# *CP*-violating super-Weyl anomaly

Koichiro Nakagawa and Yu Nakayama
*Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan*

(Received 28 February 2020; accepted 23 April 2020; published 20 May 2020)

In *CP*-violating conformal field theories in four dimensions, the Pontryagin density can appear in the Weyl anomaly. The Pontryagin density in the Weyl anomaly is consistent, but it has a peculiar feature that the parent three-point function of the energy-momentum tensor can violate *CP* only (semi)locally. In this paper, we study the supersymmetric completion of the Pontryagin density in the Weyl anomaly, where the central charge $c$ effectively becomes a complex number. The supersymmetry suggests that it accompanies the graviphoton $\theta$ term associated with the R-symmetry gauging in the Weyl anomaly. It also accompanies new *CP*-violating terms in the R-current anomaly. While there are no conclusive perturbative examples of *CP*-violating super Weyl anomaly, we construct explicit supersymmetric dilaton effective action which generates these anomalies.

DOI: 10.1103/PhysRevD.101.105013

## I. INTRODUCTION

Charge-parity (*CP*) symmetry and its breaking play fundamental roles in our understanding of our existence. It is the violation of the *CP* symmetry in the electroweak sector that enables us to generate nonzero baryon numbers, while we mysteriously observe the highly suppressed *CP* violation (beyond what we can explain by the anthropic principle) in the strong interaction sector of the standard model of particle physics. Within the Lagrangian description of quantum field theories, one can quantify the violation of the *CP* symmetry by examining the detailed structure of coupling constants: The Kobayashi-Maskawa matrix and $\theta$ term are such quantities in the electroweak and strong interaction sectors in the standard model.

It is more desirable to establish a quantification of the violation of the *CP* symmetry beyond the Lagrangian description, which can be used, for instance, in strongly coupled conformal field theories. In conformal field theories, we often use conformal data that are related to anomalies to quantify nonperturbative characteristics of the theories. For example, the number of degrees of freedom, naively counted by a number of fields, is replaced by the central charge that appears in the Weyl anomaly, for which we can rigorously prove that it decreases along the renormalization group flow. Chiral asymmetries in charged objects can be nonperturbatively quantified by the 't Hooft anomaly coefficients that are invariant under the renormalization group flow.



In Ref. [1], it was pointed out that the Weyl anomaly in four-dimensional conformal field theories may include a term that is present only in *CP*-violating theories. It is the Pontryagin density. Since the Pontryagin density is *CP* odd while the energy-momentum tensor is *CP* even (in *CP*-preserving theories), it can appear only in the *CP*-violating theories. Therefore, one hopes that the Pontryagin density in the Weyl anomaly can measure the violation of *CP* symmetry in the strongly coupled conformal field theories.

Most of the anomalies in conformal field theories can be directly computed from the conformal data, i.e., two-point functions and three-point functions. In particular, the *CP*-even terms in the Weyl anomaly, i.e., the Weyl tensor squared term (central charge $c$) and the Euler density (central charge $a$), can be read from the three-point functions of the energy-momentum tensor [2]. The Pontryagin density in the Weyl anomaly, however, has a peculiar feature that the parent three-point function of the energy-momentum tensor can violate *CP* only (semi)locally. Such anomalies are called "impossible anomalies" in Ref. [3] due to the nonexistence of conformally invariant nonlocal correlation functions.

Recently, there have been active discussions if we may realize the Pontryagin density in the Weyl anomaly in concrete field theory examples. In particular, it was claimed in Ref. [4] that a free massless Weyl fermion in four dimensions can generate the Pontryagin density in the Weyl anomaly with an imaginary coefficient (in the Lorentzian signature). If this is the case, it means that the free Weyl fermion breaks the *CP* symmetry and unitarity anomalously. The computation has been scrutinized under various regularization schemes Refs. [5–14], and the discussions seem to remain open.

In this paper, we study the supersymmetric completion of the *CP*-violating Weyl anomaly in four-dimensional superconformal field theories coupled with background superconformal supergravity. In superconformal field theories, there are various techniques, e.g., supersymmetric localization, to compute the partition function, which may eventually help us compute the $CP$-violating Weyl anomaly in the strongly coupled regime. We will show that, with the $CP$-violating Weyl anomaly, the central charge $c$, together with the Pontryagin density in the Weyl anomaly, is effectively complexified as the gauge coupling constant is complexified with the $\theta$ term in supersymmetric gauge theories. On the other hand, we will see that the central charge $a$ remains real.

As for concrete realizations of the supersymmetric $CP$-violating Weyl anomaly, we do not offer any conclusive perturbative examples, but we will discuss what would be the supersymmetric extension of the free Weyl fermion with the putative $CP$-violating Weyl anomaly. We will also show the consistency of the $CP$-violating super-Weyl anomaly explicitly by constructing the supersymmetric dilaton effective action. It can be regarded as a concrete model of the supersymmetric $CP$-violating Weyl anomaly, albeit the Weyl symmetry is spontaneously broken.

The organization of the paper is as follows. In Sec. II, we first review the possibility of the Pontryagin density in the Weyl anomaly. We show that the Pontryagin density satisfies the Wess-Zumino consistency condition, and we can construct the explicit dilaton effective action. Along the way, we also review how the Seeley-DeWitt coefficient of a chiral fermion can include the Pontryagin density. Section III is our main contribution. We show that the Wess-Zumino consistency condition for the super-Weyl anomaly allows the complexified $c$, which gives the Pontryagin density in the Weyl anomaly in its component form. We also study the structure of the supersymmetric Seeley-DeWitt coefficient. Finally, we explicitly construct the supersymmetric dilaton effective action that gives the supersymmetric $CP$-violating Weyl anomaly. In Sec. IV, we conclude with some discussions.

## II. *CP*-VIOLATING WEYL ANOMALY

### A. General structure

When we put a conformal field theory on a nontrivial four-dimensional manifold, it shows the Weyl anomaly [15]. Under the infinitesimal Weyl transformation of the metric $\delta g_{mn} = -2\sigma g_{mn}$, the (effective) action shows the variation of $\delta_\sigma S = -\int d^4x\sqrt{g}\sigma T^m_m$ with the energy-momentum tensor $T_{mn} = \frac{-2}{\sqrt{g}}\frac{\delta S}{\delta g^{mn}}$.[1]

Since the scaling dimension of the energy-momentum tensor is four in four dimensions, the most general possibility of the Weyl anomaly constructed out of the metric tensor is

$$T^m_m = \tilde{c}\mathrm{Weyl}^2 - \tilde{a}\mathrm{Euler} + \tilde{b}R^2 + \tilde{d}\Box R + \tilde{e}\mathrm{Pontryagin}. \quad (1)$$

Here,

$$\mathrm{Weyl}^2 = C^{mnrs}C_{mnrs} = R^{mnrs}R_{mnrs} - 2R^{mn}R_{mn} + \frac{1}{3}R^2 \quad (2)$$

is the Weyl tensor (denoted by $C_{mnrs}$) squared and

$$\mathrm{Euler} = R^{mnrs}R_{mnrs} - 4R^{mn}R_{mn} + R^2 \quad (3)$$

is the Euler density. The last term

$$\mathrm{Pontryagin} = R^{mnrs}\tilde{R}_{mnrs} = \frac{1}{2}\epsilon_{mnab}C^{mnrs}C^{ab}{}_{rs} \quad (4)$$

is the Pontryagin density. It is the only term that violates $CP$ in the Weyl anomaly, and it will be the main focus of the present paper.

Since the Weyl transformation is Abelian, the Weyl anomaly must satisfy the simple Wess-Zumino consistency condition [18]:

$$0 = [\delta_\sigma, \delta_\tau]S = \int d^4x(\sigma\delta_\tau - \tau\delta_\sigma)\sqrt{g}T^m_m. \quad (5)$$

It is immediate to see that the both $\mathrm{Weyl}^2$ and Pontryagin terms satisfy the condition because they are Weyl invariant themselves. On the other hand, the $\Box R$ term is consistent but trivial, because

$$\delta_\sigma \int d^4x\sqrt{g}R^2 = 12\int d^4x\sqrt{g}\sigma\Box R \quad (6)$$

and one can always remove it by adding the local counterterm $R^2$. This also shows that the $R^2$ term (alone) in the Weyl anomaly does not satisfy the Wess-Zumino consistency condition.

The consistency of the Euler density is more nontrivial. For our purpose, let us introduce the Fradkin-Tseytlin-Riegert-Paneitz [19–22] operator

$$\triangle_4 = \nabla^m\left(\nabla_m\nabla_n + 2R_{mn} - \frac{2}{3}Rg_{mn}\right)\nabla^n, \quad (7)$$

which is a Weyl-invariant generalization of the Laplacian squared in four dimensions. We may now use the identity

$$\delta_\sigma\sqrt{g}\left(\mathrm{Euler} - \frac{2}{3}\Box R\right) = -4\sqrt{g}\Delta_4\sigma \quad (8)$$

[1]In this paper, unless explicitly stated, we work in Euclidean field theories. The Euclidean action $S$ in our convention is "negative definite": $S = -\int d^4x\sqrt{g}\partial^m\phi\partial_m\phi$ for a free scalar field so that it appears to be the same in the Lorentzian signature with the $(-+++)$ convention. Except that we work in Euclidean field theories, we follow the convention of Refs. [16,17].

to show that the Euler density satisfies the Wess-Zumino consistency condition after the integration by part.

The Weyl anomaly coefficients $c$ and $a$ are related to nonlocal terms of the two-point function and three-point function of the energy-momentum tensor of a conformal field theory [23] in the flat space-time. On the other hand, the detailed analysis of the conformal Ward identity tells us that the nonlocal two-point and three-point functions of the energy-momentum tensor in four dimensions do not contain terms violating $CP$ [24,25] in the flat space-time. This means that one cannot measure the Pontryagin density in the Weyl anomaly from the nonlocal correlation functions of the energy-momentum tensor.

In contrast, the effect of the Pontryagin density in the Weyl anomaly is, if any, solely contained in the semilocal or local terms of the correlation functions of the energy-momentum tensor in the flat space-time. The semilocal terms are the correlation functions that include at least one coordinate space delta function. Such anomalies that are not supported by nonlocal conformal correlation functions are called an "impossible anomaly" in Ref. [3]. The Pontryagin density in the Weyl anomaly is such an example. We will see that the supersymmetric partner of the Pontryagin density in the Weyl anomaly also generates impossible anomalies.

### B. Seeley-DeWitt coefficient

In free field theories, one may perform the path integral explicitly to compute the effective action, and one may associate the Weyl anomaly with the Seeley-DeWitt coefficients [26–28], which we would like to review. Let us consider an elliptic operator $D$ acting on a collection of fields $\phi(x)$ (of a certain bundle). Using the complete basis of the eigenvalue equation $D\phi_\lambda(x) = \lambda\phi_\lambda(x)$, we may define the heat kernel $K(s; x, y|D)$ as

$$K(s; x, y|D) = \langle x|e^{-sD}|y\rangle = \sum \phi^\dagger_\lambda(x)e^{-s\lambda}\phi_\lambda(y). \quad (9)$$

It is called the heat kernel because it satisfies the heat equation

$$\left(\frac{\partial}{\partial s} + D\right)K(s; x, y|D) = 0 \quad (10)$$

with the initial condition $K(s = 0; x, y|D) = \delta(x - y)\mathbf{1}$.

The trace of this heat kernel has an asymptotic expansion as

$$\begin{aligned}\mathrm{Tr}(fe^{-sD}) &= \int d^dx\sqrt{g}\langle x|\mathrm{tr}f(x)e^{-sD}|x\rangle \\ &\simeq \sum_{n\in\mathbb{N}} s^{(n-d)/2}a_n(f, D). \quad (11)\end{aligned}$$

The coefficient $a_n$ is called the Seeley-DeWitt coefficient of the heat kernel associated with the elliptic operator $D$. For most of our applications, $f(x)$ is just a function, and we often use the local expression $b_n$ defined by

$$a_n(f, D) = \int d^dxf(x)\sqrt{g}b_n(D). \quad (12)$$

There is a general recipe to compute the Seeley-DeWitt coefficient. Suppose that the elliptic operator $D$ is expressed as $D = -\nabla^m\nabla_m - E$, where $\nabla_m$ is a suitable covariant derivative acting on a section of $\phi$. In such cases, it is straightforward to compute the lower-order Seeley-DeWitt coefficients. In particular, $b_4(D)$ can be explicitly computed as

$$b_4(D) = \frac{1}{(4\pi)^2 360}\mathrm{tr}(60\Box E + 60RE + 180E^2 + 12\Box R + 5R^2 - 2R^{mn}R_{mn} + 2R^{mnrs}R_{mnrs} + 30\Omega^{mn}\Omega_{mn}), \quad (13)$$

where $\Omega_{mn} = [\nabla_m, \nabla_n]$ is the curvature two-form.

For example, let us consider a conformally coupled complex scalar $\phi$ with a $U(1)$ charge $q$. Here, $D = -\nabla^m\nabla_m - \frac{1}{6}R$, and the covariant derivative includes the $U(1)$ gauge connection $A_m$: $\nabla_m\phi = \partial_m\phi + iqA_m\phi$. The above formula gives

$$b_4\left(-\nabla^2 - \frac{1}{6}R\right) = \frac{1}{(4\pi)^2 360}(-4\Box R + 6\mathrm{Weyl}^2 - 2\mathrm{Euler} - 60q^2F^{mn}F_{mn}), \quad (14)$$

where $F_{mn} = \partial_mA_n - \partial_nA_m$. Note that a bosonic field contributes to the Weyl anomaly as $T^m_m = b_4(D)$ from the path integral representation of the effective action with the zeta function regularization while a fermionic field contributes to the Weyl anomaly with $T^m_m = -b_4(D)$.

Let us now consider a Euclidean Weyl fermion $\psi_\alpha$ in the $(1/2, 0)$ representation of the Euclidean rotation group $SO(4)$, which we will call left-handed. The Euclidean action is given by

$$S = \int d^4x\bar{\psi}_{\dot{\alpha}}\nabla^{\dot{\alpha}\alpha}\psi_\alpha, \quad (15)$$

where the spinor covariant derivative $\nabla^{\dot{\alpha}\alpha} = (\bar{\sigma}^m)^{\dot{\alpha}\alpha}\nabla_m$ with respect to the spin connection $w^{ab}_m$ and the $U(1)$ gauge connection $A_m$ is given by

$$\nabla_m\psi_\alpha = \partial_m\psi_\alpha + \frac{1}{2}w_m{}^{ab}(\sigma_{ab})_\alpha{}^\beta\psi_\beta + iqA_m\psi_\alpha. \quad (16)$$

In order to define a meaningful Euclidean action, we also have to introduce an independent Weyl fermion $\bar{\psi}_{\dot{\alpha}}$ in the $(0, 1/2)$ representation of $SO(4)$ [with the $U(1)$ gauge charge $-q$], which we will call right-handed. We stress that it is mandatory to introduce the fermions with both chiralities to write down any sensible action for a Euclidean Weyl fermion.

The classical equations of motion give the left-handed Weyl equation

$$\nabla^{\dot{\alpha}\alpha}\psi_\alpha = 0 \quad (17)$$

as well as the right-handed Weyl equation

$$\nabla^{\dot{\alpha}\alpha}\bar{\psi}_{\dot{\alpha}} = 0. \quad (18)$$

While each of the Weyl equations is independently Euclidean invariant and not related with each other, the action principle demands the existence of both simultaneously.

For the left-handed Euclidean Weyl fermion, the natural second-order differential operator to compute the Seeley-DeWitt coefficient is given by

$$D_\alpha{}^\beta = \nabla_{\alpha\dot{\alpha}}\nabla^{\dot{\alpha}\beta}. \quad (19)$$

This operator can be rewritten in the standard form as

$$D_\alpha{}^\beta = \frac{1}{2}((\sigma^m\bar{\sigma}^n + \sigma^n\bar{\sigma}^m)_\alpha{}^\beta + (\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)_\alpha{}^\beta)\nabla_m\nabla_n \quad (20)$$

$$= -\nabla^m\nabla_m\delta_\alpha^\beta - \left(-\frac{R}{4}\delta_\alpha^\beta + iqF_{mn}(\sigma^{mn})_\alpha{}^\beta\right). \quad (21)$$

To compute the Seeley-DeWitt coefficient, we also need the curvature two-form

$$(\Omega_{mn})_\alpha{}^\beta = \frac{1}{2}R_{mnab}(\sigma^{ab})_\alpha{}^\beta + iqF_{mn}\delta_\alpha^\beta. \quad (22)$$

Substituting these into the general formula (13) and evaluating the spinor trace, we obtain

$$-b_4(\nabla_{\alpha\dot{\alpha}}\nabla^{\dot{\alpha}\beta}) = \frac{1}{(4\pi)^2 360}\left(6\Box R + 9\mathrm{Weyl}^2 - \frac{11}{2}\mathrm{Euler} + \frac{15}{2}\mathrm{Pontryagin} - 120q^2F^{mn}F_{mn} - 180q^2F^{mn}\tilde{F}_{mn}\right), \quad (23)$$

where $\tilde{F}_{mn} = \frac{1}{2}\epsilon_{mnrs}F^{rs}$ as usual. We have put the minus sign explicitly here to emphasize that it is a fermionic field.

Similarly, for the right-handed Euclidean Weyl fermion, the natural second-order differential operator is

$$D^{\dot{\beta}}{}_{\dot{\alpha}} = \nabla^{\dot{\beta}\alpha}\nabla_{\alpha\dot{\alpha}} \quad (24)$$

and the corresponding Seeley-DeWitt coefficient is

$$-b_4(\nabla^{\dot{\beta}\alpha}\nabla_{\alpha\dot{\alpha}}) = \frac{1}{(4\pi)^2 360}\left(6\Box R + 9\mathrm{Weyl}^2 - \frac{11}{2}\mathrm{Euler} - \frac{15}{2}\mathrm{Pontryagin} - 120q^2F^{mn}F_{mn} + 180q^2F^{mn}\tilde{F}_{mn}\right). \quad (25)$$

It is important to realize that the Seeley-DeWitt coefficient itself can be defined without introducing a fermion with the other chirality.

As already observed in Refs. [1,29,30], if we identified the Seeley-DeWitt coefficient of the Euclidean Weyl fermion of one chirality as the Weyl anomaly $T_m^m \overset{?}{=} -\ b_4(\nabla_{\alpha\ddot{\alpha}}\nabla^{\ddot{\alpha}\beta})$, we would obtain the Pontryagin density of the tangent bundle and the $U(1)$ gauge bundle. On the other hand, the sum of (23) and (25) does not contain the $CP$-violating terms. Note also that, in the Lorentzian signature, the Pontryagin density gives an extra factor of $i$:

$$-b_4(\nabla_{\alpha\ddot{\alpha}}\nabla^{\ddot{\alpha}\beta})|_{\mathrm{Lorentz}} = \frac{1}{(4\pi)^2 360}\left(6\Box R + 9\mathrm{Weyl}^2 - \frac{11}{2}\mathrm{Euler} + i\frac{15}{2}\mathrm{Pontryagin} - 120q^2F^{mn}F_{mn} - i180q^2F^{mn}\tilde{F}_{mn}\right), \quad (26)$$

in the analytically continued Seeley-DeWitt coefficient. If we interpreted it as the Weyl anomaly of a physical theory, it would imply the violation of unitarity.

### C. Effective action

Since the Pontryagin density in the Weyl anomaly satisfies the Wess-Zumino consistency condition, it should be integrable. Accordingly, it should be possible to construct an effective action that reproduces the Pontryagin density in the Weyl anomaly as a classical variation. The question of whether this is possible or not is distinct from the origin of the Pontryagin term in the Weyl anomaly and can be independently studied.

At the minimal level, the effective action that gives the Pontryagin term in the Weyl anomaly turns out to be

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}\phi\Delta_4\phi + Q\mathcal{Q}\phi - \phi\text{Pontryagin}\right). \qquad (27)$$

This classical action is also known as the dilaton effective action, because it could appear as the Nambu-Goldstone action for the spontaneous breaking of the Weyl (or conformal) symmetry.

Here, in addition to the Weyl-invariant Fradkin-Tseytlin-Riegert-Paneitz operator $\triangle_4$, we introduced the so-called $\mathcal{Q}$ curvature [31]:

$$\mathcal{Q} = -\frac{1}{6}\Box R - \frac{1}{2}R^{mn}R_{mn} + \frac{1}{6}R^2 = \frac{1}{4}\left(\text{Euler} - \text{Weyl}^2 - \frac{2}{3}\Box R\right), \qquad (28)$$

which shows a remarkable property under the Weyl transformation

$$\delta_\sigma\sqrt{g}\mathcal{Q} = -\sqrt{g}\triangle_4\sigma. \qquad (29)$$

This results in the classical violation of the Weyl symmetry

$$T^m_m = +Q\triangle_4\phi = -\frac{Q^2}{4}\left(\text{Euler} - \text{Weyl}^2 - \frac{2}{3}\Box R\right) + Q\text{Pontryagin}, \qquad (30)$$

where we have used the classical equation of motion for $\phi$. We see that the classical variation contains the Pontryagin density as we claimed.

Alternatively, one may further assign the Weyl transformation of $\phi$ as $\phi \to \phi + Q\sigma$. Under this compensated Weyl transformation,[2] the effective action changes as

$$\delta S = \int d^4x\sqrt{g}\sigma(Q^2\mathcal{Q} - Q\text{Pontryagin}), \qquad (31)$$

which directly gives the Weyl anomaly including the Pontryagin density.

One may compute the correlation functions of the energy-momentum tensor in flat space by using this dilaton effective action. We first solve $\phi$ by using the equations of motion $\triangle_4\phi_c(x) = -Q\mathcal{Q} + \text{Pontryagin}$ from the Green function $\triangle_4 G_4(x-y) = \delta(x-y)$. Then we substitute it back into the action to evaluate the on-shell action. The three-point function of the energy-momentum tensor in flat space can be computed as

$$\langle T_{mn}(x)T_{rs}(y)T_{ab}(z)\rangle = \frac{(-2)^3}{\sqrt{g}\sqrt{g}\sqrt{g}}\frac{\delta}{\delta g^{mn}(x)}\frac{\delta}{\delta g^{rs}(y)}\frac{\delta}{\delta g^{ab}(z)}S[g,\phi=\phi_c]\Big|_{g_{mn}=\delta_{mn}}. \qquad (32)$$

Since the result is lengthy, we focus on only the *CP*-violating part, which is our main focus. It is given by

$$\begin{aligned}\langle T_{mn}(x)T_{rs}(y)T_{ab}(z)\rangle_{\text{odd}} = \frac{1}{3}Q\epsilon_{knlb}(&\partial^k\partial_a[\partial^l\partial_m\delta(x-z)(\partial_r\partial_s - \delta_{rs}\Box)G_2(x-y)] \\ &- \delta_{ma}\partial^k\partial^t[\partial^l\partial_t\delta(x-z)(\partial_r\partial_s - \delta_{rs}\Box)G_2(x-y)]) + \text{sym}.\end{aligned} \qquad (33)$$

Here $F_{mn}$ is the field strength for the R-symmetry gauge field (i.e. graviphoton field strength). See e.g. Appendix A of [32] for the detailed derivation.[3]

In particular, its trace gives

$$\langle T^m_m(x)T_{rs}(y)T_{ab}(z)\rangle_{\text{odd}} = -2Q\varepsilon_{kslb}(\partial^z_r\partial^y_a - \delta_{ra}\partial^z_t\partial^{y\,t})\partial^l\delta(x-z)\partial^k\delta(x-y) + \text{sym}, \qquad (34)$$

which correctly reproduces the shape of the three-point function expected from the Pontryagin density in the Weyl anomaly.

A couple of comments are now in order. First, we may introduce the additional term $\phi\text{Weyl}^2$ to generate the independent $\text{Weyl}^2$ term in $T^m_m$. Second, as one can directly see, the three-point function computed from the dilaton effective action is semilocal. It was under active debate if this could serve as the "correct" Wess-Zumino action for the Weyl anomaly

[2]Due to the linear shift under the Weyl transformation, the Weyl symmetry is spontaneously broken in this model.

[3]The formula is corrected in [33]. We would like to thank I. Papadimitriou for the correspondence.

[2,34,35] because of the lack of nonlocal contributions to the correlation functions.[4] As for the Pontryagin density, however, there is no nonlocal term from the beginning, so the debate was irrelevant for us. Finally, the dilaton effective action can be modified in various manners by adding more Weyl-invariant terms. They give different-looking effective actions for various purposes. For example, Ref. [36] studied a variant with the two-derivative kinetic term to prove a theorem, while Ref. [37] studied the quantum nature by adding the Liouville potential.

## III. *CP*-VIOLATING SUPER-WEYL ANOMALY

### A. General structure

In order to study the supersymmetric extension of the Pontryagin density in the Weyl anomaly, we first review the structure of the super-Weyl anomaly. Let us consider the supercurrent multiplet with the supersymmetric conservation law

$$\bar{\mathcal{D}}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} + \frac{2}{3}\mathcal{D}_\alpha T = 0, \tag{35}$$

where $T$ is a chiral superfield known as the supertrace multiplet. See e.g., [38] for a detailed analysis of this equation in flat space-time. For superconformal field theories, $T = 0$ in the trivial supergravity background. In the nontrivial supergravity background, the supertrace multiplet shows the super-Weyl anomaly [39,40]:

$$\begin{aligned} 8\pi^2 T = {} & c W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \\ & - a\left(W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)(G^m G_m + 2R\bar{R})\right) \\ & + \frac{1}{16} h(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}^2 R. \end{aligned} \tag{36}$$

Here, $W^{\alpha\beta\gamma}$ is the chiral super-Weyl tensor multiplet, and $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} - \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)(G^m G_m + 2R\bar{R})$ is the chirally projected super-Euler density (see e.g., [16]). In the literature, it is usually assumed that $a$ and $c$ are real, but we will relax the condition soon. The term proportional to the real parameter $h$ is trivial and can be removed by using an appropriate supersymmetric local counterterm.

Note that the chiral multiplet $T$ contains the trace of the energy-momentum tensor and the divergence of the R current $-\frac{1}{8}(T^m_m + \frac{3}{2} i \nabla^m J^R_m)$ in the $\theta^2$ component. See e.g., [38]. Thus, in component, assuming $a$ and $c$ are real for a moment, the super-Weyl anomaly (36) implies the Weyl anomaly (up to trivial terms)

$$T^m_m = \frac{c}{16\pi^2}\mathrm{Weyl}^2 - \frac{a}{16\pi^2}\mathrm{Euler} - \frac{c}{6\pi^2} F^{mn} F_{mn} \tag{37}$$

as well as the chiral anomaly for the R-current conservation

$$\nabla^m J^R_m|_{\mathrm{Lorentz}} = \frac{c-a}{24\pi^2} R^{mnrs}\tilde{R}_{mnrs} + \frac{5a-3c}{27\pi^2} F^{mn}\tilde{F}_{mn}. \tag{38}$$

Here $F_{mn}$ is the field strength for the R-symmetry gauge field (i.e. graviphoton field strength). See e.g. Appendix A of [32] for the detailed derivation.

Alternatively, one may study the variation of the supersymmetric effective action under the super-Weyl variation $\mathcal{E} \to e^{3\sigma}\mathcal{E}$. The anomalous variation shows

$$\delta S = \int d^4x d^2\theta \mathcal{E}\sigma T + \int d^4x d^2\bar{\theta}\bar{\mathcal{E}}\bar{\sigma}\bar{T}, \tag{39}$$

where $\sigma$ is a chiral superfield corresponding to the super-Weyl variation. In component, $\sigma = \sigma_1 + i\sigma_2 + O(\theta)$ with $-\frac{\sigma_1}{2}$ being the Weyl factor[5] and $\sigma_2$ being the gauge parameter for the R-symmetry transformation.

It is known that (up to terms that can be removed by local counterterms) these are the only available super-Weyl anomaly (see e.g., [41] and older references therein). However, there is one fine print that has not been discussed very much in the literature, namely, the reality condition on $a$ and $c$ (as well as $h$). One may try to imagine what happens if $a$ and $c$ are not real but complex numbers.

First of all, the Wess-Zumino consistency condition demands that $a$ must be real [42] from the coefficients of

$$(\mathcal{D}^\alpha \sigma \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\tau} G_{\alpha\dot{\alpha}}) - (\sigma \leftrightarrow \tau). \tag{40}$$

One can also see how this must be the case from the component analysis. If $a$ were not real, then $\nabla^m J^R_m$ would include the term proportional to the Euler density whose Weyl variation is nonzero. Then it could not satisfy the mixed Wess-Zumino consistency condition for the Weyl transformation and the (gauged) R-symmetry transformation. Similarly, one can show that $h$ must be real.

However, there is no reality constraint on $c$ from the Wess-Zumino consistency condition simply because $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$ is super-Weyl invariant. Assuming $c$ is a complex number, we have the *CP*-violating Weyl anomaly

$$\begin{aligned} T^m_m|_{\mathrm{Lorentz}} = {} & \frac{\mathrm{Re}(c)}{16\pi^2}\mathrm{Weyl}^2 - \frac{a}{16\pi^2}\mathrm{Euler} - \frac{\mathrm{Re}(c)}{6\pi^2}F^{mn}F_{mn} \\ & - \frac{\mathrm{Im}(c)}{8\pi^2}\mathrm{Pontryagin} + \frac{\mathrm{Im}(c)}{6\pi^2}F^{mn}\tilde{F}_{mn} \end{aligned} \tag{41}$$

and the corresponding parity-violating chiral anomaly for the R current

[4] In two dimensions, the Liouville action or Polyakov action reproduces the local as well as nonlocal correlation functions of the energy-momentum tensor.

[5] In our convention that follows [16], the sign of the Weyl factor and the super Weyl factor is opposite.

$$\nabla^m J^R_m|_{\mathrm{Lorentz}} = \frac{\mathrm{Re}(c)-a}{24\pi^2} R^{mnrs}\tilde{R}_{mnrs} + \frac{5a-3\mathrm{Re}(c)}{27\pi^2} F^{mn}\tilde{F}_{mn} + \frac{\mathrm{Im}(c)}{24\pi^2}\mathrm{Weyl}^2 - \frac{\mathrm{Im}(c)}{9\pi^2} F^{mn}F_{mn}. \quad (42)$$

We can check that these satisfy the mixed Wess-Zumino consistency condition of the Weyl transformation and the R-symmetry transformation.

One thing to be noted here is that these two expressions are written in the Lorentzian signature and they are all real. As a consequence, they are compatible with unitarity. In the Euclidean signature, we have to put $i$ in front of the $\epsilon$ tensor so that the supersymmetric Pontryagin term in the Weyl anomaly gives a pure phase in the Euclidean partition function. Similarly, the $CP$-violating anomaly in the R-symmetry transformation is not a phase but an absolute value in the Euclidean partition function.

Note that it is not feasible to absorb the phase of $c$ into the definition of the chiral superfield $\sigma$ that defines the super-Weyl transformation. This is because $a$ must be real, and such a redefinition would violate the Wess-Zumino consistency condition. In this way, having a imaginary part of $c$ leads to a physically nontrivial effect.

Let us take a closer look at the structure of the supersymmetric partners of the Pontryagin density in the Weyl anomaly. We note that these novel anomalies break parity or at least $CP$. The Weyl anomaly contains a new term of the graviphoton $\theta$ term $F^{mn}\tilde{F}_{mn}$ associated with the R-symmetry gauging. Its existence would be related to the renormalization group beta functions for the graviphoton $\theta$ term. The R-current anomaly now contains unfamiliar $CP$-violating terms. One is the graviphoton field strength squared, and the other is the Weyl tensor squared.

Not only do they look unfamiliar, but also all of them are examples of impossible anomalies. Nonlocal terms in the three-point functions among the energy-momentum tensor and R current do not contain any terms that will directly generate these $CP$-violating anomalies. Instead, all of them are supported in the semilocal terms. For example, the R-current anomaly has the $CP$-violating semilocal terms proportional to

$$\langle J^R_\mu(x) J^R_\nu(y) J^R_\rho(z)\rangle = \partial^y_\alpha(\partial_\mu G_2(x-y)\delta_{\nu\rho}\partial^\alpha\delta(y-z)) - \partial^y_\nu(\partial_\mu G_2(x-y)\partial_\rho\delta(y-z)) + \mathrm{sym} \quad (43)$$

in a particular regularization scheme.

### B. Supersymmetric Seeley-DeWitt coefficient

Similarly to what we have studied in Sec. II C, in free supersymmetric field theories, the super-Weyl anomaly is related to the supersymmetric Seeley-DeWitt coefficients. Here, we would like to study a supersymmetric generalization of the heat kernel computation.

Let us first define the supersymmetric heat kernel for a chiral operator on the superspace

$$\frac{\partial U_c}{\partial s} = \left(\Box_+ - \frac{1}{4}(\bar{\mathcal{D}}^2\bar{R}) + R\bar{R}\right) U_c \quad (44)$$

with the initial condition

$$U_c(s=0; z, z') = \delta_+(z, z'). \quad (45)$$

Here, $\delta_+(z, z')$ is a covariantly chiral delta function, and the chiral Laplacian $\Box_+$ is defined by

$$\Box_+ = \mathcal{D}^m\mathcal{D}_m + \frac{1}{4}R\mathcal{D}^2 + iG^m\mathcal{D}_m + \frac{1}{4}(\mathcal{D}^\alpha R)\mathcal{D}_\alpha. \quad (46)$$

The supersymmetric Seeley-DeWitt coefficient $a^c_n$ for a chiral superfield is defined as an asymptotic power-series solution of its trace:

$$U_c(s; z, z) = \frac{1}{(4\pi s)^2}\sum_{n=0}^{\infty} a^c_n s^n. \quad (47)$$

The most relevant one for our discussion is $a^c_2$ and it is given by [40,43,44]

$$a^c_2 = \frac{1}{12}W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \frac{1}{48}(\bar{\mathcal{D}}^2 - 4R)G^mG_m - \frac{1}{96}(\bar{\mathcal{D}}^2 - 4R)(\mathcal{D}^2 - 4\bar{R})R. \quad (48)$$

Similarly, if we started with the antichiral operator $\Box_-$, we would end up with the supersymmetric Seeley-DeWitt coefficient $\bar{a}^a_n$ for an antichiral superfield. In particular,

$$\bar{a}^a_2 = \frac{1}{12}\bar{W}^{\alpha\beta\gamma}\bar{W}_{\alpha\beta\gamma} + \frac{1}{48}(\mathcal{D}^2 - 4\bar{R})G^mG_m - \frac{1}{96}(\mathcal{D}^2 - 4\bar{R})(\bar{\mathcal{D}}^2 - 4R)\bar{R}. \quad (49)$$

Suppose we have a supersymmetric massless Wess-Zumino model coupled with background superconformal supergravity. It is described by the classical action

$$S[\Phi] = \int d^4x d^2\theta d^2\bar{\theta} E^{-1}\bar{\Phi}\Phi, \quad (50)$$

where $\Phi$ and $\bar{\Phi}$ are the chiral and antichiral superfield, respectively. One can compute the effective action explicitly with the supersymmetric zeta function regularization of the path integral, and then one can relate the super-Weyl anomaly under the super-Weyl transformation $\mathcal{E} \to e^{3\sigma}\mathcal{E}$ with the supersymmetric Seeley-DeWitt coefficient:

$$\delta S = \frac{1}{(4\pi)^2}\left(\int d^4x d^2\theta\mathcal{E}\sigma a^c_2 + \int d^4x d^2\bar{\theta}\,\bar{\mathcal{E}}\,\bar{\sigma}\,\bar{a}^a_2\right). \quad (51)$$

This is the standard expression for the super Weyl anomaly of a free chiral multiplet (i.e. one complex scalar and one Weyl fermion) with the identification $T = \frac{1}{(4\pi)^2} a_2^c$ [40]. In particular, $a = \frac{1}{48}$ and $c = \frac{1}{24}$ are real numbers. Consequently, it does not show the Pontryagin density in the Weyl anomaly.[6]

What would be the corresponding supersymmetric Seeley-DeWitt coefficient for a Euclidean left-handed Weyl fermion (without a right-handed partner)? The idea is that, in the Euclidean signature, one may take the super-Weyl parameters $\sigma$ and $\bar{\sigma}$ independently, In particular, we can set $\bar{\sigma} = 0$ (while $\sigma$ is nonzero). This corresponds to considering the Weyl transformation on $\Phi$ (that contains the left-handed Weyl fermion) without the Weyl transformation on $\bar{\Phi}$ (that contains the right-handed Weyl fermion).

The resultant super-Weyl variation is

$$\int d^4x d^2\theta \mathcal{E}\sigma \left( \frac{1}{12} W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \frac{1}{48} (\bar{\mathcal{D}}^2 - 4R) G^m G_m - \frac{1}{96} (\bar{\mathcal{D}}^2 - 4R)(\mathcal{D}^2 - 4\bar{R}) R \right) \tag{52}$$

without the antichiral part (i.e., the terms with $\bar{\sigma}$). Suppose we would like to study the response to the Weyl transformation of this theory from the coupling to the real part of $\sigma$. Then, in components, we see that the would-be Weyl anomaly (which is given by the coupling to the real part of $\sigma$) includes the Pontryagin density with a real coefficient in the Euclidean signature.[7]

### C. Supersymmetric effective action

As we have just seen in the previous subsection, there is no simple free field computation that will give the supersymmetric generalization of the Pontryagin density in the Weyl anomaly. However, one can still construct the supersymmetric dilaton effective action to incorporate the Pontryagin density in the super-Weyl anomaly.

For this purpose, we would like to generalize the $\mathcal{N} = 1$ super-Liouville theory studied in Ref. [45] (see also [17,46]). The classical action for a chiral superfield $\Phi$ is given by

$$S[\Phi] = \int d^4x d^2\theta \mathcal{E} (\Phi \hat{\mathcal{P}} \bar{\Phi} + 4\bar{Q}\, \hat{\mathcal{Q}}\, \Phi + C\Phi W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}) + \text{H.c.} \tag{53}$$

Here the chirally projected supersymmetric Fradkin-Tseytlin-Riegert-Paneitz operator $\hat{\mathcal{P}}$ [17] is given by

$$\hat{\mathcal{P}} = -\frac{1}{64} (\bar{\mathcal{D}}^2 - 4R)(\mathcal{D}^2 \bar{\mathcal{D}}^2 + 8\mathcal{D}^\alpha (G_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}})), \tag{54}$$

and the supersymmetric $Q$-curvature chiral superfield [17,45] is given by

$$\hat{\mathcal{Q}} = -\frac{1}{8} (\bar{\mathcal{D}}^2 - 4R) \left( G^m G_m + 2R\bar{R} - \frac{1}{4} \mathcal{D}^2 R \right). \tag{55}$$

Note that, in Ref. [45], it is assumed that $Q$ is a real parameter, but here we would like to regard it as a complex parameter for the most genericity. We also note that the parameter $C$ is complex. Under the super-Weyl variation with the additional shift of $\Phi$

$$\Phi \to \Phi - 2Q\sigma, \quad \bar{\Phi} \to \bar{\Phi} - 2\bar{Q}\,\bar{\sigma}, \tag{56}$$

the supersymmetric dilaton effective action transforms as

$$S[\Phi] \to S[\Phi] - S[2Q\sigma]. \tag{57}$$

The infinitesimal variation is

$$\delta S[\Phi] = -\int d^4x d^2\theta \mathcal{E}\sigma (8\bar{Q} Q \hat{\mathcal{Q}} + 2QC W^{\alpha\beta\gamma} W_{\alpha\beta\gamma}) + \text{H.c.}, \tag{58}$$

which gives the super-Weyl variation studied e.g., in Refs. [41,42] except that in our case $QC$ can be a complex number, whose possibility has not been emphasized before. Note also that the coefficient in front of $\hat{\mathcal{Q}}$ is a real number, whose necessity is not immediately obvious, but the Wess-Zumino consistency condition of the super-Weyl anomaly demands it must be the case [42]. It is a nontrivial check that our supersymmetric dilaton effective action with a complex $Q$ consistently generates the real coefficient here.

We now discuss the component form of the super-Weyl anomaly. The real component of the chiral superfield $\sigma$ is the usual Weyl variation, while the pure imaginary component of $\sigma$ is the gauge parameter for the R symmetry. Expressing the super-Weyl anomaly in the conventional form

[6]In order to compare it with the component expression of the Weyl anomaly, we should note that the superconformal R-charge $q$ of a conformal scalar is $2/3$ and that of a Weyl fermion is $-1/3$. The result is given by $\frac{(23)+(25)}{2} + (14)$ (with appropriate $q$). Thus, identifying the supersymmetric Seeley-DeWitt coefficient with the supersymmetric trace anomaly here implies identifying the left-right average of the Seeley-DeWitt coefficient with the trace anomaly for a Weyl fermion, which does not generate the Pontryagin density.

[7]After continuing to the Lorentzian signature, this would give an imaginary coefficient in front of the Pontryagin density in the would-be Weyl anomaly, which is consistent with what we saw in Sec. II. This is to be contrasted with the manifestly real coefficient of the Pontryagin density in the Weyl anomaly discussed in Sec. III A.

$$T = \frac{c}{8\pi^2} W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - \frac{a}{8\pi^2} \left( W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} - \frac{1}{4} (\bar{\mathcal{D}}^2 - 4R)(G^m G_m + 2R\bar{R}) \right) \quad (59)$$

together with the antichiral part $\bar{T}$, we have the (classical) identification

$$c = 32\pi^2 \bar{Q} Q - 16\pi^2 QC, \quad a = 32\pi^2 \bar{Q} Q. \quad (60)$$

When $QC$ is not a real number, we have the classical Pontryagin density in the Weyl anomaly from the imaginary part of $QC$ as the variation of the supersymmetric dilaton effective action. We note that $a$ remains real even for a complex $Q$ as is expected from the Wess-Zumino consistency condition.

## IV. CONCLUSION

In this paper, we have discussed the supersymmetric completion of the $CP$-violating Pontryagin density in the Weyl anomaly. For this purpose, it was crucial to complexify the central charge $c$, where the existence of the imaginary part leads to the Pontryagin density in the Weyl anomaly as well as other $CP$-violating terms in the supersymmetric Weyl anomaly.

The consistency of the complexified $c$ can be seen from the perspective of the string theory as well. In the Calabi-Yau compactification of the type-II string theory, the effective coupling constant for the gravitational $F$ term (i.e., the Weyl tensor squared and the Pontryagin density [47]) can be obtained from

$$S = \int d^4x d^2\theta \mathcal{E} F_1(t) W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \text{H.c.}, \quad (61)$$

where $F_1(t)$ is the genus-one topological string amplitude [48,49]. Note that the gravitational coupling $F_1(t)$ is a "holomorphic" function of the moduli fields $t$. Its real part determines the coupling constant for the Weyl tensor squared, and its imaginary part determines that for the Pontryagin density.

The supersymmetric Weyl anomaly is nothing but the renormalization group beta function for $F_1(t)$, and, since $F_1(t)$ is holomorphic, it should be consistent to complexify the beta function as well. To be more precise, $F_1(t)$ has a holomorphic anomaly that reflects the nonlocality of the effective action [50]. The holomorphic anomaly equation that determines the nonholomorphicity is closely related to the supersymmetric Weyl anomaly, although there is an extra stringy contribution to it. It would be an interesting future direction to see why we obtain a "real" beta function for the complex coupling constant in most examples from the viewpoint of the string theory. Furthermore, we would like to pursue if there is any chance to obtain the imaginary part to induce the Pontryagin density in the super-Weyl anomaly in the string setup.

We should emphasize that the Pontryagin density obtained in this way is a real number in the Lorentzian signature and does not violate unitarity. In Sec. III B, we have addressed the other possibility that the super-Weyl variation of the effective action can be only holomorphic with respect to the super-Weyl parameter $\sigma$. This is feasible in the Euclidean signature, and it would lead to the Pontryagin density with an imaginary coefficient (if we naively analytically continue to the Lorentzian signature at the sacrifice of unitarity) under the holomorphic (i.e., chiral) super-Weyl variation.

We have the analogous situation in stringy-inspired theories of "nonanticommutative" field theories, or $\mathcal{N} = 1/2$ supersymmetric field theories [51,52]. They may arise from the self-dual graviphoton condensate in the Euclidean string theory. There, the available classical symmetry is the holomorphic super-Weyl variation of $\mathcal{E} \to e^{3\sigma}\mathcal{E}$ without the antiholomorphic partner. The absence of the antiholomorphic variation (i.e., $\bar{\sigma}$) is because the theory has no supersymmetry for the $\bar{\theta}$ translation, giving the name of $\mathcal{N} = 1/2$ supersymmetry. Then the natural "supersymmetric Weyl anomaly" for the $\mathcal{N} = 1/2$ supersymmetric field theories would be similar to the supersymmetric (chiral) Seeley-DeWitt coefficient studied in Sec. III B, and it could lead to the Pontryagin density with an imaginary coefficient under the holomorphic (or chiral) super-Weyl variation. The background might be closely related to the axial gravity studied in Ref. [9], and it may be worthwhile studying its connection.

## ACKNOWLEDGMENTS

One of the authors (Y. N.) thanks Keren-Zur Boaz, Sergei Kuzenko, and Yaron Oz for the correspondence. This work is in part supported by Japan Society for the Promotion of Science KAKENHI Grant No. 17K14301.

[1] Y. Nakayama, Nucl. Phys. **B859**, 288 (2012).
[2] J. Erdmenger and H. Osborn, Nucl. Phys. **B483**, 431 (1997).
[3] Y. Nakayama, Phys. Rev. D **98**, 085002 (2018).
[4] L. Bonora, S. Giaccari, and B. Lima de Souza, J. High Energy Phys. 07 (2014) 117.
[5] L. Bonora, A. D. Pereira, and B. Lima de Souza, J. High Energy Phys. 06 (2015) 024.
[6] L. Bonora and B. Lima de Souza, Bled Workshops Phys. **16**, 22 (2015).
[7] L. Bonora, M. Cvitan, P. Dominis Prester, B. Lima de Souza, and I. Smolić, J. High Energy Phys. 05 (2016) 072.
[8] F. Bastianelli and R. Martelli, J. High Energy Phys. 11 (2016) 178.
[9] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišić, and T. Štemberga, Eur. Phys. J. C **78**, 652 (2018).
[10] F. Bastianelli and M. Broccoli, Eur. Phys. J. C **79**, 292 (2019).
[11] M. B. Fröb and J. Zahn, J. High Energy Phys. 10 (2019) 223.
[12] F. Bastianelli and M. Broccoli, J. High Energy Phys. 10 (2019) 241.
[13] F. Bastianelli and M. Broccoli, Eur. Phys. J. C **80**, 276 (2020).
[14] L. Bonora and R. Soldati, arXiv:1909.11991.
[15] D. M. Capper, M. J. Duff, and L. Halpern, Phys. Rev. D **10**, 461 (1974).
[16] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity: Or a Walk Through Superspace* (IOP, Bristol, 1998), p. 656.
[17] D. Butter and S. M. Kuzenko, J. High Energy Phys. 09 (2013) 067.
[18] L. Bonora, P. Pasti, and M. Bregola, Classical Quantum Gravity **3**, 635 (1986).
[19] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **110B**, 117 (1982).
[20] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B203**, 157 (1982).
[21] R. J. Riegert, Phys. Lett. **134B**, 56 (1984).
[22] S. Paneitz, SIGMA **4**, 036 (2008).
[23] H. Osborn and A. C. Petkou, Ann. Phys. (N.Y.) **231**, 311 (1994).
[24] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, J. High Energy Phys. 11 (**2011**) 071.
[25] A. Zhiboedov, arXiv:1206.6370.
[26] B. S. DeWitt, Conf. Proc. C **630701**, 585 (1964); [Les Houches Lect. Notes **13**, 585 (1964)].
[27] P. B. Gilkey, Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem (unpublished).
[28] D. V. Vassilevich, Phys. Rept. **388**, 279 (2003).
[29] S. M. Christensen and M. J. Duff, Nucl. Phys. **B154**, 301 (1979).
[30] M. J. Duff and P. van Nieuwenhuizen, Phys. Lett. **94B**, 179 (1980).
[31] T. Branson, Math. Scand. **57**, 293 (1985).
[32] D. Anselmi, D. Z. Freedman, M. T. Grisaru, and A. A. Johansen, Nucl. Phys. **B526**, 543 (1998).
[33] D. Cassani and D. Martelli, J. High Energy Phys. 10 (2013) 025.
[34] S. Deser and A. Schwimmer, Phys. Lett. B **309**, 279 (1993).
[35] C. Coriano, M. M. Maglio, and E. Mottola, Nucl. Phys. **B942**, 303 (2019).
[36] Z. Komargodski and A. Schwimmer, J. High Energy Phys. 12 (2011) 099.
[37] T. Levy and Y. Oz, J. High Energy Phys. 06 (2018) 119.
[38] Z. Komargodski and N. Seiberg, J. High Energy Phys. 07 (2010) 017.
[39] L. Bonora, P. Pasti, and M. Tonin, Nucl. Phys. **B252**, 458 (1985).
[40] I. L. Buchbinder and S. M. Kuzenko, Nucl. Phys. **B274**, 653 (1986).
[41] L. Bonora and S. Giaccari, J. High Energy Phys. 08 (2013) 116.
[42] R. Auzzi and B. Keren-Zur, J. High Energy Phys. 05 (2015) 150.
[43] I. N. McArthur, Phys. Lett. **1B**, 194 (1983).
[44] I. N. McArthur, Class. Quant. Grav. **1**, 245 (1984).
[45] T. Levy, Y. Oz, and A. Raviv-Moshe, J. High Energy Phys. 12 (2018) 122.
[46] N. Bobev, H. Elvang, and T. M. Olson, J. High Energy Phys. 04 (2014) 157.
[47] S. Deser, M. J. Duff, and C. J. Isham, Phys. Lett. **93B**, 419 (1980).
[48] I. Antoniadis, E. Gava, and K. S. Narain, Phys. Lett. B **283**, 209 (1992).
[49] I. Antoniadis, E. Gava, and K. S. Narain, Nucl. Phys. **B383**, 93 (1992).
[50] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, Commun. Math. Phys. **165**, 311 (1994).
[51] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **7**, 405 (2003).
[52] N. Seiberg, J. High Energy Phys. 06 (2003) 010.