Precise critical exponents of the $O(N)$-symmetric quantum field model using hypergeometric-Meijer resummation

Abouzeid M. Shalaby[*]

Department of Mathematics, Statistics, and Physics, Qatar University, Al Tarfa, Doha 2713, Qatar

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In this work, we show that one can select different types of hypergeometric approximants for the resummation of divergent series with different large-order growth factors. Being of $n!$ growth factor, the divergent series for the ε expansion of the critical exponents of the $O(N)$-symmetric model is approximated by the hypergeometric functions ${}_{k+1}F_{k-1}$. The divergent ${}_{k+1}F_{k-1}$ functions are then resummed using their equivalent Meijer-G function representation. The convergence of the resummation results for the exponents ν, η, and ω has been shown to improve systematically in going from low order to the highest known six-loop order. Our six-loop resummation results are very competitive to the recent six-loop Borel with conformal mapping predictions and to recent Monte Carlo simulation results. To show that precise results extend for high N values, we listed the five-loop results for ν which are very accurate as well. The recent seven-loop order (g series) for the renormalization group functions β, γ_{ϕ^2}, and γ_{m^2} has been resummed too. Accurate predictions for the critical coupling and the exponents ν, η, and ω have been extracted from β, γ_{ϕ^2}, and γ_{m^2} approximants.

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I. INTRODUCTION

Quantum field theory (QFT) represents an important tool to study critical phenomena for different physical systems. A critical phenomenon is thus offering an indirect experimental test to the validity of QFT. The idea stems from the universal phenomenon where a number of different systems can show up the same critical behavior in spite of their different microscopic details. A very clear example is the Ising model from magnetism and the one-component ϕ^4 model from QFT [1–5]. The more general example of the ϕ^4 scalar field theory with $O(N)$ symmetry can describe the critical phenomena in many physical systems that share the same respective symmetry. Regarding the $N = 0$, for example, the theory lies in the same universality class with polymers [6] while $N = 1$ case describes the critical behavior of Ising-like models. For $N = 2$, the model describes a preferred orientation of a magnet in a plane while the case $N = 3$ can describe a rotationally invariant ferromagnet. Besides, the $N = 4$ case can mimic the phase transition in QCD at finite temperature with two light flavors [7].

The study of critical phenomena within quantum field theory has been reinforced by Wilson's introduction of the famous ε expansion [8,9]. Wilson ideas made the renormalization group functions to take a place in the heart of predicting critical exponents from the study of QFT models [1,3,4]. However, the series generated by the ε expansion is well known to be divergent [10], and thus resummation techniques are indispensable to extract reliable results from that series. In Ref. [11] (for instance), Borel transformation with conformal mapping technique has been used to resum divergent series of the critical exponents of the $O(N)$-symmetric model. Also, in Ref. [12], the five-loop ε expansion of the perturbation series for the critical exponents has been resummed using a strong-coupling resummation technique.

Resummation of the series generated by ε expansion has been shown to be slightly less precise than the resummation of renormalization group functions at fixed dimensions [11]. This fact motivated the authors of the recent work in Ref. [13] to move one step forward toward the improvement of resummation predictions of the critical exponents from ε expansion. In that reference, the six-loop perturbation series of the ε expansion for the renormalization group functions of the O(N) model have been obtained and resummed using Borel with conformal mapping resummation algorithm. They obtained accurate results for the

[*]amshalab@qu.edu.qa

exponents ν, η, and ω. However, this algorithm has three free parameters where their variations add to the uncertainty in the calculations. We will show in this work that a simple hypergeometric-Meijer resummation algorithm [14], which has no free parameters, can result in competitive approximations for the critical exponents from the ε expansion.

Methods that are using different approach (other than resummation) have been used in literature to extract accurate critical exponents of the $O(N)$ model. Among these successful methods is Monte Carlo simulation which has been used to obtain accurate critical exponents of the $O(N)$ model [15–22]. Besides, in recent years, researchers were able to extend the applicability of conformal bootstrap methods to three dimensions which in turn resulted in very accurate predictions for the critical exponents of the $O(N)$ model too [23–27]. The results of these techniques besides the recent Borel resummation results will be used for comparison with our predictions from hypergeometric-Meijer resummation of divergent series representing the critical exponents.

The divergence of perturbation series in QFT has been argued for the first time by Dyson [28]. From a mathematical point of view, singularities in the complex plane are responsible for series divergence even for a small argument [29]. The manifestation of divergence in a perturbation series appears in the form of large-order growth factors like $n!$, $(2n)!$ and $(3n)!$ (for instance). The appearance of such large-order behaviors stimulates the need for resummation of such type of perturbation series [30,31]. The most popular resummation technique is Borel and its different versions. In fact, the knowledge of the large-order behavior of a divergent series is needed not only to accelerate the convergence of resummation results but also to determine the type of the Borel transformation to be used. In our work, we will show that the large-order behavior is also important for our resummation (hypergeometric-Meijer) algorithm [14] in order to select the suitable relation between the number of numerator and denominator parameters of the used hypergeometric approximant.

Borel resummation and the hypergeometric-Meijer algorithms share the need of the large-order behavior of a divergent series to select the suitable Borel transform and the hypergeometric approximant, respectively. There exist, however, different features for both algorithms. One can get sufficient idea about the features of Borel resummation algorithm by going to its extensive use in literature. For the resummation of divergent series in QFT, one can visit some of past and recent successful studies that dealt with resummation of the divergent series of the renormalization group functions of the $O(N)$-symmetric model [1,4,11,13,32–36]. Although resummation techniques used in literature like Borel and Borel-Padé can give reasonable results for the critical exponents of the $O(N)$ model, these algorithms need a relatively high order of loop calculations which is not an easy task. To get an idea about how hard to have high orders of loops calculations, we assert that it took the researchers like 25 years to move forward from five-loop to six-loop calculations [13,37]. Even at the level of more simpler theories like the $\mathcal{PT}$-symmetric $i\phi^3$ field theory, the four-loop renormalization group functions have been just recently obtained [33]. In going to more complicated theories that have fermionic as well as gauge boson sectors, the calculation of a relatively high loop orders is not an easy task. The hypergeometric-Meijer algorithm, on the other hand, can give reasonable results even in using few orders from a perturbation series as input. It is thus very suitable for the study of nonperturbative features of a quantum field theory.

In Borel algorithms, results are always achieved via numerical calculations. This feature leads to the resummation of individual physical amplitudes one by one. The existence of a resummation algorithm that avoids this feature might help in getting other amplitudes without further resummation steps. Instead, we can obtain them from simple calculus. For instance, the vacuum energy or equivalently the effective potential is known to be the generating functional of the one-particle-irreducible amplitudes. Accordingly, getting a closed form resummation function for the effective potential enables one to get other amplitudes via functional differentiation [38,39]. The hypergeometric-Meijer resummation as we will see can give accurate results as well as being simple and of closed form. Besides, it does not have any free parameters to fix like other resummation algorithms which use optimization tools to fix the introduced free parameters.

The hypergeometric-Meijer resummation algorithm we use in this work is a development of the recently introduced simple hypergeometric resummation algorithm [40]. In the hypergeometric algorithm, the hypergeometric approximant ${}_2F_1(a,b;c;\sigma z)$ has been suggested for the resummation of a divergent series. The four parameters a, b, c, and σ are obtained by comparing the first four orders of the expansion of ${}_2F_1(a,b;c;\sigma z)$ in the variable z with the four available orders of the divergent series under consideration. To illustrate this more, consider a series representing a physical quantity $Q(z)$ as

$$Q(z) = \sum_{0}^{4} c_i z^i + O(z^5). \tag{1}$$

We have also the series expansion of $c_0\,{}_2F_1(a,b;c;\sigma z)$ as

$$c_{0\,2}F_1(a,b;c;\sigma z) = c_0 + c_0\frac{ab\sigma}{c}z + c_0\frac{a(a+1)b(b+1)\sigma^2}{2c(c+1)}z^2 + c_0\frac{a(a+1)(a+2)b(b+1)(b+2)\sigma^3}{6c(c+1)(c+2)}z^3 + c_0\frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)\sigma^4}{24c(c+1)(c+2)(c+3)}z^4 + \cdots\cdots\cdots\cdots \tag{2}$$

For $c_{0\,2}F_1(a,b;c;\sigma z)$ to serve as an approximant for $Q(x)$, we have to set

$$c_1 = c_0\frac{ab\sigma}{c} \qquad c_2 = c_0\frac{a(a+1)b(b+1)c\sigma^2}{2c(c+1)} \qquad c_3 = c_0\frac{a(a+1)(a+2)b(b+1)(b+2)\sigma^3}{6c(c+1)(c+2)}$$
$$c_4 = c_0\frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)\sigma^4}{24c(c+1)(c+2)(c+3)}, \tag{3}$$

which can be solved to determine the unknown parameters a, b, c, σ in terms of the known coefficients c_1, c_2, c_3, and c_4.

To accelerate the convergence of the algorithm, we suggested the employment of parameters from the asymptotic behavior of the perturbation series at large values of the argument z [41] or equivalently the strong-coupling data. Our suggestion is based on the realization that when $a-b$ is not an integer, the hypergeometric function has the following asymptotic form [42]:

$$_2F_1(a,b;c;g) \sim \lambda_1 g^{-a} + \lambda_2 g^{-b}, \qquad |g| \gg 1.$$

Also, the method has been generalized to accommodate higher orders from the perturbation series by using the generalized hypergeometric function $_pF_{p-1}(a_1,\ldots a_p; b_1\ldots.b_{p-1};\sigma z)$, where a_i parameters are extracted from the asymptotic behavior of the perturbation series at large z value.

The hypergeometric algorithm either the version in Ref. [40] or [41] cannot accommodate the large-order data available for many perturbation series in physics. The point is that the series expansion of the hypergeometric function $_2F_1(a,b;c;\sigma z)$ has a finite radius of convergence while it has been used for the resummation of a divergent series with zero radius of convergence. This means that the large-order behavior of the expansion of the function $_2F_1(a,b;c;\sigma z)$ cannot account explicitly for the $n!$ growth factor characterizing a perturbation series with zero radius of convergence. In fact, in the hypergeometric algorithm, the parameter σ ought to take large values to compensate for that [43,44] but itself cannot be considered as a large-order parameter. Indeed, employing parameters from large-order behavior is well known to accelerate the convergence of resummation algorithms (Borel, for instance). Moreover, one cannot apply the suitable Borel transform (divide by $n!$ for instance) unless we know the large-order behavior of the perturbation series. These facts led us to develop the hypergeometric algorithm [14] by using the approximants $_pF_{p-2}(a_1,a_2,\ldots.,a_p;b_1,b_2,\ldots.b_{p-2};\sigma z)$ instead of $_2F_1(a, b;c;\sigma z)$. The hypergeometric functions $_pF_{p-2}(a_1,a_2,\ldots., a_p;b_1,b_2,\ldots.b_{p-2};\sigma z)$ are all sharing the same analytic properties (with respect to z) and all have expansions of zero radius of convergence as well as having an $n!$ growth factor. Possessing the main features of the divergent series under consideration, the hypergeometric function $_pF_{p-2}(a_1,a_2,\ldots.,a_p;b_1,b_2,\ldots.b_{p-2};\sigma z)$ is thus an ideal candidate for the resummation of that series.

The structure of the paper is as follows. In Sec. II, we introduce the generalized hypergeometric-Meijer algorithm for the resummation of a divergent series with a growth factor of the form $((p-q-1)n)!$. In Sec. III, we use the algorithm to resum the ε expansions of the exponents $\nu(\nu^{-1}),\eta$, and ω and the critical coupling up to five loops of the $O(N)$-symmetric model. The resummation results for the recent six-loop order is presented for the exponents $\nu(\nu^{-1}),\eta$, and ω in Sec. IV. Resummation of the seven loops of the g expansion of the renormalization group functions, which have no resummation trials in literature so far, is presented in Sec. V. Summary and conclusions will follow in Sec. VI.

II. THE GENERALIZED HYPERGEOMETRIC-MEIJER RESUMMATION ALGORITHM

Consider a divergent series that represents a physical amplitude $Q(z)$ as

$$Q(z) = \sum_{n=0}^{M} c_n z^n + O(z^{M+1}), \tag{4}$$

where the first $M+1$ orders are known. Assume that the large-order behavior of that series takes the from

$$c_n \sim \alpha n!(-\sigma)^n n^b\left(1+O\left(\frac{1}{n}\right)\right), \qquad n\to\infty. \tag{5}$$

In Ref. [14], we showed that when $p=q+2$, the perturbative expansion of the hypergeometric function

${}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;-\sigma z)$ which has a zero radius of convergence can be parametrized to give the same large-order behavior of the above perturbation series. Accordingly, one sets the constraint $\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b$, besides the constraints set by matching the perturbation expansion of ${}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;-\sigma z)$ with the available orders of the divergent series. Then the parametrized divergent series of ${}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;\sigma z)$ is resummed using its representation in terms of Meijer-G function as follows [42]:

$$ {}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;z) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{p}\Gamma(a_k)} G_{p,q+1}^{1,p}\left(\begin{matrix}1-a_1,\ldots,1-a_p\\0,1-b_1,\ldots,1-b_q\end{matrix}\middle| z\right). \tag{6}$$

Note that the authors in Ref. [44] used a Borel-Padé algorithm that leads to Meijer-G approximants parametrized by weak-coupling information.

One can generalize the idea of our previous work in Ref. [14] to other types of divergent series with growth factors other than $n!$. For instance, the divergent series of the ground state energy of the sextic anharmonic oscillator has a zero radius of convergence, but the growth factor is $(2n)!$ while it is $(3n)!$ for the octic anharmonic oscillator [45]. Knowing that the asymptotic form of the ratio of two Γ functions is given by [46]

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} = n^{\alpha-\beta}\left(1+\frac{(\alpha-\beta)(-1+\alpha+\beta)}{n}+O\left(\frac{1}{n^2}\right)\right), \tag{7}$$

one can easily conclude that either the hypergeometric approximants ${}_pF_{p-1}(a_1,\ldots a_p;b_1\ldots.b_{p-1};\sigma z)$ used in Ref. [41] or ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots.b_{p-2};\sigma z)$ used in Ref. [14] cannot account for the growth factors of the sextic or octic ground state energies. Accordingly, one can accept that there exists more than one type of hypergeometric functions (different $S=p-q$) that are needed to approximate different divergent series in physics with different large-order growth factors.

Based on the idea that the large-order asymptotic behavior is responsible for the selection of the suitable hypergeometric approximant for a perturbation series, one can list different ${}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;-\sigma z)$ approximants for different growth factors as follows:

(1) For divergent series that has the large-order behavior in Eq. (5) ($n!$ growth factor), the suitable resummation function is ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots.b_{p-2};\sigma z)$.

(2) For a series that has a large-order behavior like $\gamma\Gamma(2n+\frac{1}{2})(-\sigma)^n n^b$, $n\to\infty$, the suitable one is ${}_pF_{p-3}(a_1,\ldots a_p;b_1\ldots.b_{p-3};-\sigma z)$. This is because one can easily show that for $p=q+3$, one can get a similar large-order behavior. An example of such divergent series is the ground state energy of the sextic anharmonic oscillator [45].

(3) For the ground state energy of the octic anharmonic oscillator, the large-order behavior is given by $\sim\delta\Gamma(3n+\frac{1}{2})(-\sigma)^n n^b$, $n\to\infty$, which can be reproduced by the generalized hypergeometric function ${}_pF_{p-4}(a_1,\ldots a_p;b_1\ldots.b_{p-4};-\sigma z)$.

(4) For a divergent series that has a finite radius of convergence, the suitable resummation function is ${}_pF_{p-1}(a_1,\ldots a_p;b_1\ldots.b_{p-1};\sigma z)$. An example of such series is the ground state energy of the Yang-Lee model (Eq. (86) in Ref. [2]).

Based on this classification, knowing the large-order behavior of a divergent series is essential not only to accelerate the convergence of the resummation algorithm but also to determine the suitable hypergeometric approximant. A note to be mentioned is that, for $p\geq q+2$, the hypergeometric function ${}_pF_q(a_1,\ldots a_p;b_1\ldots.b_q;\sigma z)$ has a zero radius of convergence, but it can be resumed using the closely related Meijer-G function [see Eq. (6)] which has the integral form [42]

$$G_{p,q}^{m,n}\left(\begin{matrix}c_1,\ldots,c_p\\d_1,\ldots,d_q\end{matrix}\middle| z\right) = \frac{1}{2\pi i}\int_C \frac{\prod_{k=1}^{n}\Gamma(s-c_k+1)\prod_{k=1}^{m}\Gamma(d_k-s)}{\prod_{k=n+1}^{p}\Gamma(-s+c_k)\prod_{k=m+1}^{q}\Gamma(s-d_k+1)} z^s ds. \tag{8}$$

The hypergeometric-Meijer algorithm which will be used in this work to resum the divergent series representing the critical exponents of the $O(N)$ vector model can be thus summarized in two simple steps [14]:

(1) Parametrize the hypergeometric function ${}_pF_{p-2}(a_1,\ldots a_p;b_1\ldots.b_{p-2};\sigma z)$ using both weak-coupling and large-order data of the series under consideration (for ε expansion, the strong-coupling

data represented by the numerator parameters a_i are not known yet).

(2) Resum the divergent ${}_pF_{p-2}(a_1,\ldots a_p; b_1\ldots . b_{p-2}; \sigma z)$ function using the representation in terms of the Meijer-G function in Eq. (6).

There exist some technical issues when applying the algorithm. The first issue is that for high orders, computer can take a relatively long time to solve the set of equations like the one in Eq. (3). To overcome this problem, we generated the ratio $R_n = \frac{c_n}{c_n-1}$ and then solve the following set of equations:

$$R_n = \frac{1}{n}\frac{\prod_{i=1}^{p}(a_i+n-1)}{\prod_{j=1}^{q}(b_j+n-1)}\sigma. \tag{9}$$

For example, the approximant ${}_pF_q(a_1,\ldots a_p; b_1\ldots . b_q; \sigma z)$ generates the following set of equations:

$$\begin{aligned} R_1 &= \frac{a_1 a_2\ldots\ldots\ldots\ldots a_p}{b_1 b_2\ldots\ldots\ldots\ldots b_q}\sigma \\ R_2 &= \frac{(a_1+1)(a_2+1)\ldots\ldots\ldots\ldots(a_p+1)}{2(b_1+1)\ldots\ldots\ldots\ldots(b_q+1)}\sigma \\ &. \\ &. \\ &. \\ R_{p+q} &= \frac{(a_1+p+q-1)\ldots\ldots\ldots\ldots(a_p+p+q-1)}{(p+q)(b_1+p+q-1)\ldots\ldots\ldots\ldots(b_q+p+q-1)}\sigma. \end{aligned} \tag{10}$$

This trick decreases the degree of nonlinearity in the set of equations and thus saves the computational time.

The other issue regarding the application of the hypergeometric-Meijer algorithm is that at some orders one might find no solution for the set of equations defining the parameters in the hypergeometric function. In this case, one resorts to successive subtractions of the perturbation series. This trick is well known in resummation algorithms [4,44]. However, the subtracted series will have a different large-order b parameter where it increases by one per each subtraction (see, for instance, Sec. 16.6 in Ref. [4]).

III. HYPERGEOMETRIC-MEIJER RESUMMATION FOR THE ε EXPANSION OF CRITICAL EXPONENTS AND COUPLING UP TO FIVE LOOPS

The Lagrangian density of the $O(N)$-vector model is given by

$$\mathcal{L} = \frac{1}{2}(\partial\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4, \tag{11}$$

where $\Phi = (\phi_1, \phi_2, \phi_3, \ldots\ldots\ldots . \phi_N)$ is an N-component field with $O(N)$ symmetry such that $\Phi^4 = (\phi_1^2 + \phi_2^2 + \phi_3^2 + \ldots\ldots\ldots . \phi_N^2)^2$. At the fixed point, the β function is zero which sets a critical coupling as a function of $\varepsilon = 4 - d$. Accordingly, one can obtain the renormalization group functions as power series in ε. In the following parts of this section, we list the resummation results (up to five loops) for the exponents ν, η, and ω as well as the critical coupling of that model.

A. Two-, three-, four-, and five-loop resummation for the exponent ν

Up to five loops, the power series for the reciprocal of the critical exponent ν is given by [4]

$$\nu^{-1} \approx 2 + \sum_{i=1}^{5} c_i \varepsilon^i, \tag{12}$$

where

$$c_1 = \frac{N+2}{N+8} \qquad c_2 = -\frac{(N+2)(13N+44)}{2(N+8)^3}$$
$$c_3 = \frac{(N+2)}{8(N+8)^5}\{3N^3 - 452N^2 + 96(N+8)(5N+22)\zeta(3) - 2672N - 5312\}$$

$$c_4 = \frac{(N+2)}{32(N+8)^7}\{3N^5 + 398N^4 - 12900N^3 - 1280(N+8)^2(2N^2+55N+186)\zeta(5)$$
$$+ 16(N+8)(3N^4 - 194N^3 + 148N^2 + 9472N + 19488)\zeta(3)$$
$$- 81552N^2 - 219968N + \frac{16}{5}\pi^4(N+8)^3(5N+22) - 357120\}$$
$$c_5 = \frac{(N+2)}{128(N+8)^9}\{3N^7 - 1198N^6 - 27484N^5 - 1055344N^4 - 5242112N^3$$
$$- 5256704N^2 + 56448(N+8)^3(14N^2+189N+526)\zeta(7)$$
$$+ 6999040N - 626688 - \frac{1280}{189}\pi^6(N+8)^4(2N^2+55N+186)$$
$$+ 256(N+8)^2\zeta(5)(155N^4 + 3026N^3 + 989N^2 - 66018N - 130608)$$
$$- 1024(N+8)^2(2N^4 + 18N^3 + 981N^2 + 6994N + 11688)\zeta(3)^2$$
$$+ \frac{8}{15}\pi^4(N+8)^3(3N^4 - 194N^3 + 148N^2 + 9472N + 19488)$$
$$- 16(N+8)\zeta(3)[13N^6 - 310N^5 + 19004N^4 + 102400N^3 - 381536N^2 - 2792576N - 4240640]\}. \quad (13)$$

The large-order parameters take the form in Eq. (5) where [4]

$$\sigma = \frac{3}{N+8} \quad \text{and} \quad b = 4 + \frac{N}{2}.$$

The suitable hypergeometric approximant is thus ${}_pF_{p-2}(a_1,\ldots a_p; b_1\ldots. b_{p-2}; -\sigma z)$ where it can reproduce the large-order behavior in Eq. (5). The number of unknown parameters in ${}_pF_{p-2}(a_1,\ldots a_p; b_1\ldots. b_{p-2}; -\sigma z)$ is $2p-2$, and thus we need an even number of equations to determine the unknown parameters. So, we have the following two options:

(1) Even number of loops as input: In this case, we incorporate an even number $(2p-2)$ of terms from the perturbation series to match with corresponding terms from the expansion of ${}_pF_{p-2}(a_1,\ldots a_p; b_1\ldots. b_{p-2}; -\sigma z)$.

(2) Odd number of loops as input: In this case, we take an odd number $(2p-1)$ of loops to build the odd number of equations and one equation from the large-order constraint,

$$\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b,$$

to determine the unknown numerator and denominator parameters.

So, we list resummation results that involve odd or even number of perturbative terms separately.

1. Two-loop resummation for ν

For $p = q+2$, the lowest order hypergeometric approximant for ν^{-1} is thus

$$2\,{}_2F_0\left(a_1, a_2;; -\frac{3}{N+8}\varepsilon\right) = \frac{2}{\Gamma(a_1)\Gamma(a_2)} G^{1,2}_{2,1}\left(\begin{matrix} 1-a_1, 1-a_2 \\ 0 \end{matrix}\middle| -\frac{3}{N+8}\varepsilon\right). \quad (14)$$

For this resummation function, one needs to determine the two parameters a_1 and a_2 by matching the perturbative expansion of $2\,{}_2F_0(a_1, a_2;; -\frac{3}{N+8}\varepsilon)$ with the first two terms in the perturbation series in Eq. (12). In this case, we get

$$-\frac{6a_1a_2}{N+8} = -\frac{N+2}{N+8} \qquad \frac{9a_1(a_1+1)a_2(a_2+1)}{(N+8)^2} = -\frac{(N+2)(13N+44)}{2(N+8)^3}, \quad (15)$$

from which we obtain the results,

$$a_1 = \frac{-N^2 - \sqrt{N^4 + 60N^3 + 1636N^2 + 10464N + 20032} - 42N - 152}{12(N+8)}, \quad (16)$$

$$a_2 = \frac{1}{12(N+8)}\left(\begin{array}{c} \frac{N^3}{N+8} + \frac{50N^2}{N+8} - 2N^2 + \frac{\sqrt{N^4+60N^3+1636N^2+10464N+20032}N}{N+8} \\ + \frac{8\sqrt{N^4+60N^3+1636N^2+10464N+20032}}{N+8} + \frac{488N}{N+8} - 84N + \frac{1216}{N+8} - 304 \end{array}\right). \tag{17}$$

To test the accuracy of this two-loop resummation function, let us note that for $N = 1$, the recent Monte Carlo calculation [15] gives $\upsilon = 0.63002(10)$. Our two-loop hypergeometric-Meijer resummation gives the result $\upsilon = 0.66209$. This result is very reasonable in taking into account that the algorithm is fed with only the first two orders from the perturbation series as input. For $N = 0$, a recent accurate prediction is listed in Ref. [18] as $\nu = 0.5875970(4)$ while our two-loop resummation gives $\nu = 0.60890$. For $N = 2$, Monte Carlo calculations give $\upsilon = 0.6690$ [15] while the two loops give $\nu = 0.711526$. So, it seems that the simple hypergeometric-Meijer resummation algorithm we follow in this work gives reasonable results even with very low orders of perturbation series as input. It is expected that the resummation of higher orders will improve the accuracy of the results which we will do in the following subsections.

2. Three-loop resummation for ν

For more accurate results, one can go to the higher three-loop order of hypergeometric-Meijer approximants ${}_3F_1(a_1, a_2, a_3; b_1; -\frac{3}{N+8}\varepsilon)$. Although it is parametrized by four parameters (a_1, a_2, a_3, and b_1), the use of the large-order constraint [14]

$$\sum_{i=1}^{p} a_i - \sum_{i=1}^{p-2} b_i - 2 = b$$

leads to the need of three terms only from perturbation series to determine the parameters. So, to determine them (a_1, a_2, a_3 and b_1), we solve the following set of equations:

$$\begin{aligned} c_1 &= \frac{2a_1a_2a_3}{b_1}\sigma \qquad c_2 = \frac{a_1(a_1+1)a_2(a_2+1)a_3(a_3+1)}{b_1(b_1+1)}\sigma^2 \\ c_3 &= \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)a_3(a_3+1)(a_3+2)}{3b_1(b_1+1)(b_1+2)}\sigma^3 \\ b &= a_1 + a_2 + a_3 - b_1 - 2. \end{aligned} \tag{18}$$

The predictions of this order are given in Table I for different N values and compared to two-, four-, and five-loop resummation results and to the Janke-Kleinert resummation (up to five loops) in Ref. [4] and the Borel-with conformal mapping in Refs. [11,13]. One can easily realize that the convergence has been greatly improved when moved from two-loop to the three-loop resummation.

The obvious acceleration of the convergence of the algorithm from two to three loops is strongly recommending the

TABLE I. The two-, three-, four-, and five-loop (ε expansion) hypergeometric-Meijer resummation for the critical exponent ν for the $O(N)$ model compared to the ε^5 Janke-Kleinert (JK) resummation results (sixth column) from Ref. [4] and the Borel with conformal mapping (BCM) resummation (seventh column) from Ref. [11] (first row) and recent results from Ref. [13] (second row).

	This work				JK[4]	BCM[12,14]
N	${}_2F_0$: ε^2	${}_3F_1$: ε^3	${}_3F_1$: ε^4	${}_4F_2$: ε^5	ε^5	ε^5
0	0.60890	0.58609	0.58705	0.58714	0.5865(13)	0.5875 ± 0.0018 0.5873(13)
1	0.66209	0.62502	0.62699	0.62818	0.6268(22)	0.6293 ± 0.0026 0.6290(20)
2	0.71153	0.66062	0.66103	0.667225	0.6642(111)	0.6685 ± 0.0040 0.6687(13)
3	0.75615	0.69282	0.69303	0.70364	0.6987(51)	0.7050 ± 0.0055 0.7056(16)
4	0.79557	0.72175	0.72176	0.73692	…	0.737 ± 0.008 0.7389(24)

hypergeometric-Meijer resummation algorithm to take a place among the preferred algorithms to resum divergent series with large-order behavior of the form in Eq. (5). Other features that recommend it for resummation of divergent series is that it does not include any free parameters and of closed form as well.

3. Four-loop resummation for ν

The hypergeometric approximants ${}_3F_1(a_1, a_2, a_3; b_1; -\frac{3}{N+8}\varepsilon)$ can also be used to resum the perturbation series up to four loops, but in this case, we have to solve the following set of equations:

$$c_1 = \frac{2a_1a_2a_3}{b_1}\sigma \qquad c_2 = \frac{a_1(a_1+1)a_2(a_2+1)a_3(a_3+1)}{b_1(b_1+1)}\sigma^2$$
$$c_3 = \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)a_3(a_3+1)(a_3+2)}{3b_1(b_1+1)(b_1+2)}\sigma^3$$
$$c_4 = \frac{a_1(a_1+1)(a_1+2)(a_1+3)a_2(a_2+1)(a_2+2)(a_2+3)a_3(a_3+1)(a_3+2)(a_3+3)}{12b_1(b_1+1)(b_1+2)(b_1+3)}\sigma^4. \tag{19}$$

The prediction of this order of resummation is also listed in Table I where it shows that the accuracy is improving in a systematic way when moving to higher orders.

4. Five-loop resummation for ν

In this case, we use the approximants ${}_4F_2(a_1, \ldots, a_4; b_1 \ldots b_4; -\frac{3}{N+8}\varepsilon)$ where the unknown parameters are determined from the following set of equations:

$$c_1 = \frac{2a_1a_2a_3a_4\sigma}{b_1b_2} \qquad c_2 = \frac{2a_1(a_1+1)a_2(a_2+1)a_3a_4(a_3a_4+1)\sigma^2}{b_1(b_1+1)b_2(b_2+1)}$$
$$c_3 = \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)a_3a_4(a_3a_4+1)(a_3a_4+2)\sigma^3}{3b_1(b_1+1)(b_1+2)b_2(b_2+1)(b_2+2)}$$
$$c_4 = \frac{a_1(a_1+1)(a_1+2)(a_1+3)\ldots\ldots a_4(a_4+1)(a_4+2)(a_4+3)\sigma^4}{12b_1(b_1+1)(b_1+2)(b_1+3)b_2(b_2+1)(b_2+2)(b_2+3)},$$
$$c_5 = \frac{a_1(a_1+1)(a_1+2)(a_1+3)(a_1+4)\ldots\ldots a_4(a_4+1)(a_4+2)(a_4+3)(a_4+4)\sigma^5}{60b_1(b_1+1)(b_1+2)(b_1+3)(b_1+4)b_2(b_2+1)(b_2+2)(b_2+3)(b_2+4)}$$
$$b = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - 2. \tag{20}$$

For this order, we get even more precise results for the ν exponent which are also presented in Table I and compared to the five-loop resummation from other algorithms in Refs. [4,11]. Also, to compare with other recent theoretical predictions, for $N = 0$, we get the result $\nu = 0.587142$ compared to the recent accurate Monte Carlo simulation prediction from Ref. [18] as $\nu = 0.5875970(4)$. For $N = 1$, our five-loop result gives $\nu = 0.62818$ that can be compared to Monte Carlo calculation that gives $\upsilon = 0.63002(10)$ [15]. The $N = 2$ five-loop resummation in this work gives $\nu = 0.667225$ which is competitive to Monte Carlo calculations of $\upsilon = 0.6690$ in Ref. [15]. Also, for $N = 3$, our five-loop resummation gives $\nu = 0.703644$, while the recent Monte Carlo prediction

TABLE II. The five-loop hypergeometric-Meijer resummation (${}_4F_2$ approximant) of the critical exponent ν for the $O(N)$ model for $N = 6$, 8, 10, and 12 compared to other theoretical predictions. Reference [47] used the strong-coupling resummation, and Ref. [23] is a conformal bootstrap calculation where we used $\Delta_s = 2 - 3/\nu$ to get the listed results. In Ref. [48], numerical calculations are used to predict the critical exponents and in Ref. [49] the optimally truncated direct summation of pseudo-ϵ expansion (τ, OTDS) has been used where we obtained the listed result via the relation $\alpha = 2 - D\nu$.

N	6	8	10	12
This work ${}_4F_2$: ε^5	0.79331	0.83692	0.88809	0.89472
Other calculations	0.790[50]	0.829[50]	0.866[50]	0.890[50]
	$0.78431^{+0.032}_{-0.033}$ [24]	0.818[48]	$0.88417^{+0.000}_{-0.0008}$ [24]	0.93279[49]

gives $\nu = 0.7116(10)$ [16]. These results show clearly that our five-loop resummation results are competitive either to five-loop resummation from other algorithms or to recent numerical methods.

To get an impression about the stability of the algorithm predictions for higher N values, we list in Table II our five-loop resummation (${}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -\sigma z)$) results for $N = 6, 8, 10, 12$ and compare them to other theoretical predictions.

B. Resummation of four- and five-loop series for η exponent

For the critical exponent η of the $O(N)$ model, the ε expansion up to five loops is given by [4]

$$\eta = \varepsilon^2(d_2 + d_3\varepsilon + d_4\varepsilon^2 + d_5\varepsilon^3) + O(\varepsilon^6), \qquad (21)$$

where

$$d_2 = \frac{(N+2)}{2(N+8)^2} \qquad d_3 = \frac{(N+2)(-N^2+56N+272)}{8(N+8)^4}$$
$$d_4 = \frac{(N+2)}{32(N+8)^6}\{-5N^4 - 230N^3 + 1124N^2 - 384(N+8)(5N+22)\zeta(3) + 17920N + 46144\}$$
$$d_5 = -\frac{(N+2)}{128(N+8)^8}\{13N^6 + 946N^5 + 27620N^4 + 121472N^3 - 262528N^2 - 2912768N - 5120(N+8)^2(2N^2+55N+186)\zeta(5)\frac{64}{5}\pi^4(N+8)^3(5N+22) - 5655552 - 16(N+8)(N^5+10N^4+1220N^3-1136N^2-68672N-171264)\zeta(3) - 5655552\}, \qquad (22)$$

and the large order for η of this model takes the form in Eq. (5) where [4]

$$\sigma = \frac{3}{N+8} \quad \text{and} \quad b = 3 + \frac{N}{2}.$$

Note that the factored series $(d_2 + d_3\varepsilon + d_4\varepsilon^2 + d_5\varepsilon^3) + O(\varepsilon^6)$ has the large-order parameters [4],

$$\sigma = \frac{3}{N+8} \quad \text{and} \quad b = 5 + \frac{N}{2}.$$

The lowest order approximant is thus ${}_2F_0$ which in this case is a four-loop approximant.

1. Four-loop resummation for η

The hypergeometric-Meijer approximant is then

$$\eta = d_2(N)\varepsilon^2 {}_2F_0(a_1, a_2;; -\sigma\varepsilon) = \frac{d_2(N)\varepsilon^2}{\Gamma(a_1)\Gamma(a_2)} G^{1,2}_{2,1}\left(\begin{matrix} 1-a_1, 1-a_2 \\ 0 \end{matrix}\middle| -\frac{3}{N+8}\varepsilon\right). \qquad (23)$$

The resummation results of that order are shown in Table III. The results are reasonable, but since the hypergeometric approximant ${}_2F_0$ has few number of parameters, it is expected that the improvement of the results needs higher loops to be incorporated.

TABLE III. The four- and five-loop (ε expansion) hypergeometric-Meijer resummation for the critical exponent η for the $O(N)$ model. We compared the results to Janke-Kleinert resummation for five-loop ε expansion in Ref. [4] and the Borel with conformal mapping resummation from Refs. [11] (first) and [13] (second).

	This work		JK[4]	BCM[12,14]
N	${}_2F_0$: ε^4	${}_3F_1$: ε^5	ε^5	ε^5
0	0.02804	0.03111	0.0344(42)	0.0300 ± 0.0060 0.0314(11)
1	0.03286	0.03615	0.0395(43)	0.0360 ± 0.0060 0.0366(11)
2	0.03475	0.03791	0.0412(41)	0.0385 ± 0.0065 0.0384(10)
3	0.03498	0.03781	0.0366(20)	0.0380 ± 0.0060 0.0382(10)
4	0.034274	0.03668	…	0.036 ± 0.004 0.0370(9)

2. The η five-loop resummation

In this case, the hypergeometric approximant is

$$\eta = d_2(N)\varepsilon^2 {}_3F_1(a_1, a_2, a_3; b_1; -\sigma\varepsilon). \tag{24}$$

To determine the four unknown parameters, we use the following equations:

$$\begin{aligned} d_3 &= d_2 \frac{a_1 a_2 a_3}{b_1}\sigma \qquad d_4 = d_2 \frac{a_1(1+a_1)a_2(1+a_2)a_3(1+a_3)}{b_1(1+b_1)}\sigma \\ d_5 &= d_2 \frac{a_1(1+a_1)(2+a_1)a_2(1+a_2)(2+a_2)a_3(1+a_3)(2+a_3)}{b_1(1+b_1)(2+b_1)}\sigma \\ 5+\frac{N}{2} &= a_1+a_2+a_3-b_1-2. \end{aligned} \tag{25}$$

Accordingly, the hypergeometric-Meijer approximant for this order is given by

$$\eta = d_2(N)\varepsilon^2 {}_3F_1(a_1, a_2, a_3; b_1; -\sigma\varepsilon) = d_2(N)\varepsilon^2 \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} G_{3,2}^{1,3}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3 \\ 0, 1-b_1 \end{matrix} \middle| -\frac{3}{N+8}\varepsilon\right). \tag{26}$$

Our predictions that incorporate the fourth and fifth orders of divergent series of the η exponent are listed in Table III. It is very clear that the simple algorithm we follow gives accurate results for few terms from the perturbation series as input. This can be more elaborated by looking at the large number of estimates for critical exponents in Ref. [50] too. In fact, for the same order of perturbation series involved, the precision of resummation results for η is always less than that in ν or ω because the lowest order in the perturbation series of η is ε^2 and thus always approximated by hypergeometric approximants of fewer parameters than that for ν or ω.

C. Resummation of the exponent ω

For the exponent ω, we have the five-loop perturbation series as

$$\omega = \varepsilon + e_2\varepsilon^2 + e_3\varepsilon^3 + e_4\varepsilon^4 + e_5\varepsilon^5 + O(\varepsilon^6), \tag{27}$$

where [4]

$$\begin{aligned} e_2 &= -\frac{3(3N+14)}{(N+8)^2}, \qquad e_3 = \frac{(33N^3+538N^2+4288N+9568+\zeta[3](N+8)96(5N+22))}{4(N+8)^4}, \\ e_4 &= \frac{1}{16(N+8)^6}\{5N^5-1488N^4-46616N^3-1920(N+8)^2(2N^2+55N+186)\zeta(5) \\ &\quad -419528N^2-96(N+8)(63N^3+548N^2+1916N+3872)\zeta(3)-1750080N \\ &\quad +\frac{16}{5}\pi^4(N+8)^3(5N+22)-2599552\}, \\ e_5 &= \frac{1}{64(N+8)^8}\{13N^7+7196N^6+240328N^5+3760776N^4+38877056N^3 \\ &\quad +112896(N+8)^3(14N^2+189N+526)\zeta(7)+223778048N^2+660389888N+752420864 \\ &\quad -\frac{640}{63}\pi^6(N+8)^4(2N^2+55N+186)-\frac{16}{5}\pi^4(N+8)^3(63N^3+548N^2+1916N+3872) \\ &\quad +256(N+8)^2\zeta(5)(305N^4+7386N^3+45654N^2+143212N+226992) \\ &\quad -768(N+8)^2(6N^4+107N^3+1826N^2+9008N+8736)\zeta(3)^2 \\ &\quad -16(N+8)\zeta(3)[9N^6-1104N^5-11648N^4-243864N^3-2413248N^2-9603328N-14734080]\}, \end{aligned} \tag{28}$$

and the large-order parameters for that exponent are

$$\sigma = \frac{-3}{N+8} \quad \text{and} \quad b = 5 + \frac{N}{2}.$$

The two-loop resummation gives reasonable but not precise results, so in the following, we shall list the resummation of three, four, and five loops.

1. Three-loop resummation for ω

The three-loop hypergeometric approximant is

$$\omega \approx {}_3F_1(a_1, a_2, a_3; b_1; -\sigma\varepsilon) - 1, \tag{29}$$

where

$$1 = \frac{a_1 a_2 a_3 \sigma}{b_1 b_2}, \qquad e_2 = \frac{a_1(a_1+1)a_2(a_2+1)a_3(a_4+1)\sigma^2}{2b_1(b_1+1)b_2(b_2+1)},$$
$$e_3 = \frac{a_1(a_1+1)(a_1+2)a_2(a_2+1)(a_2+2)a_3(a_3+1)(a_3+2)\sigma^3}{6b_1(b_1+1)(b_1+2)}, \qquad b = a_1 + a_2 + a_3 - b_1 - b_2 - 2. \tag{30}$$

The solutions of these equations are then substituted in the following Meijer-G function:

$$\omega \approx \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} G^{1,3}_{3,2}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3 \\ 0, 1-b_1 \end{matrix} \middle| -\frac{3}{N+8}\varepsilon \right) - 1. \tag{31}$$

2. The ω four-loop resummation

In this case also, we use the approximant ${}_3F_1(a_1, a_2, a_3; b_1; -\sigma\varepsilon)$, but we replace the fourth equation in the set in Eq. (30) by

$$e_4 = \frac{a(a+1)(a+2)(a+3)b(b+1)(b+2)(b+3)c(c+1)(c+2)(c+3)}{12d(d+1)(d+2)(d+3)}\sigma^4. \tag{32}$$

3. ω five-loop approximant

The hypergeometric function that can accommodate five loops is ${}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -\sigma\varepsilon)$ where we use the constraint on the large-order parameters,

$$b = a_1 + a_2 + a_3 + a_4 - b_1 - b_2 - 2.$$

Accordingly, the fifth order resummation for ω is

TABLE IV. The three-, four-, and five-loop hypergeometric-Meijer resummation for the critical exponent ω compared to five-loop resummation from Ref. [4] (fifth column) and the Borel with conformal mapping resummation (sixth column) from Refs. [11,13].

N	${}_3F_1$ This work: ε^3	${}_3F_1$ This work: ε^4	${}_4F_2$ This work: ε^5	JK[4]: ε^5	BCM[12,14]: ε^5
0	0.86128	0.80054	0.85086	0.817(21)	0.828 ± 0.023 0.835(11)
1	0.85628	0.79559	0.83178	0.806(13)	0.814 ± 0.018 0.818(8)
2	0.85233	0.79290	0.81329	0.800(13)	0.802 ± 0.018 0.803(6)
3	0.84979	0.79258	0.79928	0.796(11)	0.794 ± 0.018 0.797(7)
4	0.910678	0.79416	0.79249	…	0.795 ± 0.030 0.795(6)

$$\omega \approx \left(\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)} G^{1,4}_{4,3}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3, 1-a_4 \\ 0, 1-b_1, 1-b_2 \end{matrix} \middle| -\frac{3}{N+8}\right)\varepsilon - 1\right). \tag{33}$$

In Table IV, we compared our results to predictions from the Janke-Kleinert Resummation for five-loop ε expansion in Ref. [4] and Borel with conformal mapping in Refs. [11,13] for $N = 0$, 1, 2, 3, and 4. Again, the comparison shows that the algorithm we follow gives very accurate results from few orders of the perturbation series as input.

D. Resummation of the ε expansion for the critical coupling

In the way to get the ε expansion for the critical exponents, one has to obtain the dependence of the critical coupling on ε first. The expansion for the critical coupling g_c up to fifth order is given by [4]

$$\begin{aligned}
&\text{For } N = 0 \Rightarrow g_c(\varepsilon) \approx 0.375\varepsilon + 0.246\varepsilon^2 - 0.180\varepsilon^3 + 0.368\varepsilon^4 - 1.258\varepsilon^5, \\
&\text{For } N = 1 \Rightarrow g_c(\varepsilon) \approx 0.333\varepsilon + 0.210\varepsilon^2 - 0.138\varepsilon^3 + 0.269\varepsilon^4 - 0.8445\varepsilon^5, \\
&\text{For } N = 2 \Rightarrow g_c(\varepsilon) \approx 0.3\varepsilon + 0.18\varepsilon^2 - 0.108\varepsilon^3 + 0.205\varepsilon^4 - 0.591\varepsilon^5, \\
&\text{For } N = 3 \Rightarrow g_c(\varepsilon) \approx 0.273\varepsilon + 0.156\varepsilon^2 - 0.086\varepsilon^3 + 0.162\varepsilon^4 - 0.430\varepsilon^5, \\
&\text{For } N = 4 \Rightarrow g_c(\varepsilon) \approx \frac{1}{4}\varepsilon + \frac{13}{96}\varepsilon^2 - 0.0707\varepsilon^3 + 0.130\varepsilon^4 - 0.322\varepsilon^5,
\end{aligned} \tag{34}$$

while the large-order parameters are $\sigma = \frac{3}{N+8}$ and $b = 4 + \frac{N}{2}$. The third order approximation takes the form ${}_3F_1(a_1, a_2, a_3; b_1; -\sigma\varepsilon) - 1$, while the fourth order takes the same form except in the equations determining the parameters we use the large-order constraint $a_1 + a_2 + a_3 - b_1 - 2 = b$. For the five-loop resummation, we resummed the series

$$\frac{g_c(\varepsilon)}{\varepsilon} = f_1 + f_2\varepsilon + f_3\varepsilon^2 + f_4\varepsilon^3 + f_5\varepsilon^4 \tag{35}$$

for $N = 1$, 2, 3, and 4 using the hypergeometric approximant $f_1\ {}_3F_1(a_1, a_2, a_3; b_1; \sigma\varepsilon)$. For $N = 0$, however, we resummed the subtracted series $\frac{g_c(\varepsilon) - f_1\varepsilon}{f_2\varepsilon^2} = 1 + f_3\varepsilon + f_4\varepsilon^2 + f_5\varepsilon^3$ using the hypergeometric approximant

$$g_c(\varepsilon) = f_1\varepsilon + f_2\varepsilon^2\, {}_3F_1(a_1, a_2, a_3; b_1; \sigma\varepsilon), \tag{36}$$

with the constraint $a_1 + a_2 + a_3 - b_1 - 2 = b + 2$. Such technical steps are well known in resummation techniques [4,44], which can be used in case no solution has been found for the equations defining the parameters. The prediction of these orders is shown in Table V and compared with other resummation results from Refs. [4,11,35,47].

IV. SIX-LOOP HYPERGEOMETRIC-MEIJER RESUMMATION OF THE CRITICAL EXPONENTS ν, η, AND ω

In Ref. [13], the six-loop order of the renormalization group functions has been obtained and resummed using Borel with conformal mapping algorithm. The work led to the improvement of the previous resummation predictions of the five-loop order in Refs. [4,11]. This six-loop order of perturbation series represents a good test for the accuracy and stability of our resummation algorithm. We shall thus

TABLE V. The three-, four-, and five-loop hypergeometric-Meijer resummation of the critical coupling g_c for the $O(N)$ model with $N = 0$, 1, 2, 3, and 4. The result from Ref. [11] in the last column (scaled by a factor $\frac{3}{N+8}$ because of different normalizations) and SC refers to strong-coupling resummation algorithm.

N	${}_3F_1$ This work: ε^3	${}_3F_1$ This work: ε^4	${}_3F_1$ This work: ε^5	JK\SC	BCM[12]
0	0.54035	0.54684	0.49007	0.5408(83), JK[4]	0.52988 ± 0.00225
1	0.47883	0.48475	0.48462	0.4810(91), JK[4]	0.47033 ± 0.001
2	0.42779	0.43322	0.43429	0.5032(239), JK[4]	0.4209 ± 0.001
3	0.36955	0.39006	0.39214	0.3895(71), JK[4]	0.37936 ± 0.001
4	0.34921	0.35187	0.35638	0.34375, SC[50]	0.34425 ± 0.00125

extend our work in the previous section to incorporate the six-loop weak-coupling data to compare with the recent results of Borel resummation and numerical predictions.

A different ε has been used in Ref. [13] as the space-time dimension has been set as $d-2\varepsilon$. Accordingly, the nth coefficients in each perturbation series has to be divided by 2^n to keep the definition used in our work ($d-\varepsilon$). For the critical exponent ν, we then have

$$\nu^{-1} = 2 + \sum_{i=1}^{6} c_i \varepsilon^i + O(\varepsilon^7), \tag{37}$$

where the first five coefficients (c_i) are given by Eq. (13) while the sixth coefficient is given in Table VII. Accordingly, we use the approximant $2\,{}_4F_2(a_1,a_2,a_3,a_4;b_1,b_2;-\sigma\varepsilon)$ for the resummation of the ν^{-1} series above. In Table VI, one can realize that our six-loop resummation for the critical exponent ν is very competitive either to the six-loop Borel with conformal mapping algorithm in Ref. [13] or Monte Carlo calculations (ours are closer to numerical results).

For the critical exponent η, we have the series up to fifth order in Eq. (21) and we add the sixth coefficient from Ref. [13] as shown in Table VII. The hypergeometric approximant ${}_3F_1$ has been used for the resummation of the six-loop perturbation series of η and its resummation results are presented in Table VI too.

For the critical exponent ω, the sixth coefficients e_6 are listed in Table VII. In this case, we use the approximant ${}_4F_2(a_1,a_2,a_3,a_4;b_1,b_2;-\sigma\varepsilon)-1$ which in turn results in the last column in Table VI. Note that when there exist no solution for the set of equations determining the parameters, we resort to successive subtraction of the perturbation series [4,44].

TABLE VI. The six-loop Hypergeometric-Meijer resummation (first) for the critical exponent ν, η, and ω for $O(N)$ model with $N=0$, 1, 2, 3, and 4. The results are compared to recent Borel with conformal mapping (second) resummation in Ref. [13] and also recent Monte Carlo simulations methods (third).

N	ν	η	ω	Reference
	0.58744	0.03034	0.85559	This work
0	0.5874(3)	0.0310(7)	0.841(13)	[13]
	0.5875970(4)	0.031043(3)	0.904(5)	[18]
	0.62937	0.03545	0.82929	This work
1	0.6292(5)	0.0362(6)	0.820(7)	[13]
	0.63002(10)	0.03627(10)	0.832(6)	[15]
	0.66962	0.03733	0.80580	This work
2	0.6690(10)	0.0380(6)	0.804(3)	[13]
	0.6717(1)	0.0381(2)	0.785(20)	[19]
	0.70722	0.037301	0.79272	This work
3	0.7059(20)	0.0378(5)	0.795(7)	[13]
	0.7116(10)	0.0378(3)	0.791(22)	[16]
	0.74151	0.03621	0.76793	This work
4	0.7397(35)	0.0366(4)	0.794(9)	[13]
	0.750(2)	0.0360(3)	0.817 (30)	[16]

TABLE VII. The coefficients of the sixth order in the ε expansion from Ref. [13] but scaled properly to match with the choice $d-\varepsilon$ of the space-time dimension in our work, while in Ref. [13] the choice was $d-2\varepsilon$. In this table, c_6 for ν^{-1}, d_6 for η, and e_6 for ω series, respectively.

N	0	1	2	3	4
c_6	−3.856	−3.573	−3.103	−2.639	−2.234
d_6	−0.0907	−0.0813	−0.0686	−0.0570	−0.0474
e_6	−130.00	−93.111	−68.777	−52.205	−40.567

V. RESUMMATION OF THE SEVEN-LOOP COUPLING SERIES FOR β, γ_{m^2}, AND γ_ϕ RENORMALIZATION GROUP FUNCTIONS

In the minimal subtraction scheme, Schnetz obtained the seven-loop order of the renormalization group functions β, γ_{m^2}, and γ_ϕ for the $O(N)$-symmetric model [51]. Here γ_{m^2} is the mass anomalous dimension, while γ_ϕ represents the field anomalous dimension. In the following, we list our resummation results for $N=0$, 1, 2, 3, and 4 while the results are compared to recent calculations from different techniques in Tables VIII–XII. Note that for the g series, the large-order parameters for the $O(N)$-symmetric model are $\sigma=1$ and $b_\beta=3+N/2$, $b_\omega=4+N/2$, $b_{\gamma_\phi}=2+N/2$, and $b_{\gamma_{m^2}}=3+N/2$ [4], where $\omega=\beta'_g$.

A. Resummation results for self-avoiding walks ($N=0$)

For $N=0$ and in three dimensions, the seven-loop order for the β function is given by

$$\beta \approx -g + 2.667g^2 - 4.667g^3 + 25.46g^4 - 200.9g^5 + 2004g^6 - 23315g^7 + 303869g^8. \tag{38}$$

TABLE VIII. The seven-loop (7L) hypergeometric-Meijer resummation for the critical exponents ν, η, and ω of the self-avoiding walks model ($N=0$). Here we compare with our results from the previous section (ε^6), conformal bootstrap (CB) calculations [52], Monte Carlo (MC) simulation for ν from Refs. [13,17] and η from Ref. [18]. The six-loop Borel with conformal mapping (BCM) resummation (ε^6) from Ref. [13] and five loops (ε^5) from the same reference.

Method	ν	η	ω
7L: This work	0.58723	0.03129	0.85650
ε^6: This work	0.58744	0.03034	0.85559
CB	0.5877(12)	0.0282(4)	…
MC	0.5875970(4)	0.031043(3)	0.899(12)
ε^6: BCM	0.5874(3)	0.0310(7)	0.841(13)
ε^5: BCM	0.5873(13)	0.0314(11)	0.835(11)

TABLE IX. The seven-loop hypergeometric-Meijer resummation for the critical exponents ν, η, and ω of the $O(1)$-symmetric model. Here we compare with our results from the previous section (ε^6), conformal bootstrap calculations from Ref. [25], and Monte Carlo (MC) simulation from Ref. [15]. The six-loop Borel with conformal mapping (BCM) resummation (ε^6) from Ref. [13] and five loops (ε^5) from the same reference. The very recent calculations of critical exponents using nonperturbative renormalization group (NPRG) [53] are listed last where results for ν and η are up to $O(\partial^6)$ while for ω is up to $O(\partial^4)$.

Method	ν	η	ω
7L: This work	0.62934	0.03684	0.82790
ε^6: This work	0.62937	0.03545	0.82929
CB	0.62999(5)	0.03631(3)	0.8303(18)
MC	0.63002(10)	0.03627(10)	0.832(6)
ε^6: BCM	0.6292(5)	0.0362(6)	0.820(7)
ε^5: BCM	0.6290(20)	0.0366(11)	0.818(8)
NPRG	0.63012(16)	0.0361(11)	0.832(14)

TABLE X. The seven-loop hypergeometric-Meijer resummation for the critical exponents ν, η, and ω of the $O(2)$-symmetric model. For comparison, other predictions are listed from the previous section (ε^6), conformal bootstrap calculations [27] for ν and η, while ω from Refs. [13,26]. MC calculations from Ref. [54]. The six-loop BCM resummation (ε^6) from Ref. [13] and five loops (ε^5) from the same reference while NPRG results up to $O(\partial^4)$ [53] are listed last.

Method	ν	η	ω
7L: This work	0.66953	0.03824	0.80233
ε^6: This work	0.66962	0.03733	0.80580
CB	0.6719(11)	0.03852(64)	0.811(10)
MC	0.67183(18)	0.03853(48)	0.789
ε^6: BCM	0.6690(10)	0.0380(6)	0.804(3)
ε^5: BCM	0.6687(13)	0.0384(10)	0.803(6)
NPRG	0.6716(6)	0.0380(13)	0.791(8)

TABLE XI. The seven-loop hypergeometric-Meijer resummation for the critical exponents ν, η, and ω of the $O(3)$-symmetric model. The results are compared with our results from the previous section (ε^6), conformal bootstrap calculations from Ref. [27] for ν and η, while ω from Refs. [13,26]. For MC simulations, ω is taken from Ref. [20], while ν and η are taken from Ref. [16]. The six-loop BCM resummation is taken from Ref. [13] and five loops from the same reference. The very recent calculations using NPRG [53] are listed last and up to $O(\partial^4)$.

Method	ν	η	ω
7L: This work	0.70810	0.03795	0.78683
ε^6: This work	0.70722	0.037301	0.79272
CB	0.7121(28)	0.0386(12)	0.791(22)
MC	0.7116(10)	0.0378(3)	0.773
ε^6: BCM	0.7059(20)	0.0378(5)	0.795(7)
ε^5: BCM	0.7056(16)	0.0382(10)	0.797(7)
NPRG	0.7114(9)	0.0376(13)	0.769(11)

TABLE XII. The seven-loop hypergeometric-Meijer resummation for the critical exponents ν, η, and ω of the $O(4)$-symmetric model. Here we compare with our results from the previous section (ε^6), conformal bootstrap calculations [13,26] for ν and ω, while η from Ref. [24]. MC simulations for ω is taken from Ref. [20], while ν and η are from Ref. [16]. The six-loop BCM resummation (ε^6) is taken from Ref. [13] and five loops (ε^5) from the same reference. NPRG results up to $O(\partial^4)$ [53] are shown in the last row.

Method	ν	η	ω
7L: This work	0.750935	0.03740	0.80325
ε^6: This work	0.74151	0.03621	0.76793
CB	0.751(3)	0.0378(32)	0.817(30)
MC	0.750(2)	0.0360(3)	0.765 (30)
ε^6: BCM	0.7397(35)	0.0366(4)	0.794(9)
ε^5: BCM	0.7389(24)	0.0370(9)	0.795(6)
NPRG	0.7478(9)	0.0360(12)	0.761(12)

We resummed this series using the approximant $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$ which resulted in the Meijer-G approximant of the form

$$\beta = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)} G^{1,5}_{5,4}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3, 1-a_4, 1-a_5 \\ 0, 1-b_1, 1-b_2, 1-b_3 \end{matrix}\middle| -g\right) - 1. \quad (39)$$

The critical coupling is obtained from the zero of the β function where we found $g_c = 0.53430$. The series for correction to scaling critical exponent ω is obtained from differentiating the above series with respect to g, and it has been resummed using the approximant $(-{}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g_c))$ where the large-order constraint $\sum a_i - \sum b_i - 2 = b_\omega$ has been employed and we found the result $\omega = 0.85650$. This result can be compared with the recent Monte Carlo simulations calculations in Ref. [18] that predict the result $\omega = \frac{\Delta_1}{\nu} = 0.899(12)$ (see Table VIII for comparison with different methods).

The field anomalous dimension is also given by

$$\gamma_\phi \approx 0.05556g^2 - 0.03704g^3 + 0.1929g^4 - 1.006g^5 + 7.095g^6 - -57.74g^7. \quad (40)$$

The suitable hypergeometric approximant used is

$$\gamma_\phi = {}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -1) - \left(1 + g\frac{a_1a_2a_3a_4}{b_1b_2}\right). \quad (41)$$

The critical exponent η is obtained from the relation $\eta = 2\gamma_\phi(g_c)$ where we get the result $\eta = 0.03129$. In a recent conformal bootstrap calculation, the result $\eta = 2\Delta_\phi - 1 = 0.0282(4)$ has been obtained [52] while the Monte Carlo result is $\eta = 0.031043(3)$ in Refs. [13,17].

For the mass anomalous dimension γ_{m^2}, the series up to seven-loop order is given by

$$\gamma_{m^2} \approx -0.6667g + 0.5556g^2 - 2.056g^3 + 10.76g^4 - 75.70g^5 + 636.7g^6 - 6080g^7.$$

The hypergeometric approximant used is $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$ which corresponds to the following Meijer-G function:

$$\gamma_{m^2} = \left(\frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)\Gamma(a_5)} G_{5,4}^{1,5}\left(\begin{matrix} 1-a_1, 1-a_2, 1-a_3, 1-a_4, 1-a_5 \\ 0, 1-b_1, 1-b_2, 1-b_3 \end{matrix}\middle| -g\right)\right) - 1. \quad (42)$$

The critical exponent ν is then obtained as $\nu = (2 + \gamma_{m^2}(g_c))^{-1}$ which yields the result $\nu = 0.58723$. This result can be compared with conformal bootstrap prediction $\nu = 0.5877(12)$ in Ref. [52] and the Monte Carlo result $\nu = 0.5875970(4)$ in Ref. [18].

B. Resummation results for Ising universality class ($N = 1$)

For $N = 1$, the seven-loop β function that has been recently obtained [51] is given by

$$\beta \approx -\varepsilon g + 3.000g^2 - 5.667g^3 + 32.55g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + 474651g^8. \quad (43)$$

The suitable approximant for this series is $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$ which we used to obtain the critical coupling g_c at which $\beta = 0$. In three dimensions ($\varepsilon = 1$), the predicted critical coupling is $g_c = 0.47947$. This value can be compared with the five-loop resummation in Table V. The critical exponent ω also predicted to have the value 0.82790. The conformal bootstrap calculation gives the result $\omega = 0.8303(18)$ in Ref. [25] while Monte Carlo simulations result is $\omega = 0.832(6)$ [15].

The seven-loop perturbation series for the anomalous mass dimension γ_{m^2} has been obtained in the same reference [51], where

$$\gamma_{m^2} \approx -g + 0.8333g^2 - 3.500g^3 + 19.96g^4 - 150.8g^5 + 1355g^6 - 13760g^7. \quad (44)$$

We used $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$ too for the resummation of this series. The ν exponent is then

$$\nu = (2 + \gamma_{m^2}(g_c))^{-1} = 0.62934.$$

The recent Monte Carlo prediction gives the value $\nu = 0.63002(10)$ in Ref. [15], while in Ref. [25] one can find the result $\nu = 0.62999(5)$ using conformal bootstrap calculations.

The seven-loop order of the perturbation series for the field anomalous dimension γ_ϕ is also obtained in Ref. [51] as

$$\gamma_\phi \approx 0.08333g^2 - 0.06250g^3 + 0.3385g^4 - 1.926g^5 + 14.38g^6 - 124.2g^7. \quad (45)$$

We used the "hypergeometric approximant",

$$\gamma \approx {}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; (-g)) - \left(1 - \frac{a_1a_2a_3a_4}{b_1b_2}(-g)\right), \quad (46)$$

to resum that series and the exponent η is obtained from the relation $\eta = 2\gamma(g_c)$. We get the result $\eta = 0.03684$. This result is compatible with the recent conformal bootstrap calculation of $\eta = 0.03631(3)$ [25] and Monte Carlo simulation result of $\eta = 0.03627(10)$ in Ref. [15].

C. Resummation results for $N = 2$ (XY universality class)

In this case, the seven-loop β function is given by

$$\beta \approx -g + 3.333g^2 - 6.667g^3 + 39.95g^4 - 350.5g^5 + 3845g^6 - 48999g^7 + 696998g^8. \tag{47}$$

This series is resummed using the approximant $(-g({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g))$ which gives the critical coupling value $g_c = 0.43292$. Resuming the g-differentiated series yields the result $\omega = 0.80233$. The value $\omega = 0.789$ has been adopted using a recent high-precision Monte Carlo calculations [54] while the conformal bootstrap calculations give $\omega = 0.811(10)$ [13,26].

The mass anomalous dimension has the seventh-loop result as

$$\gamma_{m^2} \approx -1.333g + 1.111g^2 - 5.222g^3 + 31.87g^4 - 255.8g^5 + 2434g^6 - 26086g^7, \tag{48}$$

where we resummed it using $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$. This led to the result $\nu = 0.66953$. The recent Monte Carlo result is $\nu = 0.67183(18)$ [54] while the conformal bootstrap gives $\nu = 0.6719(11)$ [27].

For the field anomalous dimension γ_ϕ, we have

$$\gamma_\phi \approx 0.11111g^2 - 0.09259g^3 + 0.5093g^4 - 3.148g^5 + 24.71g^6 - 224.6g^7. \tag{49}$$

The corresponding hypergeometric approximant is $0.11111g^2({}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -g))$ with the result $\eta = 0.03824$. For that exponent, the recent Monte Carlo simulations in Ref. [54] give $\eta = 0.03853(48)$ while conformal bootstrap gives the result $\eta = 0.03852(64)$ [27].

D. Resummation results for Heisenberg universality class ($N = 3$)

The seven-loop β function for $N = 3$ is given by

$$\beta \approx -g + 3.667g^2 - 7.667g^3 + 47.65g^4 - 437.6g^5 + 4999g^6 - 66243g^7 + 978330g^8. \tag{50}$$

To resum this series, we used the hypergeometric approximant $(-g + 3.667g^2 - 7.667g^3({}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -g))$ which predicts the critical coupling value $g_c = 0.39363$ while the resummation of the ω series gives the value 0.78683. Conformal bootstrap result is $\omega = 0.791(22)$ [13,26] and the Monte Carlo result is $\omega = 0.773$ [20].

The series representing the mass anomalous dimension up to seven-loop order is

$$\gamma_{m^2} \approx -1.667g + 1.389g^2 - 7.222g^3 + 46.64g^4 - 394.9g^5 + 3950^6 - 44412g^7, \tag{51}$$

which has been resummed using $({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1)$ that gives the result $\nu = 0.70810$. In Ref. [27], conformal bootstrap calculations give the value $\nu = 0.7121(28)$ and the Monte Carlo simulations in Ref. [16] give $\nu = 0.7116(10)$.

The field anomalous dimension γ_ϕ has the seventh order perturbative form,

$$\gamma_\phi \approx 0.1389g^2 - 0.1273g^3 + 0.6993g^4 - 4.689g^5 + 38.44g^6 - 365.9g^7, \tag{52}$$

which approximated by $(g({}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -g) - 1))$ and gives the result $\eta = 0.03795$. To compare with other recent results, the bootstrap calculations in Ref. [27] give $\eta = 0.0386(12)$ and the Monte Carlo results give $\eta = 0.0378(3)$ [16].

E. Resummation results for the $O(4)$-symmetric case

The seven-loop β function for $N = 4$ is shown to be

$$\beta \approx -g + 4.000g^2 - 8.667g^3 + 55.66g^4 - 533.0g^5 + 6318g^6 - 86768g^7 + 1.326 \times 10^6 g^8. \tag{53}$$

The corresponding approximant is $(-g({}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g))$ which yields $g_c = 0.36662$ while resumming the ω series gives the result $\omega = 0.80325$. Monte Carlo methods in Ref. [20] give $\omega = 0.765$ while conformal bootstrap calculations predict the result $\omega = 0.817(30)$ [13,26].

The anomalous mass dimension is given by

$$\gamma_{m^2} \approx -2.000g + 1.667g^2 - 9.500g^3 + 64.39g^4 - 571.9g^5 + 5983g^6 - 70240g^7, \quad (54)$$

which has been approximated by ${}_5F_3(a_1, a_2, a_3, a_4, a_5; b_1, b_2, b_3; -g) - 1$ and gives $\nu = 0.75093$. This result is very close to the Monte Carlo result $\nu = 0.750(2)$ in Ref. [16] and the conformal bootstrap result $\nu = 0.751(3)$ in Ref. [26].

Likewise, the field anomalous dimension up to seven loops is given by

$$\gamma_\phi \approx 0.1667g^2 - 0.1667g^3 + 0.9028g^4 - 6.563g^5 + 55.93g^6 - 555.2g^7, \quad (55)$$

which is approximated by $g({}_4F_2(a_1, a_2, a_3, a_4; b_1, b_2; -g) - 1)$ and gives the result $\eta = 0.03740$. Again, the Monte Carlo simulations in Ref. [16] give the values $\eta = 0.0365(3)$. Also, Monte Carlo simulations and finite-size scaling of three-dimensional Potts models in Ref. [22] give the result $\eta = 5 - 2y_h = 0.036(6)$ and the conformal bootstrap calculation is 0.0378(32) [24].

A note to be mentioned is that one should not judge the convergence of the seven-loop resummation results by comparing with six-loop resummation or lower order resummation in this work. The point is that the seven-loop resummation in this work applied for the g series but for the other orders we resummed the ε series. Our aim behind resumming both available series is to test our algorithm using different types of perturbation series. To have an idea about the good convergence of our algorithm for the resummation of the g series, one should look at different orders of resummation of the g series itself. For instance, for $N = 4$, we get $\omega = 0.77963$ from five-loop resummation of the g series, $\omega = 0.78162$ from six loops [14] compared to the seven-loop result in Table XII as $\omega = 0.80325$.

VI. SUMMARY AND CONCLUSIONS

We show that divergent series with different large-order behaviors can be approximated by different generalized hypergeometric functions ${}_pF_q(a_1, \ldots a_p; b_1 \ldots . b_q; \sigma z)$. The relation between the number of numerator and denominator parameters (p and q) is determined from the growth factor in the large-order behavior of the divergent series. For a divergent series with a growth factor $n!$, the series expansion of the hypergeometric function ${}_pF_q(a_1, \ldots a_p; b_1 \ldots . b_q; \sigma z)$ where $p = q + 2$ can reproduce a large-order behavior with same growth factor. Accordingly, the hypergeometric function ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots . b_{p-2}; \sigma z)$ is the suitable candidate to approximate such type of divergent series. Since the function ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots . b_{-p-2}; \sigma z)$ possesses an expansion of zero radius of convergence, a representation in terms of Meijer-G function is capable to resum the divergent hypergeometric series.

For divergent series that have growth factors $(2n)!$ and $(3n)!$, hypergeometric functions with $p = q + 3$ and $p = q + 4$, respectively, can reproduce such large-order behaviors and thus are suitable approximants for such perturbation series. On the other hand, one might have a divergent series with finite radius of convergence which has a large-order behavior with a growth factor of 1. To mimic such type of large-order behavior, the hypergeometric function ${}_pF_{p-1}(a_1, \ldots a_p; b_1 \ldots b_{p-1}; \sigma z)$ can be used as suitable approximant for such kind of divergent series.

The large-order behavior of the ε expansion of the renormalization group functions for the $O(N)$-symmetric model has a growth factor of $n!$. Accordingly, we used the hypergeometric function ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots . b_{p-2}; \sigma z)$ to approximate the respective divergent series. Since the strong-coupling data are not yet known for such expansion, we use weak-coupling and large-order data to parametrize the hypergeometric function ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots . b_{p-2}; \sigma z)$. The parametrization of the hypergeometric function is then followed by the resummation step of using a representation in terms of Meijer-G function. We applied the algorithm to resum the divergent series representing critical exponents $\nu(\nu^{-1})$, η, and ω, as well as the critical coupling up to ε^5 order as input. For N equals 0,1,2,3, and 4, the results ought to be reasonable even for very low order of perturbation used to parametrize the hypergeometric approximant. The results are greatly improved in using third order and being more precise in going to fourth order while the fifth order offers very competitive predictions when compared to other resummation algorithms in literature.

To show that the precise results extend to higher N values, we resummed the perturbation series for the exponent ν for $N = 6$, 8, 10 and 12. The precision of the results can be seen from Table II where we listed the fifth order resummation results for the exponent ν and compared it with other methods.

All the hypergeometric functions ${}_pF_{p-2}(a_1, \ldots a_p; b_1 \ldots . b_{p-2}; \sigma z)$ share the same analytic behavior. Accordingly, one expects no surprises in going to higher orders of resummation. To test this clear fact, as well as to seek more improved results, we resummed the six-loop order for the perturbation series for the exponents ν, η, and ω for $N = 1$, 2, 3, and 4. The results are showing improved predictions for those exponents. When compared to other calculations, our results for the critical exponents are compatible with the recent six-loop BC resummation method in Ref. [13], MC simulations calculations [15,16,19–22,54], and conformal bootstrap methods [23,24,26,27,52].

The very recent seven-loop order (coupling-series) for the renormalization group functions β, γ_{ϕ}, and γ_{m^2} has been resummed too. Up to the best of our knowledge, no other resummation algorithm has been used to resum this order. Very accurate results for the critical coupling and the exponent ν have been extracted from the resummed functions.

In all of our calculations, we used weak-coupling and large-order data as input. The a_i parameters in the hypergeometric functions ${}_pF_q(a_1, \ldots a_p; b_1 \ldots . b_{p-2}; \sigma z)$ are well known to represent the strong-coupling data [14]. However, the strong-coupling expansion for the series under consideration has not been obtained yet (up to the best of our knowledge). Accordingly, we cannot get benefited from this fact in further acceleration of the convergence of the resummation algorithm. However, the expansion coefficients of the hypergeometric function depend on the strong-coupling parameters and they in turn constrained to match the weak-coupling and large-order data. Accordingly, this algorithm is linking the unknown strong-coupling parameters to the known weak-coupling and large-order data. Thus, the algorithm has the ability to predict the nonperturbative asymptotic strong-coupling behavior of a quantum field theory from knowing the weak-coupling and large-order data. In other algorithms, this asymptotic behavior is predicted from optimization techniques and different optimizations can even lead to different results for the same theory.

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