \quad \text{with}$$
$$c_s^2 \equiv \frac{\dot{p}}{\dot{\varrho}} = -1 - \frac{2\ddot{\phi}}{3H\dot{\phi}}. \tag{36}$$

Using this e and Eq. (18), Eq. (20) gives

$$\dot{\Phi} = \frac{Hc_A^2 c^2}{4\pi G(\varrho + \frac{p}{c^2})}\frac{\Delta}{a^2}\varphi_\chi \quad \text{with} \quad c_A^2\Delta \equiv \Delta + 3(1 - c_s^2)K. \tag{37}$$

Thus, the equations in Sec. II B with $e = 0 = \Pi$ are valid with c_s^2 replaced by c_A^2, see Sec. III of [23]; in the absence of K, we have $c_A^2 = 1$ [26].

III. THE PHYSICAL NATURE OF THE $k^2 = 0$ AND $k^2 = 3K$ MODES

In the spherical geometry the mode function has discrete wave numbers with $k^2 \equiv (n^2 - 1)K$ and $n = 1, 2, \ldots$; we often keep K explicitly even though we normalized it earlier as $K = 1$. In the literature the two lowest wave numbers with $n = 1$ and 2, thus $k^2 = 0$ and $3K$, are claimed to be fictitious perturbations [1,2,28]. Our review in the previous section shows no particular trouble for $\Delta = 0$ and $-3K$ cases. In this section we study the individual case in more detail and show the physical nonfictitious nature of these two modes.

A. $k^2 = 0$ (homogeneous) modes

First, we consider the $k^2 = 0$ mode. By setting $\Delta = 0$ our basic equations in (9)–(15) become a set of ordinary differential equations depending only on time and are therefore spatially homogeneous.

For $\Delta = 0$ we have $\Phi = \varphi + \delta/[3(1 + w)]$ and from Eqs. (9) and (14) we can show

$$\left(\varphi + \frac{\delta}{3(1 + w)}\right)^{\cdot} + \frac{He}{\varrho c^2 + p} = 0. \tag{38}$$

Thus, for $e = 0$, we have

$$\varphi+\frac{\delta}{3(1+w)}=C. \tag{39}$$

In the UDG ($\delta\equiv 0$), which is possible for $H\neq 0$, we have (for $e=0$)

$$\varphi_\delta\equiv\varphi+\frac{\delta}{3(1+w)}=C. \tag{40}$$

From Eqs. (14) and (10), we have

$$\kappa_\delta=-\frac{3Kc^2}{a^2H}C,\qquad \alpha_\delta=-\frac{Kc^2}{a^2H^2}C, \tag{41}$$

and the solutions for other variables in the same gauge follow from Eqs. (13) and (11):

$$\begin{aligned}\chi_\delta&=\frac{c}{a}\int^t a\left[\left(1-\frac{Kc^2}{a^2H^2}\right)C+\frac{8\pi G}{c^4}\Pi\right]dt,\\ \frac{4\pi G}{c^2}\left(\varrho+\frac{p}{c^2}\right)av_\delta&=-\frac{Kc^2}{a^2H}C+\frac{Kc^2}{a^3}\int^t a\left[\left(1-\frac{Kc^2}{a^2H^2}\right)C\right.\\ &\quad\left.+\frac{8\pi G}{c^4}\Pi\right]dt.\end{aligned} \tag{42}$$

As we have solutions for a complete set of variables in the UDG, the solutions in any other gauge can be derived using the gauge transformation properties in Eq. (16). As an example, density perturbation in the CG, δ_v, and the curvature perturbation in the ZSG, φ_χ, can be derived in the following way. From Eq. (16) we have

$$\begin{aligned}\delta_v&\equiv\delta+\frac{3}{c^2}aH(1+w)v\equiv\frac{3}{c^2}aH(1+w)v_\delta,\\ \varphi_\chi&\equiv\varphi-\frac{1}{c}H\chi=\varphi_\delta-\frac{1}{c}H\chi_\delta.\end{aligned} \tag{43}$$

For comparison with exact solutions in the zero-pressure case presented in Sec. II C and Appendix A, it is convenient to have

$$\frac{a}{H}-\int^t a\left(1-\frac{Kc^2}{\dot a^2}\right)dt=4\pi G\varrho a^3\int^t\frac{dt}{\dot a^2}. \tag{44}$$

For a static background we have $H=0$ so we have to go back to the original equations in (9)–(15) and the gauge transformation properties in Eq. (16); as δ is naturally gauge invariant, we cannot take the UDG. Although we cannot construct φ_δ, the combination $\varphi+\delta/[3(1+w)]$ is fine and still gives C, which in fact vanishes, see Eq. (10). The static situation will be studied in Sec. IV C.

B. $k^2=3K$ modes

For $k^2=3K$ we have $\Delta+3K=0$. As we have $\Delta+3K$ terms often appearing in our basic equations, many variables vanish in some gauges. Following Bardeen [28] we consider the case of the UEG with $\kappa\equiv 0$. Equations (10)–(12) give $\delta=0$, $v=0$ and $\alpha=-e/(\varrho c^2+p)$, respectively; thus, the UDG and the CG also give the identical results. Despite this simplification in the three gauges (UEG, UDG and CG), Eqs. (9) and (13) give the following equations:

$$\begin{aligned}&\dot\varphi=\frac{Kc}{a^2}\chi-\frac{He}{\varrho c^2+p},\\ &\frac{1}{ca}(a\chi)^{\cdot}=\varphi-\frac{e}{\varrho c^2+p}+\frac{8\pi G}{c^4}\Pi;\\ &\frac{1}{a^3}\left[a^3\left(\dot\varphi+\frac{He}{\varrho c^2+p}\right)\right]^{\cdot}-\frac{Kc^2}{a^2}\left(\varphi-\frac{e}{\varrho c^2+p}\right)=\frac{8\pi G}{c^2}\frac{K}{a^2}\Pi,\end{aligned} \tag{45}$$

which are valid for the three gauge conditions. For $e=0=\Pi$ we have

$$\dot\varphi=\frac{Kc}{a^2}\chi,\qquad\frac{1}{ca}(a\chi)^{\cdot}=\varphi;\qquad\frac{1}{a^3}(a^3\dot\varphi)^{\cdot}-\frac{Kc^2}{a^2}\varphi=0. \tag{46}$$

These are nontrivial equations and as the gauge modes are completely fixed in all three gauges, the variables cannot be removed by gauge transformation. Equations in the other remaining gauges (the ZSG and UCG) are more nontrivial. Each variable in all these gauge conditions has a unique gauge-invariant combination.

An exception is the SG where even after fixing $\alpha\equiv 0$ in all coordinates we have nonvanishing ξ^t, where $\xi^t(\mathbf{x})$ is the remnant gauge mode. From Eqs. (10) and (12), for $\delta p=c_s^2\delta\varrho$ with $c_s^2\equiv\dot p/\dot\varrho$, we have the solution $\delta\varrho\propto H(\varrho+p/c^2)$ which is exactly the behavior of the gauge mode as we have $\delta\hat\varrho=\delta\varrho+c^{-1}3H(\varrho+p/c^2)\xi^t(\mathbf{x})$. Thus, this solution can be *removed* as a fictitious gauge mode. By removing this gauge mode, and by *setting* $\xi^t(\mathbf{x})=0$ so $\delta=0$, the result becomes *identical* to taking the UDG. The complication caused by the remnant gauge mode in the SG can be overcome by removing the gauge mode (as we just did and Lifshitz did in [1]), by transforming to another gauge, or by constructing the gauge-invariant combinations in the SG as Field and Shepley did in [20]. In any case the results are the same as our analysis made above under the other fundamental gauge conditions: without any remnant gauge mode.

In the zero-pressure fluid with general K and Λ, the complete solutions in all fundamental gauge conditions are presented in Appendix A for these two modes.

C. Disagreements in the literature

The original comments on the fictitious nature of $k^2=0$ and $3K$ modes in spherical geometry were made by Lifshitz

[1,2] and Bardeen [28]; see also [29]. Here we analyze their arguments.

Lifshitz has introduced a scalar harmonic function Q and constructed the vector and tensor harmonic functions as [Eqs. (3.4), (3.10) and (3.11) in [1]]

$$\Delta Q = -(n^2-1)Q, \qquad P_i \equiv \frac{1}{n^2-1}Q_{,i},$$
$$P_{ij} \equiv \frac{1}{n^2-1}Q_{,i|j} + \frac{1}{3}\gamma_{ij}Q. \tag{47}$$

Thus, $k^2 = n^2 - 1$ with $n = 1, 2, \ldots$; P_{ij} is traceless with $P^i_i = 0$. In a footnote below this equation Lifshitz mentions that P_{ij} *cannot* be constructed for $n = 1$ and 2; P_i and P_{ij} diverge (indeterminate as we have $Q_{,i} = 0$ for $n = 1$ with $Q = \sum_{\ell,m} Q_{1\ell m} = Q_{100} =$ constant), and P_{ij} vanishes for $n = 2$ (we have $Q_{,i|j} = -\gamma_{ij}Q$ for $n = 2$ with $Q = \sum_{\ell,m} Q_{2\ell m}$); see Eq. (88). Thus, the vector and tensor harmonic functions P_i and P_{ij} constructed in this way have trouble in handling $n = 1$ and 2 modes properly.

Lifshitz has taken the synchronous gauge setting $\alpha \equiv 0 \equiv \beta$ in our notation; thus we have $\alpha = 0$ and $\chi = \frac{a^2}{c}\dot{\gamma}$ in our equations. The spatial metric is expanded using the tensor harmonic function as [Eq. (4.1) in [1]]

$$\tilde{g}_{ij} \equiv a^2(\gamma_{ij} + h_{ij}), \quad h_{ij} \equiv \mu\frac{1}{3}\gamma_{ij}Q + \lambda P_{ij}, \quad h \equiv h^i_i = \mu Q. \tag{48}$$

Compared with our metric convention in Eq. (4), we have

$$\varphi = \frac{1}{6}(\mu + \lambda)Q, \qquad \gamma = \frac{\lambda}{2(n^2-1)}Q. \tag{49}$$

Notice that λ is involved with a $1/(n^2-1)$ factor which is troublesome for $n = 1$.

Below Eq. (4.5) Lifshitz has made a major comment concerning our issue: "For $n = 1$, 2, we must put $\lambda = 0$ because the tensor P_{ij} does not exist for these values of n." By setting $\lambda = 0$ we have $\gamma = 0$ in our notation, thus together with $\beta = 0$ we have $\chi = 0$. Thus, for these two modes Lifshitz sets $\alpha = 0$ and $\chi = 0$ simultaneously. As we set $\alpha = 0$ and $\chi = 0$ simultaneously, with $e = 0 = \Pi$, all variables in Eqs. (9)–(15) vanish. Thus, all perturbations disappear for these two modes $n = 1$ and 2. Setting $\alpha = 0 = \chi$, however, is like imposing two temporal gauge (hypersurface, slicing) conditions simultaneously which is *not* allowed even for $n = 1$ and 2. Although P_i and P_{ij} cannot be constructed for $n = 1$ and/or 2 it is merely due to the way of constructing P_i and P_{ij} in Eq. (47) which can be avoided by simply not introducing such vector and tensor harmonics, see our Eq. (4).

Bardeen has similarly introduced harmonic functions as [Eqs. (2.7)–(2.9) in [28]]

$$\Delta Q = -k^2 Q, \qquad Q_i \equiv -\frac{1}{k}Q_{,i}, \qquad Q_{ij} \equiv \frac{1}{k^2}Q_{,i|j} + \frac{1}{3}\gamma_{ij}Q. \tag{50}$$

Compared with Lifshitz's notation we have $Q_{ij} = P_{ij}$ and $Q_i = -kP_i$; thus Q_i and Q_{ij} diverge (are indeterminate) for $n = 1$, and Q_{ij} vanishes for $n = 2$. The metric tensor is expanded using the harmonic functions as [Eq. (2.14) in [28]]

$$\tilde{g}_{00} = -a^2(1 + 2AQ), \qquad \tilde{g}_{0i} = -a^2 BQ_i,$$
$$\tilde{g}_{ij} = a^2[(1 + 2H_L Q)\gamma_{ij} + 2H_T Q_{ij}]. \tag{51}$$

Compared with our metric convention (which is in fact the convention of Bardeen's other work [16]), in Eq. (4), we have

$$\alpha = AQ, \qquad \beta = -\frac{1}{k}BQ,$$
$$\varphi = \left(H_L + \frac{1}{3}H_T\right)Q, \qquad \gamma = \frac{1}{k^2}H_T Q, \tag{52}$$

thus B and H_T are involved with $1/k$ and $1/k^2$ factors which are troublesome for vanishing k; see below Eq. (211) in [30].

Concerning $n = 1$ and 2 modes, below Eq. (4.9) Bardeen mentions: "A spatially homogeneous perturbation or the lowest inhomogeneous mode $k^2 = 3K$ in a closed universe require special treatment in that Q_i and/or Q_{ij} vanish identically, Φ_H, Φ_A, and v_s are no longer gauge-invariant. … A homogeneous scalar perturbation is really no perturbation at all, but an inappropriate choice of background." Compared with our notation, we have [Eqs. (3.9)–(3.11) in [28]]

$$\varphi_\chi \equiv \varphi - \frac{H}{c}\chi = \left[H_L + \frac{1}{3}H_T + H\left(\frac{a}{kc}B - \frac{a^2}{k^2c^2}\dot{H}_T\right)\right]Q$$
$$\equiv \Phi_H Q,$$
$$\alpha_\chi \equiv \alpha - \frac{1}{c}\dot{\chi} = \left[A + \frac{1}{kc}(aB)^{\cdot} - \frac{1}{k^2c^2}(a^2\dot{H}_T)^{\cdot}\right]Q$$
$$\equiv \Phi_A Q,$$
$$v_\chi \equiv v - \frac{c}{a}\chi = \frac{1}{k}\left(v_B - \frac{a}{k}\dot{H}_T\right)Q \equiv \frac{1}{k}v_s Q, \tag{53}$$

where v_B is Bardeen's v; compared with our v we have

$$\tilde{u}_i \equiv -\frac{a}{c}v_{,i},$$
$$\frac{\tilde{u}^i}{\tilde{u}^0} = \left(-\frac{1}{c}v + \beta\right)^{|i} \equiv \frac{1}{c}v_B Q^i = -\frac{1}{kc}v_B Q^{|i}, \quad \text{thus,}$$
$$v = \frac{1}{k}(v_B - cB)Q. \tag{54}$$

Although Bardeen's definitions of Φ_H, Φ_A and v_s in Eq. (53) have problems for $k=0$, in our notation, φ_χ, α_χ and v_χ, thus Φ_H, Φ_A and v_s, are gauge invariant *independently* of the value of k.

Concerning the homogeneous perturbation: the $n=1$ (homogeneous) mode scalar perturbation is a correct result from the perturbation and is handled properly while maintaining the homogeneous and isotropic nature of the background. This becomes clear in the Einstein static model as presented in Eq. (75). Compared with this proper treatment, the perturbation of the background equations, although it happens to give the same result as in Eq. (78), is a hand-waving procedure. Rigorously, we have to perturb the original Einstein's equation around the background, instead of simply perturbing the background equations, see Sec. IV D.

Below Eq. (6.26) of [28] Bardeen addresses the $n=2$ mode by analyzing it in the UEG, and states "Since Q_{ij} vanishes identically, Eq. (6.23) no longer applies." Bardeen's Eq. (6.23) is a combination of our Eqs. (9) and (11) which do apply and gives $v_\kappa=0$. Although many variables vanish in this gauge, there are still surviving ones in the same gauge (these are φ_κ and χ_κ) as in Eq. (45). Concerning these variables, below Eq. (6.27), Bardeen mentions "The amplitude [φ_κ] now depends on the way spatial coordinates are propagated from one hypersurface to the next through the hypersurface condition Eq. (5.22). The traceless part of the metric tensor perturbation and the spatial curvature perturbation vanish. The absence of any physical *adiabatic* mode when $k^2=3K$ was first recognized by Lifshitz and Khalatnikov."

Bardeen's Eq. (5.22) is our Eq. (9). Together with Eq. (13) it gives Eqs. (45) and (46). We find no reason why these equations and behaviors of the gauge-invariant variables φ_κ and χ_κ should be regarded as coordinate effects. In Bardeen's statement "the traceless part of the metric tensor perturbation" is H_T in Eq. (51), thus our γ, which can be set zero as the spatial gauge condition (without any effect in our formulation as we are using χ, which remains the same). However, his "spatial curvature perturbation" is our φ_κ which follows Eq. (45) and has *no* reason to vanish; if it vanishes χ_κ should vanish as well which implies that all perturbation variables in the UEG vanish. Vanishing φ after imposing the UEG is like imposing two gauge conditions (the UEG and the UCG) simultaneously, which is *not* allowed for general perturbation including the $k^2=3K$ mode.

IV. STABILITY OF THE EINSTEIN STATIC MODEL WITH PRESSURE

Einstein proposed in 1917 a static and closed world model by employing the cosmological constant in a closed universe [31], for a centennial review see [32]. Here, we consider the presence of an additional pressure. In the static background, Eq. (8) gives field equations:

$$\frac{8\pi G}{3}\varrho_0=\frac{Kc^2}{a_0^2}-\frac{\Lambda c^2}{3},$$
$$4\pi G\left(\varrho_0+\frac{3p_0}{c^2}\right)=\Lambda c^2,$$
$$4\pi G\left(\varrho_0+\frac{p_0}{c^2}\right)=\frac{Kc^2}{a_0^2}, \tag{55}$$

thus

$$a_0^2=\frac{Kc^2}{4\pi G(1+w)\varrho_0}=\frac{1+3w}{1+w}\frac{K}{\Lambda}. \tag{56}$$

We have $w>-1/3$ for $K>0$ and $\Lambda>0$.

To the linear order, ignoring stress, Eqs. (9)–(15) give

$$\kappa=-3\dot\varphi-c\frac{\Delta}{a_0^2}\chi, \tag{57}$$

$$4\pi G\delta\varrho+c^2\frac{\Delta+3K}{a_0^2}\varphi=0, \tag{58}$$

$$\kappa+c\frac{\Delta+3K}{a_0^2}\chi-\frac{12\pi G}{c^2}a_0\left(\varrho_0+\frac{p_0}{c^2}\right)v=0, \tag{59}$$

$$\dot\kappa+c^2\frac{\Delta}{a_0^2}\alpha=4\pi G\left(\delta\varrho+\frac{3\delta p}{c^2}\right), \tag{60}$$

$$\varphi+\alpha-\frac{1}{c}\dot\chi=0, \tag{61}$$

$$\delta\dot\varrho-\left(\varrho_0+\frac{p_0}{c^2}\right)\left(\kappa+\frac{\Delta}{a_0}v\right)=0, \tag{62}$$

$$\dot v=\frac{1}{a_0}\left(\frac{\delta p}{\varrho_0+\frac{p_0}{c^2}}+c^2\alpha\right). \tag{63}$$

The gauge transformation properties in Eq. (16) become

$$\hat\alpha=\alpha-\frac{1}{c}\dot\xi^t,\qquad \hat\varphi=\varphi,\qquad \hat\chi=\chi-\xi^t,$$
$$\hat\kappa=\kappa+c\frac{\Delta}{a_0^2}\xi^t,\quad \delta\hat\varrho=\delta\varrho,\quad \delta\hat p=\delta p,\quad \hat v=v-\frac{c}{a_0}\xi^t,$$
$$\hat\Pi=\Pi,\qquad \delta\hat\phi=\delta\phi-\frac{1}{c}\dot\phi_0\xi^t. \tag{64}$$

Thus, $\delta\varrho$ and φ are naturally gauge invariant, and κ becomes gauge invariant for $\Delta=0$. As we have $\Phi=0$ we cannot use the equations and solutions in Secs. II B and II C.

Without imposing the gauge condition, from Eqs. (60), (62) and (63), we can derive [except for the $\Delta+3K=0$ case, where we have $\delta\varrho=0$ from Eq. (58), see below] the following second-order density perturbation equation:

$$\delta\ddot{\varrho} = \left[4\pi G(1+w)(1+3c_s^2)\varrho_0 + c_s^2c^2\frac{\Delta}{a_0^2}\right]\delta\varrho = \frac{c^2}{a_0^2}[K + c_s^2(\Delta+3K)]\delta\varrho. \tag{65}$$

The solution is [Eq. (226) in [29] and Eq. (34) in [33]]

$$\delta\varrho \propto \varphi \propto e^{\pm\sqrt{4\pi G(1+w)(1+3c_s^2)\varrho_0 - c_s^2c^2k^2/a_0^2}t} = e^{\pm\sqrt{K-c_s^2(k^2-3K)}ct/a_0} \propto e^{\pm\sqrt{[1-(L+3)(L-1)c_s^2]K}ct/a_0}, \tag{66}$$

where we set

$$\Delta = -k^2 = -(n^2-1)K = -L(L+2)K, \quad (L = n-1 = 0,1,2,\ldots). \tag{67}$$

For a stable solution we need

$$c_s^2 \geq \frac{1}{(L+3)(L-1)}, \tag{68}$$

and for $L \geq 2$ we have $c_s^2 \geq 1/5$ [7,29,33,34]. The $L = 0$ and 1 modes are always unstable; although these two modes were often ignored in the literature as fictitious perturbations, we have shown that these are physical.

Equation (65) is *not* applicable for $L = 1$, thus $k^2 = 3K$. In this case from Eq. (58) we have $\delta\varrho = 0$, but as φ obeys the same equation, the solution in terms of φ remains valid, see Sec. IV B.

A. Complete solutions for general $k^2 = L(L+2)K$

The variables δ and φ are gauge invariant, and the solutions for them are given in Eq. (66). Solutions for other variables in all other gauges can be derived from a known solution. From Eqs. (57)–(63), or using the gauge transformation properties in Eq. (64), for the relatively growing mode, we can show

$$\begin{aligned}
&\varphi \propto e^{+\sqrt{[1-(L+3)(L-1)c_s^2]K}ct/a_0}, \qquad \delta = (1+w)(L+3)(L-1)\varphi; \qquad \kappa_\chi = -3\sqrt{[1-(L+3)(L-1)c_s^2]K}\frac{c}{a_0}\varphi,\\
&v_\chi = -c\sqrt{\frac{1-(L+3)(L-1)c_s^2}{K}}\varphi, \qquad \alpha_\chi = -\varphi; \qquad \kappa_v = (L+3)(L-1)\sqrt{[1-(L+3)(L-1)c_s^2]K}\frac{c}{a_0}\varphi\\
&\chi_v = a_0\sqrt{\frac{1-(L+3)(L-1)c_s^2}{K}}\varphi, \qquad \alpha_v = -(L+3)(L-1)c_s^2\varphi; \qquad \chi_\kappa = 3a_0\frac{\sqrt{1-(L+3)(L-1)c_s^2}}{L(L+2)\sqrt{K}}\varphi,\\
&v_\kappa = -c\frac{(L+3)(L-1)}{L(L+2)\sqrt{K}}\sqrt{1-(L+3)(L-1)c_s^2}\varphi, \qquad \alpha_\kappa = -\frac{(L+3)(L-1)}{L(L+2)}(1+3c_s^2)\varphi.
\end{aligned} \tag{69}$$

These are complete solutions. Since all these variables are proportional to φ, they share the same equation in (65).

The solutions for v_κ, α_κ and χ_κ diverge when $L = 0$. This happens because we *cannot* take the UEG for $L = 0$ and κ becomes gauge invariant for this situation. As we have $\varphi = \sum_{L,\ell,m}\varphi_{L\ell m}Q_{L\ell m}$ [see Eq. (85) for a proper expansion], the breakdown of the UEG for $L = 0$ implies the breakdown of the UEG in general in the Einstein static model. Thus the UEG is *not* available for the Einstein static model. Excluding solutions in the UEG, the above solutions are generally *valid* for all L including $L = 0$ and 1; the latter special cases will be displayed below.

B. $k^2 = 3K$ mode ($L = 1$)

From Eq. (58) we have $\delta\varrho = 0$ which is naturally gauge invariant. From Eqs. (57), (60) and (61) another naturally gauge invariant variable φ follows:

$$\ddot{\varphi} = \frac{Kc^2}{a_0^2}\varphi = 4\pi G(1+w)\varrho_0\varphi, \quad \text{thus,}$$
$$\varphi \propto e^{\pm\sqrt{K}ct/a_0} \propto e^{\pm\sqrt{4\pi G(1+w)\varrho_0}t}, \tag{70}$$

which is exponentially unstable; although $\delta\varrho = 0$, the solution for φ in Eq. (66) remains valid. By taking either the UEG ($\kappa \equiv 0$) or the CG ($v \equiv 0$) we have $\kappa = v = \alpha = 0$, thus

$$v_\kappa = 0 = \alpha_\kappa, \qquad \kappa_v = 0 = \alpha_v. \tag{71}$$

In both gauges we still have nontrivial equations in (57) and (61)

$$\dot{\varphi} = \frac{Kc}{a_0^2}\chi, \qquad \varphi = \frac{1}{c}\dot{\chi}, \tag{72}$$

a combination of which leads to Eq. (70). For relatively growing solutions we can show

$$\varphi \propto e^{+\sqrt{K}ct/a_0}, \qquad \delta = 0, \qquad v_\chi = -\frac{c}{\sqrt{K}}\varphi,$$
$$\kappa_\chi = -3\frac{\sqrt{K}c}{a_0}\varphi, \qquad \alpha_\chi = -\varphi, \qquad \chi_v = \chi_\kappa = \frac{a_0}{\sqrt{K}}\varphi, \tag{73}$$

so the general solutions in Eq. (69) remain *valid* for $L = 1$.

C. $k^2 = 0$ mode ($L = 0$)

Equation (64) shows that besides $\delta\varrho$ and φ, the variable κ is also gauge invariant. Equation (65) gives

$$\delta\ddot{\varrho} = 4\pi G(1+w)(1+3c_s^2)\varrho_0\delta\varrho = (1+3c_s^2)\frac{Kc^2}{a_0^2}\delta\varrho, \tag{74}$$

with the solution

$$\delta\varrho \propto e^{\pm\sqrt{4\pi G(1+w)(1+3c_s^2)\varrho_0}t} = e^{\pm\sqrt{(1+3c_s^2)K}ct/a_0}. \tag{75}$$

This mode is exponentially unstable for $c_s^2 > -1/3$. All surviving variables satisfy the same equation as $\delta\varrho$ in Eq. (74). For relatively growing modes we have

$$\varphi \propto e^{+\sqrt{(1+3c_s^2)K}ct/a_0}, \qquad \delta = -3(1+w)\varphi,$$
$$\kappa = -3\sqrt{(1+3c_s^2)K}\frac{c}{a_0}\varphi, \qquad \alpha_v = 3c_s^2\varphi, \qquad \alpha_\chi = -\varphi,$$
$$v_\chi = -c\sqrt{\frac{1+3c_s^2}{K}}\varphi, \qquad \chi_v = a_0\sqrt{\frac{1+3c_s^2}{K}}\varphi. \tag{76}$$

Thus, the general solutions in Eq. (69) remain *valid* for $L = 0$. Since κ is gauge invariant, we do not have variables like v_κ, α_κ and χ_κ which display divergent behavior in Eq. (69).

D. Stability of the static background

By directly perturbing the background equations to the linear order, with $a = a_0 + \delta a$, $\varrho = \varrho_0 + \delta\varrho$ and $p = p_0 + \delta p$, the first two in Eq. (8) give

$$\frac{\delta\ddot{a}}{a_0} = -\frac{4\pi G}{3}\left(\delta\varrho + \frac{3\delta p}{c^2}\right),$$
$$\frac{4\pi G}{3}\delta\varrho = -\frac{Kc^2}{a_0^2}\frac{\delta a}{a_0}, \quad \text{thus,}$$
$$\delta\ddot{a} = (1+3c_s^2)\frac{Kc^2}{a_0^2}\delta a, \tag{77}$$

with the solution [Eq. (225) in [29]]:

$$\delta a \propto \delta\varrho \propto e^{\pm\sqrt{(1+3c_s^2)K}ct/a_0}. \tag{78}$$

Thus, the static background is exponentially unstable for $c_s^2 > -1/3$.

The behavior of $\delta\varrho$ happens to *coincide* with the homogeneous perturbation mode in Eq. (75). We note, however, that in our proper perturbation of Einstein equations with the results in Eq. (76), even for the homogeneous mode, besides δg_{ij}, the quantity δg_{00} is excited in the ZSG ($\delta g_{0i} \equiv 0$), and both δg_{00} and δg_{0i} are excited in the CG. By perturbing the background metric with $a = a_0 + \delta a$ we effectively have $\alpha = \delta a/a_0 = \varphi$ which *differs* from our result in the ZSG with $\alpha = -\varphi$, and in the CG with $\alpha = 3c_s^2\varphi$. Thus, despite a coincidence in $\delta\varrho$, perturbing the Friedmann equation alone [35] is a hand-waving procedure. The correct way is to perturb the original Einstein equations, as we did in Sec. IV C; see Sec. 5.7 in [29].

E. Einstein static background with scalar field

In the Einstein static background, we set $\ddot{\phi}_0 \equiv 0$ but $\dot{\phi}_0 \neq 0$ [7]; if $\dot{\phi}_0 = 0$ we have $w = -1$, $\delta\varrho = 0 = \delta p$ and the case becomes trivial. From Eq. (33) we have $V_{,\phi} = 0$ and so $V = V_0$. Thus, we have

$$\varrho_0 = \frac{1}{2}\dot{\phi}_0^2 + V_0, \qquad p_0 = \frac{1}{2}\dot{\phi}_0^2 - V_0, \tag{79}$$

for the background, and

$$\delta\varrho = \delta p = \dot{\phi}_0\delta\dot{\phi} - \dot{\phi}_0^2\alpha, \quad (\varrho_0 + p_0)v = \frac{1}{a_0}\dot{\phi}_0\delta\phi, \tag{80}$$

$$\delta\ddot{\phi} - \frac{\Delta}{a^2}\delta\phi = \dot{\phi}_0(\kappa + \dot{\alpha}), \tag{81}$$

for the perturbation. Equations (55) and (56) for the background and Eqs. (57)–(63) for the perturbation remain valid with the fluid quantities replaced as above.

Equations (55) and (56) give

$$\frac{8\pi G}{3}\varrho_0 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}_0^2 + V_0\right) = \frac{K}{a_0^2} - \frac{\Lambda}{3},$$
$$4\pi G(\varrho_0 + 3p_0) = 8\pi G(\dot{\phi}_0^2 - V_0) = \Lambda,$$
$$4\pi G(\varrho_0 + p_0) = 4\pi G\dot{\phi}_0^2 = \frac{K}{a_0^2}. \tag{82}$$

a. In the presence of Λ, we have

$$\frac{K}{a_0^2} = \Lambda\frac{1+w}{1+3w} = \frac{\Lambda}{2}\frac{\dot{\phi}_0^2}{\dot{\phi}_0^2 - V_0}, \quad w \equiv \frac{p_0}{\varrho_0} = \frac{\dot{\phi}_0^2 - 2V}{\dot{\phi}_0^2 + 2V}, \tag{83}$$

thus, for $K > 0$ and $\Lambda > 0$ we have $\dot{\phi}_0^2 > V_0$ and $-1/3 < w < 1$.

b. In the absence of Λ, we have

$$\dot{\phi}_0^2 = V_0 = \frac{K}{4\pi G a_0^2}, \qquad w = -\frac{1}{3}. \tag{84}$$

As we have $\delta p = \delta\varrho$ the rest of the analyses made for the fluid case remains valid with $c_s^2 \equiv \delta p/\delta\varrho = 1$; notice that, as we have $V_{,\phi} = 0$ for the background, see Eqs. (33) and (34), this is true *independently* of the scalar field potential. Thus, *all $L \geq 2$ modes are stable, but $L = 0$ and 1 modes are unstable*, as in the fluid case. The equation of motion in Eq. (81) is consistent with Eqs. (62) and (63), and $\delta\phi$ is determined by $\delta\phi = a_0\dot{\phi}_0 v$.

V. THE TAUB CONSTRAINT

Losic and Unruh [8] state in their conclusions that "the requirement that the second order Einstein constraint equations be integrable demands that any inhomogenous linear mode perturbations of the Einstein static universe must be accompanied by the homogenous linear mode with comparable amplitude." In order for a solution of the linearized equation to be a proper solution of the exact equation, it should satisfy a constraint to the second order. We may call it the Taub constraint [36]. Here we evaluate the Taub constraint based on a timelike Killing vector in an Einstein static background. The Taub constraint is derived in Appendix B considering the general background metric, see Eqs. (B9) and (B15). Our result confirms Losic and Unruh's above conclusion, but only for $c_s^2 = 0$, whereas they were claiming it was true for arbitrary c_s^2.

A. Spherical harmonic expansion

When we consider the spherical background geometry we need harmonics in spherical geometry. We expand

$$\varphi(t,\chi,\theta,\phi) \equiv \varphi_{\mathbf{k}}(t)Q(\mathbf{k};\chi,\theta,\phi) \equiv \sum_{n,\ell,m}\varphi_{n\ell m}(t)Q_{n\ell m}(\chi,\theta,\phi) \equiv \sum_{n=1,2,\ldots}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell}\varphi_{n\ell m}(t)\Pi_{n\ell}(\chi)Y_\ell^m(\theta,\phi),$$
$$\varphi_{n\ell m}(t) = \int \varphi(t,\chi,\theta,\phi)\Pi_{n\ell}(\chi)Y_\ell^{m*}(\theta,\phi)\sqrt{\gamma}d^3x, \tag{85}$$

where $\Pi_{n\ell}(\chi)$ is an associated Legendre function with proper normalization [29]:

$$\int_0^\pi \Pi_{n\ell}(\chi)\Pi_{n'\ell}(\chi)\sin^2\chi d\chi = \delta_{nn'}, \qquad \Pi_{n\ell}(\chi) = \sqrt{\frac{n\Gamma(n+\ell+1)}{\Gamma(n-\ell)}}\frac{1}{\sqrt{\sin\chi}}P_{n-1/2}^{-\ell-1/2}(\cos\chi). \tag{86}$$

Using the spatial metric in Eq. (7) it is convenient to have

$$\gamma_{\chi\chi} = 1, \qquad \gamma_{\theta\theta} = \sin^2\chi, \qquad \gamma_{\phi\phi} = \sin^2\chi\sin^2\theta; \qquad \gamma^{\chi\chi} = 1, \qquad \gamma^{\theta\theta} = \frac{1}{\sin^2\chi}, \qquad \gamma^{\phi\phi} = \frac{1}{\sin^2\chi\sin^2\theta};$$
$$\Gamma^{(\gamma)\,\chi}_{\theta\theta} = -\sin\chi\cos\chi, \qquad \Gamma^{(\gamma)\,\chi}_{\phi\phi} = -\sin\chi\cos\chi\sin^2\theta, \qquad \Gamma^{(\gamma)\,\theta}_{\chi\theta} = \frac{\cos\chi}{\sin\chi}, \qquad \Gamma^{(\gamma)\,\theta}_{\phi\phi} = -\sin\theta\cos\theta,$$
$$\Gamma^{(\gamma)\,\phi}_{\chi\phi} = \frac{\cos\chi}{\sin\chi}, \qquad \Gamma^{(\gamma)\,\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}, \quad \text{or} \quad 0 \text{ otherwise.} \tag{87}$$

For $n = 1$ and 2 the harmonic functions Q are

$$Q|_{n=1} = Q_{100} = \frac{1}{\sqrt{2\pi}}, \qquad Q|_{n=2} = Q_{200} + Q_{21-1} + Q_{210} + Q_{211} = \frac{\sqrt{2}}{\pi}[\cos\chi + \sin\chi(\cos\theta - 2i\sin\theta\sin\phi)]. \tag{88}$$

We have $Q_{,i} = 0$ for $n = 1$, and $Q_{,i|j} = -\gamma_{ij}Q$ for $n = 2$.

Using $n = L + 1$, thus $k^2 = n^2 - 1 = L(L+2)$, we have

$$\varphi(t,\mathbf{x}) = \varphi(t,\chi,\theta,\phi) = \sum_{L=0,1,\ldots}\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}\varphi_{L\ell m}(t)\Pi_{L\ell}(\chi)Y_\ell^m(\theta,\phi),$$
$$\Delta\varphi(t,\mathbf{x}) = -\sum_{n,\ell,m}(n^2-1)\varphi_{n\ell m}(t)\Pi_{n\ell}Y_\ell^m = -\sum_{L,\ell,m}L(L+2)\varphi_{L\ell m}(t)\Pi_{L\ell}Y_\ell^m. \tag{89}$$

It is convenient to have

$$\int \varphi^2\sqrt{\gamma}d^3x = \sum_{L,\ell,m}|\varphi_{n\ell m}|^2, \qquad \int \varphi^{|i}\varphi_{,i}\sqrt{\gamma}d^3x = -\int \varphi\Delta\varphi\sqrt{\gamma}d^3x = \sum_{L,\ell,m}L(L+2)|\varphi_{L\ell m}|^2,$$
$$\int \sqrt{\gamma}\varphi^{|ij}\varphi_{,i|j}d^3x = \int \varphi\Delta(\Delta+2K)\varphi\sqrt{\gamma}d^3x = \sum_{L,\ell,m}(L^2+2L-2)L(L+2)K^2|\varphi_{n\ell m}|^2. \tag{90}$$

B. The Taub constraint in two gauges

Complete solutions are presented in Eq. (69); φ and δ are gauge invariant and solutions in the UEG are not valid. Using the mode expansion in Eq. (89), the solutions can be written as

$$\varphi = \sum_{L,\ell,m}\varphi_{L\ell m}(t)\Pi_{L\ell}Y^m_\ell = \sum_{L,\ell,m}e^{\sqrt{[1-(L+3)(L-1)c_s^2]K}c(t-t_0)/a_0}\varphi_{L\ell m}(t_0)\Pi_{L\ell}Y^m_\ell;$$
$$\delta_{L\ell m} = (1+w)(L+3)(L-1)\varphi_{L\ell m}; \qquad \kappa_{\chi L\ell m} = -3\sqrt{[1-(L+3)(L-1)c_s^2]K}\frac{c}{a_0}\varphi_{L\ell m},$$
$$v_{\chi L\ell m} = -c\sqrt{\frac{1-(L+3)(L-1)c_s^2}{K}}\varphi_{L\ell m}, \qquad \alpha_{\chi L\ell m} = -\varphi_{L\ell m}; \qquad \chi_{vL\ell m} = a_0\sqrt{\frac{1-(L+3)(L-1)c_s^2}{K}}\varphi_{L\ell m},$$
$$\kappa_{vL\ell m} = (L+3)(L-1)\sqrt{[1-(L+3)(L-1)c_s^2]K}\frac{c}{a_0}\varphi_{L\ell m}, \qquad \alpha_{vL\ell m} = -c_s^2(L+3)(L-1)\varphi_{L\ell m}. \tag{91}$$

We have shown that these solutions are valid for all L values. For the $L=0$ mode, the temporal dependence of all variables is $\propto e^{\sqrt{(1+3c_s^2)K}ct/a_0}$, and so is unstable for $c_s^2 > -1/3$. For the $L=1$ mode the temporal dependence of all variables becomes $\propto e^{\sqrt{K}ct/a_0}$, and so is unstable, independently of c_s^2. For $c_s^2 > 1/5$ all modes with $L \geq 2$ become stable as presented in Eq. (68) [7,29,33,34].

Now, we can evaluate the Taub constraint in Eq. (B17) using Eq. (B20) and the linear solutions in Eq. (91). In the ZSG and the CG, respectively, we have

$$\mathcal{T}_{\rm ZSG} = \sum_{L=0,1,\ldots}\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}[7L^2+14L-15-(2L^2+4L-3)(L+3)(L-1)c_s^2]K|\varphi_{L\ell m}|^2 = 0, \tag{92}$$

$$\mathcal{T}_{\rm CG} = \sum_{L=0,1,\ldots}\sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell}[3(2L^2+4L-5)-(L+3)^2(L-1)^2c_s^2]K|\varphi_{L\ell m}|^2 = 0. \tag{93}$$

The Taub constraint apparently depends on the gauge condition. Taub constraint on the ZSG applies to the linear solutions in the ZSG, and likewise for the CG. We have $\mathcal{T}_{L=1} > 0$ and $\mathcal{T}_{L=0} < 0$ in both gauges.

For $c_s^2 = 0$, in both gauges we have $\mathcal{T}_{\rm L\geq 1} > 0$ and $\mathcal{T}_{\rm L=0} < 0$, thus the Taub constraint demands that the homogeneous perturbation ($L=0$ mode) should be excited as long as we have nonvanishing inhomogeneous perturbation; the solutions in Eq. (91) show that for $c_s^2 = 0$, all perturbations are unstable proportional to $e^{\sqrt{K}ct/a_0}$ independently of L.

Similarly, in the ZSG, for $c_s^2 > 41/65$ we have $\mathcal{T}_{\rm L=1} > 0$ while all contributions from the other modes are pure negative. Thus, in this case, the $L=1$ mode should be excited as long as we have any other perturbation mode in the ZSG. Concerning the $L=1$ mode, a similar conclusion does not follow in the CG, as we need $c_s^2 \leq 1$.

VI. DISCUSSION

We have shown that the lowest two ($L=0$ and 1) perturbation modes in closed universes are not fictitious perturbations, see Secs. II and III. However, the case is more subtle than normally considered as we often have linearization stability issues in a closed space with the Killing symmetry. The Einstein static model is such a closed space with a timelike Killing vector. The Taub constraint provides a constraint on quadratic combinations of linear order variables for linearization stability to hold. We have derived the Taub constraint in general background

metric in Appendix B; these are Eqs. (B9) and (B15) for general background, and Eqs. (B16) and (B17) for cosmological background. We have evaluated the Taub constraint in the Einstein static model with a timelike Killing vector, see Secs. IV and V. The results are presented in Eqs. (92) and (93) for two fundamental gauge conditions available in the Einstein static model with pressure. The result can be compared with other works as follows.

According to Losic and Unruh [8] $\mathcal{T}_{L\geq 2}$ should be pure positive; they ignored the $L=1$ mode as a gauge mode. But in such a case the Taub constraint *demands* the presence of the $L=0$ mode which is negative. Although Losic and Unruh have claimed this is the case for general c_s^2, Eqs. (92) and (93) show that this is true only for $c_s^2=0$ in both the ZSG and the CG. The gauge condition adopted by Losic and Unruh, and whether they were using the same fluid as ours are unclear to us.

For $c_s^2>1/5$ we have that the $L\geq 2$ modes are stable, while $L=0$ and 1 modes are unstable. Although Losic and Unruh have stated that the perturbation should accompany the unstable $L=0$ mode, our result does not confirm the case at least in our two gauge conditions; it is true only for $c_s^2=0$ and in this case all modes are unstable.

Studying a conformal variation Gibbons [33,34] concluded that $\mathcal{T}_{L\geq 2}>0$, $\mathcal{T}_{L=1}=0$ and $\mathcal{T}_{L=0}<0$ for $c_s^2=0$. Although we expressed Gibbons's result using $\mathcal{T}$, his method is based on second-order variation of entropy and the exact relation to our method is unclear. The conformal variation, $\delta g_{ab}=\phi g_{ab}$, implies $\alpha=\varphi$ and $\chi=0$ in our notation, and this *differs* from the ZSG where $\alpha=-\varphi$ and $\chi=0$. The conformal variation is not available in a proper perturbation theory.

The presence of stable perturbation modes with $L\geq 2$ for $c_s^2>1/5$ has suggested the Einstein static model with pressure might be a potential evolutionary stage in the early universe, before inflation without singularity [7,37]. An Einstein static phase supported by a massless scalar field belongs to this case with $c_s^2=1$, see below Eq. (84). Although it has been suggested that the excitation of $L\geq 2$ modes should accompany the homogeneous ($L=0$) mode [8], which is always unstable, our result shows that this applies only for $c_s^2=0$. For $c_s^2>1/5$, the Taub constraint in two gauge conditions in Eqs. (92) and (93) shows that it is not necessary to accompany $L=0$ and/or $L=1$ modes, both of which are unstable; for $c_s^2=1/5$, we can show that the $L=0$ mode is negative, the $L=1$ to 3 modes are positive and the $L\geq 4$ modes are negative again in the ZSG, whereas the $L=0$ mode is negative; the $L=1$–4 modes are positive and the $L\geq 5$ modes are negative again in the CG. As both the $L=0$ and $L=1$ modes are unstable even for $c_s^2>1/5$, these two modes must be suppressed to have a stable Einstein static stage. How to avoid exciting these lowest two modes for a successful realization is a question yet to be answered.

Previously the $L=0$ and 1 (thus $n=1$ and 2) modes were generally regarded as fictitious, thus largely ignored in the literature, see [38–40]. The physical nature of these two modes with newly restored honor in this paper implies that one needs to properly take into account of these two modes in the future cosmological calculation of closed Friedmann world model. These include the full sky galaxy correlation function and power spectrum (see [41,42] in flat background), the CMB (cosmic microwave background radiation) anisotropy power spectra (see [43,44] for contributions of monopole and dipole in flat background), and others like the luminosity distance and redshift (see [45,46] in flat background). Recent measurement of cosmological parameters shows a tendency of favoring slightly positive curvature [47–52].

Although he missed the opportunity to predict the dynamic universe, Einstein's legacy of establishing modern cosmology over 100 years ago by introducing the cosmological principle and the enigmatic cosmological constant may yet be extended by his choice of the spherical geometry with closed topology [31].

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APPENDIX A: EXACT SOLUTIONS FOR A ZERO-PRESSURE FLUID

Here, we present a complete set of exact solutions including the cases of $k^2=0$ and $3K$. We consider a zero-pressure fluid ($p=0=\delta p$, $\Pi=0$) with the cosmological constant and the background curvature. Relatively decaying solutions are absorbed in the lower bound of integration, and $g(\mathbf{x})$ is the remnant gauge mode in the SG.

	General k^2	$k^2=0$	$k^2=3K$
v_χ	$C\frac{c^2}{aH}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{c^2}{aH}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{c^2}{aH}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
v_κ	$C\frac{(k^2-3K)c^2}{k^2c^2-3a^2\dot{H}}\frac{c^2}{aH}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{Kc^2}{a^2\dot{H}}\frac{c^2}{aH}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	0
v_α	$g\frac{c}{a}$	$g\frac{c}{a}$	$g\frac{c}{a}$
v_φ	$C\frac{c^2}{aH}(1+Kc^2H\int^t\frac{dt}{a^2})$	$C\frac{c^2}{aH}(1+Kc^2H\int^t\frac{dt}{a^2})$	$C\frac{c^2}{aH}(1+Kc^2H\int^t\frac{dt}{a^2})$
v_δ	$-C(k^2-3K)\frac{c^4}{3a}\int^t\frac{dt}{a^2}$	$C\frac{Kc^4}{a}\int^t\frac{dt}{a^2}$	0
χ_v	$C\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
χ_κ	$C\frac{12\pi G\varrho a^2}{k^2c^2-3a^2\dot{H}}\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$-C\frac{4\pi G\varrho}{\dot{H}}\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$C\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
χ_α	χ_v-g	χ_v-g	χ_v-g
χ_φ	$-C4\pi G\varrho a^2c\int^t\frac{dt}{a^2}$	$-C4\pi G\varrho a^2c\int^t\frac{dt}{a^2}$	$-C4\pi G\varrho a^2c\int^t\frac{dt}{a^2}$
χ_δ	$C\frac{c}{H}[1+(\frac{k^2c^2}{3}-4\pi G\varrho a^2)H\int^t\frac{dt}{a^2}]$	$C\frac{c}{H}(1-4\pi G\varrho a^2H\int^t\frac{dt}{a^2})$	$C\frac{c}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
κ_v	$C\frac{(k^2-3K)c^2}{a^2H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$-C\frac{3Kc^2}{a^2H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	0
κ_χ	$-C\frac{12\pi G\varrho}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$-C\frac{12\pi G\varrho}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$	$-C\frac{12\pi G\varrho}{H}(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
κ_α	$\kappa_v+g\frac{1}{c}(3\dot{H}-\frac{k^2c^2}{a^2})$	$\kappa_v+g\frac{1}{c}3\dot{H}$	$-g\frac{1}{c}12\pi G\varrho$
κ_φ	$-C\frac{4\pi G\varrho}{H}(3+k^2c^2H\int^t\frac{dt}{a^2})$	$-C\frac{12\pi G\varrho}{H}$	$-C\frac{12\pi G\varrho}{H}(1+Kc^2H\int^t\frac{dt}{a^2})$
κ_δ	$C\frac{(k^2-3K)c^2}{a^2H}(1+\frac{1}{3}k^2c^2H\int^t\frac{dt}{a^2})$	$-C\frac{3Kc^2}{a^2H}$	0
α_v	0	0	0
α_χ	$-C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$	$-C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$	$-C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$
α_κ	$-C\frac{(k^2-3K)c^2}{(k^2c^2-3a^2\dot{H})^2}4\pi G\varrho a^2(3+k^2c^2H\int^t\frac{dt}{a^2})$	$C\frac{Kc^2}{a^2}\frac{4\pi G\varrho}{\dot{H}}$	0
α_φ	$-C\frac{4\pi G\varrho}{H^2}$	$-C\frac{4\pi G\varrho}{H^2}$	$-C\frac{4\pi G\varrho}{H^2}$
α_δ	$C\frac{(k^2-3K)c^2}{3a^2H^2}$	$-C\frac{Kc^2}{a^2H^2}$	0
φ_v	$C(1+Kc^2H\int^t\frac{dt}{a^2})$	$C(1+Kc^2H\int^t\frac{dt}{a^2})$	$C(1+Kc^2H\int^t\frac{dt}{a^2})$
φ_χ	$C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$	$C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$	$C4\pi G\varrho a^2H\int^t\frac{dt}{a^2}$
φ_κ	$C\frac{4\pi G\varrho a^2}{k^2c^2-3a^2\dot{H}}(3+k^2c^2H\int^t\frac{dt}{a^2})$	$-C\frac{4\pi G\varrho}{\dot{H}}$	$C(1+Kc^2H\int^t\frac{dt}{a^2})$
φ_α	$\varphi_v-g\frac{H}{c}$	$\varphi_v-g\frac{H}{c}$	$\varphi_v-g\frac{H}{c}$
φ_δ	$C(1+\frac{1}{3}k^2c^2H\int^t\frac{dt}{a^2})$	C	$C(1+Kc^2H\int^t\frac{dt}{a^2})$
δ_v	$C(k^2-3K)c^2H\int^t\frac{dt}{a^2}$	$-3CKc^2H\int^t\frac{dt}{a^2}$	0
δ_χ	$C[3+(k^2c^2-12\pi G\varrho a^2)H\int^t\frac{dt}{a^2}]$	$3C(1-4\pi G\varrho a^2H\int^t\frac{dt}{a^2})$	$3C(1+a^2H\dot{H}\int^t\frac{dt}{a^2})$
δ_κ	$C\frac{(k^2-3K)c^2}{k^2c^2-3a^2\dot{H}}(3+k^2c^2H\int^t\frac{dt}{a^2})$	$C\frac{3Kc^2}{a^2\dot{H}}$	0
δ_α	$\delta_v+3g\frac{H}{c}$	$\delta_v+3g\frac{H}{c}$	$3g\frac{H}{c}$
δ_φ	$C(3+k^2c^2H\int^t\frac{dt}{a^2})$	$3C$	$3C(1+Kc^2H\int^t\frac{dt}{a^2})$

APPENDIX B: TAUB CONSTRAINT

1. Derivation

Einstein's equations are

$$\tilde{E}^{ab}\equiv\tilde{R}^{ab}-\frac{1}{2}\tilde{g}^{ab}\tilde{R}+\Lambda\tilde{g}^{ab}-\frac{8\pi G}{c^4}\tilde{T}^{ab}=0. \tag{B1}$$

To second order in perturbation, the metric tensor and its inverse are

$$\tilde{g}_{ab}\equiv g_{ab}+h_{ab},\qquad \tilde{g}^{ab}=g^{ab}-h^{ab}+h^a_ch^{cb}, \tag{B2}$$

where h_{ab} includes the second order and its indices are raised and lowered using the background metric g_{ab} and its inverse metric g^{ab}. The connection is

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{2}(h^a_{b:c} + h^a_{c:b} - h_{bc}{}^{:a}) - \frac{1}{2}h^{ad}(h_{bd:c} + h_{cd:b} - h_{bc:d}), \tag{B3}$$

where a colon indicates the covariant derivative using the background metric g_{ab}. The curvatures are

$$\begin{aligned}
\tilde{R}^a{}_{bcd} &= R^a{}_{bcd} + h^a_{b:[dc]} + h^a_{[d:\{b\}c]} - h_{b[d}{}^{:a}{}_{c]} - h^{ae}(h_{eb:[dc]} + h_{e[d:\{b\}c]} - h_{b[d:\{e\}c]}) \\
&\quad + \frac{1}{2}(h^e_{b:[d} + h^e_{[d:\{b\}} - h_{b[d}{}^{:e})(h^a_{c]:e} - h^a_{\{e\}:c]} - h_{c]e}{}^{:a}), \\
\tilde{R}_{ab} &= R_{ab} + \frac{1}{2}(h^c_{a:bc} + h^c_{b:ac} - h_{ab}{}^{:c}{}_c - h_{:ab}) - \frac{1}{2}h^{ce}(h_{ea:bc} + h_{eb:ac} - h_{ec:ab} - h_{ab:ec}) \\
&\quad + \frac{1}{4}(h^e_{a:b} + h^e_{b:a} - h_{ab}{}^{:e})(h_{,e} - 2h^c_{e:c}) + \frac{1}{4}h^{cd}{}_{:a}h_{cd:b} + \frac{1}{2}h^{c:d}_a(h_{bc:d} - h_{bd:c}) \equiv R_{ab} + R^L_{ab} + R^Q_{ab}, \\
\tilde{R} &= R - h^{ab}R_{ab} + h^{ab}{}_{:ab} - h^{:a}{}_a + h^a_c h^{cb}R_{ab} + h^{ab}(-h^c_{a:cb} + h_{ab}{}^{:c}{}_c - h^c_{a:bc} + h_{,a:b}) \\
&\quad - h^{ab}{}_{:b}h^c_{a:c} + h^{ab}{}_{:b}h_{,a} - \frac{1}{4}h^{:a}h_{,a} + \frac{1}{4}h^{ab;c}(3h_{ab:c} - 2h_{ac:b}) \equiv R + R^L + R^Q,
\end{aligned} \tag{B4}$$

where $h \equiv h^c_c$ and we have $A_{[ab]} \equiv \frac{1}{2}(A_{ab} - A_{ba})$. The indices L and Q indicate the linear and quadratic parts, respectively. The quadratic part includes the terms with quadratic combination of two first (linear) order terms. The linear part can be decomposed into the first-order and second-order perturbations, like $R^L_{ab} = R^{(1)}_{ab} + R^{(2)}_{ab}$; for example, to the second order, we have $h_{ab} = h^{(1)}_{ab} + h^{(2)}_{ab} \equiv h^L_{ab}$.

The background and first-order Einstein equation give $E^{ab} = 0$ and $E^{(1)ab} = 0$. The equation to the second order can be arranged as

$$E^{Lab} \equiv \tilde{R}^{Lab} - \frac{1}{2}g^{ab}\tilde{R}^L + h^{ab}\left(\frac{1}{2}R - \Lambda\right) - \frac{8\pi G}{c^4}T^{Lab} = -E^{Qab} \equiv \frac{8\pi G}{c^4}t^{ab}, \tag{B5}$$

with

$$\begin{aligned}
\frac{8\pi G}{c^4}t^{ab} &= \frac{1}{2}h^{cd}(h^{a:b}_c{}_d + h^{b:a}_c{}_d - h_{cd}{}^{:ab} - h^{ab}{}_{:cd}) - \frac{1}{4}(h^{ca:b} + h^{cb:a} - h^{ab:c})(h_{,c} - 2h^d_{c:d}) - \frac{1}{4}h^{cd:a}h_{cd}{}^{:b} \\
&\quad - \frac{1}{2}h^{ac:d}(h^b_{c:d} - h^b_{d:c}) + h^{c(a}(h^{d:b)}_c{}_d + h^{b)d}{}_{:cd} - h^{b):d}_c{}_d - h^{:b)}{}_c) - h^{ac}h^{bd}R_{cd} - 2h^{cd}h^{(a}_c R^{b)}_d \\
&\quad + \frac{1}{2}g^{ab}\Big[h^c_e h^{ed}R_{cd} - h^{cd}(h^e_{c:de} + h^e_{c:ed} - h_{cd}{}^{:e}{}_e - h_{,c:d}) - \frac{1}{4}h^{:c}h_{,c} + h^{:c}h^d_{c:d} - h^{cd}{}_{:d}h^e_{c:e} \\
&\quad + \frac{1}{4}h^{cd:e}(3h_{cd:e} - 2h_{de:c})\Big] + \frac{1}{2}h^{ab}(h^{cd}R_{cd} + h^{:c}{}_c - h^{cd}{}_{:cd}) + h^a_c h^{bc}\left(\frac{1}{2}R - \Lambda\right) + \frac{8\pi G}{c^4}T^{Qab}.
\end{aligned} \tag{B6}$$

This was presented by Taub in Eq. (3.5) of [36] for the Minkowski background. Here we consider a general background metric g_{ab}.

From $\tilde{E}^{ab}{}_{;b} \equiv 0$, we have $E^{ab}{}_{:b} = 0 = E^{(1)ab}{}_{:b}$ and $E^{(2)ab}{}_{:b} = 0$. Thus $E^{Lab}{}_{:b} = 0$, and we have $E^{Qab}{}_{:b} = 0 = t^{ab}{}_{:b}$. For a Killing vector ξ_a, where $\xi_{a:b} + \xi_{b:a} \equiv 0$, we have

$$0 = (\sqrt{-g}t^{ab}\xi_b)_{:a} = (\sqrt{-g}t^{ab}\xi_b)_{,a}. \tag{B7}$$

thus [see Eq. (4.7) of [36]]

$$0 = \int (\sqrt{-g}t^{ab}\xi_b)_{:a}d^4x = \int \sqrt{-g}t^{ab}\xi_b d^3\sigma_a = \int \sqrt{-g}t^{ab}\xi_b n_a d^3x, \tag{B8}$$

with n_a the timelike normal ($n_i \equiv 0$) four vector. Therefore, we define

$$\mathcal{T} \equiv -\frac{8\pi G}{c^4}\int \sqrt{-g}t^{0b}\xi_b d^3x = \int \sqrt{-g}E^{Q0b}\xi_b d^3x = 0, \tag{B9}$$

and call this the Taub constraint. In the presence of the Killing vectors in the background metric g_{ab}, Fischer, Marsden and Moncrief [9,10] have proved the violation of this condition as the criterion of linearization instability for the *vacuum* case. Similar results hold for Einstein field equations coupled with matter fields such as scalar fields, electromagnetic fields and Yang-Mills fields [11,53,54].

2. ADM constraint formulation

Evaluation of Eq. (B9) with Eq. (B6) needs complicated algebra. There is a simpler formulation using the constraint equations. The ADM (Arnowitt-Deser-Misner) energy and momentum constraint equations can be written as [Eq. (3.14) in [55]],

$$\mathcal{E}^0 \equiv K^{ij}K_{ij} - K^2 - R^{(h)} + \frac{16\pi G}{c^4}E + 2\Lambda = 0, \tag{B10}$$

$$\mathcal{E}^i \equiv K^{ij}{}_{\|j} - K^{\|i} - \frac{8\pi G}{c^4}J^i = 0. \tag{B11}$$

The indices and the covariant derivatives ($\|$) in the ADM notation are based on the ADM metric $h_{ij} \equiv \tilde{g}_{ij}$. From Eq. (B1), we can show

$$\tilde{E}^{00} = -\frac{1}{2N^2}\mathcal{E}^0, \qquad \tilde{E}^{0i} = \frac{1}{N}\mathcal{E}^i + \frac{N^i}{4N^4}\mathcal{E}^0. \tag{B12}$$

To the second order, we have

$$\tilde{E}^{00} = E^{(0)00} + E^{L00} + E^{Q00}, \qquad \tilde{E}^{0i} = E^{L0i} + E^{Q0i};$$
$$\mathcal{E}^0 \equiv \mathcal{E}^{(0)0} + \mathcal{E}^{L0} + \mathcal{E}^{Q0}, \qquad \mathcal{E}^i \equiv \mathcal{E}^{Li} + \mathcal{E}^{Qi}. \tag{B13}$$

As we have $E^{(0)00} = 0 = \mathcal{E}^{(0)0}$ for the background, and $E^{(1)00} = 0 = \mathcal{E}^{(1)0}$ and $E^{(1)0i} = 0 = \mathcal{E}^{(1)i}$ for the first-order perturbation, the quadratic parts become

$$E^{Q00} = -\frac{1}{2(N^{(0)})^2}\mathcal{E}^{Q0}, \qquad E^{Q0i} = \frac{1}{N^{(0)}}\mathcal{E}^{Qi}. \tag{B14}$$

Using this Eq. (B9) gives for the Taub constraint

$$\begin{aligned}\mathcal{T} &= \int \sqrt{-g}\xi_b E^{Q0b}d^3x \\ &= \int \sqrt{h^{(0)}}N^{(0)}(\xi_0 E^{Q00} + \xi_i E^{Q0i})d^3x \\ &= \int \sqrt{h^{(0)}}\left(-\frac{1}{2N^{(0)}}\xi_0\mathcal{E}^{Q0} + \xi_i\mathcal{E}^{Qi}\right)d^3x,\end{aligned} \tag{B15}$$

where we used $\sqrt{-g} = N^{(0)}\sqrt{h^{(0)}}$. This is an alternative presentation of the Taub constraint to Eq. (B9) which needs only the energy and momentum constraint equations.

In the cosmological background, the Taub constraint derived in Eq. (B15) yields

$$\begin{aligned}\mathcal{T} &= \int \sqrt{\gamma}a^4(\xi_0 E^{Q00} + \xi_i E^{Q0i})d^3x \\ &= \int \sqrt{\gamma}a^3\left(-\frac{1}{2a}\xi_0\mathcal{E}^{Q0} + \xi_i\mathcal{E}^{Qi}\right)d^3x,\end{aligned} \tag{B16}$$

where γ is the determinant of γ_{ij}. The Friedmann metric has six spacelike Killing vectors [56]. Einstein's static model has an additional timelike Killing vector with $\xi^a \equiv \delta^a_0$. We will consider the Taub constraint based on this timelike Killing vector. Using $\xi_a = -\delta^0_a$, Eq. (B16) gives

$$\begin{aligned}\mathcal{T} &= \int \sqrt{-g}\xi_b E^{Q0b}d^3x \\ &= -a_0^4\int \sqrt{\gamma}E^{Q00}d^3x = \frac{1}{2}a_0^2\int \sqrt{\gamma}\mathcal{E}^{Q0}d^3x.\end{aligned} \tag{B17}$$

Thus, for evaluation of the Taub constraint in our case, we only need the energy constraint equation to second order.

3. The energy constraint equation to second order

The fully nonlinear and exact perturbation equations in the presence of background curvature were presented in [57]; the equations are derived by taking a spatial gauge $\gamma \equiv 0$ in the metric in Eq. (4) and replacing $a\beta_{,i}$ and $-v_{,i}$ by χ_i and v_i, respectively, now including the vector-type perturbation as well. The ADM energy constraint equation gives [Eq. (3.2) in [57]]

$$\begin{aligned}\mathcal{E}^0 = &-\frac{6}{c^2}\left(H^2 - \frac{8\pi G}{3}\tilde{\varrho} + \frac{Kc^2}{a^2(1+2\varphi)} - \frac{\Lambda c^2}{3}\right) + \frac{4}{c^2}H\kappa \\ &+ \frac{4\Delta\varphi}{a^2(1+2\varphi)^2} - \frac{2}{3c^2}\kappa^2 + \frac{16\pi G}{c^2}\left(\tilde{\varrho} + \frac{\tilde{p}}{c^2}\right)(\gamma^2 - 1) \\ &- \frac{6\varphi^{|i}\varphi_{,i}}{a^2(1+2\varphi)^3} + \bar{K}^i_j\bar{K}^j_i,\end{aligned} \tag{B18}$$

with γ the Lorentz factor, and N the lapse function, where

$$\gamma \equiv \frac{1}{\sqrt{1-\frac{v^k v_k}{c^2(1+2\varphi)}}}, \qquad N \equiv a\mathcal{N} \equiv a\sqrt{1+2\alpha+\frac{\chi^k\chi_k}{a^2(1+2\varphi)}},$$
$$\bar{K}^i_j\bar{K}^j_i = \frac{1}{a^4\mathcal{N}^2(1+2\varphi)^2}\left\{\frac{1}{2}\chi^{i|j}(\chi_{i|j}+\chi_{j|i}) - \frac{1}{3}\chi^i{}_{|i}\chi^j{}_{|j} - \frac{4}{1+2\varphi}\left[\frac{1}{2}\chi^i\varphi^{|j}(\chi_{i|j}+\chi_{j|i}) - \frac{1}{3}\chi^i{}_{|i}\chi^j\varphi_{,j}\right]\right.$$
$$\left. + \frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_i\varphi^{|j}\varphi_{,j} + \frac{1}{3}\chi^i\chi^j\varphi_{,i}\varphi_{,j}\right)\right\}. \tag{B19}$$

To second order, we have

$$\mathcal{E}^0 = -\frac{6}{c^2}\left(H^2 - \frac{8\pi G}{3}\varrho + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3}\right) + \frac{4}{c^2}\left(4\pi G\delta\varrho + H\kappa + c^2\frac{\Delta+3K}{a^2}\varphi\right)$$
$$+ \frac{16\pi G}{c^2}\left(\varrho + \frac{p}{c^2}\right)\frac{v^i v_i}{c^2} - \frac{2}{3c^2}\kappa^2 - \frac{2}{a^2}[3\varphi^{|i}\varphi_{,i} + 4\varphi(2\Delta+3K)\varphi] + \frac{1}{a^4}\left[\frac{1}{2}\chi^{i|j}(\chi_{i|j}+\chi_{j|i}) - \frac{1}{3}\chi^i{}_{|i}\chi^j{}_{|j}\right]$$
$$\equiv \mathcal{E}^{(0)0} + \mathcal{E}^{L0} + \mathcal{E}^{Q0}. \tag{B20}$$

For the scalar perturbation, we have $v_i \equiv -v_{,i}$ and $\chi_i \equiv \chi_{,i}$. The evaluation of the Taub constraint in Eq. (B17) using Eq. (B20) in a couple of gauge conditions in Einstein's static model is presented in Sec. V.

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