Patterns of *CP* violation from mirror symmetry breaking in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot

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A violation of mirror symmetry in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot has long been recognized as a signal of C and CP violation. Here we show how the isospin of the underlying C- and CP-violating structures can be reconstructed from their kinematic representation in the Dalitz plot. Our analysis of the most recent experimental data reveals, for the first time, that the C- and CP-violating amplitude with total isospin $I = 2$ is much more severely suppressed than that with total isospin $I = 0$.

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I. INTRODUCTION

The decay $\eta \to 3\pi$ first came to prominence after the observation of $K_L \to \pi^+\pi^-$ decay and the discovery of CP violation in 1964 [1], because it could be used to test whether $K_L \to \pi^+\pi^-$ decay was generated by CP violation in the weak interactions [2,3]. Rather, CP violation could arise from the interference of the CP-conserving weak interaction with a new, "strong" interaction that breaks C and CP; this new interaction could be identified through the appearance of a charge asymmetry in the momentum distribution of π^+ and π^- in $\eta \to \pi^+\pi^-\pi^0$ decay [2,4,5]. Since $\eta \to \pi^+\pi^-\pi^0$ breaks G parity, isospin I and/or charge-conjugation C must be broken in order for the process to occur. Thus a charge asymmetry could arise from the interference of a C-conserving, but isospin-breaking amplitude with a isospin-conserving, but C-violating one [4]. Numerical estimates were made by assuming that the isospin-violating contributions were driven by electromagnetism [4–6]. Since that early work, our understanding of these decays within the Standard Model (SM) has changed completely: the weak interaction does indeed break CP symmetry, through flavor-changing transitions characterized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Moreover, isospin breaking in the strong interaction, mediated by the up-down quark mass difference [7–9], is now known to provide the driving effect in mediating $\eta \to 3\pi$ decay [10–13], with isospin-breaking, electromagnetic effects playing a much smaller role [14–17].

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Modern theoretical studies of $\eta \to 3\pi$ decay focus on a complete description of the final-state interactions within the SM, in order to extract the isospin-breaking, light-quark mass ratio $Q \equiv \sqrt{(m_s^2 - \hat{m}^2)/(m_d^2 - m_u^2)}$, with $\hat{m} = (m_d + m_u)/2$, precisely [11–13,18–23]. There has been no further theoretical study of CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay since 1966. Since the η meson carries neither spin nor flavor, searches for new physics in this system possess special features. For example, $\eta \to \pi^+\pi^-\pi^0$ decay must be parity P conserving if the π and η mesons have the same intrinsic parity, so that C violation in this process implies that CP is violated as well. There has been, moreover, much effort invested in the possibility of flavor-diagonal CP violation via a nonzero permanent electric dipole moment (EDM), which is P and time-reversal T violating, or P and CP violating if CPT symmetry is assumed. Studies of flavor-diagonal, C and CP violating processes are largely lacking. We believe that the study of the Dalitz plot distribution in $\eta \to \pi^+(p_{\pi^+})\pi^-(p_{\pi^-})\pi^0(p_{\pi^0})$ decay is an ideal arena in which to search for C and CP violation beyond the SM. Were we to plot the Dalitz distribution in terms of the Mandelstam variables $t \equiv (p_{\pi^-} + p_{\pi^0})^2$ and $u \equiv (p_{\pi^+} + p_{\pi^0})^2$, the charge asymmetry we have noted corresponds to a failure of mirror symmetry, i.e., of $t \leftrightarrow u$ exchange, in the Dalitz plot. In contrast to that C and CP violating observable, a nucleon EDM could be mediated by a minimal P- and T-violating interaction, the mass-dimension-four $\bar{\theta}$ term of the SM, and not new weak-scale physics. The $\bar{\theta}$ term can also generate $\eta \to \pi\pi$ and $\eta/\eta' \to 4\pi^0$ decay, breaking P and CP explicitly, so that limits on the decay rate constrains the square of a CP-violating parameter [24–27]. Since the $\bar{\theta}$ term is C even, it cannot contribute to the charge asymmetry, at least at tree level. Moreover, SM weak interactions do not support flavor-diagonal C and CP violation.

Note that the charge asymmetry is linear in CP-violating parameters.

The appearance of a charge asymmetry and thus of C (and CP) violation in $\eta \to \pi^+(p_{\pi^+})\pi^-(p_{\pi^-})\pi^0(p_{\pi^0})$ decay can be probed experimentally through the measurement of a left-right asymmetry, A_{LR} [28]:

$$A_{LR} \equiv \frac{N_+ - N_-}{N_+ + N_-} \equiv \frac{1}{N_{\rm tot}}(N_+ - N_-), \qquad (1)$$

where $N_\pm$ is the number of events with $u \gtrless t$, so that the π^+ has more (less) energy than the π^- if $u > (<) t$ in the η rest system. A number of experiments have been conducted over the years to test for a charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$ decay, with early experiments finding evidence for a nonzero asymmetry [29–31], but with possible systematic problems becoming apparent only later, as, e.g., in Ref. [32]. Other experiments find no evidence for a charge asymmetry and C violation [28,32–36], and we note that new, high-statistics experiments are planned [37–39]. It is also possible to form asymmetries that probe the isospin of the C-violating final state: a sextant asymmetry A_S, sensitive to the $I = 0$ state [4,5], and a quadrant asymmetry A_Q, sensitive to the $I = 2$ final state [4,28]. These asymmetries are more challenging to measure and are only poorly known [28]. In this paper we develop a method to discriminate between the possible $I = 0$ and $I = 2$ final states by considering the pattern of mirror-symmetry-breaking events they engender in the Dalitz plot. Mirror-symmetry breaking as a probe of CP violation has also been studied in untagged, heavy-flavor decays [40–43], with Ref. [42] analyzing how different CP-violating mechanisms populate the Dalitz plot. We also note Refs. [44,45] for Dalitz studies of CP violation in heavy-flavor decays.

II. THEORETICAL FRAMEWORK

The $\eta \to 3\pi$ decay amplitude in the SM can be expressed as [10,11]

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u), \qquad (2)$$

where we employ the Mandelstam variables u, t, and $s = (p_{\pi^+} + p_{\pi^-})^2$ and work to leading order in strong-interaction isospin breaking. Since $C = -(-1)^I$ in $\eta \to 3\pi$ decay [4], the C- and CP-even transition amplitude with a $\Delta I = 1$ isospin-breaking prefactor must have $I = 1$. The amplitude $M(s,t,u)$ thus corresponds to the total isospin $I = 1$ component of the $\pi^+\pi^-\pi^0$ state and can be expressed as [11,46]

$$M_1^C(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s), \qquad (3)$$

where $M_I(z)$ is an amplitude with $\pi - \pi$ rescattering in the z-channel with isospin I. This decomposition can be recovered under isospin symmetry in chiral perturbation theory (ChPT) up to next-to-next-to-leading order (NNLO), $\mathcal{O}(p^6)$, because the only absorptive parts that can appear are in the $\pi - \pi$ S-and P-wave amplitudes [13]. An analogous relationship exists in $\eta \to 3\pi^0$ decay [11], though there is no Dalitz plot asymmetry and hence no effect linear in CP violation in that case because the final-state particles are all identical.

Since we are considering C and CP violation, additional amplitudes can appear—namely, total $I = 0$ and $I = 2$ amplitudes. The complete amplitude is thus

$$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M_1^C(s,t,u) + \alpha M_0^{\not C}(s,t,u) + \beta M_2^{\not C}(s,t,u), \qquad (4)$$

where α and β are unknown, low-energy constants—complex numbers to be determined by fits to the experimental event populations in the Dalitz plot. If they are determined to be nonzero, they signal the appearance of C- and CP-violation. To construct A_{LR} in Eq. (1), we compute

$$N_\pm = \frac{1}{256\pi^3 M_\eta^3}\int_{u \gtrless t} dt du |A(s,t,u)|^2, \qquad (5)$$

using Eq. (4) and working to leading order in CP violation. Since the phase space is symmetric and the CP-violating terms are antisymmetric under $u \leftrightarrow t$ exchange, we see that the CP-violating terms leave the total decay rate unchanged in $\mathcal{O}(\alpha)$, $\mathcal{O}(\beta)$.

We now turn to the amplitudes $M_{0,2}^{\not C}(s,t,u)$. Here, too, we introduce functions $M_I(z)$ for amplitudes that contain $\pi - \pi$ scattering in the z channel with isospin I. After using angular-momentum conservation and the Clebsch-Gordon coefficients for the addition of the possible isospin states, as shown in the Appendix, we have

$$M_0^{\not C}(s,t,u) = (s-t)M_1'(u) + (u-s)M_1'(t) - (u-t)M_1'(s) \qquad (6)$$

and

$$M_2^{\not C}(s,t,u) = (s-t)M_1''(u) + (u-s)M_1''(t) + 2(u-t)M_1''(s) + \sqrt{5}[M_2''(u) - M_2''(t)], \qquad (7)$$

where the superscripts distinguish the functions that appear in each state of *total* isospin. In what follows we do not compute $M_I'(z)$ and $M_I''(z)$ explicitly, but, rather, estimate them. With this, we can use the experimental studies we consider in this paper to set limits on the possibilities, by constraining α and β. For context we note that the particular new-physics operators that would give C- and CP-violation are not well established, though examples have been discussed in the literature [47–51]. From the viewpoint of SM effective field theory (SMEFT) [52,53], we also know that there are many more examples, even in leading-mass dimension, than have been discussed thus far [54]. Nevertheless, we can draw conclusions about $M_I'(z)$ and $M_I''(z)$ irrespective of the choice of new-physics operator. In particular, since the operators that mediate $I = 0$ or $I = 2$ amplitudes break C, they cannot mediate a $\eta \to \pi^0$ transition, as we suppose that the neutral meson states remain of definite C-parity in the presence of C-violation. Thus if we were to evaluate the decay diagrams in NLO ChPT in these exotic cases, they would have the same decay topology as the diagrams that appear in that order in the SM amplitude for $\eta \to \pi^+\pi^-\pi^0$ decay. Thus there is a one-to-one map of the two-body rescattering terms in the SM to the C- and CP-violating amplitudes. To proceed, we assume that the phases of the functions $M_I(z)$, $M_I'(z)$ and $M_I''(z)$ arise from the strong-interaction dynamics of final-state, $\pi - \pi$ scattering of isospin I in channel z, making the phase of each function common to all three isospin amplitudes. Such treatments are familiar from the search for non-SM CP violation, such as in the study of $B \to \pi(\rho \to \pi\pi)$ decays [55–58]. Moreover, at the low energies we consider here, the scattering of the two-pions in the final state is predominantly elastic, as mixing with other final-states can only occur through G-parity breaking. Regardless of the total isospin of the final state pions, the effective Hamiltonian that mediates the decay separates into a C- and/or I-breaking piece and a C- and I-conserving piece. Working to leading order in C- and/or I-breaking, and assuming that the final-state interactions are two-body only, Watson's theorem [59], familiar from $K \to \pi\pi$ decays [57], also applies to this case and makes the phase of the function $M_I(z)$ common to the three cases. However, the functions $M_I(z)$, $M_I'(z)$ and $M_I''(z)$ could differ by polynomial prefactors that depend on z. Nevertheless, we believe these effects are relatively negligible, because the energy release in $\eta \to \pi^+\pi^-\pi^0$ decay is small. We illustrate this explicitly later in this section.

We wish to study the possible patterns of C- and CP-violation across the Dalitz plot, so that we now turn to the explicit evaluation of Eq. (4) and its associated Dalitz distribution. Much effort has been devoted to the evaluation of the SM contribution, with work in ChPT [10,13,60], as well as in frameworks tailored to address various final-state-interaction effects [11,18–23,61–65]. In what follows we employ a next-to-leading-order (NLO) ChPT analysis [10,13] because it is the simplest choice in which the C- and CP-violating coefficients α and β can have both real and imaginary parts. A comparison of the NLO and NNLO analyses of Bijnens *et al.* [13], noting Table I of Ref. [32], shows that this is an acceptable choice. We thus think it is rich enough to give a basic view as to how our idea works. To compute the C-violating amplitudes, we decompose the $I = 1$ amplitude into the isospin basis $M_I(z)$. As well known [13,21,22,66], the isospin decompositions involving the $\pi - \pi$ rescattering functions $J^r_{PQ}(s)$ are unique, whereas the polynomial parts of the amplitude are not, due to the relation $s + t + u = 3s_0$, where $s_0 = (M_\eta^2 + 2M_{\pi^+}^2 + M_{\pi^0}^2)/3$. Thus there are $M_I(z)$ redefinitions that leave the $I = 1$ amplitude invariant, as discussed in Ref. [66]. However, since we assume that strong rescattering effects dominate $M_I(x)$, we can demand that the $I = 0$, 2 amplitudes remain invariant also. As a result, only the redefinition $M_0(s) - \frac{4}{3}\delta_1$ and $M_2(z) + \delta_1$, with δ_1 an arbitrary constant, survives. In what follows we adopt the NLO analyses of Refs. [10,13], and our isospin decomposition of Ref. [10] is consistent with that in Bijnens and Ghorbani [13]—its detailed form can be found in the information in the Appendix. Small differences in the numerical predictions exist, however, due to small differences in the inputs used [10,13], and we study their impact explicitly. Returning to the would-be NLO ChPT computation of the total $I = 0$, 2 amplitudes, we note that C- and CP-violating four-quark operators are generated by operators in mass dimension 8 in SMEFT [54], so that these amplitudes start beyond $\mathcal{O}(p^4)$, though this is not at odds with pulling out a strong rescattering function. The p^2-dependence of a C- and CP-violating operator from physics beyond the SM would in part be realized as dimensionless ratios involving the new physics scale and would appear in the prefactors α or β as appropriate.

Irrespective of the particular new-physics operator, we note, by analyticity, that the $M_I(x)$ for the total $I = 0$, 2 amplitudes could differ from the SM form, which is driven by the strong $\pi - \pi$ phase shifts, by a multiplicative

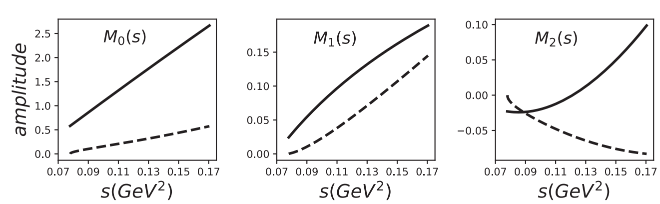

FIG. 1. Amplitudes of $M_I(s)$ from Eqs. (A9), (A10) and (A11). The solid lines represent the real part of $M_I(s)$ and the dashed lines denote the imaginary part. Their s-dependence is driven by that of the $\pi - \pi$ phase shift [10]. Note that the $M_0(s)$ and $M_2(s)$ amplitudes are dimensionless, whereas $M_1(s)$ has units of GeV^{-2}. Since the form $(u - t)M_1(s)$ appears in the final C- and CP-violating amplitudes, we note that $M_2(s)$ is typically a factor of a few larger across the Dalitz plot.

polynomial factor, nominally of form $1 + C_1^I x/M_\pi^2 + C_2^I x^2/M_\pi^4 + \cdots$, where C_1^I and C_2^I are constants. (We note polynomials of similar origin appear in the time-like pion form factor [67].) We emphasize that in assuming that strong-interaction phases dominate we suppose these corrections to be unimportant. We believe this to be an excellent approximation, which we illustrate through a plot of the functions $M_I(s)$, as shown in Fig. 1. The physics of $\pi - \pi$ scattering make the functions $M_I(s)$ vary substantially with s, whereas s itself only changes by about a factor of 2 in $\eta \to 3\pi$ decay. As a result we expect that the ignored polynomial factors are numerically unimportant, so that their neglect does not impact the conclusions of this paper.

III. RESULTS

The Dalitz distribution in $\eta \to \pi^+\pi^-\pi^0$ is usually described in terms of variables X and Y [68]:

$$X \equiv \sqrt{3}\frac{T_{\pi^+} - T_{\pi^-}}{Q_\eta} = \frac{\sqrt{3}}{2M_\eta Q_\eta}(u - t), \quad (8)$$

$$Y \equiv \frac{3T_{\pi^0}}{Q_\eta} - 1 = \frac{3}{2M_\eta Q_\eta}[(M_\eta - M_{\pi^0})^2 - s] - 1, \quad (9)$$

where $Q_\eta = T_{\pi^+} + T_{\pi^-} + T_{\pi^0} = M_\eta - 2M_{\pi^+} - M_{\pi^0}$, and T_{π^i} is the π^i kinetic energy in the η rest frame. The decay probability can be parametrized in a polynomial expansion around the center point $(X, Y) = (0, 0)$ [32]:

$$|A(s,t,u)|^2 = N(1 + aY + bY^2 + cX + dX^2 + eXY + fY^3 + gX^2Y + hXY^2 + lX^3 + \ldots). \quad (10)$$

Since the C transformation on the decay amplitude is equivalent to $t \leftrightarrow u$ exchange [42], we see that the appearance of terms that are odd in X would indicate both C and CP violation. The KLOE-2 collaboration [32] has provided a more precise estimate of the C-even parameters in Eq. (10) and bounded the C-odd ones. Returning to Eq. (4), we see that the C- and CP-violating contributions to the decay probability are

$$\begin{aligned}
&\frac{1}{\xi}|A(s,t,u)|^2_{\not C} \\
&= M_1^C[\alpha M_0^{\not C} + \beta M_2^{\not C}]^* + \text{H.c.} \\
&= 2\mathrm{Re}(\alpha)[\mathrm{Re}(M_1^C)\mathrm{Re}(M_0^{\not C}) + \mathrm{Im}(M_1^C)\mathrm{Im}(M_0^{\not C})] \\
&\quad - 2\mathrm{Im}(\alpha)[\mathrm{Re}(M_1^C)\mathrm{Im}(M_0^{\not C}) - \mathrm{Im}(M_1^C)\mathrm{Re}(M_0^{\not C})] \\
&\quad + 2\mathrm{Re}(\beta)[\mathrm{Re}(M_1^C)\mathrm{Re}(M_2^{\not C}) + \mathrm{Im}(M_1^C)\mathrm{Im}(M_2^{\not C})] \\
&\quad - 2\mathrm{Im}(\beta)[\mathrm{Re}(M_1^C)\mathrm{Im}(M_2^{\not C}) - \mathrm{Im}(M_1^C)\mathrm{Re}(M_2^{\not C})], \quad (11)
\end{aligned}$$

where $\xi \equiv -(M_K^2/M_\pi^2)(M_K^2 - M_\pi^2)/(3\sqrt{3}F_\pi^2 Q^2)$, and the existing experimental assessments of $|A(s,t,u)|^2_{\not C}$ correspond to the set of odd X polynomials in $|A(s,t,u)|^2$. The parameter N drops out in the evaluation of the asymmetries, and the parameters c, e, h, and l are taken from the first line of Table 4.6 in the Ph.D. thesis of Caldeira Balkeståhl [69],

$$\begin{aligned}
c &= (-4.34 \pm 3.39) \times 10^{-3}, \quad & e &= (2.52 \pm 3.20) \times 10^{-3}, \\
h &= (1.07 \pm 0.90) \times 10^{-2}, \quad & l &= (1.08 \pm 6.54) \times 10^{-3},
\end{aligned} \quad (12)$$

which fleshes out Ref. [32]—the results emerge from a global fit to the Dalitz distribution. There is a typographical error in the sign of c in Ref. [32]. We now turn to the extraction of $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, $\mathrm{Re}(\beta)$, and $\mathrm{Im}(\beta)$ using the experimental data and Eqs. (3), (6), (7) using the $M_I(z)$ amplitudes from $\mathcal{O}(p^4)$ ChPT [10,13]. We evaluate the denominators of the possible charge asymmetries by computing $\xi^2|M_1^C(z,t,u)|^2$ only.

Herewith we collect the parameters needed for our analysis. We compute the phase space with physical masses, so that $s + t + u = 3s_0$, but the decay amplitudes [10,13] on which we rely, namely, $M(s,t,u)$ in Eq. (2), should be in the isospin limit, implying some adjustment of the input parameters may be needed. We adopt the hadron masses and $\sqrt{2}F_\pi = (130.2 \pm 1.7) \times 10^{-3}$ GeV from Ref. [70] for both amplitudes. For the Gasser and Leutwyler (GL) amplitude [10] we use $M_\pi \equiv \sqrt{(2M_{\pi^\pm}^2 + M_{\pi^0}^2)/3}$, $M_K \equiv \sqrt{(M_{K^+}^2 + M_{K^0}^2)/2}$, where we discuss our treatment of the two-particle thresholds in the supplement, with $F_0 = F_\pi$, $F_K/F_\pi = 1.1928 \pm 0.0026$ [70], and $L_3 = (-3.82 \pm 0.30) \times 10^{-3}$ from the NLO fit with the scale $\mu = 0.77$ GeV [71]. We use these parameters in the prefactor in Eq. (2) also, as well as $Q = 22.0$ [22], to find $\xi = -0.137$. For the Bijnens and Ghorbani (BG) amplitude through $\mathcal{O}(p^4)$ [13], we use $M(s,t,u) = M^{(2)}(s) + M^{(4)}(s,t,u)$ and multiply the prefactor in Eq. (2) by $-(3F_\pi^2)/(M_\eta^2 - M_\pi^2)$ to yield that in Ref. [13]. In the $\mathcal{O}(p^2)$ term, which contributes to $M_0(s)$, $M^{(2)}(s) = (4M_\pi^2 - s)/F_\pi^2$, and we use M_π and F_π as defined for the GL amplitude [10]. In the $\mathcal{O}(p^4)$ term, we use M_{π^0} and M_{K^0} as indicated, as well as $\Delta = M_\eta^2 - M_{\pi^0}^2$ and L_3, L_5, L_7, L_8 from fit 10 of Ref. [72].

We solve for α and β in two different ways for each of the decay amplitudes [10,13]. We begin with the GL amplitude [10], first making a Taylor expansion of Eq. (11) to cubic power in X and Y about $(X, Y) = (0, 0)$. We then equate coefficients associated with the X, XY, XY^2, and X^3 terms to c, e, h and l, respectively, and then solve the four equations to obtain $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, $\mathrm{Re}(\beta)$, and $\mathrm{Im}(\beta)$. The resulting values of α and β are

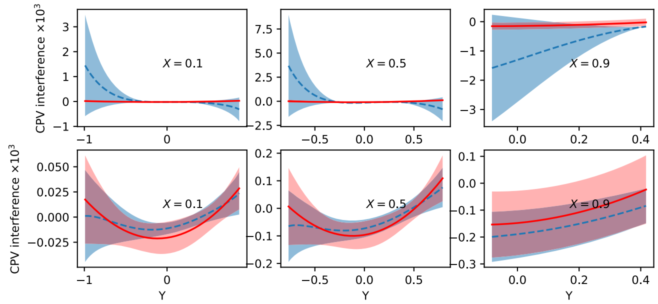

FIG. 2. Results for the C- and CP-violating (CPV) interference term, $|A(s,t,u)|^2_{\not{C}}$ in Eq. (11), using the GL amplitude [10] and two methods for the determination of α and β: (i) a Taylor expansion (top row) and (ii) a global fit (bottom row) as described in text using the GL decay amplitude [10]. The blue dashed lines with a one-σ error band (dark) are our results, and the red solid lines with a one-σ error band (light) are the KLOE-2 results, as per Eq. (12) [69].

$$\begin{aligned} \mathrm{Re}(\alpha) &= 16 \pm 24, \\ \mathrm{Re}(\beta) &= (-1.5 \pm 2.7) \times 10^{-3}, \\ \mathrm{Im}(\alpha) &= -20 \pm 29, \\ \mathrm{Im}(\beta) &= (-1.3 \pm 4.7) \times 10^{-3}. \end{aligned} \tag{13}$$

In the first row of Fig. 2 we compare the resulting assessment of Eq. (11) with the KLOE-2 results. Large discrepancies exist, particularly at large values of X and/or Y, where the empirical Dalitz plot [32] shows considerable stength. Thus we turn to a second procedure, in which we make a global fit of α and β in Eq. (11) to the KLOE-2 results. That is, we assess the Dalitz distribution $N(X,Y)$ and its error by using the Dalitz plot parameters in Eq. (12), discretized onto a (X,Y) mesh with 682 points. To determine $N(X,Y)$ and its error we use the odd-X terms in Eq. (10) with the normalization factor $N = 0.0474$ as per the GL amplitude [10] and compute the covariance matrix using Eq. (12) and the correlation matrix given in Table 4.3 of Ref. [69]. We then fit $|A(s,t,u)|^2_{\not{C}}$ using the GL amplitude [10] to $N(X,Y)$ using a χ^2 optimization to find

$$\begin{aligned} \mathrm{Re}(\alpha) &= -0.65 \pm 0.80, \\ \mathrm{Im}(\alpha) &= 0.44 \pm 0.74, \\ \mathrm{Re}(\beta) &= (-6.3 \pm 14.7) \times 10^{-4}, \\ \mathrm{Im}(\beta) &= (2.2 \pm 2.0) \times 10^{-3}, \end{aligned} \tag{14}$$

and we show the results of this method in the second row of Fig. 2. Enlarging the (X,Y) mesh to 1218 points incurs changes within ± 1 of the last significant figure. The comparison with experiment shows that the fitting procedure is the right choice. We draw the same conclusion from the use of the BG amplitude [13], noting that the global fit in that case (with $N = 0.0508$) gives

$$\begin{aligned} \mathrm{Re}(\alpha) &= -0.79 \pm 0.91, \\ \mathrm{Im}(\alpha) &= 0.61 \pm 0.93, \\ \mathrm{Re}(\beta) &= (-1.4 \pm 2.3) \times 10^{-3}, \\ \mathrm{Im}(\beta) &= (2.3 \pm 1.4) \times 10^{-3}, \end{aligned} \tag{15}$$

so that the results are compatible within errors. Using these solutions, we obtain $A_{LR} = (-7.18 \pm 4.51) \times 10^{-4}$ using Ref. [10] and $A_{LR} = (-7.20 \pm 4.52) \times 10^{-4}$ using Ref. [13]. These compare favorably with $A_{LR} = (-7.29 \pm 4.81) \times 10^{-4}$ that we determine using the complete set of Dalitz plot parameters and the covariance matrix we construct given the information in Ref. [69]. We note that our A_{LR} as evaluated from the Dalitz plot parameters, which are fitted from the binned data, is a little different from the reported value using the unbinned data, i.e., $(-5.0 \pm 4.5^{+5.0}_{-11}) \times 10^{-4}$, reported by KLOE-2 [32]. The discrepancy is not significant, and we suppose its origin could arise from the slight mismatch between the theoretically accessible phase space and the experimentally probed one, or other experimental issues. Although NLO ChPT does not describe the CP-conserving Dalitz distribution well [13], we find it can confront the existing CP-violating observables successfully.

We have shown that the empirical Dalitz plot distribution can be used to determine α and β. These, in turn, limit the strength of C-odd and CP-odd operators that can arise from physics beyond the SM [47–51,54]. That β is so much smaller than α can be, in part, understood from the differing behavior of the $M_I(z)$, as illustrated in Fig. 1, which

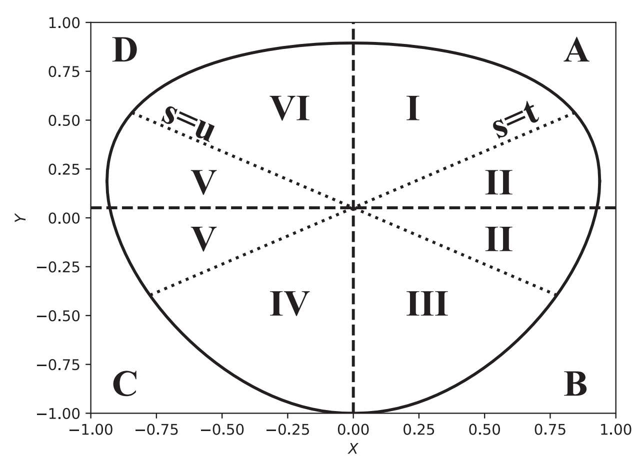

FIG. 3. The Dalitz plot geometry in $\eta \to \pi^+\pi^-\pi^0$ decay, broken into regions for probes of its symmetries. The solid line is the boundary of the physically accessible region. The asymmetry A_{LR}, Eq. (1), compares the population N_+ $(X > 0)$ with N_- $(X < 0)$. The quadrant asymmetry A_Q probes $I = 2$ contributions, $N_{\mathrm{tot}} A_Q \equiv N(\mathrm{A}) + N(\mathrm{C}) - N(\mathrm{B}) - N(\mathrm{D})$ [4], and the sextant asymmetry A_S probes $I = 0$ contributions, $N_{\mathrm{tot}} A_S \equiv N(\mathrm{I}) + N(\mathrm{III}) + N(\mathrm{V}) - N(\mathrm{II}) - N(\mathrm{IV}) - N(\mathrm{VI})$ [4,5]. All asymmetries probe C and CP violation.

follows because the $L = 0$, $I = 2$ $\pi - \pi$ phase shift is larger than the $L = 1$, $I = 1$ one for the kinematics of interest [10,73–75], making it easier to veto the $I = 2$ operators. Crudely, the ratio of β to α we have found is that of the SM electromagnetic interactions that would permit a $I = 2$ amplitude to appear in addition to a $I = 0$ one. The utility of our Dalitz analysis is underscored by our results for the quadrant asymmetry A_Q and sextant asymmetry A_S defined in Fig. 3. Using Ref. [10] and Eq. (14), e.g., we find $A_Q = (2.85 \pm 3.72) \times 10^{-4}$, and $A_S = (3.87 \pm 4.04) \times 10^{-4}$; the asymmetries by themselves hide the nature of the underlying strong amplitudes. For reference we note the KLOE-2 results using unbinned data [32]: $A_Q = (1.8 \pm 4.5^{+4.8}_{-2.3}) \times 10^{-4}$ and $A_S = (-0.4 \pm 4.5^{+3.1}_{-3.5}) \times 10^{-4}$, with which our results are compatible within errors.

IV. SUMMARY

We propose an innovative way of probing C- and CP-violation in the $\eta \to \pi^+\pi^-\pi^0$ Dalitz plot. Working to leading order in charge conjugation C and isospin I breaking, we have shown that the strong amplitudes associated with the appearance of C- and CP-violation can be estimated from the SM amplitude for $\eta \to \pi^+\pi^-\pi^0$ if the decomposition of Eq. (3) holds [11]. We have illustrated this in NLO ChPT, though the use of more sophisticated theoretical analyses would also be possible. New-physics contributions that differ in their isospin can thus be probed through the kinematic pattern they imprint in the Dalitz plot. Our method opens a new window on the study of C- and CP-violation in $\eta \to \pi^+\pi^-\pi^0$ decay, and it holds promise for the high-statistics experiments of the future [37–39].

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APPENDIX: CALCULATIONAL DETAILS

We begin by showing how Eqs. (6) and (7) emerge from elementary considerations. Working in the isospin limit, a $|\pi^+\pi^-\pi^0\rangle$ state with $J = 0$ must obey Bose symmetry, so that it is proportional to

$$|\pi^+\pi^-\rangle_s|\pi^0\rangle + |\pi^+\pi^0\rangle_s|\pi^-\rangle + |\pi^-\pi^0\rangle_s|\pi^+\rangle, \quad \text{(A1)}$$

where "s" denotes a symmetrized combination of distinct pion states. In what follows, as in the CP-conserving case, Eq. (3) [11], we consider S- and P-wave $\pi - \pi$ amplitudes only. The symmetrized two-pion states can be written as a S-wave $I = 0$ or $I = 2$ state or as a P-wave $I = 1$ state. We choose $|\pi^i\rangle \equiv |I = 1, I_3 = i\rangle$. For S-waves, we write

$$|\pi^i\pi^j\rangle_s \equiv \frac{1}{\sqrt{2}}\{|\pi^i\pi^j\rangle + |\pi^j\pi^i\rangle\}, \quad \text{(A2)}$$

whereas for P-waves we note

$$\underbrace{|\pi^i\pi^j\rangle_{I=1,L=1}|\pi^k\rangle}_{L=1} \quad \text{(A3)}$$

with $i + j + k = 0$ contributes to the $J = 0$ state. Defining

$$|\pi^i\pi^j\rangle_a \equiv \frac{1}{\sqrt{2}}\{|\pi^i\pi^j\rangle - |\pi^j\pi^i\rangle\}, \quad \text{(A4)}$$

we see, e.g.,

$$|(\pi^+\pi^-)_{I=1}\rangle_s|\pi^0\rangle = |\pi^+\pi^-\rangle_a(p_{\pi^+} - p_{\pi^-}) \cdot p_{\pi^0}|\pi^0\rangle. \quad \text{(A5)}$$

We can also label particular $\eta \to \pi^+\pi^-\pi^0$ decay amplitudes by the isospin of the two-pion state, as used in Eq. (3). Enumerating the possibilities, we find

$$|(\pi^+\pi^-)_{I=0}\rangle|\pi^0\rangle \to M_0(s), \quad \text{(A6)}$$

which contributes to the total $I = 1$ amplitude, M_1^C, only, as well as

$$\begin{aligned} &|(\pi^+\pi^-)_{I=1}\rangle|\pi^0\rangle(p_{\pi^+} - p_{\pi^-}) \cdot p_{\pi^0} \to M_1(s)\frac{u-t}{2}, \\ &|(\pi^+\pi^0)_{I=1}\rangle|\pi^-\rangle(p_{\pi^+} - p_{\pi^0}) \cdot p_{\pi^-} \to M_1(u)\frac{s-t}{2}, \\ &|(\pi^-\pi^0)_{I=1}\rangle|\pi^+\rangle(p_{\pi^0} - p_{\pi^-}) \cdot p_{\pi^+} \to M_1(t)\frac{u-s}{2}, \end{aligned} \quad \text{(A7)}$$

which contribute to the amplitudes with total $I = 0$, 1, and 2, and

$$\begin{aligned} &|(\pi^+\pi^-)_{I=2}\rangle|\pi^0\rangle \to M_2(s), \\ &|(\pi^+\pi^0)_{I=2}\rangle|\pi^-\rangle \to M_2(u), \\ &|(\pi^-\pi^0)_{I=2}\rangle|\pi^+\rangle \to M_2(t), \end{aligned} \quad \text{(A8)}$$

which contribute to the total $I = 1$ and 2 amplitudes. Using the Clebsch-Gordan coefficients tabulated in Ref. [70], we find, after redefining $M_1/2\sqrt{2} \to M_1$ and $M_2\sqrt{3/10} \to M_2$, that $M_1^C(s,t,u)$, $M_0^{\not{C}}(s,t,u)$, and $M_2^{\not{C}}(s,t,u)$ are precisely as given in Eqs. (3) [11], (6), and (7). Note that only the C-odd amplitudes are odd under $t \leftrightarrow u$ as needed. Adding the possible total I amplitudes in leading order in C, CP, and I violation, with their associated coefficients, yields Eq. (4).

We now turn to our isospin decomposition of the $\eta \to \pi^+\pi^-\pi^0$ amplitude of Gasser and Leutwyler through $\mathcal{O}(p^4)$ [10]:

$$\begin{aligned} M_0(s) = {} & \left[\frac{2(s-s_0)}{\Delta}+\frac{5}{3}\right]\frac{1}{2F_\pi^2}(2s-M_\pi^2)J^r_{\pi\pi}(s)+\frac{1}{6F_\pi^2\Delta}(4M_K^2-3M_\eta^2-M_\pi^2)(s-2M_\pi^2)J^r_{\pi\pi}(s) \\ & +\frac{1}{4F_\pi^2\Delta}\left[-6s^2+s(5M_\pi^2+4M_K^2+3M_\eta^2)-4M_K^2\left(M_\eta^2+\frac{1}{3}M_\pi^2\right)\right]J^r_{KK}(s) \\ & +\frac{M_\pi^2}{3F_\pi^2\Delta}\left(2s-\frac{11}{3}M_\pi^2+M_\eta^2\right)J^r_{\pi\eta}(s)-\frac{M_\pi^2}{2F_\pi^2}J^r_{\eta\eta}(s)-\frac{3s}{8F_\pi^2}\frac{(3s-4M_K^2)}{(s-4M_K^2)}\left(J^r_{KK}(s)-J^r_{KK}(0)-\frac{1}{8\pi^2}\right) \\ & +\left[1+a_1+3a_2\Delta+a_3(9M_\eta^2-M_\pi^2)+\frac{2}{3}d_1+\frac{8M_\pi^2}{3\Delta}d_2\right]\left(1+3\frac{s-s_0}{\Delta}\right)+a_4-\frac{8}{3}\frac{M_\pi^2}{\Delta}d_1 \\ & -\frac{3}{\Delta}(2\mu_\pi+\mu_K)(s-s_0)+\left(\frac{4L_3}{F_0^2\Delta}-\frac{1}{64\pi^2F_\pi^2\Delta}\right)\left(\frac{4}{3}s^2-9s_0s+9s_0^2\right) \\ & -\frac{1}{64\pi^2F_\pi^2\Delta}3(s-s_0)(4M_\pi^2+2M_K^2), \end{aligned} \tag{A9}$$

$$M_1(z)=\frac{1}{4\Delta F_\pi^2}\left[(z-4M_\pi^2)J^r_{\pi\pi}(z)+\left(\frac{1}{2}z-2M_K^2\right)J^r_{KK}(z)\right], \tag{A10}$$

and

$$\begin{aligned} M_2(z) = {} & \left(1-\frac{3}{2}\frac{z-s_0}{\Delta}\right)\left[-\frac{1}{2F_\pi^2}(z-2M_\pi^2)J^r_{\pi\pi}(z)+\frac{1}{4F_\pi^2}(3z-4M_K^2)J^r_{KK}(z)+\frac{M_\pi^2}{3F_\pi^2}J^r_{\pi\eta}(z)\right] \\ & +\left(\frac{1}{64\pi^2F_\pi^2\Delta}-\frac{4L_3}{F_0^2\Delta}\right)z^2, \end{aligned} \tag{A11}$$

where $\Delta = M_\eta^2 - M_\pi^2$, $M_\pi^2 = (2M_{\pi^+}^2 + M_{\pi^0}^2)/3$, and $M_K^2 = (M_{K^+}^2 + M_{\bar{K}^0}^2)/2$. We refer to Ref. [76] for $J^r_{PQ}(z)$, noting Eqs. (8.8–8.10) and (A.11), where P and Q denote the mesons π, K, or η, and to Ref. [10] for a_i and d_i. We note that the $J^r_{PQ}(z)$ carry renormalization-scale μ dependence, though cancelling that dependence is beyond the scope of our current approach—we note a similar issue arises in the use of the pion form factor in the analysis of $B \to \pi(\rho \to \pi\pi)$ decay [58]. For this choice of M_π and the use of physical phase space we need to evaluate the possible two-particle thresholds with care. The rescattering function $J^r_{\pi\pi}(z)$ contains $\sigma(s) = \sqrt{1-4m_\pi^2/z}$. If we use $m_\pi^2 = M_\pi^2$, then for $M_I(z)$ with $z = t$ or u evaluated at its minimum value the argument of the square root is less than zero. To avoid this problem, we use $\sigma(z) = \sqrt{1-(M_{\pi^\pm}+M_{\pi^0})^2/z}$ for $z = t$ or u. For $M_I(s)$, though, $s_{\min} = 4M_{\pi^+}^2$ and this problem does not occur. However, for consistency we use $\sigma(s) = \sqrt{1-4M_{\pi^+}^2/s}$ for $M_I(s)$. Moreover, we note $J^r_{\pi\eta}(s)$ contains $\nu(s) = \sqrt{(s-(M_\eta-m_\pi)^2)(s-(M_\eta+m_\pi)^2)}$. If we use $m_\pi = M_\pi$, then for $M_I(s)$ at the maximum of s, we once again find the argument of the square root to be less than zero. To avoid this, we use $\nu(s) = \sqrt{(s-(M_\eta-M_{\pi^0})^2)(s-(M_\eta+M_{\pi^0})^2)}$ for $M_I(s)$. To be consistent, we use $\nu(z) = \sqrt{(z-(M_\eta-M_{\pi^+})^2)(z-(M_\eta+M_{\pi^+})^2)}$ for $M_I(z)$ with $z = t$ or u. As a check of our assessments we have extracted the C- and CP-conserving Dalitz plot parameters from this amplitude. Describing the CP-conserving piece of $|A(s,t,u)|^2$ by $N(1+aY+bY^2+dX^2+fY^3+gX^2Y)$, recalling Eq. (10), we find using a global fit that $a = -1.326$, $b = 0.426$, $d = 0.086$, $f = 0.017$, and $g = -0.072$. These results compare favorably to the global fit results of Ref. [21]; namely, $a = -1.328$, $b = 0.429$, $d = 0.090$, $f = 0.017$, and $g = -0.081$. That work also uses the decay amplitude of Ref. [10] through $\mathcal{O}(p^4)$ and the same value of L_3 but includes electromagnetic corrections through $\mathcal{O}(e^2p^2)$ as well.

In evaluating the BG amplitude [13] we note that an overall 2 should not appear on the second right-hand side of Eq. (3.23); this is needed for the result to agree with that of Ref. [10].

Values of the strong functions associated with the CP-violating parameters $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Im}(\beta)$ in Eq. (11) on our analysis grids in (X, Y) are available upon request.

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