$\mathcal{N} = (2,0)$ non-Abelian Proca-Stückelberg theory in six dimensions

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We formulate a $N = (2,0)$ supersymmetric non-Abelian Proca-Stückelberg theory in six space-time dimensions (6D). As the foundation of our construction, we start with our recent work on $N = 1$ supersymmetric Proca-Stückelberg formulation in 4D with a Yang-Mills (YM) multiplet $(A_\mu{}^I, \lambda^I)$ and a chiral multiplet $(\varphi^I, \chi^I, \phi^I)$, where the index $I = 1, 2, \ldots, \dim G$ is for the adjoint representation of a non-Abelian group G, while φ^I parametrizes the coordinates of the group manifold G. Since φ^I and ϕ^I transform differently under G, the conventional global R symmetry is lost. Next, we apply this mechanism to 6D with the two multiplets: a YM multiplet $(A_\mu{}^I, \lambda^{\underline{\alpha} I})$ and a hypermultiplet (HM) $(\phi^{iJ}, \chi_{\underline{\alpha}}{}^I, \varphi^I)$. The index $i = 1, 2, 3$ is for the $\mathbf{3}$ of $Sp(1)$. The spinorial index $\underline{\alpha} = (\alpha, A)$ $(\alpha = 1, \ldots, 4)$ is for the Majorana-Weyl spinor index for $D = 5 + 1$ with $A = 1, 2$ for the $\mathbf{2}$ of $Sp(1)$. As opposed to the common notion that *all* four scalars in a HM in 6D must form the $(\mathbf{2}, \mathbf{2})$ of global $Sp(1) \times Sp(1)$, we can use a scalar φ^I in the $(\mathbf{1}, \mathbf{1})$ of $Sp(1) \times Sp(1)$ as a Nambu-Goldstone boson absorbed into the longitudinal component of $A_\mu{}^I$, separated from the remaining three scalars ϕ^{iI} in the $(\mathbf{3}, \mathbf{1})$ of $Sp(1) \times Sp(1)$. Similar to our recent result in 4D with broken automorphism R symmetry, the new feature of our result is that all four scalars in the HM in 6D do *not* have to form the $(\mathbf{2}, \mathbf{2})$ of $Sp(1) \times Sp(1)$.

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I. INTRODUCTION

There has been a considerable number of applications of the so-called "tensor-hierarchy" formulations [1,2] to the consistent interactions of *non-Abelian* tensors. Explicit examples are models such as the *supersymmetrization* [3] of Jackiw-Pi model [4], the *supersymmetrization* [5,6] of Proca-Stückelberg formulation [7], supersymmetric composite gauge models [8], and the supersymmetric Cremmer-Scherk theory [9]. The common feature among these formulations is the Chern-Simon-like modifications of the conventional field strengths of *non-Abelian* tensors, such as the field-strength $G_{\mu\nu\rho}{}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I + 3f^{IJK}F_{[\mu\nu}{}^J C_{\rho]}{}^K$ of a second-rank tensor $B_{\mu\nu}{}^I$ [1,2].

Before the works [5,6] on supersymmetric non-Abelian Proca-Stückelberg theory [7] in four dimensions (4D), there were already some works [10] in similar directions. However, these works are limited to $U(1)$ Abelian gauge groups. The new feature of our papers [5,6] is the simultaneous accomplishment of *both* the *supersymmetrization and "non-Abelianization"* of the Proca-Stückelberg theory [7].

In our recent paper [9], we have presented a Proca-Stueckelberg mechanism [7] for non-Abelian gauge symmetry in 4D. Its *Abelian limit* is shown to correspond to the Proca-Stueckelberg type breaking of R symmetry in 4D. In [11], it is concluded that R-symmetry breaking is closely related to supersymmetry breaking in 4D. Moreover, supersymmetry breaking in F theory [12] by an instanton is associated with R-symmetry breaking [13]. In other words, there is a natural link between supersymmetry breaking and R-symmetry breaking. As such, it is important to investigate R-symmetry-breaking models, such as our formulation in 4D [9], where Proca-Stueckelberg mechanism [7] results in R-symmetry breaking.

The basic ingredient of our supersymmetric non-Abelian Proca-Stückelberg formulations [5,6,9] is described as follows: We introduce the scalars $\varphi \equiv \varphi^I T^I$ with the anti-Hermitian generators T^I $(I = 1, 2, \ldots, \dim G)$ parametrizing the coordinates of the group manifold G. The conventional Yang-Mills (YM) field strength $F_{\mu\nu}{}^I$ is modified to [9]

$$\mathcal{F}_{\mu\nu}{}^I \equiv F_{\mu\nu}{}^I + m^{-1} f^{IJK} P_\mu{}^J P_\nu{}^K, \qquad (1.1)$$

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where $F_{\mu\nu}{}^I \equiv 2\partial_{[\mu}A_{\nu]}{}^I + mf^{IJK}A_\mu{}^J A_\nu{}^K$, while $P_\mu{}^I$ is the covariant field strength of φ made from the representative e^φ of the group manifold G as

$$P_\mu{}^I \equiv [(D_\mu e^\varphi)e^{-\varphi}]^I \equiv [(\partial_\mu e^\varphi)e^{-\varphi}]^I + mA_\mu{}^I. \qquad (1.2)$$

The modification (1.1) is associated with the modified Bianchi identities (BIds),

$$D_{[\mu}P_{\nu]}{}^I \equiv +\frac{1}{2}m\mathcal{F}_{\mu\nu}{}^I, \qquad (1.3a)$$

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^I \equiv +f^{IJK}\mathcal{F}_{[\mu\nu}{}^J P_{\rho]}{}^K. \qquad (1.3b)$$

The modifications of field strengths in tensor-hierarchy formulations [1,2] are with Chern-Simons terms as the combination of a field strength and a potential field, or just the single factor of a potential field [1,2], while the modification (1.1) involves two field strengths. Although they look different, there is similarity between tensor-hierarchy [1,2] and a non-Abelian Proca-Stückelberg formulation [5,6] as the "generalized Chern-Simons term," so that the latter can be interpreted as a "generalized" tensor-hierarchy formulation. To be more specific, we can regard any modification of BIds (1.3) by the wedge product of field strengths as the "generalized" tensor-hierarchy formulations.

In our recent paper [6] on supersymmetric Proca-Stückelberg formulations in 4D, we introduced a tensor multiplet $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ and an extra vector multiplet $(K_\mu{}^I, \rho^I, C_{\mu\nu\rho}{}^I)$, in addition to the YM multiplet $(A_\mu{}^I, \lambda^I)$. The scalar φ^I is used as a Nambu-Goldstone scalar absorbed into the longitudinal component of $A_\mu{}^I$. However, the field content in [6] is rather involved, because of the extra vector multiplet. One clue to exclude such extra vector multiplets is found in the superspace formulations [14–16] of Freedman-Townsend theory [17], where a chiral-superfield is introduced. In such formulations, due to the chiral superfields introduced, scalar and pseudoscalar fields transform in the same way under the gauge group G. In such *conventional chiral-superfield formulations*, the so-called global R symmetry [18,19] is *conserved, by definition*.

However, as a different approach, if we try to use the chiral multiplet (CM) (A, B, χ), in which *only* the scalar A (but *not* B) is used for a Nambu-Goldstone field as an alternative formulation of supersymmetric Proca-Stückelberg theory, it seems inevitable to *break* global R symmetry. In fact, in our recent paper [9], we have presented such a formulation, in which the scalar A is transforming as the coordinates of the group manifold G, while the pseudoscalar B is transforming as the usual adjoint representation. In a sense, this formulation has more potential applications to other dimensions, such as 6D, where there are plural scalars in a hypermultiplet (HM), and it is more advantageous to separate one scalar as a Nambu-Goldstone scalar from the remaining scalars.

From this viewpoint, it seems to be a new feature in supersymmetry that a single scalar is separated from other (pseudo)scalars and is absorbed into the longitudinal part of a vector. In $N = 1$ locally supersymmetric systems in 4D, a scalar and a pseudoscalar fields in a chiral multiplet form the coordinates of complex Kähler manifolds [20]. This is further generalized to a hyper-Kähler manifold [21] for $N = 2$ supersymmetry in 4D. Our objective in [9], therefore, was to look for a formulation in which a scalar is separated from the remaining scalars in a CM in 4D and is used as a Nambu-Goldstone field.

In our present paper, we first review the supersymmetric formulation for non-Abelian Proca-Stückelberg theory in 4D *only* with the YM-multiplet $(A_\mu{}^I, \lambda^I)$ and the CM $(\varphi^I, \chi^I, \phi^I)$, presented in [9]. Following this result in 4D, we formulate a super-Proca-Stückelberg theory in 6D. We consider $N = (2, 0)$ supersymmetry[1] in 6D, with a YM multiplet $(A_\mu{}^I, \lambda^I)$, and a hypermultiplet (HM) $(\phi^{iI}, \chi^I, \varphi^I)$, where λ^I and χ^I are Majorana-Weyl spinors with the chiralities $\gamma_7(\lambda^I, \chi^I) = (+\lambda^I, -\chi^I)$. The crucial technique here is that out of the four scalars in HM, we single out one scalar φ^I separated from other remaining three scalars ϕ^{iI} $(i = 1, 2, 3)$. These are, respectively, in the **1** and **3** of the global $Sp(1)$ gauge group. As such, the conventional global symmetry among the original four scalars is lost in 6D, as global R symmetry [18,19] is lost in 4D in Sec. III. This further implies that the conventional superfield approach in 6D is *not* the most general formulation, due to the lack of uniform treatment of scalars in the HM [23], or in terms of 4D chiral superfields [24].

This paper is organized as follows: In Sec. II, we give the preliminary analysis related to a purely bosonic case of Proca-Stückelberg formulation. We also review our recent supersymmetric Proca-Stückelberg formulation in 4D *only* with the a YM multiplet $(A_\mu{}^I, \lambda^I)$ and the CM $(\varphi^I, \chi^I, \phi^I)$ [9], *lacking* global R symmetry [18,19]. In Sec. III, we give our action in 6D with a Lagrangian invariant under $N = (2, 0)$ supersymmetry with technical details. Section IV is devoted to concluding remarks, while in Appendix we give the superspace [25,26,23] reconfirmation of our system. Since our system has the peculiar separation $\mathbf{1} + \mathbf{3}$ in the HM, the conventional superspace method in 6D [23,24] does *not* directly apply. This situation is also similar to the 4D case that the *chiral-superfield* formulation [24] does *not* apply to our system [9].

[1]We use the symbol $N = (2, 0)$ based on Majorana-Weyl spinors, counting each component of **2** of $Sp(1)$ separately for supersymmetry. This counting based on a Majorana-Weyl spinor complies with that by J. Strathdee in [22] in 6D.

II. N = 1 SUPERSYMMETRIC PROCA-STÜCKELBERG FORMULATION IN 4D

We first give some preliminaries to review the Proca-Stückelberg formulation. Let a scalar φ^I ($I = 1, 2, \ldots, \dim G$) carry the adjoint index of any non-Abelian gauge group G. We identify φ^I with the g-dimensional coordinates of the group manifold G. We use also the symbol $\varphi \equiv \varphi^I T^I$ with the anti-Hermitian generators T^I of G.

Consider the *finite* gauge transformation for the group-manifold representative [27],

$$e^\varphi \longrightarrow (e^\varphi)' = e^{-\Lambda} e^\varphi, \tag{2.1}$$

and that of the gauge field $A_\mu \equiv A_\mu{}^I T^I$,

$$A_\mu \longrightarrow A'_\mu = e^{-\Lambda} A_\mu e^\Lambda + m^{-1} e^{-\Lambda} (\partial_\mu e^\Lambda), \tag{2.2}$$

where $\Lambda \equiv \Lambda^I T^I$ is a finite gauge-transformation parameter. The conventional field strength $F_{\mu\nu}{}^I \equiv 2\partial_{[\mu} A_{\nu]}{}^I + m f^{IJK} A_\mu{}^J A_\nu{}^K$ transforms as

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = e^{-\Lambda} F_{\mu\nu} e^\Lambda. \tag{2.3}$$

Accordingly, the field strength of φ,

$$P_\mu \equiv (D_\mu e^\varphi) e^{-\varphi} \equiv (\partial_\mu e^\varphi) e^{-\varphi} + m A_\mu, \tag{2.4}$$

transforms "left and right covariantly" [28,27] as

$$P_\mu \longrightarrow P'_\mu = e^{-\Lambda} P_\mu e^\Lambda. \tag{2.5}$$

The definition of D_μ on e^φ is $D_\mu e^\varphi \equiv \partial_\mu e^\varphi + m A_\mu e^\varphi$, because this combination transforms "left covariantly": $D_\mu e^\varphi \longrightarrow e^{-\Lambda} D_\mu e^\varphi$ like (2.1), with *no* derivative on the finite parameter Λ.

The field strength $P_\mu{}^I$ plays a crucial role for the group-manifold σ model, and Proca-Stückelberg formulation, as well. Its BId is

$$D_{[\mu} P_{\nu]}{}^I \equiv +\frac{1}{2} m \mathcal{F}_{\mu\nu}{}^I, \tag{2.6}$$

where $\mathcal{F}$ is the modified field strength of F defined by [9]

$$\begin{aligned} \mathcal{F}_{\mu\nu}{}^I &\equiv F_{\mu\nu}{}^I + m^{-1} f^{IJK} P_\mu{}^J P_\nu{}^K, \\ F_{\mu\nu}{}^I &\equiv 2\partial_{[\mu} A_{\nu]}{}^I + m f^{IJK} A_\mu{}^J A_\nu{}^K. \end{aligned} \tag{2.7}$$

The P-BId (2.6) gives the necessity and justification of the modified $\mathcal{F}$ instead of F. As such, the modification of $F_{\mu\nu}{}^I$ into the peculiar $\mathcal{F}_{\mu\nu}{}^I$ is *not* based on the authors' subjective tastes, but on the naturalness of P-BIds. Of course, in principle, we can separate the $P \wedge P$ term in the P-BId (2.6), but it unnecessarily increases terms, as elucidated in the supersymmetric-invariance confirmation of our action in the *non-Abelian case*, as has been also confirmed in 4D [9].

In terms of two field strengths $\mathcal{F}$ and P, the Lagrangian for *non-Abelian* Proca-Stückelberg theory is [5,6]

$$\mathcal{L}_{\mathrm{PS}} = -\frac{1}{4} (\mathcal{F}_{\mu\nu}{}^I)^2 - \frac{1}{2} (P_\mu{}^I)^2. \tag{2.8}$$

Needless to say, by the field redefinition $A_\mu{}^I \equiv \tilde{A}_\mu{}^I - m^{-1} [(\partial_\mu e^\varphi) e^{-\varphi}]^I$, the φ-kinetic term in (2.8) becomes the mass term of $\tilde{A}_\mu{}^I$ as $-(1/2)(P_\mu{}^I)^2 = -(1/2) m^2 (\tilde{A}_\mu{}^I)^2$. This is nothing but the non-Abelian version of the original Proca-Stückelberg mechanism [7].

The $N = 1$ supersymmetrization of the non-Abelian Proca-Stückelberg Lagrangian (2.8) in 4D [9] is reviewed as follows. Even before [9], we performed similar supersymmetrizations in [5,6]. However, there are certain drawbacks in those formulations. For example, the formulation in [5] needs the extra auxiliary field $C_{\mu\nu\rho}{}^I$ together with the tensor multiplet $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$. In other words, there are two extra bosonic fields $C_{\mu\nu\rho}{}^I$ and $B_{\mu\nu}{}^I$ needed. Similarly, [6] needs an extra vector multiplet $(K_\mu{}^I, \rho^I, C_{\mu\nu\rho}{}^I)$, with *two additional* tensor fields $K_\mu{}^I$ and $C_{\mu\nu\rho}{}^I$ other than the original $A_\mu{}^I$ and φ^I.

To improve on these drawbacks, we minimized in [9] the number of multiplets *without* extra tensor multiplet or vector multiplet. In other words, we use the field content more economical than those in [5,6] with only two multiplets: the non-Abelian YM multiplet $(A_\mu{}^I, \lambda^I)$ and the CM $(\varphi^I, \chi^I, \phi^I)$, *without any other multiplet*. The scalar φ^I parametrizes the coordinates of the gauge group G, while a pseudoscalar ϕ^I is in the adjoint representation. Thus, the two spin-zero fields 0^+ and 0^- within a CM play different roles under the same group G. Accordingly, the conventional global R symmetry [18,19] is lost in this CM.

Our action $I_{\mathrm{4D}} \equiv \int d^4x \mathcal{L}_{\mathrm{4D}}$ for $N = 1$ supersymmetric Proca-Stückelberg theory in 4D [9] has the Lagrangian,

$$\begin{aligned} \mathcal{L}_{\mathrm{4D}} = &-\frac{1}{4} (\mathcal{F}_{\mu\nu}{}^I)^2 + \frac{1}{2} (\bar{\lambda}^I \not{D} \lambda^I) - \frac{1}{2} (P_\mu{}^I)^2 + \frac{1}{2} (\bar{\chi}^I \not{D} \chi^I) - \frac{1}{2} (D_\mu \phi^I)^2 \\ &+ m (\bar{\lambda}^I \chi^I) - \frac{1}{2} m^2 (\phi^I)^2 - i m f^{IJK} (\bar{\lambda}^I \gamma_5 \chi^J) \phi^K - \frac{1}{2} f^{IJK} (\bar{\lambda}^I \gamma^\mu \lambda^J) P_\mu{}^K, \end{aligned} \tag{2.9}$$

where the field strength P is defined by (2.4) and $\mathcal{F}$ by (2.7), while

$$D_\mu \lambda^I \equiv \partial_\mu \lambda^I + m f^{IJK} A_\mu{}^J \lambda^K, \quad \text{(idem for } D_\mu \chi^I \text{ and } D_\mu \phi^I). \tag{2.10}$$

The field strength P and $\mathcal{F}$ satisfy their BIds (1.3). The $P \wedge P$ term in $\mathcal{F}$ is *not* directly required by supersymmetry, but it *is* by the *non-Abelianization* of the original Proca-Stückelberg formulation [7]. However, it is also closely related to the consistency with supersymmetry, as is confirmed in terms of superspace language in the Appendix.

Our action $I_{4\mathrm{D}}$ is invariant under $N = 1$ supersymmetry,

$$\delta_Q A_\mu{}^I = +(\bar\epsilon \gamma_\mu \lambda^I) - m^{-1} f^{IJK} (\bar\epsilon \chi^J) P_\mu{}^K, \tag{2.11a}$$

$$\delta_Q \lambda^I = + \frac{1}{2} (\gamma^{\mu\nu} \epsilon) \mathcal{F}_{\mu\nu}{}^I + i m (\gamma_5 \epsilon) \phi^I + f^{IJK} \lambda^J (\bar\epsilon \chi^K), \tag{2.11b}$$

$$[(\delta_Q e^{\varphi}) e^{-\varphi}]^I = +(\bar\epsilon \chi^I), \tag{2.11c}$$

$$\delta_Q \chi^I = -(\gamma^\mu \epsilon) P_\mu{}^I + i (\gamma_5 \gamma^\mu \epsilon) D_\mu \phi^I, \tag{2.11d}$$

$$\delta_Q \phi^I = + i (\bar\epsilon \gamma_5 \chi^I). \tag{2.11e}$$

Despite the loss of the conventional global R symmetry [18,19], the total consistency is reconfirmed in superspace in the Appendix of [9]. The loss of global R symmetry is one of the reasons why we can *no* longer use the conventional *chiral superfields* [14–16] in superspace.

The supersymmetric invariance $\delta_Q I_{4\mathrm{D}} = 0$ has been confirmed in [9]. However, due to the subtlety of our system, we give rather detailed review of the confirmation. There are in total six sectors arising in the variation $\delta_Q I_{\mathrm{SPS}}$ up to $\mathcal{O}(\Phi^4)$[2]: (i) $m^0\Phi^2$, (ii) $m^1\Phi^2$, (iii) $m^2\Phi^2$, (iv) $m^0\Phi^3$, (v) $m^1\Phi^3$, and (vi) $m^2\Phi^3$ up to $\mathcal{O}(\Phi^4)$.

The sector (i) is rather a routine confirmation at the bilinear order whose details we skip. So is the sector (ii), because it is at the bilinear order without subtlety whose details are skipped. The sector (iii) has only one sort of term: $m^2\chi\phi$, which is straightforward.

The sector (iv) is nontrivial with three subsectors: (a) $\chi\bar\lambda D\lambda$, (b) $\lambda P \mathcal{F}$, and (c) $\chi \mathcal{F}^2$. The subsector (a) needs Fierz-rearrangements and the $\lambda\chi$ term in $\delta_Q\lambda$ for the cancellation of all terms. The subsector (c) of (iv) for χF^2 terms is straightforward whose details are skipped.

The subsector (b) for $\lambda P \mathcal{F}$ terms is the most crucial one. As has been promised with (2.7), the importance of the *modified field strength* $\mathcal{F}_{\mu\nu}{}^I$ instead of $F_{\mu\nu}{}^I$ will be elucidated in this subsector (b).[3] There are three terms contributing to this sector, resulting in their cancellation,[4]

$$\begin{aligned}
0 \stackrel{?}{=} & \; \delta_Q \left[-\frac{1}{2} f^{IJK} (\bar\lambda^I \gamma^\mu \lambda^J) P_\mu{}^K - \frac{1}{4} (\mathcal{F}_{\mu\nu}{}^I)^2 + \frac{1}{2} (\bar\lambda^I \not{D} \lambda^I) \right] \Bigg|_{\lambda P \mathcal{F}} \\
= & - f^{IJK} \left[\left(-\frac{1}{2} \bar\epsilon \gamma^{\rho\sigma} \mathcal{F}_{\rho\sigma}{}^I \right) \gamma^\mu \lambda^J \right] P_\mu{}^K \\
& - \frac{1}{2} [+2 f^{IJK} (\bar\epsilon \gamma_\mu \lambda^J) P_\nu{}^K] \mathcal{F}^{\mu\nu\, I} \\
& + \left[-\frac{1}{2} (\bar\epsilon \gamma^{\rho\sigma}) \mathcal{F}_{\rho\sigma}{}^I \right] \gamma^\mu D_\mu \lambda^I \Bigg|_{\lambda P \mathcal{F}}
\end{aligned} \tag{2.12a}$$

$$\begin{aligned}
\stackrel{\nabla}{=} & + \frac{1}{2} f^{IJK} (\bar\epsilon \gamma^{\mu\rho\sigma} \lambda^I) P_\mu{}^J \mathcal{F}_{\rho\sigma}{}^K + f^{IJK} (\bar\epsilon \gamma^\rho \lambda^I) P^{\sigma J} \mathcal{F}_{\rho\sigma}{}^K \\
& - f^{IJK} (\bar\epsilon \gamma_\mu \lambda^I) P_\nu{}^J \mathcal{F}^{\mu\nu K} + \frac{1}{2} f^{IJK} (\bar\epsilon \gamma^{\rho\sigma\mu} \lambda^I) P_\sigma{}^K \mathcal{F}_{\mu\rho}{}^J
\end{aligned} \tag{2.12b}$$

$$= 0 \quad \text{(Q.E.D.)}, \tag{2.12c}$$

where $\stackrel{\nabla}{=}$ is an equality up to a surface term. The cancellations occurred between the first and fourth as well as second and third terms in (2.12b). In these manipulations, the $\mathcal{F}$-BId (2.6) has been used. Note that if the field strength $(\mathcal{F}_{\mu\nu}{}^I)^2$ in the A_μ-kinetic term *were* replaced by the conventional one $(F_{\mu\nu}{}^I)^2$, there *would* arise *no* $\lambda\mathcal{F}P$ term from $\delta_Q(\mathcal{F}_{\mu\nu}{}^I)^2$ *via* $(\delta_Q P) \wedge P$. Therefore, the $\lambda P F$ terms *would not* be canceled. In a nonsupersymmetric case, there was *no* such necessity. As has been also explained in [5], it is *not* the authors' "subjective taste" to use the *modified* field strength $\mathcal{F}_{\mu\nu}{}^I$, but is required by superinvariance $\delta_Q I_{4\mathrm{D}} = 0$.

The sector (v) has six subsectors: (a) $m\lambda\phi P$, (b) $m\lambda\chi^2$, (c) $m\lambda^3$, (d) $m\lambda\phi D\phi$, (e) $m\chi\phi^2$, and (f) $m\chi\phi\mathcal{F}$. While the subsector (b) needs Fierz rearrangements, the remaining subsectors (a), (c) through (f) are straightforward to handle. The sector (vi) has only one kind of term: $m^2\chi\phi^2$, whose cancellation is straightforward and whose detail is skipped here. The sectors (v) for $m^1\Phi^3$ and (vi) for $m^2\Phi^3$ are rather straightforward with Fierz arrangements, which are *not* peculiar to our system.

The total consistency of our new multiplet $(\varphi^I, \chi^I, \phi^I)$, where only φ^I works as the group-manifold coordinates can be also confirmed by the closures of supersymmetry. This can be accomplished also by the use of the field equations of *all* fields. Since they have been given by (3.9) in [9], we do *not* list them up.

First, the *closure of supersymmetry* on $A_\mu{}^I$ is confirmed as

[2]The symbol Φ stands for any fundamental fields in our multiplets.

[3]This feature has been already mentioned in our papers [5].

[4]The symbol $\stackrel{?}{=}$ stands for an equality that is to be confirmed.

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]A_\mu{}^I = \delta_P A_\mu{}^I + \delta_T^{(0)}A_\mu{}^I + \delta_T^{(1)}A_\mu{}^I = \delta_P A_\mu{}^I + \delta_T A_\mu{}^I, \tag{2.13a}$$

$$\delta_1 \equiv \delta_Q(\epsilon_1), \qquad \delta_2 \equiv \delta_Q(\epsilon_2), \qquad \xi^\mu \equiv +2(\epsilon_1\gamma^\mu\epsilon_2), \qquad \delta_P A_\mu{}^I = \xi^\nu \partial_\nu A_\mu{}^I, \tag{2.13b}$$

$$\delta_T^{(0)} \equiv \delta_T(\alpha^{(0)}), \qquad \delta_T^{(1)} \equiv \delta_T(\alpha^{(1)}), \qquad \delta_T \equiv \delta_T^{(0)} + \delta_T^{(1)}, \tag{2.13c}$$

$$\alpha^I \equiv \alpha^{(0)I} + \alpha^{(1)I}, \qquad \alpha^{(0)I} \equiv -\xi^\nu A_\nu{}^I, \qquad \alpha^{(1)I} \equiv m^{-1} f^{IJK}(\bar\epsilon_1\chi^J)(\bar\epsilon_2\chi^K), \tag{2.13d}$$

where δ_P stands for a translation operation with the parameter ξ^μ. We skip the details, which are explained in [9].

Second, the closure on e^φ is [9]

$$\begin{aligned}([\delta_1,\delta_2]e^\varphi)e^{-\varphi} &= \{\delta_1[(\delta_2 e^\varphi)e^{-\varphi}] - (\delta_2 e^\varphi)(\delta_1 e^{-\varphi})\} - (1 \leftrightarrow 2)\\ &= +(\delta_P e^\varphi)e^{-\varphi} + (\delta_T e^\varphi)e^{-\varphi}, \quad \text{(Q.E.D.)}\end{aligned} \tag{2.14}$$

where $\delta_1 \equiv \delta_Q(\epsilon_1)$, $\delta_2 \equiv \delta_Q(\epsilon_2)$, and $(\delta_T e^\varphi)e^{-\varphi} \equiv -\alpha^{(0)} - \alpha^{(1)} = +\xi^\mu A_\mu{}^I T^I - m^{-1}[(\bar\epsilon_1\chi),(\bar\epsilon_2\chi)]$. Here, we omit the generators T^I, regarding each term as generator valued. This peculiar closure for generator-valued quantities provides supporting evidence for the consistency. In [9], this component result has been further reconfirmed in superspace.

Third, the closure on ϕ^I is

$$[\delta_1,\delta_2]\phi^I = \delta_P\phi^I + \delta_T\phi^I. \quad \text{(Q.E.D.)} \tag{2.15}$$

Note that this closure works, despite the different δ_T transformations of ϕ^I and φ^I.

Fourth, the closure on λ^I contains the λ-field equation,[5]

$$\begin{aligned}[\delta_1,\delta_2]\lambda^I &= +\xi^\mu D_\mu\lambda^I + \delta_T\lambda^I - \frac{1}{4}\xi^\mu\gamma_\mu\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar\lambda^I}\right)\\ &\quad -\frac{1}{4}\zeta^{\mu\nu}\gamma_{\mu\nu}\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar\lambda^I}\right)\end{aligned} \tag{2.16a}$$

$$\doteq \delta_P\lambda^I + \delta_T\lambda^I \quad \text{(Q.E.D.)} \quad \zeta^{\mu\nu} \equiv (\bar\epsilon_2\gamma^{\mu\nu}\epsilon_1). \tag{2.16b}$$

In (2.16b), we have used the λ-field equation.

Fifth, the closure on χ^I is

$$[\delta_1,\delta_2]\chi^I = +\xi^\mu D_\mu\chi^I - \frac{1}{2}\xi^\mu\gamma_\mu\left(\frac{\delta\mathcal{L}_{\rm SPS}}{\delta\bar\chi^I}\right) \tag{2.17a}$$

$$\doteq +\delta_P\chi^I + \delta_T\chi^I. \quad \text{(Q.E.D.)} \tag{2.17b}$$

We have used the χ-field equation.

For the validity of our *unconventional* CM, we mention the following three points: The first reason is rather logical: We already know that a similar situation with a tensor multiplet was presented in [5]. The tensor multiplet (TM) in [5] has the component fields $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$ in terms of the notation in [5]. The reason why the TM in [5] does *not* follow the conventional tensor (linear) multiplet [25,29], i.e., why it can *not* be described in terms of a scalar superfield L is as follows: On the scalar superfield L [25,29], the *commutator* (*but not anticommutator*) of two spinorial derivatives gives

$$[\nabla_\alpha, \bar\nabla_{\dot\beta}]L = c_1(\sigma^{cde})_{\alpha\dot\beta}G_{cde} + c_2\,{\rm tr}(W_\alpha\bar W_{\dot\beta}), \tag{2.18}$$

where α (or $\dot\beta$) is for the positive (or negative) chirality. Note that L is a *singlet* under the YM group, *without* an adjoint index. Obviously, this is *impossible* for *non-Abelian* TM in [5], because the G term in (2.18) should carry the adjoint index, while the ${\rm tr}(W\bar W)$ term does *not*, due to its trace operation. The attempt to make the $W\bar W$ term to be replaced by something like $f^{IJK}(W_\alpha{}^J\bar W_{\dot\beta}{}^K)$ does *not* work either, because such a term vanishes for an *Abelian* case. Because of this lack of fundamental scalar superfield, we do *not* have superspace action formulation at the present time.

The second reason is rather intuitive. Since the spin-zero fields φ^I and ϕ^I play *different roles* under G, it is obvious that this multiplet can *not* be described in terms of a common superfield, such as the scalar superfield L^I carrying the common index for φ^I and ϕ^I. The third reason is based on the analogy of higher-dimensional supersymmetry, e.g., 11D [30] or 10D [31] with *no explicit action formulation in superspace* in terms of *off shell* superfields. In view of this analogy, the lack of *action formulation in superspace* for our *on shell* system is nothing unusual.

Note that our results above are highly sophisticated, so that their cancellations are *neither trivial results nor accidental coincidences*. In particular, the sophisticated cancellations of quadratic-order terms in the closure on φ has *not* been well presented by papers in the past

[5] The symbol $\doteq$ stands for any equality that holds by the use of field equation(s).

before our paper [9]. These computational and intuitive considerations provide the supporting evidence for two important aspects:

(1) $N = 1$ supersymmetry necessitates the *modified* field strength $\mathcal{F}_{\mu\nu}{}^I$ instead of the conventional one $F_{\mu\nu}{}^I$.

(2) Our *nonconventional* CM $(\varphi^I, \chi^I, \phi^I)$ with φ^I and ϕ^I transforming *differently* under δ_T is consistent with $N = 1$ supersymmetry. This has been confirmed with couplings to YM multiplet $(A_\mu{}^I, \lambda^I)$.

Our supersymmetric Proca-Stückelberg theory given by (2.9) through (2.11) is more economical than our previous formulations [5,6]. Notice that in our CM $(\varphi^I, \chi^I, \phi^I)$, the spin-zero fields φ^I and ϕ^I play completely different roles, because the former is for the coordinates of the group manifold G, while the latter is in the adjoint representation of G. To our knowledge, this supersymmetric Proca-Stückelberg theory has *not* been presented before in the past.

III. THE LAGRANGIAN WITH N = (2,0) SUPERSYMMETRY IN 6D

With our successful supersymmetrization of the Proca-Stückelberg theory in 4D [9], we are ready to consider its application to 6D. Similar to 4D, we introduce two $N = (2,0)$ multiplets in our system in 6D [22,28,32], a YM multiplet $(A_\mu{}^I, \lambda^{\underline{\alpha} I})$ and a HM $(\phi^{iI}, \chi_{\underline{\alpha}}{}^I, \varphi^I)$. The $\lambda^{\underline{\alpha} I} \equiv \lambda^{\alpha AI}$ (or $\chi_{\underline{\alpha}}{}^I \equiv \chi_{\alpha A}{}^I$) are Majorana-Weyl spinors with the positive (or negative) chirality: $\gamma_7(\lambda^I, \chi^I) = (+\lambda^I, -\chi^I)$. The index $A = 1, 2$ is for the **2** of $Sp(1)$, which is lowered (or raised) by the antisymmetric $Sp(1)$ metric $\epsilon_{AB} = -\epsilon_{BA}$ (or $\epsilon^{AB} = -\epsilon^{BA}$) [28,32].

As was already stated, the original four scalars in the conventional HM $(\varphi^{aI}, \chi_{\underline{\alpha}}{}^I)$ $(a = 1.2, 3, 4)$ are separated into the singlet **1** and triplet **3** of $Sp(1)$ in our formulation. The former **1** is a Nambu-Goldstone scalar to be absorbed into the longitudinal components of $A_\mu{}^I$.

Our experience with the $N = 1$ supersymmetric case in 4D [9] leads us to the total action, $I_{6\mathrm{D}} \equiv \int d^6x \mathcal{L}_{6\mathrm{D}}$ for our $N = (2,0)$ supersymmetric Proca-Stückelberg theory, where

$$\begin{aligned}\mathcal{L}_{6\mathrm{D}} = &-\frac{1}{4}(\mathcal{F}_{\mu\nu}{}^I)^2 + \frac{1}{2}(\bar{\lambda}^I \not{D} \lambda^I) - \frac{1}{2}(P_\mu{}^I)^2 + \frac{1}{2}(\bar{\chi}^I \not{D} \chi^I) \\ &- \frac{1}{2}(D_\mu \phi^{iI})^2 + m(\bar{\lambda}^I \chi^I) - \frac{1}{2} m^2 (\phi^{iI})^2 \\ &- \frac{1}{2} m^2 f^{IJK} \epsilon^{ijk} \phi^{iI} \phi^{jJ} \phi^{kK} \\ &- m f^{IJK} (\bar{\lambda}^I \tau^i \chi^J) \phi^{iK} - \frac{1}{2} f^{IJK} (\bar{\lambda}^I \gamma^\mu \lambda^J) P_\mu{}^K, \end{aligned} \tag{3.1}$$

up to $\mathcal{O}(\Phi^4)$. The 2×2 matrices $(\tau^i)_A{}^B$ $(i = 1, 2, 3;\ A, B = 1, 2)$ are all anti-Hermitian, satisfying $\tau^i \tau^j = -\delta^{ij} + \epsilon^{ijk} \tau^k$. Or equivalently, $(\tau^i)_A{}^B (\tau^j)_B{}^C = -\delta^{ij} \delta_A{}^C + \epsilon^{ijk} (\tau^k)_A{}^C$. The contracted spinorial and $Sp(1)$ indices are omitted, e.g., $(\bar{\lambda}^I \not{D} \lambda^I) \equiv \lambda^{\alpha AI} (\gamma^\mu)_{\alpha\beta} D_\mu \lambda^{\beta BI} \epsilon_{BA}$, etc.[6]

The covariant derivative D_μ acts with the minimal coupling, like $D_\mu \phi^{iJ} \equiv \partial_\mu \phi^{iJ} + m f^{IJK} A_\mu{}^J \phi^{iK}$ or $D_\mu \lambda^{\underline{\alpha} I} \equiv \partial_\mu \lambda^{\underline{\alpha} I} + m f^{IJK} A_\mu{}^J \lambda^{\underline{\alpha} K}$. The field strengths $\mathcal{F}$ and P are defined similar to (2.7) and (2.4) by

$$\begin{aligned}\mathcal{F}_{\mu\nu}{}^I &\equiv F_{\mu\nu}{}^I + m^{-1} f^{IJK} P_\mu{}^J P_\nu{}^K \\ &\equiv 2\partial_{[\mu} A_{\nu]}{}^I + m f^{IJK} A_\mu{}^J A_\nu{}^K + m^{-1} f^{IJK} P_\mu{}^J P_\nu{}^K, \end{aligned} \tag{3.2a}$$

$$P_\mu{}^I \equiv [(D_\mu e^\varphi) e^{-\varphi}]^I \equiv [(\partial_\mu e^\varphi) e^{-\varphi}]^I + m A_\mu{}^I, \tag{3.2b}$$

where $\varphi \equiv \varphi^I T^I$ with the generators T^I of the gauge group G. These field strengths satisfy the BIds,

$$D_{[\mu} \mathcal{F}_{\nu\rho]}{}^I \equiv + f^{IJK} \mathcal{F}_{[\mu\nu}{}^J P_{\rho]}{}^K, \tag{3.3a}$$

$$D_{[\mu} P_{\nu]}{}^I \equiv + \frac{1}{2} m \mathcal{F}_{\mu\nu}{}^I. \tag{3.3b}$$

Our action $I_{6\mathrm{D}}$ is invariant up to $\mathcal{O}(\Phi^4)$ under $N = (2, 0)$ supersymmetry,

$$\delta_Q A_\mu{}^I = +(\bar{\epsilon} \gamma_\mu \lambda^I) - m^{-1} f^{IJK} (\bar{\epsilon} \chi^J) P_\mu{}^K, \tag{3.4a}$$

$$\begin{aligned}\delta_Q \lambda^I = &+\frac{1}{2}(\gamma^{\mu\nu}\epsilon) \mathcal{F}_{\mu\nu}{}^I - m(\tau^i \epsilon) \phi^{iI} \\ &+ f^{IJK} \lambda^J (\bar{\epsilon} \chi^K) - \frac{1}{2} m f^{IJK} \epsilon^{ijk} (\tau^i \epsilon) \phi^{jJ} \phi^{kK}, \end{aligned} \tag{3.4b}$$

$$\delta_Q \phi^{iI} = +(\bar{\epsilon} \tau^i \chi^I), \tag{3.4c}$$

$$\delta_Q \chi^I = +(\gamma^\mu \tau^i \epsilon) D_\mu \phi^{iI} - (\gamma^\mu \epsilon) P_\mu{}^I, \tag{3.4d}$$

$$[(\delta_Q e^\varphi) e^{-\varphi}]^I = +(\bar{\epsilon} \chi^I). \tag{3.4e}$$

Useful relationships are such as the arbitrary variations,

$$\begin{aligned}\delta \mathcal{F}_{\mu\nu}{}^I = &\ 2D_{[\mu} (\tilde{\delta} A_{\nu]}{}^I) + 2 f^{IJK} (\tilde{\delta} A_{[\mu}{}^J) P_{\nu]}{}^K \\ &- f^{IJK} [(\delta e^\varphi) e^{-\varphi}]^J \mathcal{F}_{\mu\nu}{}^K, \end{aligned} \tag{3.5a}$$

$$\delta P_\mu{}^I = D_\mu [(\delta e^\varphi) e^{-\varphi}]^I + m (\tilde{\delta} A_\mu{}^I) + f^{IJK} [(\delta e^\varphi) e^{-\varphi}]^J P_\mu{}^K, \tag{3.5b}$$

$$\tilde{\delta} A_\mu{}^I \equiv \delta A_\mu{}^I + m^{-1} f^{IJK} [(\delta e^\varphi) e^{-\varphi}]^J P_\mu{}^K. \tag{3.5c}$$

[6]Note that the metric $C_{\alpha\dot{\beta}}$ for 6D spinors are *symmetric*, but it changes the *dottedness* of spinors. For example, we can express $\lambda_{\dot{\alpha}}{}^{AI} \equiv \lambda^{\beta AI} C_{\beta\dot{\alpha}}$; we consistently avoid using the lower dot index ${}_{\dot{\alpha}}$ for a λ field, (or the upper dotted index ${}^{\dot{\alpha}}$ for a χ field).

Note that $\tilde{\delta}_Q A_\mu{}^I$ does *not* have the second term in (3.4a), because of (3.5c). Accordingly, we have

$$\delta_Q \mathcal{F}_{\mu\nu}{}^I = -2(\bar{\epsilon}\gamma_{[\mu}D_{\nu]}\lambda^I) + 2f^{IJK}(\bar{\epsilon}\gamma_{[\mu}\lambda^J)P_{\nu]}{}^K - f^{IJK}(\bar{\epsilon}\chi^J)\mathcal{F}_{\mu\nu}{}^K, \tag{3.6a}$$

$$\delta_Q P_\mu{}^I = (\bar{\epsilon}D_\mu\chi^I) + m(\bar{\epsilon}\gamma_\mu\lambda^I) + f^{IJK}(\bar{\epsilon}\chi^J)P_\mu{}^K. \tag{3.6b}$$

Let us next describe the peculiar properties of our Lagrangian (3.1). First of all, the gauge-symmetry breaking occurs with the effective mass term for $A_\mu{}^I$, arising from the kinetic term $-(1/2)(P_\mu{}^I)^2$ of φ^I. As in 4D, this is because the finite gauge transformation of A_μ in (2.1) can absorb the derivative term on φ as $\tilde{A}_\mu{}^I \equiv A_\mu{}^I + m^{-1}[(\partial_\mu e^{\varphi})e^{-\varphi}]^I$ as $P_\mu{}^I = m\tilde{A}_\mu{}^I$. Thereby, the kinetic term $-(1/2)(P_\mu{}^I)^2$ becomes the mass term $-(1/2)m^2(\tilde{A}_\mu{}^I)^2$.

Second, there is a mixture mass term between λ and χ. This is expected, because of the Proca-Stückelberg mechanism [7]. Because of this mixture mass term, the original gaugino λ becomes massive as a Majorana spinor combined with χ. The $A_\mu{}^I$ absorbs φ^I to be massive, with the original degrees of freedom (d.o.f.) 4 changed to 5 of a massive vector. The original two Majorana-Weyl-fermions λ^I and χ^I, each having 4 d.o.f. are combined to form a Majorana-fermion with $2+2=4$ d.o.f. The remaining ϕ^{iI} field remains as massive carrying 3 d.o.f. All of these are summarized in the following table:

TABLE I. Physical d.o.f. of our fields *modulo* dim G for their adjoint indices.

Before absorptions	$A_\mu{}^I$	λ^I	ϕ^{iI}	χ^I	φ^I
Physical d.o.f.	4	4	3	4	1
After absorptions	$A_\mu{}^I$	λ^I	ϕ^{iI}	χ^I	φ^I
Physical d.o.f.	5	8	3	0	0

Third, some readers may develop a reasonable skepticism against our formulation for the following reason: The fact that the two scalars ϕ^{iI} and φ^I transform differently under G leads to the suspicion that they actually belong to two *different supermultiplets*. There are two reasons why such possibility does *not* make sense. A short reason is that their d.o.f. do *not* match between bosons and fermions. A long reason is as follows: Since ϕ^{iI} has 3 d.o.f., while φ^I has 1 d.o.f. *modulo* the adjoint indices, we need in total 4 d.o.f. from fermions. However, as the general representation analysis in diverse dimensions shows [33], the acceptable minimal units of fermions in $D=5+1$ are either *symplectic* (pseudo) Majorana-Weyl spinors in the **2** of $Sp(1)$ [33]. Each of the pair in the **2** of $Sp(1)$ carries $2^{6/3-1}=2$ d.o.f. as a *Majorana-Weyl* spinor. The subtraction of one in the exponent of 2 here is due to the *Majorana-Weyl* condition. So, the minimal unit of a fermion in $D=5+1$ is a *Majorana-Weyl* spinor with $2\times 2=4$ d.o.f. in total. In other words, any *Majorana-Weyl* fermion in a given supermultiplet should carry 4 d.o.f. If there *were* two scalar supermultiplets, where two scalars ϕ^{iI} and φ^I separately belong to, there *would* be two distinct fermions with 8 d.o.f. in total, each carrying 4 d.o.f. in each scalar multiplet. If so, we *would* be short of bosons. As our Table I and our transformation rule (3.4) show, there is *only one Majorana-Weyl* fermion χ with 4 d.o.f. *modulo* adjoint index. An additional confirmation is the closure of two supersymmetries on all fields φ^I, ϕ^{iI}, and χ^I, as in (3.11) and (3.12) to be shown below.

Fourth, the *modified* field strength $\mathcal{F}_{\mu\nu}{}^I$ is involved as the kinetic term for the YM field $A_\mu{}^I$. This is nothing but the supersymmetrization and 6D generalization of the purely bosonic case (2.8) in 4D.

Fifth, there are terms like $m(\bar{\lambda}\tau\chi)\phi$ and $(\bar{\lambda}\gamma\lambda)P$. These are understandable, because the former is the conventional interaction term between the gaugino and the three scalar fields in the HM. However, the $(\bar{\lambda}\gamma\lambda)P$ term is peculiar to our Proca-Stückelberg formulation, because the scalars φ^I are coordinates of the group manifold G, so that its *bare* field φ^I should *not* appear directly in the Lagrangian. Its involvement is only through its field strength $P_\mu{}^I$, similar to [5,6].

Six, there is a peculiar cubic-potential term: $m^2 f^{IJK}\epsilon^{ijk}\phi^{iI}\phi^{jJ}\phi^{kK}$. The nontriviality of this term is that it contains the structure constants of two different groups: G and $Sp(1)$.

We next describe the details of the invariance confirmation of our action $\delta_Q I = \mathcal{O}(\Phi^4)$. We investigate six sectors: $\mathcal{O}(m^0\Phi^2)$, $\mathcal{O}(m^1\Phi^2)$, $\mathcal{O}(m^2\Phi^2)$, $\mathcal{O}(m^0\Phi^3)$, $\mathcal{O}(m^1\Phi^3)$, and $\mathcal{O}(m^2\Phi^3)$. Let us describe how the confirmation works at each of these sectors, in turn.

(i) At $\mathcal{O}(m^0\Phi^2)$, there are three independent sectors: (a) $\lambda D\mathcal{F}$, (b) χDP, and (c) $\chi D^2\phi$. These sectors are rather routine at the free-field level, whose details we skip here.

(ii) At $\mathcal{O}(m^1\Phi^2)$, there are three sectors: (a) $m\lambda P$, (b) $m\chi\mathcal{F}$, and (c) $m\lambda\phi$. For sector (a), there are two contributions from the φ-kinetic term and the $m\bar{\lambda}\chi$ term. These cancel with no problem. For sector (b), the contributions from the χ-kinetic term and the $m\bar{\lambda}\chi$ term cancel each other, by the use of the P-BId (3.3b). Similarly for sector (c), the λ-kinetic term and the $m\bar{\lambda}\chi$ term cancel each other.

(iii) At $\mathcal{O}(m^2\Phi^2)$, there is only one sector $m^2\chi\phi$. The contributions are from the $m\bar{\lambda}\chi$ and $m^2\phi^2$ terms, which cancel each other.

(iv) At $\mathcal{O}(m^0\Phi^3)$, there are three sectors: (a) $\lambda P\mathcal{F}$, (b) $\chi\mathcal{F}^2$, and (c) $\lambda^2 D\chi$. For sector (a), three terms, $(\bar{\lambda}\gamma\lambda)P$ and the kinetic terms of A and λ, contribute. It is crucial to use the $\mathcal{F}$-BId (3.3a). For sector (b), there is only the potential contribution from the A-kinetic term, which vanishes by itself due to $f^{IJK}\mathcal{F}_{\mu\nu}{}^J\mathcal{F}^{\mu\nu K} \equiv 0$. For sector (c), two contributions are from the λ-kinetic term and the $(\bar{\lambda}\gamma\lambda)P$ term. It is easy to see their cancellation, even without any Fierz identity.

(v) At $\mathcal{O}(m^1\Phi^3)$, there are five sectors: (a) $m\lambda\phi P$, (b) $m\lambda^3$, (c) $m\lambda\phi D\phi$, (d) $m\chi\phi\mathcal{F}$, and (e) $m\lambda\chi^2$. For sector (a), two Lagrangian terms, $(\bar{\lambda}\gamma\lambda)P$ and $(\lambda\tau\chi)\phi$, contribute, but their cancellation is straightforward. For sector (b), the λ-kinetic term and the $(\bar{\lambda}\gamma\lambda)P$ term contribute. They double their contribution of the type,

$$f^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I)(\bar{\lambda}^J\gamma^\mu\lambda^K) \equiv f^{IJK}(\bar{\epsilon}^A\gamma_\mu\lambda_A{}^I)(\bar{\lambda}^{BJ}\gamma^\mu\lambda_B{}^K) \equiv 0, \qquad (3.7)$$

which vanishes *by itself* as a Fierz identity. This identity itself is confirmed by the basic Fierz identity [32],

$$(\bar{\psi}_1\psi_2)(\bar{\psi}_3\psi_4) = -\frac{1}{8}(\bar{\psi}_1\gamma_\mu\psi_4)(\bar{\psi}_3\gamma^\mu\psi_2) + \frac{1}{8}(\bar{\psi}_1\gamma_\mu\tau^i\psi_4)(\bar{\psi}_3\gamma^\mu\tau^i\psi_2) + \frac{1}{96}(\bar{\psi}_1\gamma_{\mu\nu\rho}\psi_4)(\bar{\psi}_3\gamma^{\mu\nu\rho}\psi_2) - \frac{1}{96}(\bar{\psi}_1\gamma_{\mu\nu\rho}\tau^i\psi_4)(\bar{\psi}_3\gamma^{\mu\nu\rho}\tau^i\psi_2), \qquad (3.8)$$

where each of the four spinors ψ_i $(i = 1, 2, \ldots, 4)$ is a *Majorana-Weyl* spinor in the **2** of $Sp(1)$ with the chiralities $\gamma_7(\psi_1, \psi_2, \psi_3, \psi_4) = (-\psi_1, +\psi_2, +\psi_3, -\psi_4)$. The contractions of their **2** indices $A, B, \ldots$ are omitted, e.g., the $(\bar{\psi}_1\psi_2) \equiv (\bar{\psi}_1^A\psi_{2A})$. For sector (c), the kinetic term of ϕ, that of χ, and the $m(\bar{\lambda}\tau\chi)\phi$ term contribute. The crucial relationship is $\tau^i\tau^j = -\delta^{ij} + \epsilon^{ijk}\tau^k$. For sector (d), the contributions from the χ-kinetic term and the $(\bar{\lambda}\tau\chi)\phi$ term cancel each other. For sector (e), the χ-kinetic-term, $m(\bar{\lambda}\chi)$ term, and the $(\bar{\lambda}\tau\chi)\phi$ term contribute. After appropriate Fierzings, there remain only two independent structures: $f^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^I) \times (\bar{\chi}^J\gamma^\mu\chi^K)$ and $f^{IJK}(\bar{\epsilon}\gamma_{[3]}\tau^i\lambda^I)(\bar{\chi}^J\gamma^{[3]}\tau^i\chi^K)$. Fortunately, each of these two structures add up to zero.

(vi) At $\mathcal{O}(m^2\Phi^3)$, there is only one sector $m^2\chi\phi^2$. There are three contributions from the $m(\bar{\lambda}\chi)$, $m(\bar{\lambda}\tau\chi)\phi$, and $m^2\phi^3$ terms. They are shown to cancel each other by the use of $\tau^I\tau^j = -\delta^{ij} + \epsilon^{ijk}\tau^k$. This provides the good confirmation of the peculiar potential term: $(1/2)m^2 f^{IJK}\epsilon^{ijk}\phi^{iI}\phi^{jJ}\phi^{kK}$.

Our field equations for all fields are listed up as

$$\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^I} = +\not{D}\lambda^I + m\chi^I - mf^{IJK}(\tau^i\chi^J)\phi^{iK} - f^{IJK}(\gamma^\mu\lambda^J)P_\mu{}^K \doteq 0, \qquad (3.9a)$$

$$\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\chi}^I} = +\not{D}\chi^I + m\lambda^I - mf^{IJK}(\tau^i\lambda^J)\phi^{iK} \doteq 0, \qquad (3.9b)$$

$$\frac{\delta\mathcal{L}_{6D}}{\delta A_\mu{}^I} = -D_\nu\mathcal{F}^{\mu\nu\, I} - mf^{IJK}(\bar{\lambda}^J\gamma^\mu\lambda^K) - mP^{\mu I} - f^{IJK}P_\nu{}^J\mathcal{F}^{\mu\nu\, K} - \frac{1}{2}f^{IJK}(\bar{\chi}^J\gamma^\mu\chi^K) - mf^{IJK}\phi^{iJ}D_\mu\phi^{iK} \qquad (3.9c)$$

$$\frac{\delta\mathcal{L}_{6D}}{\delta\phi^{iI}} = +D_\mu^2\phi^{iI} - m^2\phi^{iI} - mf^{IJK}(\bar{\lambda}^J\tau^i\chi^K) - \frac{3}{2}f^{IJK}\epsilon^{ijk}\phi^{jJ}\phi^{kK} \doteq 0, \qquad (3.9d)$$

$$\left[\frac{\delta\mathcal{L}_{6D}}{(\delta e^\varphi)e^{-\varphi}}\right]^I \doteq +D_\mu P^{\mu I} - mf^{IJK}(\bar{\lambda}^J\chi^K) \doteq 0, \qquad (3.9e)$$

up to $\mathcal{O}(\Phi^3)$. Note that the φ-field equation (3.9e) directly from the Lagrangian is originally

$$\left[\frac{\delta\mathcal{L}_{6D}}{(\delta e^\varphi)e^{-\varphi}}\right]^I = +D_\mu P^{\mu I} + f^{IJK}(\bar{\lambda}^J\not{D}\lambda^K) - f^{IJK}D_\mu(P_\nu{}^J\mathcal{F}^{\mu\nu K}) \doteq 0. \qquad (3.10)$$

However, for the second term in the right-hand side of (3.10), we can use the λ-field equation yielding the second term $mf(\bar{\lambda}\chi)$ of (3.9e). For this reason, we put the symbol $\doteq$ for the first equality in (3.9e). The third term $D(P\mathcal{F})$ in (3.10) vanishes up to $\mathcal{O}(\Phi^3)$ because of the P-BId (3.3b), and the $A_\mu{}^I$-field equation (3.9c).

We mention one subtlety related to the closure of supersymmetry. The most nontrivial closure of two supersymmetries is on φ^I because of the nonlinear feature of the coordinates φ^I. However, this is just parallel to the 4D case (2.14) [9]. To be more specific, it goes as

$$
\begin{aligned}
([\delta_1,\delta_2]e^{\varphi})e^{-\varphi} &= \{\delta_1[(\delta_2 e^{\varphi})e^{-\varphi}] - (\delta_2 e^{\varphi})(\delta_1 e^{-\varphi})\} - (1\leftrightarrow 2)\\
&= [\delta_1(\bar\epsilon_2\chi^I) + (\delta_2 e^{\varphi})e^{-\varphi}(\delta_1 e^{\varphi})e^{-\varphi}] - (1\leftrightarrow 2)\\
&= \bar\epsilon_2(-\gamma^\mu\epsilon_1 P_\mu{}^I + i\gamma_5\gamma^\mu\epsilon_1 D_\mu\phi^I) - (1\leftrightarrow 2) + [(\bar\epsilon_2\chi),(\bar\epsilon_1\chi)]\\
&= +\xi^\mu P_\mu + f^{IJK}(\bar\epsilon_2\chi^J)(\bar\epsilon_1\chi^K)T^I\\
&= +(\delta_P e^{\varphi})e^{-\varphi} + (\delta_T e^{\varphi})e^{-\varphi}. \quad \text{(Q.E.D.)}
\end{aligned} \tag{3.11}
$$

Here, as in (2.14), $\alpha^I \equiv -\xi^\mu A_\mu{}^I + m^{-1} f^{IJK}(\bar\epsilon_1\chi^J)(\bar\epsilon_2\chi^K)$. As is clearly seen, this is nothing but the same pattern as in 4D in (2.14). This closure works despite the special role as the coordinates of φ of the group manifold of G. Nevertheless, we need to emphasize that the peculiar role played by the scalar φ^I singlet under $Sp(1)$ is consistent with the closure of supersymmetries.

Similarly, we can confirm the closure of supersymmetries both on ϕ^{iI} and χ^I as

$$
[\delta_1,\delta_2]\phi^{iI} = +\xi^\mu D_\mu\phi^{iI} = \delta_P\phi^{iI} + \delta_T\phi^{iI} \quad \text{(Q.E.D.)}, \tag{3.12a}
$$

$$
\begin{aligned}
[\delta_1,\delta_2]\chi^I &= \xi^\mu D_\mu\chi^I - \frac{1}{2}\xi^\mu(\gamma_\mu \not{D}\chi^I) - \frac{1}{2}m\xi^\mu(\gamma_\mu\lambda^I) + \frac{1}{2}mf^{IJK}(\xi^\mu\gamma_\mu\tau^i\lambda^J)\phi^{iK}\\
&= \delta_P\chi^I + \delta_T\chi^I - \frac{1}{2}\xi^\mu\left[\gamma_\mu\left(\frac{\delta\mathcal{L}_{,6D}}{\delta\bar\chi^I}\right)\right] \doteq \delta_P\chi^I + \delta_T\chi^I \quad \text{(Q.E.D.)}.
\end{aligned} \tag{3.12b}
$$

Note that the nontrivial feature of the closure on χ^I with sophisticated Fierzings needed to reach the final form in (3.12b) with the correct coefficients.

As the additional confirmation of the validity of the whole system, we look into the mutual consistency of field equations. We will perform three confirmations: (i) The supersymmetry variation of the λ-field equation (3.9a). (ii) The supervariation of the χ-field equation (3.9b). The divergence of the A_μ-field equation (3.9c). Even though these confirmations are parallel to the corresponding 4D case in [9], we give their details in order to resolve any doubt on our system.

(i) The supervariation of the λ-field equation (3.9a) works as follows:

$$
\begin{aligned}
0 \overset{?}{=} \delta_Q\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar\lambda^I}\right) &= \delta_Q[+\not{D}\lambda^I + m\chi^I - mf^{IJK}(\tau^i\chi^J)\phi^{iK} - f^{IJK}(\gamma^\mu\lambda^J)P_\mu{}^K]\\
&= +\gamma^\mu D_\mu\left[+\frac{1}{2}(\gamma^{\rho\sigma}\epsilon)\mathcal{F}_{\rho\sigma}{}^I - m(\tau^i\epsilon)\phi^{iI} + f^{IJK}\lambda^J(\bar\epsilon\chi^K) - \frac{1}{2}m\epsilon^{ijk}(\tau^i\epsilon)\phi^{jJ}\phi^{kK}\right]\\
&\quad + mf^{IJK}(\bar\epsilon\gamma_\mu\lambda^J)(\gamma_\mu\lambda^K)\\
&\quad + m[-(\gamma^\mu\epsilon)P_\mu{}^I + (\gamma^\mu\tau^i\epsilon)D_\mu\phi^{iJ}]\\
&\quad - mf^{IJK}\tau^i[-(\gamma^\mu\epsilon)P_\mu{}^J + (\gamma^\mu\tau^k\epsilon)D_\mu\phi^{kJ}]\phi^{iK} - mf^{IJK}(\tau^i\chi^J)(\bar\epsilon\tau^i\chi^K)\\
&\quad - f^{IJK}\gamma^\mu\left[+\frac{1}{2}(\gamma^{\rho\sigma}\epsilon)\mathcal{F}_{\mu\nu}{}^J - m(\tau^i\epsilon)\phi^{iJ}\right]P_\mu{}^K\\
&\quad - f^{IJK}(\gamma^\mu\lambda^J)[(\bar\epsilon D_\mu\chi^K) + m(\bar\epsilon\gamma_\mu\lambda^K)] + \mathcal{O}(\Phi^3),
\end{aligned} \tag{3.13}
$$

where each term is evaluated as

$$
+\frac{1}{2}(\gamma^\mu\gamma^{\rho\sigma}\epsilon)D_\mu\mathcal{F}_{\rho\sigma}{}^I = +\frac{1}{2}f^{IJK}(\gamma^{\rho\sigma\tau}\epsilon)\mathcal{F}_{\rho\sigma}{}^J P_\tau{}^K - (\gamma^\mu\epsilon)D_\nu\mathcal{F}^{\mu\nu\,I}, \tag{3.14a}
$$

$$
\begin{aligned}
&+ f^{IJK}(\gamma^\mu D_\mu\lambda^J)(\bar\epsilon\chi^K) + f^{IJK}(\gamma^\mu\lambda^J)(\bar\epsilon D_\mu\chi^K)\\
&= +f^{IJK}\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar\lambda^J}\right)(\bar\epsilon\chi^K) - \frac{1}{8}mf^{IJK}(\gamma_\mu\epsilon)(\bar\chi^J\gamma^\mu\chi^K) + \frac{1}{96}mf^{IJK}(\gamma_{\mu\nu\rho}\tau^i\epsilon)(\bar\chi^J\gamma^{\mu\nu\rho}\tau^i\chi^K)\\
&\quad + f^{IJK}(\bar\epsilon D_\mu\chi^K)(\gamma^\mu\lambda^J),
\end{aligned} \tag{3.14b}
$$

$$+ mf^{IJK}(\bar{\epsilon}\gamma_\mu\lambda^J)(\gamma^\mu\lambda^K) = -\frac{1}{2}mf^{IJK}(\gamma_\nu\epsilon)(\bar{\lambda}^J\gamma^\nu\lambda^K), \tag{3.14c}$$

$$- mf^{IJK}(\tau^i\chi^J)(\bar{\epsilon}\tau^i\chi^K) = -\frac{3}{8}mf^{IJK}(\gamma_\mu\epsilon)(\bar{\chi}^J\gamma^\mu\chi^K) - \frac{1}{96}mf^{IJK}(\gamma_{\mu\nu\rho}\tau^i\epsilon)(\bar{\chi}^J\gamma^{\mu\nu\rho}\tau^i\chi^K), \tag{3.14d}$$

$$-\frac{1}{2}f^{IJK}(\gamma^\mu\gamma^{\rho\sigma}\epsilon)\mathcal{F}_{\mu\rho}{}^J P_\sigma{}^K = -\frac{1}{2}f^{IJK}(\gamma^{\mu\rho\sigma}\epsilon)\mathcal{F}_{\mu\rho}{}^J P_\sigma{}^K + f^{IJK}(\gamma^\rho\epsilon)\mathcal{F}_{\rho\sigma}{}^J P^{\sigma K}. \tag{3.14e}$$

In (3.14c) and (3.14d), we have used the Fierz-identity formula (3.8). Using these in (3.13), we get considerable cancellations, being left only with terms that vanish by the use of field equations,

$$\begin{aligned} 0 \stackrel{?}{=} \delta_Q\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^I}\right) &= (\gamma^\mu\epsilon)\Big[-D_\nu\mathcal{F}_\mu{}^{\nu I} - mf^{IJK}(\bar{\lambda}^J\gamma_\mu\lambda^K) - mP_\mu{}^I \\ &\quad -\frac{1}{2}mf^{IJK}(\bar{\chi}^J\gamma_\mu\chi^K) + f^{IJK}\mathcal{F}_{\mu\nu}{}^J P^{\nu K} - mf^{IJK}\phi^{iJ}D_\mu\phi^{iK}\Big] + f^{IJK}\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^I}\right)(\bar{\epsilon}\chi^K) \\ &= +(\gamma^\mu\epsilon)\left(\frac{\delta\mathcal{L}_{6D}}{\delta A_\mu{}^I}\right) + f^{IJK}\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^J}\right)(\bar{\epsilon}\chi^K) \doteq 0. \quad \text{(Q.E.D.)} \end{aligned} \tag{3.15}$$

Special attention should be paid to the fact that our $\mathcal{F}$-BId (3.3a) has been used in (3.14a), which shows the crucial role played by the *modified* field strength $\mathcal{F}_{\mu\nu}{}^I$ instead of the original $F_{\mu\nu}{}^I$. This also shows that its usage is *not* based on the authors' *"subjective taste"* for the whole formulation.

(ii) The supervariation of the χ-field equation (3.9b) works in a way parallel to that of the λ-field equation, so we give only the result,

$$0 \stackrel{?}{=} \delta_Q\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\chi}^I}\right) = \delta_Q[+\not{D}\chi^I + m\lambda^I - mf^{IJK}(\tau^i\lambda^J)\phi^{iK}] \tag{3.16a}$$

$$= -\epsilon\left[\frac{\delta\mathcal{L}_{6D}}{(\delta e^\varphi)e^{-\varphi}}\right]^I + f^{IJK}\epsilon\left[\bar{\lambda}^J\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^K}\right)\right] + (\tau^i\epsilon)\left(\frac{\delta\mathcal{L}_{6D}}{\delta\phi^{iI}}\right) \doteq 0. \quad \text{(Q.E.D.)} \tag{3.16b}$$

In this confirmation, the validity of the peculiar $\phi\wedge\phi$ term in $\delta_Q\lambda$ in (3.4b) has been reconfirmed, because $\phi\wedge\phi$ term is produced by the variation $\delta_Q\lambda$ in the second term in the rhs of (3.16a).

(iii) The divergence of the $A_\mu{}^I$-field equation (3.9c) works in a way parallel to Eq. (3.19) of [9]. Because of this cancellation structure, we skip the details, giving only the result,

$$\begin{aligned} 0 \stackrel{?}{=} D_\mu\left(\frac{\delta\mathcal{L}_{6D}}{\delta A_\mu{}^I}\right) &= -mf^{IJK}\left[\bar{\lambda}^J\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\lambda}^K}\right)\right] - mf^{IJK}\left[\bar{\chi}^J\left(\frac{\delta\mathcal{L}_{6D}}{\delta\bar{\chi}^K}\right)\right] \\ &\quad - m\left[\frac{\delta\mathcal{L}_{6D}}{(\delta e^\varphi)e^{-\varphi}}\right]^I - mf^{IJK}\phi^{iJ}\left(\frac{\delta\mathcal{L}_{6D}}{\delta\phi^{iK}}\right) \doteq 0. \quad \text{(Q.E.D.)} \end{aligned} \tag{3.17}$$

Since *all* fields $A_\mu{}^I$, λ^I, ϕ^{iI}, χ^I, and φ^I in our system couple to $A_\mu{}^I$, the verification (3.17) provides nontrivial cross confirmation of the consistency of our system. The most crucial aspect is that our two different scalars in two different representations of $Sp(1)$, i.e., φ^I in the **1** of $Sp(1)$ and ϕ^{iI} in the **3** of $Sp(1)$, do *not* pose any problem in our system. This is reflected in the two distinct terms at the end of (3.17).

Other nontrivial confirmations are such as the coefficient of the λ^2 term and χ^2 term in the $A_\mu{}^I$-field equation (3.9c) that differs by a factor of 2. This originates from the extra $f^{IJK}(\bar{\lambda}^I\gamma^\mu\lambda^J)P_\mu{}^K$ term in the Lagrangian (3.1). The validity of this statement has been also confirmed by the divergence confirmation $D_\mu(\delta\mathcal{L}_{6D}/\delta A_\mu{}^I) \stackrel{?}{=} 0$.

These confirmations provide additional supporting evidence for the consistency of our total system. We do *not* have to stress that so many mutual cross confirmations of our system resolve any possible doubt of the validity of our supersymmetric system. Especially, the consistency with the unconventional scalars φ^I and ϕ^{iI} in the two different representations of $Sp(1)$ has been verified. Needless to say, these component-level confirmations have been also reconfirmed in terms of superspace language in the Appendix.

We have so far skipped fixing higher-order terms, such as $\mathcal{O}(\Phi^4)$ at the Lagrangian level, corresponding to $\mathcal{O}(\Phi^3)$ at the field-equation level. This principle is similar to the corresponding case in 4D [9]. A natural question that may be raised is whether such a prescription is really valid. Our short answer is that according to our past experience, once a system with the lowest order but nontrivial supersymmetric interactions is established, it is supposed to work at higher orders.

IV. SUMMARY AND CONCLUDING REMARKS

In this paper, we have presented an economical formulation of $N = (2,0)$ supersymmetric non-Abelian Proca-Stückelberg theory in $D = 5 + 1$ that was *not* known before. We have succeeded in separating the scalar φ^I from the remaining scalars ϕ^{iI} and use the former as the Nambu-Goldstone boson to be absorbed into ${A_\mu}^I$ as the supersymmetric Proca-Stückelberg mechanism. The success of our formulation in 6D is also based on the similar formulation in 4D [9], only with the YM multiplet $({A_\mu}^I, \lambda^I)$ and the CM $(\varphi^I, \chi^I, \phi^I)$, where the CM is used for formulating the supersymmetric Proca-Stückelberg theory, and accordingly, conventional R symmetry is lost in 4D.

This result is unexpected and nontrivial from the following viewpoint: In $D = 5 + 1$, a conventional HM is like $(\phi^a, \psi^{\underline{\alpha}})$ $a = 1, \ldots, 4$; $\underline{\alpha} = (\alpha A)$; $a = 1, 2$; $A = 1, 2$, where all scalars transform uniformly under the gauge group G [23,24]. However, in our present formulation, one single scalar φ^I transforms as the coordinates of the group manifold, while other three remaining scalars ϕ^{iI} transform as the adjoint representation. Under the global automorphism symmetry $Sp(1)$, the former (or the latter) transforms as the $\mathbf{1}$ (or the $\mathbf{3}$) of $Sp(1)$.

As our objective in this paper, we applied our 4D result to 6D with the parallel field content: YM multiplet $({A_\mu}^I, \lambda^I)$ and HM $(\varphi^I, \chi^I, \phi^{iI})$. This is an unexpected result, due to the different transformations of φ^I and $\phi^{iI} i = 1, 2, 3$, respectively, in the $\mathbf{1}$ and $\mathbf{3}$ of $Sp(1)$. Because in the conventional HM, they together form the $\mathbf{4} = \mathbf{2} \times \mathbf{2}$ of $Sp(1) \times Sp(1)$.

This feature further indicates a general conjecture that the scalars in a supermultiplet do *not* have to realize a *uniform representation* or *maximal symmetry*, such as global R symmetry in 4D, or $\mathbf{2} \times \mathbf{2}$ of $Sp(1) \times Sp(1)$ in 6D.

There are differences as well as similarities of our result compared with our previous tensor-hierarchy formulations [5,6]. One example of the latter is the term $f^{IJK}(\bar{\lambda}^I \gamma^\mu \lambda^J){P_\mu}^K$. One example of the former is the cubic-potential term $m^2 f^{IJK} \epsilon^{ijk} \phi^{iI} \phi^{jJ} \phi^{kK}$, which is very peculiar to our present formulation. The latter coupling is possible because of the three scalars ϕ^{iI} in the $\mathbf{3}$ of $Sp(1)$ after the separation of one scalar φ.

Due to the peculiar single scalar φ^I separated from the remaining three scalars ϕ^{iI} in our HM, we have a nonconventional superspace reformulation in the Appendix. This implies that the conventional method using scalar superfields [23,24] is *neither* the only way *nor* the most-general way to describe supersymmetric systems. We have established this fact by explicit supporting evidence in 6D in this paper, in addition to the recent result in 4D [9].

The YM field strength (2.7) for non-Abelian Proca-Stückelberg formulations [7] is modified by a "generalized" Chern-Simon-like term [5,6]. This is interpreted as an "generalized" tensor-hierarchy formulation [1,2], in the sense that BIds of field strengths are modified by the products of field strengths like (1.3) or (3.3).

In our present work, we have presented arguments and provided ample computational details to prove the consistency of our model and the procedures adapted for proving the consistency. Admittedly, our computations have only been carried up to third order in the fields. One concern is that the consistency of our assigning the two scalars, one representing/serving as coordinates and the other conventional, to one superfield may actually not go through once higher orders in the fields that are considered. If this were to be the case, then one way out is to assign one scalar to one superfield (with its own superpartners) and the other scalar to a second superfield (with its own superpartners). Thus, the system would have two superfields with the doubling of the superpartners. We would then recover our present system once the new additional superpartners are either set to zero or are expressed in terms of the known fields through the equations of motion. This procedure will at least overcome the controversial unconventional assignment of the scalar fields as has been used in the presentation and at the same time save the model.

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APPENDIX: SUPERSPACE REFORMULATION

In this Appendix, we give the superspace reformulation of our component results. This reformulation is *not* a routine task, but has two important missions. First, our system has the peculiar split $\mathbf{1} + \mathbf{3}$ in the HM that can *not* be simply described by conventional superfields [23]. This pattern seems to apply also to the method of higher-dimensional superspace mimicking the 4D *chiral superfields* [24]. Second, it provides good supporting evidence of the total consistency of our component formulation, including representation-related subtleties.

The first point can be elucidated by the 4D-based chiral-superfield formulation of HM in 6D with the action of the HM [24],

$$I_{\rm HM} = \int d^6x d^4\theta(\bar{\Phi}_+\Phi_+ + \bar{\Phi}_-\Phi_-) + \int d^6x d^2\theta(\Phi_+\partial\Phi_-) + \int d^6x d^2\bar{\theta}(\bar{\Phi}_+\bar{\partial}\bar{\Phi}_-), \tag{A1a}$$

$$z \equiv \frac{1}{2}(x^4 + ix^5), \qquad \bar{z} \equiv \frac{1}{2}(x^4 - ix^5), \qquad \partial \equiv \frac{\partial}{\partial z} = \partial_4 - i\partial_5, \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \partial_4 + i\partial_5. \tag{A1b}$$

The 4D-based "chiral superfields" Φ and $\bar{\Phi}$ have their R symmetry, just as in 4D [34]. In contrast, in our formulation, the original four scalars are decomposed into the $\mathbf{1} + \mathbf{3}$ of $Sp(1)$, so that it can *not* be described in terms of chiral superfields $\Phi_\pm$ and $\bar{\Phi}_\pm$ in 4D.

Our formulation therefore does *not* rely on 4D-based "chiral superfields" [24]. Instead, our formulation is controlled by the $\mathcal{F}$ and P-BIds,[7]

$$+\frac{1}{2}\nabla_{[A}\mathcal{F}_{BC)}{}^I - \frac{1}{2}T_{[AB|}{}^D\mathcal{F}_{D|C)}{}^I - \frac{1}{2}f^{IJK}\mathcal{F}_{[AB}{}^J P_{C)}{}^K \equiv 0, \tag{A2a}$$

$$+\nabla_{[A}P_{B)}{}^I - T_{AB}{}^C P_C{}^I - m\mathcal{F}_{AB}{}^I \equiv 0. \tag{A2b}$$

These corresponds to the component-BIds (1.3). The superspace constraints at engineering dimensions $0 \le d \le 1$ are

$$T_{\underline{\alpha}\underline{\beta}}{}^c = +2(\gamma^c)_{\underline{\alpha}\underline{\beta}} \equiv +2(\gamma^c)_{\alpha\beta}\epsilon_{AB}, \tag{A3a}$$

$$\mathcal{F}_{\underline{\alpha}b}{}^I = -(\gamma_b\lambda^I)_{\underline{\alpha}} \equiv +(\gamma_b)_{\underline{\alpha}\underline{\beta}}\lambda^{\underline{\beta}I} \equiv +(\gamma_b)_{\alpha\beta}\epsilon_{AB}\lambda^{\beta BI} \equiv -(\gamma_b)_{\alpha\beta}\lambda^{\beta I}{}_A, \tag{A3b}$$

$$P_{\underline{\alpha}}{}^I = -\chi_{\underline{\alpha}}{}^I \equiv -\chi_{\alpha A}{}^I, \tag{A3c}$$

$$\nabla_{\underline{\alpha}}\phi^{iI} = -(\tau^i\chi^I)_{\underline{\alpha}} \equiv -(\tau^i\chi^I)_{\alpha A} \equiv -(\tau^i)_A{}^B\chi_{\alpha B}{}^I, \tag{A3d}$$

$$\nabla_{\underline{\alpha}}\lambda^{\underline{\beta}I} = +\frac{1}{2}(\gamma^{cd})_{\underline{\alpha}}{}^{\underline{\beta}}\mathcal{F}_{cd}{}^I - m(\tau^i)_{\underline{\alpha}}{}^{\underline{\beta}}\phi^{iI} + f^{IJK}\chi_{\underline{\alpha}}{}^J\lambda^{\underline{\beta}K} - \frac{1}{2}mf^{IJK}\epsilon^{ijk}(\tau^i)_{\underline{\alpha}}{}^{\underline{\beta}}\phi^{jJ}\phi^{kK}, \tag{A3e}$$

$$\nabla_{\underline{\alpha}}\chi_{\underline{\beta}}{}^I = -(\gamma^c\tau^i)_{\underline{\alpha}\underline{\beta}}\nabla_c\phi^{iI} - (\gamma^c)_{\underline{\alpha}\underline{\beta}}P_c{}^I. \tag{A3f}$$

[7] We use the superspace-coordinate indices $A = (a, \underline{\alpha}) = (a, \alpha, A), B = (b, \underline{\beta}) = (b, \beta, B), \ldots$, where $a, b, \ldots = 0, 1, \ldots, 5$ (or $\alpha, \beta, \ldots = 1, 2, 3, 4$; $A, B, \ldots = 1, 2$) are for bosonic (or fermionic) coordinates. Our antisymmetrization in superspace is normalized as $M_{[AB)} \equiv M_{AB} - (-1)^{AB}M_{BA}$, *without* the factor of $1/2$. We use this superspace notation only in this section. Other notations in superspace also comply with that in [35].

All other *independent* components, such as $\mathcal{F}_{\underline{\alpha}\underline{\beta}}{}^I$ or $T_{\underline{\alpha}b}{}^c$, are all zero. Even though we are using the $A, B, \ldots = 1, 2$ indices for the $\mathbf{2}$ of $Sp(1)$ that are the same as the superspace indices $A, B, \ldots$, they are distinguished from the context. These are all consistent with the component transformation rule (3.4a) through (3.4e).

The constraints at $d = 3/2$ corresponding to (3.6) are

$$\nabla_{\underline{\alpha}}\mathcal{F}_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\lambda^I)_{\underline{\alpha}} - f^{IJK}(\gamma_{[b|}\lambda^J)_{\underline{\alpha}}P_{|c]}{}^K - f^{IJK}\mathcal{F}_{bc}{}^J\chi_{\underline{\alpha}}{}^K, \tag{A4a}$$

$$\nabla_{\underline{\alpha}}P_b{}^I = -\nabla_b\chi_{\underline{\alpha}}{}^I - m(\gamma_b\lambda^I)_{\underline{\alpha}} - f^{IJK}\chi_\alpha{}^J P_b{}^K. \tag{A4b}$$

As usual, the fermionic superfield equations are obtained by the use of (A4). For example, the λ-field equation is obtained by

$$\begin{aligned}-(\nabla\!\!\!\!/\,\lambda^I)_{\underline{\alpha}} &= +(\gamma^c)_{\underline{\alpha}\underline{\beta}}\nabla_c\lambda^{\underline{\beta}I} = \frac{1}{2}\{\nabla_{\underline{\alpha}}, \nabla_{\underline{\beta}}\}\lambda^{\underline{\beta}I} \\ &= +\frac{1}{2}\nabla_{\underline{\alpha}}(\nabla_{\underline{\beta}}\lambda^{\underline{\beta}I}) + \frac{1}{2}\nabla_{\underline{\beta}}(\nabla_{\underline{\alpha}}\lambda^{\underline{\beta}I}) \\ &= +\frac{1}{2}\nabla_{\underline{\alpha}}[-f^{IJK}(\bar{\chi}^J\lambda^K)] + \frac{1}{2}\nabla_{\underline{\beta}}\Big[+\frac{1}{2}(\gamma^{cd})_{\underline{\alpha}}{}^{\underline{\beta}}\mathcal{F}_{cd}{}^I \\ &\quad - m(\tau^i)_{\underline{\alpha}}{}^{\underline{\beta}}\phi^{iI} + f^{IJK}\chi_{\underline{\alpha}}{}^J\lambda^{\underline{\beta}K} \\ &\quad - \frac{1}{2}mf^{IJK}\epsilon^{ijk}(\tau^i)_{\underline{\alpha}}{}^{\underline{\beta}}\phi^{jJ}\phi^{kK}\Big].\end{aligned} \tag{A5}$$

Evaluating the right-hand side, we get the λ-field equation (A7a) below. For the χ-field equation, due to the negative chirality of χ, the procedure is a little different,

$$\begin{aligned}+(\nabla\!\!\!\!/\,\chi^I)^{\underline{\alpha}} &= (\gamma^c)^{\underline{\alpha}\underline{\beta}}\nabla_c\chi_{\underline{\beta}}{}^I \\ &= +(\gamma^c)^{\underline{\alpha}\underline{\beta}}\left(-\frac{1}{16}\right)(\gamma_c)^{\underline{\delta}\underline{\epsilon}}\{\nabla_{\underline{\delta}}, \nabla_{\underline{\epsilon}}\}\chi_{\underline{\beta}}{}^I \\ &= -\frac{1}{8}(\gamma^c)^{\underline{\alpha}\underline{\beta}}(\gamma_c)^{\underline{\delta}\underline{\epsilon}}\nabla_{\underline{\delta}}(\nabla_{\underline{\epsilon}}\chi_{\underline{\beta}}{}^I) \\ &= -\frac{1}{8}(\gamma^c)^{\underline{\alpha}\underline{\beta}}(\gamma_c)^{\underline{\delta}\underline{\epsilon}}\nabla_{\underline{\delta}}[-(\gamma^d\tau)_{\underline{\epsilon}\underline{\beta}}\nabla_d\phi^{iI} - (\gamma_d)_{\underline{\epsilon}\underline{\beta}}P_c{}^I].\end{aligned} \tag{A6}$$

Evaluating the right-hand side, we get the χ-field equation (A7b) below. As usual, the bosonic field equations for $A_a{}^I, \phi^{iI}$, and φ^I are obtained by applying spinorial derivatives on these fermionic field equations.

We thus reach the field equations of all fields,

$$+\nabla\!\!\!\!/\,\lambda^I + m\chi^I - mf^{IJK}(\tau^i\chi^J)\phi^{iK} - f^{IJK}(\gamma^b\lambda^J)P_b{}^K \doteq 0, \tag{A7a}$$

$$+\nabla\!\!\!\!/\,\chi^I + m\lambda^I - mf^{IJK}(\tau^i\lambda^J)\phi^{iK} \doteq 0, \tag{A7b}$$

$$-\nabla_b \mathcal{F}^{abI} - \frac{1}{2} m f^{IJK} (\bar{\lambda}^J \gamma^a \lambda^K) - m P^{aI} - f^{IJK} P_b{}^J \mathcal{F}^{ab\,K} - \frac{1}{2} m f^{IJK} (\bar{\chi}^J \gamma^a \chi^K) - m f^{IJK} \phi^{iJ} \nabla^a \phi^{iK} \doteq 0, \tag{A7c}$$

$$+\nabla_a^2 \phi^{iI} - m^2 \phi^{iI} - m f^{IJK} (\bar{\lambda}^J \tau^i \chi^K) - \frac{3}{2} f^{IJK} \epsilon^{ijk} \phi^{jJ} \phi^{kK} \doteq 0, \tag{A7d}$$

$$+\nabla_a P^{aI} - m f^{IJK} (\bar{\lambda}^J \chi^K) \doteq 0, \tag{A7e}$$

up to $\mathcal{O}(\Phi^3)$. Here, we omitted the spinorial indices, e.g., $\nabla\!\!\!\!/\,\lambda^I$ instead of $(\nabla\!\!\!\!/\,\lambda^I)_{\underline{\alpha}} \equiv -(\gamma^c)_{\underline{\alpha}\underline{\beta}} \nabla_c \lambda^{\underline{\beta} I}$ etc. Our field equations in (A7) are consistent with the component field equations (3.9). In contrast to the component case in Sec. III, the second term $mf(\bar{\lambda}\chi)$ in (A7e) arises *without* the explicit use of λ- field equation.

As the final remark, we stress the importance of an *on shell* superspace formulation based on BIds such as (A2), instead of using *off shell* chiral superfields in 4D [34]. This has been also emphasized in [9], citing the cases of 10D or 11D superspace [30,31]. One of the reasons is that *off shell* superfield formulation has a certain limit for describing a supersymmetric system. Typical examples are the supersymmetric tensor-hierarchy systems [2], and supergravity in 10D [31] or 11D [30]. When the gauge group is *non-Abelian* like [2,9], the conventional superfield description fails because of the problem mentioned with (2.18).

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