Chiral random matrix theory for single-flavor spin-one Cooper pairing

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We propose a new non-Hermitian chiral random matrix model that describes single-flavor spin-one Cooper pairing of quarks. For three colors the model shows spontaneous breaking of color $SU(3)_C$ and spin $SO(3)_J$ symmetries down to the diagonal $SO(3)_{C+J}$ subgroup, in striking analogy to the color-spin locked phase of one-flavor QCD at high density. For two colors, color-singlet spin-one diquarks condense and trigger symmetry breaking $U(1) \times SO(3)_J \to SO(2)_J$. In both cases the microscopic large-N limit is rigorously taken and the effective theory of Nambu-Goldstone modes is derived.

I. INTRODUCTION

In relativistic cold ultradense matter, the Fermi surface is destabilized by attractive interactions between quarks and the ground state is likely to exhibit color superconductivity [1]. For three flavors, the most stable state at asymptotically high density is believed to be the so-called color-flavor locked (CFL) phase in which color and flavor symmetries are all spontaneously broken [2], while for two flavors the 2SC phase breaks color down to SU(2) without chiral symmetry breaking [1,3,4]. In reality, however, constraints from color and electric charge neutrality and β-equilibrium as well as the large strange quark mass enhance the mismatch of Fermi momenta of different flavors, and make the CFL and 2SC phases less favored at moderate densities that can be reached in the interior of compact stars. Various alternative phases are proposed in the literature [5–12]. The possibility of Cooper pairing between equal flavors is particularly interesting because it is free from stress under the mismatch of Fermi momenta. Due to the Pauli principle, the Cooper pair wave function must be antisymmetric under the exchange of quantum numbers of quarks. The perturbative one-gluon exchange provides an attractive interaction in the color antisymmetric channel, so it seems reasonable to assume that the Cooper pair is antisymmetric in colors. Then Lorentz-scalar pairing is forbidden and the Cooper pair is forced to carry a nonzero total angular momentum J. Studies by various authors [1,13–27] suggest that the ground state for $J = 1$ at asymptotically high density is the color-spin locked (CSL) phase [1,15], where the color $SU(3)_C$ and the spin $SO(3)_J$ symmetries are dynamically broken to the diagonal $SO(3)_{C+J}$ subgroup. This phase retains invariance under a combination of color and spatial rotation.

Random matrix theory (RMT) [28–30] offers a powerful approach to probing nonperturbative dynamics of gauge theories. It is well established that statistical fluctuations of the Dirac operator eigenvalues in the microscopic domain are exactly described by RMT with chiral symmetry [31,32]; the Dirac eigenvalues of order $1/V_4$ with V_4 the Euclidean spacetime volume correspond to the hard-edge limit of the Wishart-Laguerre ensemble. This correspondence generalizes to a weakly non-Hermitian Dirac operator at small chemical potential μ of order $1/\sqrt{V_4}$ [33–35]. Moreover, in QCD-like theories (i.e., two-color QCD, QCD with adjoint quarks, and QCD with isospin chemical potential), there exist strongly non-Hermitian random matrix ensembles that govern the Dirac spectrum at high density $\mu \gg m_\pi$ [36–39]. The typical scale of the microscopic domain in these theories is set by the BCS gap of quarks [40]. The question whether universality of the microscopic limit of RMT holds for color-superconducting phases of dense three-color QCD has remained open [41]. Recently a suitable matrix model was constructed for two and three flavors [47].

In this paper, we propose a new non-Hermitian RMT that describes single-flavor Cooper pairing for two and three colors. By taking the microscopic large-N limit with N the matrix size we show, for three colors, that diquark condensation occurs in the color-triplet spin-one channel and spontaneously breaks $U(1)_B \times U(1)_A \times SU(3)_C \times SO(3)_J$ symmetry down to $SO(3)_{C+J}$, which is the diagonal subgroup of $SU(3)_C \times SO(3)_J$. This is the first-ever realization of color-spin locking in RMT. For two colors, we show that color-singlet spin-one diquarks condense and break $U(1)_B \times U(1)_A \times SO(3)_J$ spontaneously. There seems to be no previous work on one-flavor two-color QCD at high density and this work provides the first systematic study based on symmetries.

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This paper is organized as follows. In Sec. II our new matrix model is defined and basic properties are summarized. In Sec. III the large-N limit is taken for three colors and color-spin locking is demonstrated. Similarities and differences between RMT and QCD are clarified. In Sec. IV the analysis for two colors is presented. The breaking of rotational symmetry by diquark condensates is demonstrated and the effective theory of soft modes is rigorously derived. We conclude in Sec. V.

II. THE MATRIX MODEL

The partition function of our random matrix model for N_c colors is defined by

$$
Z = \int \mathrm{d}X \int \mathrm{d}Y \int \prod_{A=0}^{N_c^2-1} \mathrm{d}V^A \int \prod_{A=0}^{N_c^2-1} \mathrm{d}W^A
$$
$$
\times \exp\{-N\mathrm{Tr}(X^T X) - N\mathrm{Tr}(Y^T Y)
$$
$$
- 2N\mathrm{Tr}[(V^A)^T V^A] - 2N\mathrm{Tr}[(W^A)^T W^A]\}
$$
$$
\times \det(\mathcal{D} + m\mathbb{1}_{4N_c N}) \tag{1}
$$

where $X, Y, V^A, W^A (A = 0, 1, ..., N_c^2 - 1)$ are $N \times N$ real matrices and $\mathrm{d}X, \mathrm{d}Y, \mathrm{d}V^A, \mathrm{d}W^A$ are flat Cartesian measures over $\mathbb{R}^{N \times N}$. The "Dirac operator" $\mathcal{D}$ is a $4N_c N \times 4N_c N$ matrix defined as

$$
\mathcal{D} = \begin{pmatrix} 0 & D_R \\ D_L & 0 \end{pmatrix} \tag{2}
$$

$$
D_L \equiv (V^A \otimes t^A + iX \otimes \mathbb{1}_{N_c}) \otimes \mathbb{1}_2 \tag{3}
$$

$$
D_R \equiv (W^A \otimes t^A + iY \otimes \mathbb{1}_{N_c}) \otimes \mathbb{1}_2 \tag{4}
$$

where $t^A (A = 0, 1, ..., N_c^2 - 1)$ are the generators of $\mathrm{U}(N_c)$ in the fundamental representation, normalized as $\mathrm{Tr}(t^A t^B) = 2\delta_{AB}$. The last factor $\mathbb{1}_2$ in (3) and (4) indicates that $D_{L,R}$ are diagonal in the spin space. In the chiral limit, there is a symmetry

$$
[\mathrm{SU}(N_c) \times \mathrm{SU}(2)_J]_L \times [\mathrm{SU}(N_c) \times \mathrm{SU}(2)_J]_R \tag{5}
$$

which is reduced to the vectorial subgroup $[\mathrm{SU}(N_c) \times \mathrm{SU}(2)_J]_V$ by nonzero quark mass. There is also an orthogonal symmetry

$$
V^A \to g_1 V^A g_2^T, \qquad X \to g_1 X g_2^T,
$$
$$
W^A \to g_2 W^A g_1^T, \qquad Y \to g_2 Y g_1^T \tag{6}
$$

with $g_{1,2} \in \mathrm{O}(N)$.

Let us introduce quarks ψ_{ai}^α and antiquarks $\bar{\psi}_{ai}^\alpha$, where $a \in \{1, ..., N_c\}$ is color, $i \in \{1, 2\}$ is spin and $\alpha \in \{1, 2, ..., N\}$. Then

$$
Z = \int \mathrm{d}\bar{\psi}_R \mathrm{d}\bar{\psi}_L \mathrm{d}\psi_R \mathrm{d}\psi_L \int \mathrm{d}X \int \mathrm{d}Y \int \prod_{A=0}^{N_c^2-1} \mathrm{d}V^A \int \prod_{A=0}^{N_c^2-1} \mathrm{d}W^A
$$
$$
\times \exp\{-N\mathrm{Tr}(X^T X) - N\mathrm{Tr}(Y^T Y) - 2N\mathrm{Tr}[(V^A)^T V^A] - 2N\mathrm{Tr}[(W^A)^T W^A]\}
$$
$$
\times \exp\left[\begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_L \end{pmatrix}_{ai}^\alpha \begin{pmatrix} m\delta_{\alpha\beta}\delta_{ab} & W_{\alpha\beta}^A t_{ab}^A + iY_{\alpha\beta}\delta_{ab} \\ V_{\alpha\beta}^A t_{ab}^A + iX_{\alpha\beta}\delta_{ab} & m^*\delta_{\alpha\beta}\delta_{ab} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}_{bi}^\beta \right] \tag{7}
$$

$$
= \int \mathrm{d}\bar{\psi}_R \mathrm{d}\bar{\psi}_L \mathrm{d}\psi_R \mathrm{d}\psi_L \int \mathrm{d}X \int \mathrm{d}Y \int \prod_{A=0}^{N_c^2-1} \mathrm{d}V^A \int \prod_{A=0}^{N_c^2-1} \mathrm{d}W^A
$$
$$
\times \exp[-NX_{\alpha\beta}^2 - NY_{\alpha\beta}^2 - 2N(V_{\alpha\beta}^A)^2 - 2N(W_{\alpha\beta}^A)^2 + m\bar{\psi}_{Rai}^\alpha \psi_{Lai}^\alpha + m^*\bar{\psi}_{Lai}^\alpha \psi_{Rai}^\alpha
$$
$$
+ \bar{\psi}_{Lai}^\alpha V_{\alpha\beta}^A t_{ab}^A \psi_{Lbi}^\beta + i\bar{\psi}_{Lai}^\alpha X_{\alpha\beta} \psi_{Lai}^\beta + \bar{\psi}_{Rai}^\alpha W_{\alpha\beta}^A t_{ab}^A \psi_{Rbi}^\beta + i\bar{\psi}_{Rai}^\alpha Y_{\alpha\beta} \psi_{Rai}^\beta]. \tag{8}
$$

It is now straightforward to integrate out the Gaussian random matrices. Using the completeness relation $t_{ab}^A t_{cd}^A = 2\delta_{ad}\delta_{bc}$ we obtain

$$
Z \propto \int \mathrm{d}\bar{\psi}_R \mathrm{d}\bar{\psi}_L \mathrm{d}\psi_R \mathrm{d}\psi_L \exp\left[m\bar{\psi}_{Rai}^\alpha \psi_{Lai}^\alpha + m^*\bar{\psi}_{Lai}^\alpha \psi_{Rai}^\alpha - \frac{1}{4N}\bar{\psi}_{Lai}^\alpha \bar{\psi}_{Lbj}^\beta \psi_{Lbj}^\beta \psi_{Lai}^\alpha + \frac{1}{4N}\bar{\psi}_{Lai}^\alpha \bar{\psi}_{Lbj}^\beta \psi_{Laj}^\beta \psi_{Lbi}^\alpha + (L \leftrightarrow R) \right] \tag{9}
$$

$$
= \int \mathrm{d}\bar{\psi}_R \mathrm{d}\bar{\psi}_L \mathrm{d}\psi_R \mathrm{d}\psi_L \exp\left[m\bar{\psi}_{Rai}^\alpha \psi_{Lai}^\alpha + m^*\bar{\psi}_{Lai}^\alpha \psi_{Rai}^\alpha + \frac{1}{4N}\bar{\psi}_{Lai}^\alpha \bar{\psi}_{Lbj}^\beta \psi_{Lcj}^\beta \psi_{Ldi}^\beta (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (L \leftrightarrow R) \right] \tag{10}
$$

$$= \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp\left[m\bar{\psi}^{\alpha}_{Rai}\psi^{\alpha}_{Lai} + m^*\bar{\psi}^{\alpha}_{Lai}\psi^{\alpha}_{Rai} + \frac{1}{8N}\bar{\psi}^{\alpha}_{Lai}\bar{\psi}^{\alpha}_{Lbj}\psi^{\beta}_{Lck}\psi^{\beta}_{Ld\ell} \cdot 2\delta_{jk}\delta_{i\ell} \cdot (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (L \leftrightarrow R)\right].$$

(11)

For the Pauli matrices $\sigma_M \equiv (\mathbb{1}_2, \sigma_i)$ there is the identity

$$\sum_{M=0}^{3}(\sigma_M\sigma_2)_{ij}(\sigma_2\sigma_M)_{k\ell} = 2\delta_{jk}\delta_{i\ell}.$$

(12)

It allows us to write

$$Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp\left[m\bar{\psi}^{\alpha}_{Rai}\psi^{\alpha}_{Lai} + m^*\bar{\psi}^{\alpha}_{Lai}\psi^{\alpha}_{Rai} + \frac{1}{8N}(\bar{\psi}^{\alpha}_{La}\sigma_M\sigma_2\bar{\psi}^{\alpha}_{Lb})(\psi^{\beta}_{Lc}\sigma_2\sigma_M\psi^{\beta}_{Ld})(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (L \leftrightarrow R)\right]$$

(13)

where we have omitted the spinor indices for brevity. Noting that the $M = 0$ term vanishes identically due to the Pauli principle, we are left with

$$Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp\left[m\bar{\psi}^{\alpha}_{Rai}\psi^{\alpha}_{Lai} + m^*\bar{\psi}^{\alpha}_{Lai}\psi^{\alpha}_{Rai} + \frac{1}{8N}(\bar{\psi}^{\alpha}_{La}\sigma_m\sigma_2\bar{\psi}^{\alpha}_{Lb})(\psi^{\beta}_{Lc}\sigma_2\sigma_m\psi^{\beta}_{Ld})(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (L \leftrightarrow R)\right]$$

(14)

with $m \in \{1, 2, 3\}$, which should not be confused with the quark mass. The treatment henceforth depends on the number of colors and is discussed in the following sections.

III. THREE COLORS

For $N_c = 3$ the identity $\varepsilon_{abe}\varepsilon_{cde} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ holds. Hence we have from (14)

$$Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp\left[m\bar{\psi}^{\alpha}_{Rai}\psi^{\alpha}_{Lai} + m^*\bar{\psi}^{\alpha}_{Lai}\psi^{\alpha}_{Rai} + \frac{1}{8N}(\bar{\psi}^{\alpha}_{La}\sigma_m\sigma_2\bar{\psi}^{\alpha}_{Lb}\varepsilon_{abe})(\psi^{\beta}_{Lc}\sigma_2\sigma_m\psi^{\beta}_{Ld}\varepsilon_{cde}) + (L \leftrightarrow R)\right].$$

(15)

To bilinearize the quartic interaction we insert the constant factor

$$\int d\Delta_L \exp\left[-8N\left\{(\Delta_L)_{me} - \frac{1}{8N}\bar{\psi}^{\alpha}_{La}\sigma_m\sigma_2\bar{\psi}^{\alpha}_{Lb}\varepsilon_{abe}\right\}\left\{(\Delta_L^*)_{me} - \frac{1}{8N}\psi^{\beta}_{Lc}\sigma_2\sigma_m\psi^{\beta}_{Ld}\varepsilon_{cde}\right\}\right] \times (L \leftrightarrow R)$$

(16)

where $\Delta_{L,R}$ are complex 3×3 matrices that transform as triplet under color SU(3) and spin SO(3). They are also charged under U(1)$_B$ and U(1)$_A$. Then

$$Z \propto \int d\Delta_L \int d\Delta_R \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp[-8N\text{Tr}(\Delta_L^\dagger\Delta_L) - 8N\text{Tr}(\Delta_R^\dagger\Delta_R) + m\bar{\psi}^{\alpha}_{Rai}\psi^{\alpha}_{Lai} + m^*\bar{\psi}^{\alpha}_{Lai}\psi^{\alpha}_{Rai}$$
$$+ \psi^{\alpha}_{La}\sigma_2\sigma_m\psi^{\alpha}_{Lb}\varepsilon_{abc}\Delta_{Lmc} + \bar{\psi}^{\alpha}_{La}\sigma_m\sigma_2\bar{\psi}^{\alpha}_{Lb}\varepsilon_{abc}\Delta_{Lmc}^* + (L \leftrightarrow R)]$$

(17)

$$
= \int d\Delta_L \int d\Delta_R \exp[-8N\mathrm{Tr}(\Delta_L^\dagger \Delta_L) - 8N\mathrm{Tr}(\Delta_R^\dagger \Delta_R)] \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L
$$

$$
\times \exp\left[\begin{pmatrix} \bar{\psi}_L \\ \bar{\psi}_R \\ \psi_L \\ \psi_R \end{pmatrix}_{ai}^\alpha \begin{pmatrix} (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{Lmc}^* & 0 & 0 & m^*\delta_{ab}\delta_{ij}/2 \\ 0 & (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{Rmc}^* & m\delta_{ab}\delta_{ij}/2 & 0 \\ 0 & -m\delta_{ab}\delta_{ij}/2 & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{Lmc} & 0 \\ -m^*\delta_{ab}\delta_{ij}/2 & 0 & 0 & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{Rmc} \end{pmatrix} \begin{pmatrix} \bar{\psi}_L \\ \bar{\psi}_R \\ \psi_L \\ \psi_R \end{pmatrix}_{bj}^\alpha \right]
$$

$$
= \int d\Delta_L \int d\Delta_R \exp[-8N\mathrm{Tr}(\Delta_L^\dagger \Delta_L) - 8N\mathrm{Tr}(\Delta_R^\dagger \Delta_R)]
$$

$$
\times \mathrm{Pf}^N \begin{pmatrix} (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{Rmc}^* & m\delta_{ab}\delta_{ij}/2 \\ -m\delta_{ab}\delta_{ij}/2 & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{Lmc} \end{pmatrix} \mathrm{Pf}^N \begin{pmatrix} (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{Lmc}^* & m^*\delta_{ab}\delta_{ij}/2 \\ -m^*\delta_{ab}\delta_{ij}/2 & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{Rmc} \end{pmatrix}. \tag{18}
$$

This is an exact rewriting of the original partition function and so far no approximation has been made. Now we shall let N large with $m \sim 1/\sqrt{N}$. To nail down the saddle point of the integral, let us take the chiral limit using the Pfaffian formulas [48]

$$
\mathrm{Pf}[(\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{mc}] = 2\det\Delta, \tag{19}
$$

$$
\mathrm{Pf}[(\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{mc}^*] = 2\det\Delta^\dagger. \tag{20}
$$

Here, $(\sigma_2\sigma_m)_{ij}\varepsilon_{abc}\Delta_{mc}$ and $(\sigma_m\sigma_2)_{ij}\varepsilon_{abc}\Delta_{mc}^*$ are treated as 6×6 matrices with left index (a, i) and right index (b, j). The partition function in the chiral limit therefore reads

$$
Z \propto \int d\Delta_L \int d\Delta_R \exp[-8N\mathrm{Tr}(\Delta_L^\dagger \Delta_L) - 8N\mathrm{Tr}(\Delta_R^\dagger \Delta_R)]
$$

$$
\times \det^N(\Delta_L^\dagger \Delta_L)\det^N(\Delta_R^\dagger \Delta_R). \tag{21}
$$

With a quick calculation involving the singular value decomposition of $\Delta_{L,R}$ one can verify that

$$
\Delta_L = \Delta_R = \frac{1}{2\sqrt{2}}\mathbb{1}_3 \tag{22}
$$

maximizes the integrand. It is not invariant under separate rotations by color SU(3) and spin SO(3) but is invariant under their simultaneous rotation. Namely color and spin are locked:

$$
\mathrm{U}(1)_B \times \mathrm{U}(1)_A \times \mathrm{SU}(3)_C \times \mathrm{SO}(3)_J \to \mathrm{SO}(3)_{C+J}. \tag{23}
$$

The soft fluctuations around (22) can be parametrized as

$$
\Delta_L = U, \qquad \Delta_R = V \tag{24}
$$

with $U, V \in \mathrm{U}(3)$. (This V should not be confused with the Gaussian random matrix V^A inside $\mathcal{D}$.) Let us define

$$
e^{i\phi_U} \equiv \det U, \qquad e^{i\phi_V} \equiv \det V,
$$

$$
\mathcal{Q}_{ai,bj} \equiv \varepsilon_{abm}(\sigma_2\sigma_m)_{ij}. \tag{25}
$$

The large-N partition function can be evaluated as follows:

$$
Z \sim \int_{\mathrm{U}(3)} dU \int_{\mathrm{U}(3)} dV \det^{N/2}\begin{pmatrix} (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}V_{mc}^* & \sqrt{2}m\delta_{ab}\delta_{ij} \\ -\sqrt{2}m\delta_{ab}\delta_{ij} & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}U_{mc} \end{pmatrix}\det^{N/2}\begin{pmatrix} (\sigma_m\sigma_2)_{ij}\varepsilon_{abc}U_{mc}^* & \sqrt{2}m^*\delta_{ab}\delta_{ij} \\ -\sqrt{2}m^*\delta_{ab}\delta_{ij} & (\sigma_2\sigma_m)_{ij}\varepsilon_{abc}V_{mc} \end{pmatrix} \tag{26}
$$

$$
= \int_{\mathrm{U}(3)} dU \int_{\mathrm{U}(3)} dV \det^{N/2}\left[\begin{pmatrix} V^* \otimes \mathbb{1}_2 & 0 \\ 0 & U \otimes \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \varepsilon_{abc}V_{mc}^*(\sigma_m\sigma_2)_{ij} & \sqrt{2}m\delta_{ab}\delta_{ij} \\ -\sqrt{2}m\delta_{ab}\delta_{ij} & \varepsilon_{abc}U_{mc}(\sigma_2\sigma_m)_{ij} \end{pmatrix}\begin{pmatrix} V^\dagger \otimes \mathbb{1}_2 & 0 \\ 0 & U^T \otimes \mathbb{1}_2 \end{pmatrix}\right]
$$

$$
\times \det^{N/2}\left[\begin{pmatrix} U^* \otimes \mathbb{1}_2 & 0 \\ 0 & V \otimes \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \varepsilon_{abc}U_{mc}^*(\sigma_m\sigma_2)_{ij} & \sqrt{2}m^*\delta_{ab}\delta_{ij} \\ -\sqrt{2}m^*\delta_{ab}\delta_{ij} & \varepsilon_{abc}V_{mc}(\sigma_2\sigma_m)_{ij} \end{pmatrix}\begin{pmatrix} U^\dagger \otimes \mathbb{1}_2 & 0 \\ 0 & V^T \otimes \mathbb{1}_2 \end{pmatrix}\right] \tag{27}
$$

$$\propto \int_{U(3)} dU \int_{U(3)} dV \, \det^{N/2} \begin{pmatrix} e^{-i\phi_V}\mathcal{Q}^* & \sqrt{2}m(V^*U^T)\otimes\mathbb{1}_2 \\ -\sqrt{2}m(UV^\dagger)\otimes\mathbb{1}_2 & e^{i\phi_U}\mathcal{Q} \end{pmatrix}$$

$$\times \det^{N/2} \begin{pmatrix} e^{-i\phi_U}\mathcal{Q}^* & \sqrt{2}m^*(U^*V^T)\otimes\mathbb{1}_2 \\ -\sqrt{2}m^*(VU^\dagger)\otimes\mathbb{1}_2 & e^{i\phi_V}\mathcal{Q} \end{pmatrix} \tag{28}$$

$$\propto \int_{U(3)} dU \int_{U(3)} dV \, \det^{N/2} \begin{pmatrix} \mathbb{1}_6 & \sqrt{2}m[(V^*U^T)\otimes\mathbb{1}_2]e^{-i\phi_U}\mathcal{Q}^{-1} \\ -\sqrt{2}m[(UV^\dagger)\otimes\mathbb{1}_2]e^{i\phi_V}\mathcal{Q}^{*-1} & \mathbb{1}_6 \end{pmatrix}$$

$$\times \det^{N/2} \begin{pmatrix} \mathbb{1}_6 & \sqrt{2}m^*[(U^*V^T)\otimes\mathbb{1}_2]e^{-i\phi_V}\mathcal{Q}^{-1} \\ -\sqrt{2}m^*[(VU^\dagger)\otimes\mathbb{1}_2]e^{i\phi_U}\mathcal{Q}^{*-1} & \mathbb{1}_6 \end{pmatrix} \tag{29}$$

$$= \int_{U(3)} dU \int_{U(3)} dV \, \det^{N/2}(\mathbb{1}_6 + 2m^2 e^{-i(\phi_U-\phi_V)}[(V^*U^T)\otimes\mathbb{1}_2]\mathcal{Q}^{-1}[(UV^\dagger)\otimes\mathbb{1}_2]\mathcal{Q}^{*-1})$$

$$\times \det^{N/2}(\mathbb{1}_6 + 2m^{*2}e^{i(\phi_U-\phi_V)}[(U^*V^T)\otimes\mathbb{1}_2]\mathcal{Q}^{-1}[(VU^\dagger)\otimes\mathbb{1}_2]\mathcal{Q}^{*-1}) \tag{30}$$

$$\sim \int_{U(3)} d\hat{U} \exp[\hat{m}^2 \text{Tr}([\hat{U}\otimes\mathbb{1}_2]\mathcal{Q}^{-1}[\hat{U}^T\otimes\mathbb{1}_2]\mathcal{Q}^{*-1}) + \hat{m}^{*2}\text{Tr}([\hat{U}^\dagger\otimes\mathbb{1}_2]\mathcal{Q}^{-1}[\hat{U}^*\otimes\mathbb{1}_2]\mathcal{Q}^{*-1})] \tag{31}$$

where we have defined

$$\hat{U} \equiv e^{-i(\phi_U-\phi_V)/2}V^*U^T,$$
$$\hat{m} \equiv \sqrt{N}m. \tag{32}$$

Using a software [49] we verified

$$\text{Tr}([\hat{U}\otimes\mathbb{1}_2]\mathcal{Q}^{-1}[\hat{U}^T\otimes\mathbb{1}_2]\mathcal{Q}^{*-1})$$
$$= \frac{1}{2}\{\text{Tr}(\hat{U}^2) - (\text{Tr}\hat{U})^2 - \text{Tr}(\hat{U}^T\hat{U})\}. \tag{33}$$

Upon substituting (33) into (31), we finally arrive at the low-energy effective theory

$$Z \sim \int_{U(3)} d\hat{U} \exp\left[\frac{\hat{m}^2}{2}\{\text{Tr}(\hat{U}^2) - (\text{Tr}\hat{U})^2 - \text{Tr}(\hat{U}^T\hat{U})\} + \text{c.c.}\right]. \tag{34}$$

Notice that (34) respects invariance under SO(3): $\hat{U}$ transforms as $\hat{U} \to g\hat{U}g^T$ for $g \in$ SO(3). The U(1) part of $\hat{U}$ is the Nambu-Goldstone boson associated with the U(1)$_A$ symmetry [50]. On the other hand, the SU(3) part of $\hat{U}$ describes the Nambu-Goldstone bosons associated with the symmetry breaking SU(3)$_C \times$ SO(3)$_J \to$ SO(3)$_{C+J}$. At this point the qualitative difference between QCD and RMT becomes evident. In one-flavor QCD, these 8 modes are absorbed into gluons via the Anderson-Higgs mechanism and disappear from the physical spectrum [15,27]. By contrast, there is no *local* gauge invariance in RMT and the Anderson-Higgs mechanism is not operative, leading consequently to the appearance of these 8 modes in the low-energy effective theory (34).

We also wish to point out that while our RMT concerns pairing of quarks with the same chirality, it is known that in one-flavor QCD pairing of quarks of opposite chirality can take place [15,19,52]. However, this is beyond the scope of this paper.

IV. TWO COLORS

Due to its unique features and theoretical simplicity, two-color QCD has long been studied as a prototype of strongly coupled non-Abelian gauge theories. A particularly pleasant feature of two-color QCD is that for an even number of flavors with degenerate masses, the path-integral measure is positive definite even at nonzero quark chemical potential and can be simulated on a lattice [53–56]. Interestingly, the color-singlet scalar diquarks in two-color QCD are degenerate with pions and condense when the chemical potential exceeds $m_\pi/2$, breaking U(1)$_B$ spontaneously. The phase structure of two-color QCD has been studied with chiral perturbation theory [57–65] and low-energy effective models [66–74].

It seems that almost all previous work focused on two-color QCD with an even number of flavors [75], but there is a qualitative difference between even and odd flavors. (Note that we refer to the number of *Dirac* fermions here. The famous Witten's SU(2) anomaly [76], which asserts that an SU(2) gauge theory coupled to an odd number of *Weyl* fermions in the spin-1/2 representation of SU(2) is inconsistent, does not affect our present discussion.) It is easy to understand this intuitively. If we assume that quarks form pairs in the Lorentz-scalar color-antisymmetric channel, the Pauli principle stipulates that the quantum number of flavors must be antisymmetric. For even flavors all quarks can participate in this pairing, while for odd

flavors a single flavor remains gapless (recall that an antisymmetric matrix of odd dimension always has a zero eigenvalue). Then what option is available for this isolated quark? One option is to form a color-adjoint diquark condensate in the Lorentz-scalar channel, causing color superconductivity. At least in the high-density limit we regard this possibility as unlikely, because the single-gluon exchange interaction is repulsive in the color-symmetric channel. Another option is to form a color-singlet diquark condensate in the channel of total angular momentum $J = 1$. It inevitably breaks rotational symmetry. Neither a low-energy effective theory nor RMT is known for this phase.

In order to understand implications of our RMT for dense two-color QCD, let us return to (14). For $N_c = 2$ the identity $\varepsilon_{ab}\varepsilon_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ holds. Therefore

$$Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp\left[m\bar{\psi}_{Rai}^\alpha \psi_{Lai}^\alpha + m^*\bar{\psi}_{Lai}^\alpha \psi_{Rai}^\alpha + \frac{1}{8N}(\bar{\psi}_{La}^\alpha \sigma_m \sigma_2 \varepsilon_{ab} \bar{\psi}_{Lb}^\alpha)(\psi_{Lc}^\beta \sigma_2 \sigma_m \varepsilon_{cd} \psi_{Ld}^\beta) + (L \leftrightarrow R)\right]. \tag{35}$$

The ensuing analysis is routine. To bilinearize the quartic interaction we insert the constant factor

$$\int_{\mathbb{C}^3} d\vec{\Omega}_L \exp\left[-8N\left(\Omega_{Lm} - \frac{1}{8N}\bar{\psi}_{La}^\alpha \sigma_m \sigma_2 \varepsilon_{ab} \bar{\psi}_{Lb}^\alpha\right) \times \left(\Omega_{Lm}^* - \frac{1}{8N}\psi_{Lc}^\beta \sigma_2 \sigma_m \varepsilon_{cd} \psi_{Ld}^\beta\right)\right] \times (L \leftrightarrow R) \tag{36}$$

where $\vec{\Omega}_{L,R}$ are three-component complex vectors that transform as triplet under SO(3) and are also charged under $U(1)_B \times U(1)_A$. Then

$$Z \propto \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2) + m\bar{\psi}_{Rai}^\alpha \psi_{Lai}^\alpha + m^*\bar{\psi}_{Lai}^\alpha \psi_{Rai}^\alpha$$
$$+ \Omega_{Lm}\psi_{La}^\alpha \sigma_2 \sigma_m \varepsilon_{ab} \psi_{Lb}^\alpha + \Omega_{Lm}^* \bar{\psi}_{La}^\alpha \sigma_m \sigma_2 \varepsilon_{ab} \bar{\psi}_{Lb}^\alpha + (L \leftrightarrow R)] \tag{37}$$

$$= \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2)]$$
$$\times \exp\left[\begin{pmatrix}\bar{\psi}_L \\ \bar{\psi}_R \\ \psi_L \\ \psi_R\end{pmatrix}_{ai}^\alpha \begin{pmatrix} \Omega_{Lm}^*(\sigma_m \sigma_2)_{ij}\varepsilon_{ab} & 0 & 0 & m^*\delta_{ab}\delta_{ij}/2 \\ 0 & \Omega_{Rm}^*(\sigma_m \sigma_2)_{ij}\varepsilon_{ab} & m\delta_{ab}\delta_{ij}/2 & 0 \\ 0 & -m\delta_{ab}\delta_{ij}/2 & \Omega_{Lm}(\sigma_2 \sigma_m)_{ij}\varepsilon_{ab} & 0 \\ -m^*\delta_{ab}\delta_{ij}/2 & 0 & 0 & \Omega_{Rm}(\sigma_2 \sigma_m)_{ij}\varepsilon_{ab} \end{pmatrix} \begin{pmatrix}\bar{\psi}_L \\ \bar{\psi}_R \\ \psi_L \\ \psi_R\end{pmatrix}_{bj}^\alpha\right] \tag{38}$$

$$= \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \ \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2)]$$
$$\times \mathrm{Pf}^N\begin{pmatrix} \Omega_{Rm}^*(\sigma_m \sigma_2)_{ij}\varepsilon_{ab} & m\delta_{ab}\delta_{ij}/2 \\ -m\delta_{ab}\delta_{ij}/2 & \Omega_{Lm}(\sigma_2 \sigma_m)_{ij}\varepsilon_{ab} \end{pmatrix} \mathrm{Pf}^N\begin{pmatrix} \Omega_{Lm}^*(\sigma_m \sigma_2)_{ij}\varepsilon_{ab} & m^*\delta_{ab}\delta_{ij}/2 \\ -m^*\delta_{ab}\delta_{ij}/2 & \Omega_{Rm}(\sigma_2 \sigma_m)_{ij}\varepsilon_{ab} \end{pmatrix} \tag{39}$$

$$= \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \ \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2)]$$
$$\times \det^N\begin{pmatrix} \Omega_{Rm}^*(\sigma_m \sigma_2)_{ij} & m\delta_{ij}/2 \\ m\delta_{ij}/2 & \Omega_{Lm}(\sigma_2 \sigma_m)_{ij} \end{pmatrix} \det^N\begin{pmatrix} \Omega_{Lm}^*(\sigma_m \sigma_2)_{ij} & m^*\delta_{ij}/2 \\ m^*\delta_{ij}/2 & \Omega_{Rm}(\sigma_2 \sigma_m)_{ij} \end{pmatrix} \tag{40}$$

$$= \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2)]\left\{\vec{\Omega}_R^{*2}\vec{\Omega}_L^2 - \frac{m^2}{2}\vec{\Omega}_R^* \cdot \vec{\Omega}_L + \frac{m^4}{16}\right\}^N \left\{\vec{\Omega}_L^{*2}\vec{\Omega}_R^2 - \frac{m^{*2}}{2}\vec{\Omega}_L^* \cdot \vec{\Omega}_R + \frac{m^{*4}}{16}\right\}^N. \tag{41}$$

We are now ready to take the microscopic large-N limit with $m \propto 1/\sqrt{N}$. In the chiral limit we find

$$Z \propto \int_{\mathbb{C}^3} d\vec{\Omega}_L \int_{\mathbb{C}^3} d\vec{\Omega}_R \exp[-8N(|\vec{\Omega}_L|^2 + |\vec{\Omega}_R|^2)]|\vec{\Omega}_L^2|^{2N}|\vec{\Omega}_R^2|^{2N} \tag{42}$$

$$= \left\{\int_{\mathbb{C}^3} d\vec{\Omega}(e^{-8|\vec{\Omega}|^2}|\vec{\Omega}^2|^2)^N\right\}^2. \tag{43}$$

Thus the problem of finding the saddle point at large N boils down to extremizing the function $f(\vec{\Omega}) \equiv e^{-8|\vec{\Omega}|^2}|\vec{\Omega}^2|^2$. To simplify the problem we shall draw upon the fact that by a suitable $U(1) \times SO(3)$ rotation, an arbitrary complex 3-vector $\vec{\Omega}$ can be brought to the form $(a + ib, c, 0)$ with $a, b, c \in \mathbb{R}$. Then

$$f(\vec{\Omega}) = e^{-8(a^2+b^2+c^2)}[(a^2 - b^2 + c^2)^2 + 4a^2b^2]. \quad (44)$$

This function is maximized when $b = 0$ and $a^2 + c^2 = 1/4$, or when $c = 0$ and $a^2 + b^2 = 1/4$. To show this, first note that $f = e^{-8(a^2+b^2+c^2)}[(a^2 + b^2 + c^2)^2 - 4b^2c^2]$. This means that f can be increased by rotating (b, c) into either $(\sqrt{b^2 + c^2}, 0)$ or $(0, \sqrt{b^2 + c^2})$. Thus one can safely assume $bc = 0$. If $b = 0$, the vector $(a, c, 0)$ can be rotated to $(\sqrt{a^2 + c^2}, 0, 0)$ by the action of $SO(3)$. If $c = 0$, the vector $(a + ib, 0, 0)$ can be rotated to $(\sqrt{a^2 + b^2}, 0, 0)$ by the action of $U(1)$. In either case, a quick calculation shows that $(1/2, 0, 0)$ maximizes f.

The above analysis reveals that the manifold of maxima of f is given by $U(1) \times SO(3)$ rotations of $(1/2, 0, 0)$. Picking out any single point on this manifold as a ground state breaks global symmetries according to

$$U(1) \times SO(3) \rightarrow SO(2). \quad (45)$$

Note that this breaking occurs for each chirality independently.

The low-energy fluctuations in the ground state can be parametrized as

$$\vec{\Omega}_L = \frac{1}{2}e^{i\phi_L}\mathbf{n}_L \quad \text{and} \quad \vec{\Omega}_R = \frac{1}{2}e^{i\phi_R}\mathbf{n}_R \quad (46)$$

where the unit vectors $\mathbf{n}_{L,R} \in \mathbb{R}^3$ ($\mathbf{n}_{L,R}^2 = 1$) stand for the directions of diquarks in the spin space. For $N \gg 1$ with $\hat{m} \equiv \sqrt{N}m \sim O(1)$, we have from (41)

$$Z \sim \int_{U(1)} d\phi_L \int_{U(1)} d\phi_R \int_{S^2} d\mathbf{n}_L \int_{S^2} d\mathbf{n}_R$$
$$\times \left\{ \frac{1}{16}e^{2i(\phi_L-\phi_R)} - \frac{m^2}{8}e^{i(\phi_L-\phi_R)}\mathbf{n}_L \cdot \mathbf{n}_R \right\}^N$$
$$\times \left\{ \frac{1}{16}e^{-2i(\phi_L-\phi_R)} - \frac{m^{*2}}{8}e^{-i(\phi_L-\phi_R)}\mathbf{n}_L \cdot \mathbf{n}_R \right\}^N \quad (47)$$
$$\sim \int_{U(1)} d\phi_L \int_{U(1)} d\phi_R \int_{S^2} d\mathbf{n}_L \int_{S^2} d\mathbf{n}_R$$
$$\times \exp[-2\hat{m}^2 e^{-i(\phi_L-\phi_R)}\mathbf{n}_L \cdot \mathbf{n}_R + \text{c.c.}]$$
$$\propto \int_{U(1)} d\phi \int_{S^2} d\mathbf{n}_L \int_{S^2} d\mathbf{n}_R \exp(4|\hat{m}|^2\mathbf{n}_L \cdot \mathbf{n}_R \cos\phi).$$
$$(48)$$

The ϕ field is nothing but the Nambu-Goldstone mode for the $U(1)_A$ symmetry. Integrating out ϕ yields

$$Z \propto \int_{S^2} d\mathbf{n}_L \int_{S^2} d\mathbf{n}_R \ I_0(4|\hat{m}|^2\mathbf{n}_L \cdot \mathbf{n}_R) \quad (49)$$

$$\propto \int_0^\pi d\theta \sin\theta \ I_0(4|\hat{m}|^2 \cos\theta) \quad (50)$$

$$\propto \int_0^1 dx \ I_0(4x|\hat{m}|^2), \quad (51)$$

where I_0 is the modified Bessel function of the first kind. This is the simplest form we found for the microscopic partition function. An important observation is that a nonzero quark mass tends to make the diquarks of opposite chiralities parallel to each other. In fact, for a large mass $|\hat{m}| \gg 1$, the leading asymptotic behavior $I_0(z) \sim e^{|z|}/\sqrt{2\pi|z|}$ for $|z| \gg 1$ [77], as applied to (49), shows that a dominant contribution to Z comes from the region with $\mathbf{n}_L \cdot \mathbf{n}_R \sim \pm 1$, i.e., $\mathbf{n}_L \| \mathbf{n}_R$.

V. CONCLUSIONS

In this paper we reported an attempt to extend chiral random matrix theory (RMT) [31] to cold and dense one-flavor quark matter, in which quarks are thought to form Cooper pairs with nonzero total angular momentum [1]. For three colors the putative ground state is the color-spin locked (CSL) phase [15] that breaks color and rotational symmetries but retains invariance under their combination, bearing a close resemblance to the B phase of ^{3}He and the CFL phase of three-flavor quark matter [2]. In this paper we have formulated a new non-Hermitian chiral RMT and showed that in the microscopic large-N limit it exhibits the same symmetry breaking pattern as the CSL phase of QCD. This work can be viewed as a natural sequel to our previous work [47] that put forward chiral RMT for Cooper pairing in two and three-flavor dense QCD. A downside of our RMT is the existence of 8 unphysical Nambu-Goldstone modes that enter into the large-N low-energy effective theory. They are associated with the breaking of color SU(3) symmetry. Although they are "eaten" by gluons in the CSL phase of QCD, they do represent physical fluctuations in RMT because the Anderson-Higgs mechanism does not operate in a zero-dimensional theory with no local gauge invariance. In the second half of this paper we also discussed the application of our RMT to two colors. In this case the symmetry breaking pattern of RMT was found to be $U(1) \times SO(3) \rightarrow SO(2)$, which matches physical expectations for two-color one-flavor QCD at high density. It would be worthwhile to perform weak-coupling calculations of gap equations for the latter in order to precisely check the validity of correspondence between our RMT and two-color QCD. Analytical calculation of the microscopic spectral density of the Dirac operator $\mathcal{D}$ in RMT is quite challenging and we leave it to future work.

[1] D. Bailin and A. Love, Superfluidity and superconductivity in relativistic fermion systems, Phys. Rep. **107**, 325 (1984).

[2] M. G. Alford, K. Rajagopal, and F. Wilczek, Color flavor locking and chiral symmetry breaking in high density QCD, Nucl. Phys. **B537**, 443 (1999).

[3] M. G. Alford, K. Rajagopal, and F. Wilczek, QCD at finite baryon density: Nucleon droplets and color superconductivity, Phys. Lett. B **422**, 247 (1998).

[4] R. Rapp, T. Schäfer, E. V. Shuryak, and M. Velkovsky, Diquark Bose Condensates in High Density Matter and Instantons, Phys. Rev. Lett. **81**, 53 (1998).

[5] K. Rajagopal and F. Wilczek, The condensed matter physics of QCD, in *At the Frontier of Particle Physics. Handbook of QCD*, edited by M. Shifman and B. Ioffe (World Scientific, Singapore, 2000), Vol. 1–3, pp. 2061–2151.

[6] M. G. Alford, Color superconducting quark matter, Annu. Rev. Nucl. Part. Sci. **51**, 131 (2001).

[7] M. Buballa, NJL model analysis of quark matter at large density, Phys. Rep. **407**, 205 (2005).

[8] I. A. Shovkovy, Two lectures on color superconductivity, Found. Phys. **35**, 1309 (2005).

[9] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schäfer, Color superconductivity in dense quark matter, Rev. Mod. Phys. **80**, 1455 (2008).

[10] K. Fukushima and T. Hatsuda, The phase diagram of dense QCD, Rep. Prog. Phys. **74**, 014001 (2011).

[11] K. Fukushima and C. Sasaki, The phase diagram of nuclear and quark matter at high baryon density, Prog. Part. Nucl. Phys. **72**, 99 (2013).

[12] R. Casalbuoni, Lecture notes on superconductivity: Condensed matter and QCD, arXiv:1810.11125.

[13] M. Iwasaki and T. Iwado, Superconductivity in the quark matter, Phys. Lett. B **350**, 163 (1995).

[14] T. Schäfer and F. Wilczek, Quark description of hadronic phases, Phys. Rev. D **60**, 074014 (1999).

[15] T. Schäfer, Quark hadron continuity in QCD with one flavor, Phys. Rev. D **62**, 094007 (2000).

[16] M. G. Alford, J. A. Bowers, J. M. Cheyne, and G. A. Cowan, Single color and single flavor color superconductivity, Phys. Rev. D **67**, 054018 (2003).

[17] A. Schmitt, Q. Wang, and D. H. Rischke, Electromagnetic Meissner Effect in Spin One Color Superconductors, Phys. Rev. Lett. **91**, 242301 (2003).

[18] A. Schmitt, Spin-one color superconductivity in cold and dense quark matter, Ph.D. Thesis, Frankfurt University, 2004.

[19] A. Schmitt, The ground state in a spin-one color superconductor, Phys. Rev. D **71**, 054016 (2005).

[20] A. Schmitt, I. A. Shovkovy, and Q. Wang, Pulsar Kicks via Spin-1 Color Superconductivity, Phys. Rev. Lett. **94**, 211101 (2005).

[21] D. N. Aguilera, D. Blaschke, M. Buballa, and V. L. Yudichev, Color-spin locking phase in two-flavor quark matter for compact star phenomenology, Phys. Rev. D **72**, 034008 (2005).

[22] F. Marhauser, D. Nickel, M. Buballa, and J. Wambach, Color-spin locking in a self-consistent Dyson-Schwinger approach, Phys. Rev. D **75**, 054022 (2007).

[23] B. Feng, D.-f. Hou, and H.-c. Ren, Angular momentum mixing in a non-spherical color superconductor, Nucl. Phys. **B796**, 500 (2008).

[24] B. Feng, D. Hou, and H.-c. Ren, Angular momentum mixing in single flavor color superconductivity with transverse pairing, Nucl. Phys. **B813**, 408 (2009).

[25] T. Brauner, Helical ordering in the ground state of spin-one color superconductors as a consequence of parity violation, Phys. Rev. D **78**, 125027 (2008).

[26] B. Feng, D. Hou, H.-c. Ren, and P.-p. Wu, The Single Flavor Color Superconductivity in a Magnetic Field, Phys. Rev. Lett. **105**, 042001 (2010).

[27] J.-y. Pang, T. Brauner, and Q. Wang, Spin-one color superconductors: Collective modes and effective Lagrangian, Nucl. Phys. **A852**, 175 (2011).

[28] M. Mehta, *Random Matrices*, 3rd ed. (Academic Press, New York, 2004).

[29] G. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices* (Cambridge University Press, Cambridge, England, 2009).

[30] G. Akemann, J. Baik, and P. Di Francesco, *The Oxford Handbook of Random Matrix Theory*, Oxford Handbooks in Mathematics (Oxford University Press, Oxford, 2011).

[31] J. Verbaarschot and T. Wettig, Random matrix theory and chiral symmetry in QCD, Annu. Rev. Nucl. Part. Sci. **50**, 343 (2000).

[32] J. Verbaarschot, QCD, chiral random matrix theory and integrability, in *Application of Random Matrices in Physics. Proceedings, NATO Advanced Study Institute, Les Houches, France, 2004* (Springer, Dordrecht, 2005), pp. 163–217, https://www.springer.com/gp/book/9781402045295.

[33] J. C. Osborn, Universal Results from an Alternate Random Matrix Model for QCD with a Baryon Chemical Potential, Phys. Rev. Lett. **93**, 222001 (2004).

[34] G. Akemann, J. Osborn, K. Splittorff, and J. Verbaarschot, Unquenched QCD Dirac operator spectra at nonzero baryon chemical potential, Nucl. Phys. **B712**, 287 (2005).

[35] G. Akemann, Matrix models and QCD with chemical potential, Int. J. Mod. Phys. A **22**, 1077 (2007).

[36] T. Kanazawa, T. Wettig, and N. Yamamoto, Chiral random matrix theory for two-color QCD at high density, Phys. Rev. D **81**, 081701 (2010).

[37] G. Akemann, T. Kanazawa, M. Phillips, and T. Wettig, Random matrix theory of unquenched two-colour QCD with nonzero chemical potential, J. High Energy Phys. 03 (2011) 066.

[38] T. Kanazawa, *Dirac Spectra in Dense QCD* (Springer, Japan, 2013).

[39] T. Kanazawa and T. Wettig, Stressed Cooper pairing in QCD at high isospin density: Effective Lagrangian and random matrix theory, J. High Energy Phys. 10 (2014) 055.

[40] T. Kanazawa, T. Wettig, and N. Yamamoto, Banks-Casher-type relation for the BCS gap at high density, Eur. Phys. J. A **49**, 88 (2013).

[41] Phenomenological applications of RMT to color-superconducting phases of QCD can be found in [42–46]. These works employed RMT only to study the general topology of the QCD phase diagram and did not discuss the microscopic large-N limit.

[42] B. Vanderheyden and A. Jackson, A random matrix model for color superconductivity at zero chemical potential, Phys. Rev. D **61**, 076004 (2000).

[43] B. Vanderheyden and A. Jackson, Random matrix model for chiral symmetry breaking and color superconductivity in QCD at finite density, Phys. Rev. D **62**, 094010 (2000).

[44] S. Pepin and A. Schafer, QCD at high baryon density in a random matrix model, Eur. Phys. J. A **10**, 303 (2001).

[45] B. Vanderheyden and A. Jackson, Random matrix models for phase diagrams, Rep. Prog. Phys. **74**, 102001 (2011).

[46] T. Sano and K. Yamazaki, Random matrix model for chiral and color-flavor locking condensates, Phys. Rev. D **85**, 094032 (2012).

[47] T. Kanazawa, Color-flavor locking and 2SC pairing in random matrix theory, arXiv:2004.04483.

[48] To verify these formulas we used SymPy, a Python library for symbolic mathematics [49].

[49] A. Meurer *et al.*, SymPy: symbolic computing in Python, PeerJ Comput. Sci. **3**, e103 (2017).

[50] Axial anomaly is suppressed at high density [51].

[51] T. Schäfer, Instanton effects in QCD at high baryon density, Phys. Rev. D **65**, 094033 (2002).

[52] R. D. Pisarski and D. H. Rischke, Color superconductivity in weak coupling, Phys. Rev. D **61**, 074017 (2000).

[53] S. Hands, J. B. Kogut, M.-P. Lombardo, and S. E. Morrison, Symmetries and spectrum of SU(2) lattice gauge theory at finite chemical potential, Nucl. Phys. **B558**, 327 (1999).

[54] S. Hands, I. Montvay, S. Morrison, M. Oevers, L. Scorzato, and J. Skullerud, Numerical study of dense adjoint matter in two color QCD, Eur. Phys. J. C **17**, 285 (2000).

[55] J. Kogut, D. Sinclair, S. Hands, and S. Morrison, Two color QCD at nonzero quark number density, Phys. Rev. D **64**, 094505 (2001).

[56] J. B. Kogut, D. Toublan, and D. Sinclair, The phase diagram of four flavor SU(2) lattice gauge theory at nonzero chemical potential and temperature, Nucl. Phys. **B642**, 181 (2002).

[57] J. Kogut, M. A. Stephanov, and D. Toublan, On two color QCD with baryon chemical potential, Phys. Lett. B **464**, 183 (1999).

[58] J. Kogut, M. A. Stephanov, D. Toublan, J. Verbaarschot, and A. Zhitnitsky, QCD-like theories at finite baryon density, Nucl. Phys. **B582**, 477 (2000).

[59] K. Splittorff, D. Son, and M. A. Stephanov, QCD—like theories at finite baryon and isospin density, Phys. Rev. D **64**, 016003 (2001).

[60] K. Splittorff, D. Toublan, and J. Verbaarschot, Diquark condensate in QCD with two colors at next-to-leading order, Nucl. Phys. **B620**, 290 (2002).

[61] M. A. Metlitski and A. R. Zhitnitsky, Theta-parameter in 2 color QCD at finite baryon and isospin density, Nucl. Phys. **B731**, 309 (2005).

[62] T. Brauner, On the chiral perturbation theory for two-flavor two-color QCD at finite chemical potential, Mod. Phys. Lett. A **21**, 559 (2006).

[63] T. Kanazawa, T. Wettig, and N. Yamamoto, Chiral Lagrangian and spectral sum rules for dense two-color QCD, J. High Energy Phys. 08 (2009) 003.

[64] T. Kanazawa, T. Wettig, and N. Yamamoto, Singular values of the Dirac operator in dense QCD-like theories, J. High Energy Phys. 12 (2011) 007.

[65] P. Adhikari, S. B. Beleznay, and M. Mannarelli, Finite density two color chiral perturbation theory revisited, Eur. Phys. J. C **78**, 441 (2018).

[66] B. Vanderheyden and A. Jackson, Random matrix study of the phase structure of QCD with two colors, Phys. Rev. D **64**, 074016 (2001).

[67] B. Klein, D. Toublan, and J. Verbaarschot, Diquark and pion condensation in random matrix models for two color QCD, Phys. Rev. D **72**, 015007 (2005).

[68] C. Ratti and W. Weise, Thermodynamics of two-colour QCD and the Nambu Jona-Lasinio model, Phys. Rev. D **70**, 054013 (2004).

[69] T. Brauner, K. Fukushima, and Y. Hidaka, Two-color quark matter: U(1)(A) restoration, superfluidity, and quarkyonic phase, Phys. Rev. D **80**, 074035 (2009); Erratum, Phys. Rev. D **81**, 119904 (2010).

[70] J. O. Andersen and T. Brauner, Phase diagram of two-color quark matter at nonzero baryon and isospin density, Phys. Rev. D **81**, 096004 (2010).

[71] L. He, Nambu-Jona-Lasinio model description of weakly interacting Bose condensate and BEC-BCS crossover in dense QCD-like theories, Phys. Rev. D **82**, 096003 (2010).

[72] N. Strodthoff, B.-J. Schaefer, and L. von Smekal, Quark-meson-diquark model for two-color QCD, Phys. Rev. D **85**, 074007 (2012).

[73] S. Imai, H. Toki, and W. Weise, Quark-Hadron matter at finite temperature and density in a two-color PNJL model, Nucl. Phys. **A913**, 71 (2013).

[74] N. Strodthoff and L. von Smekal, Polyakov-Quark-Meson-Diquark model for two-color QCD, Phys. Lett. B **731**, 350 (2014).

[75] In [54] two-color QCD with a single flavor in the *adjoint* representation of SU(2) was investigated.

[76] E. Witten, An SU(2) anomaly, Phys. Lett. B **117B**, 324 (1982).

[77] DLMF, NIST Digital Library of Mathematical Functions, edited by f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, http://dlmf.nist.gov/.