Dynamical analysis of self-gravitating stars in modified Gauss-Bonnet gravity

M. Z. Bhatti,* Z. Yousaf,† and A. Khadim‡

Department of Mathematics, University of the Punjab, Quaid-i-Azam Campus, Lahore-54590, Pakistan

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In this paper, we have continued the work of Herrera *et al.* [1] in $f(G)$ gravitational theory. For this purpose, a spherically symmetric fluid exhibiting locally anisotropic pressure along with the energy density, is taken under consideration. The perturbation scheme is imposed on modified field equations and the dynamical equations. The collapse equation is derived from these perturbed equations which assists to disclose the instability zone under both Newtonian and post-Newtonian constraints. It is wrapped up by concluding that dynamical instability is interpreted by the adiabatic index Γ which relies on the anisotropic pressure, energy density, and the dark source terms due to $f(G)$ gravity.

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I. INTRODUCTION

General relativity (GR) has become a vital constituent of the present era which demonstrates the zigzagging of space time. This is a most alluring theory which explains all the gravitationally affected phenomena, such as motion of the galaxy cluster, behavior of the black holes, and many such mysteries. General relativity lends a different approach to visualizing the Universe and different phenomena happening in the cosmos, such as motion of the planets, expansion of the Universe, and theoretical explanation of black holes. Various experiments, such as gravitational lensing, gravitational redshift, etc., were implemented to test GR, and this theory astonished us by maintaining its standard. Despite a number of large applications, however, there are some phenomena which are not fully illustrated by GR. The interior of a black hole is one such limitation, since all physical laws are declined at the singularity of the black hole, as well as GR having no quantum limit and being unable to clarify the dark matter and dark energy. So, collectively, all these shortcomings paved a path to the development of modified theories of gravity which could help to explain all the queries related to dark energy and the current cosmological model. Another incentive behind the modified gravity is to consolidate different theories such as Kaluza-Klein theory and string theory.

The modified Gauss-Bonnet gravity is one of the compelling modified gravity theories. In modified Gauss-Bonnet gravity, $f(G)$ is the general function of the Gauss-Bonnet with the Gauss-Bonnet invariant G, and it helps to answer the queries related to late time cosmic expansion. The $f(G)$ gravity theory illustrates dark energy with much freedom as compared to other gravity theories. For a suitable choice of function f, this theory can justify the transition from a phantom to a nonphantom state. The $f(G)$ gravity model was initially proposed by Nojiri and Odintsov [2] by adding some terms of Gauss-Bonnet function $f(G)$ to the Hilbert action which helps to analyze various current features of cosmos. Baojiu *et al.* [3] investigated the covariant and gauge invariant perturbation equations and found that cosmological data put few constraints on $f(G)$ models. Felice and Tsujikawa [4] studied the solar system constraints on a cosmologically feasible $f(G)$ gravity model and calculated some corrections to the vacuum Schwarzschild solution. Also, they performed some experiments for the assessment of stability for the modifications to GR. Goheer *et al.* [5] presented the decelerating power-law solutions for the particular form of $f(G)$ theories.

Garcia *et al.* [6] dealt with a special model $f(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}$ of Gauss-Bonnet gravity to investigate the late time cosmic acceleration by imposing weak energy conditions. Zhao *et al.* [7] computed field equations and the equations of motion to figure out a nonconserved energy momentum tensor with the effects of the specified model, having the product of Lagrangian density and an arbitrary function of the Gauss-Bonnet term. Furthermore, they considered two particular models, $f(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}$ and $f(G) = a_3 G^n (1 + b_3 G^m)$, of $f(G)$ gravity to analyze the energy conditions by the means of the power law solution and equation of state of matter with ω smaller than $-1/3$. Bamba *et al.* [8] probed the bouncing cosmology and stability conditions for its solution by remodeling $f(G)$ gravity. It was also observed that unified model $F(G) = P(t)G + Q(t)$ helps to scrutinize the late time cosmic

*mzaeem.math@pu.edu.pk
†zeeshan.math@pu.edu.pk
‡ammarakhadim4@gmail.com

acceleration along with the early time bounce. Nojiri *et al.* [9] constructed two action integrals, with and without auxiliary scalars, to study cosmological reconstruction and deceleration-acceleration transition in modified Gauss-Bonnet gravity. It was found that this action integral corresponds to the cosmological solutions having big bang and big rip singularity, which contains an auxiliary field.

The constraints under which $f(G)$ becomes cosmologically worthwhile were extracted by Felice and Tsujikawa [10]. One of the imperative conditions for stability of late-time de Sitter solution is $d^2f/d^2G > 0$, and it was found that $d^2f/d^2G \to +0$ for $|G| \to \infty$ is asymptotic behavior of feasible models. Zhou *et al.* [11] made a study of $f(G)$ models and tested several toy models to derive the conditions for cosmologically feasible $f(G)$ dark energy models as geometrical constraints. As well as useful trajectories, aping ΛCDM models in radiation and a matter dominated era were also attained. Mohseni [12] studied the effects of force acting along the four velocities of dynamic particles in the background of modified Gauss-Bonnet gravity. Fayaz *et al.* [13] examined the Gauss-Bonnet gravity in a nonisotropic universe with the help of power law solutions. They have explored the criteria for transition to phantom phase and studied the stability issues of modified gravitational models.

Rastkar *et al.* [14] proposed that when the Universe enters a phantom phase, a peculiar class of $f(G)$ has power-law solutions irrespective of the matter dominated and accelerating power-law solutions. Bamba *et al.* [15] calculated viability conditions of some particular $f(G)$ models induced by energy conditions with the help of current estimated data of deceleration parameters. Bhatti *et al.* [16] performed computational simulations to check the stable regions of some strange stars with the help of logarithmic $f(G,T)$ gravity, where T indicates the trace of matter tensor. Yousaf [17,18] examined the stability of collapsing stars, and, after evaluating structure scalars, the role of the Raychaudhuri equation is examined in this context.

A physical model is worthwhile only if it is stable. The preeminent eagerness of a star is to maintain its stability and so it undergoes the collapse or expansion due to the struggle between the interstellar pressure and gravitational pull in order to linger in hydrostatic equilibrium. Chan *et al.* [19] studied the influence of anisotropy and radiation on dynamical instability of a spherical system and found that stability of the system is highly affected due to the influence of anisotropy and radiations. Chan *et al.* [20] explored the influence of shear and shearing viscosity on dynamical instability of the spherical fluid. They found that the fluid with shear collapses rapidly where viscosity decreases the instability. Herrera *et al.* [21] calculated the dynamical instability ranges for the spherically symmetric nonadiabatic fluid configuration in terms of Γ. Moreover, they formulated that instability of the fluid increases or decreases due to the Newtonian (N) corrections and the relativistic terms due to dissipation, respectively.

Chan *et al.* [22] calculated how the dynamical instability is affected by the heat flow and found the range of stability affected by the dissipation and relativistic corrections. Alonso *et al.* [23] investigated the heat conduction for the Lorentz gas by both the analytical and numerical method and selected such crucial dynamical characters which are related to the entire hyperbolic dynamics. Herminghaus [24] discussed that if the working scenario of the two conducting plates is different then the dielectric between them undergoes dynamical instability. Moreover, he suggested that predicted output is a result of the wave intensifying. Herrera *et al.* [1] performed stability analysis in order to calculate stability regimes through a perturbation scheme. Bhatti with his collaborators extended their results and investigated the collapse rate for those relativistic systems that maintain plane [25,26], spherically symmetric [27,28], cylindrically symmetric [29,30], and axially symmetric [31] geometries in their evolution with modified gravity.

The formulation of the paper is as follows. In Sec. II, the field equations and dynamical equations for spherically symmetric anisotropic fluid in the background of $f(G)$ gravity are explored. The perturbation scheme is introduced for metric and material variables and applied to field equations in Sec. III. In Sec. IV, the perturbation is applied on the dynamical equations, and consequently the collapse equation is calculated. In Sec. V, the range of stability is discussed at N and post-Newtonian (pN) eras.

II. FLUID DISTRIBUTION AND FIELD EQUATIONS

In order to continue a systematic investigation of dynamical instability, we assume the fluid distribution to be spherically symmetric and anisotropic. For such configuration, the line element can be written as

$$ds_-^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $A(t,r), B(t,r)$, and $R(t,r)$ are metric coefficients. The energy-momentum tensor depicts the anisotropic fluid distribution of the normal matter, mathematically given as

$$T^-_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r + P_\perp)\chi_\alpha\chi_\beta, \quad (2)$$

where μ is the energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, V^α the four-velocity of the fluid, and χ_α is unit four-vector along the radial direction. The unit four-vectors satisfy the following identities

$$V^\alpha V_\alpha = -1, \qquad \chi^\alpha\chi_\alpha = 1, \qquad \chi^\alpha V_\alpha = 0. \quad (3)$$

We define the four-acceleration and the expansion of the fluid as

$$a_\alpha = V_{\alpha;\beta}V^\beta, \qquad \Theta = V^\alpha{}_{;\alpha}, \tag{4}$$

where; represents covariant derivative. The shearing motion in the fluid can be characterized by the shear tensor whose mathematical form is given by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta). \tag{5}$$

The four-vectors in comoving formalism satisfy

$$V^a = A^{-1}\delta^a_0, \qquad \chi^a = B^{-1}\delta^a_1. \tag{6}$$

The nonzero components of four acceleration are evaluated to be

$$a_1 = \frac{A'}{A},$$

where prime represents the derivative with respect to r while the expression for expansion scalar is found to be

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \tag{7}$$

where the dot represents the derivative with regard to t. In a similar manner, the nonvanishing components of shear tensor are found as

$$\sigma_{11} = \frac{2}{3}B^2\sigma \qquad \sigma_{22} = \sigma_{33}\sin^{-2}\theta = -\frac{1}{3}R^2\sigma, \tag{8}$$

while the shear scalar turns out to be

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right). \tag{9}$$

Now, we will take the modified Gauss-Bonnet gravity [$f(G)$] to study its effects on the dynamical analysis of celestial objects. This theory is viable because it passes the solar system tests under some reasonable choice of $f(G)$ model and may describe the late-time cosmic expansion [2]. This theory manages to discuss the quintessence, crossing acceleration, cosmological constant with a chance to transit between accelerated and decelerated phases in a quiet comprehensive way. The Einstein-Hilbert action can be modified in this gravity theory as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2k}R + f(G)\right) + S_M(g_{\mu\nu}, \psi), \tag{10}$$

where $\kappa = 8\pi$, and S_M is the matter action indicating the matter field depending upon the geometry, while $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$ is the 4-dimensional topological invariant named as Gauss-Bonnet invariant. Here, a specific function $f(G) = G + \alpha G^2$ of family $f(G)$ is taken under consideration. Now, by varying the above action integral corresponding to the metric tensor, we obtain following modified field equations as:

$$G^-_{\alpha\beta} + D^-_{\alpha\beta} = T^-_{\alpha\beta}, \tag{11}$$

where $G^-_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$ is the Einstein tensor, and

$$\begin{aligned} D^-_{\alpha\beta} = {} & 4R_{\alpha\rho\beta\sigma}\nabla^\rho\nabla^\sigma f_G + 4(R_{\rho\beta}g_{\sigma\alpha} - R_{\rho\sigma}g_{\alpha\beta} - R_{\alpha\beta}g_{\sigma\rho} + R_{\alpha\sigma}g_{\beta\rho})\nabla^\rho\nabla^\sigma f_G \\ & + 2R(g_{\alpha\beta}g_{\sigma\rho} - g_{\alpha\sigma}g_{\beta\rho})\nabla^\rho\nabla^\sigma f_G + \frac{1}{2}(Gf_G - f)g_{\alpha\beta}, \end{aligned} \tag{12}$$

with Ricci tensor $R_{\alpha\beta}$, Ricci scalar R, Riemannian tensor $R_{\alpha\rho\beta\sigma}$, and the metric tensor $g_{\alpha\beta}$. The nontrivial components of modified field equations with Eqs. (1), (2), (6), and (11) are given as

$$\begin{aligned} T^-_{00} = \mu A^2 = {} & \left(2\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R} - \left(\frac{A}{B}\right)^2\left[2\frac{R''}{R} + \left(\frac{R'}{R}\right)^2 - 2\frac{B'}{B}\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right] \\ & - \frac{A^2}{2}(Gf_G - f) + \chi_1\dot{f}'_G + \chi_2 f''_G + \chi_3 f'_G + \chi_4\dot{f}_G, \end{aligned} \tag{13}$$

$$T^-_{01} = 0 = -2\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right) + \eta_1\dot{f}_G + \eta_2 f'_G + \eta_3\dot{f}'_G, \tag{14}$$

$$\begin{aligned} T^-_{11} = P_r B^2 = {} & -\left(\frac{B}{A}\right)^2\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} \\ & - \left(\frac{B}{R}\right)^2 + \frac{B^2}{2}(Gf_G - f) + \phi_1\dot{f}'_G + \phi_2\ddot{f}_G + \phi_3\dot{f}_G + \phi_4 f'_G, \end{aligned} \tag{15}$$

$$T^{-}_{22} = T^{-}_{33}\sin^{-2}\theta = P_\perp R^2 = -\left(\frac{R}{A}\right)^2\left[\frac{\ddot{B}}{B}+\frac{\ddot{R}}{R}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)+\frac{\dot{B}}{B}\frac{\dot{R}}{R}\right]$$
$$+\left(\frac{R}{B}\right)^2\left[\frac{A''}{A}+\frac{R''}{R}-\frac{A'}{A}\left(\frac{B'}{B}-\frac{R'}{R}\right)-\frac{B'}{B}\frac{R'}{R}\right]+\frac{R^2}{2}(Gf_G-f)+\psi_1\ddot{f}_G+\psi_2 f''_G+\psi_3\dot{f}_G+\psi_4 f'_G, \tag{16}$$

where χ_i, η_i, ϕ_i, and ψ_i are defined in the Appendix. The corresponding Ricci scalar is given as

$$R = -\frac{2}{R^2B^3A^3}(2R''BRA^3 - 2\ddot{R}B^3RA - \ddot{B}B^2R^2A + A''BR^2A^2 - \dot{R}^2B^3A - 2\dot{R}\,\dot{B}\,RAB^2 + 2\dot{R}\,\dot{A}\,B^3R + R'^2BA^3 - 2R'B'RA^3$$
$$+\dot{B}\,\dot{A}\,R^2B^2 - B'A'R^2A^2 - A^3B^3 + 2R'A'BRA^2). \tag{17}$$

The mass function $m(t,r)$ which describes the total energy inside the radius r is provided by Misner-Sharp [32]. For our line element, it turns out to be

$$m = \frac{R}{2}\left[\left(\frac{\dot{R}}{A}\right)^2 - \left(\frac{R'}{B}\right)^2 + 1\right]. \tag{18}$$

The nontrivial components of Bianchi identities using Eq. (11) turn out to be

$$(T^{-\alpha\beta}_{;\beta} - D^{-\alpha\beta}_{;\beta})V_\alpha = -\frac{1}{A}\left[\dot{\mu} + (\mu+P_r)\frac{\dot{B}}{B} + 2(\mu+P_\perp)\frac{\dot{R}}{R}\right] - \frac{1}{A}\left[\left(-2\frac{\dot{A}}{A^3} + \frac{\dot{B}}{A^2B} + 2\frac{\dot{R}}{RA^2}\right)\right.$$
$$\times\left[\frac{A^2}{2}(Gf_G-f) - \chi_1\dot{f}' - \chi_2 f''_G - \chi_3 f'_G - \chi_4\dot{f}_G\right] + \frac{1}{A^2}\left[A\dot{A}(Gf_G-f) + \frac{A^2}{2}(\dot{G}f_G + G\dot{f}_G - \dot{f})\right.$$
$$\left. - \dot{\chi}_1\dot{f}'_G - \chi_1\ddot{f}'_G - \dot{\chi}_2 f''_G - \chi_2\dot{f}''_G - \dot{\chi}_3 f'_G - \chi_3\dot{f}'_G - \dot{\chi}_4\dot{f}_G - \chi_4\ddot{f}_G\right]$$
$$+\left(\frac{B'}{B^3} - 2\frac{R'}{B^2R}\right)[-\eta_1\dot{f}_G - \eta_2 f'_G - \eta_3\dot{f}'_G] - \frac{1}{B^2}[-\eta'_1\dot{f}_G - \eta_1\dot{f}'_G - \eta'_2 f'_G - \eta_2 f''_G - \eta'_3\dot{f}'_G - \eta_3\dot{f}''_G]$$
$$+\frac{\dot{B}}{B^3}\left[-\frac{B^2}{2}(Gf_G - \dot{f}) - \phi_1\dot{f}'_G - \phi_2\ddot{f}_G - \phi_3\dot{f}_G - \phi_4 f'_G\right]$$
$$\left.+\frac{2\dot{R}}{R^3}\left[-\frac{R^2}{2}(Gf_G-f) - \psi_1\ddot{f}_G - \psi_2 f''_G - \psi_3\dot{f}_G - \psi_4 f'_G\right]\right] = 0, \tag{19}$$

$$(T^{-\alpha\beta}_{;\beta} - D^{-\alpha\beta}_{;\beta})\chi_\alpha = \frac{1}{B}\left[P'_r + (\mu+P_r)\frac{A'}{A} + 2(P_r - P_\perp)\frac{R'}{R}\right] + \frac{1}{B}\left[\frac{1}{A^2}[\dot{\eta}_1\dot{f}_G + \eta_1\ddot{f}_G + \dot{\eta}_2 f'_G + \eta_2\dot{f}'_G + \dot{\eta}_3\dot{f}'_G + \eta_3\ddot{f}'_G]\right.$$
$$+\left(\frac{\dot{A}}{A^3} - 2\frac{\dot{R}}{RA^2}\right)[-\eta_1\dot{f}_G - \eta_2 f'_G - \eta_3\dot{f}'_G] + \left(-\frac{2B'}{B^3} + \frac{A'}{AB^2} + \frac{2R'}{RB^2}\right)$$
$$\times\left(-\frac{B^2}{2}(Gf_G - f) - \phi_1\dot{f}'_G - \phi_2\ddot{f}_G - \phi_3\dot{f}_G - \phi_4 f'_G\right)$$
$$+\frac{1}{B^2}\left(-BB'(Gf_G-f) - \frac{B^2}{2}(G'f_G - Gf'_G - f')\right.$$
$$\left. - \phi'_1\dot{f}'_G - \phi_1\dot{f}''_G - \phi'_2\ddot{f}_G - \phi_2\ddot{f}'_G - \phi_3\dot{f}'_G - \phi'_3\dot{f}_G - \phi_4 f''_G\right)$$
$$+\frac{A'}{A^3}\left(\frac{A^2}{2}(Gf_G-f) - \chi_1\dot{f}'_G - \chi_2 f''_G - \chi_3 f'_G - \chi_4\dot{f}_G\right)$$
$$\left.+2\frac{R'}{R^3}\left(\frac{R^2}{2}(Gf_G - G) + \psi_1\ddot{f}_G + \psi_2 f''_G + \psi_3\dot{f}_G + \psi_4 f'_G\right)\right] = 0. \tag{20}$$

The extra curvature ingredients appearing in the above equations are given in the Appendix.

III. THE PERTURBATION SCHEME

In this section, we will apply perturbation on the field equations and the dynamical equations, which is the small change in the physical system. Here, our concern is check the stability of the system against the radial perturbation scheme. So, initially, we consider that fluid and geometry is described only by radial dependent coordinates; i.e., the system is in static equilibrium. We also assume that with the passage of time all the quantities have both radial and time dependence. Also, we consider that $0 < \epsilon \ll 1$.

We are dealing with the nonlinear partial differential equations. In order to have deep analysis to see the role of radially dependent variables on the dynamical instability of relativistic matter content, we use a particular mathematical method to solve the equations. In literature, there are very few methods of solving nonlinear differential equations; among them there is a method of transforming subsequent equations into separable forms [33,34]. The mathematical profile of the $f(G)$ model after a time $t = 0$ can be expressed in a separable form. Therefore, we have

$$A(t,r) = A_0 r + \epsilon T(t) a(r), \tag{21}$$

$$B(t,r) = B_0(r) + \epsilon T(t) b(r), \tag{22}$$

$$R(t,r) = R_0(r) + \epsilon T(t) c(r), \tag{23}$$

$$\mu(t,r) = \mu_0(r) + \epsilon \bar{\mu}(t,r), \tag{24}$$

$$P_r(t,r) = Pr0(r) + \epsilon \bar{P}_r(t,r), \tag{25}$$

$$P_\perp(t,r) = P_{\perp 0}(r) + \epsilon \bar{P}_\perp(t,r), \tag{26}$$

$$m(t,r) = m_0(r) + \epsilon \bar{m}(t,r), \tag{27}$$

$$\Theta(t,r) = \epsilon \bar{\Theta}(t,r), \tag{28}$$

$$\sigma(t,r) = \epsilon \bar{\sigma}(t,r), \tag{29}$$

$$G(t,r) = G_0(r) + \epsilon T(t) g(r), \tag{30}$$

$$f(t,r) = G_0(1 + \alpha G_0) + \epsilon T g(1 + 2\alpha G_0). \tag{31}$$

We assume that the spherically symmetric anisotropic system with an environment of $f(G)$ gravity is in a static state at a very large past time that can be described through an equation $T(-\infty) = 0$, thereby putting $f(G)$ as a radial dependent function. After this, the system equipped with $f(G)$ dynamics enters in the present state with the passage of time and continues to collapse, and moves on by decreasing its areal radius.

Where the quantities with subscript zero only depend on r, Eqs. (13)–(16) by using Eqs. (21)–(31) for static configuration turn out to be

$$\mu_0 = \frac{1}{(rB_0)^2}\left(2r\frac{B'_0}{B_0} + {B_0}^2 - 1\right) - \frac{\alpha {G_0}^2}{2} + \frac{8\alpha G''_0}{r^2 {B_0}^4}(1 - {B_0}^2) + \frac{8B'_0 \alpha G'_0}{r^2 {B_0}^5}({B_0}^2 - 3), \tag{32}$$

$$P_{r0} = \frac{1}{(rB_0)^2}\left(2r\frac{A'_0}{A_0} - {B_0}^2 + 1\right) + \frac{\alpha {G_0}^2}{2} + \frac{8\alpha G'_0}{r^2 {B_0}^4}\left(-3\frac{A'_0}{A_0} + \frac{{B_0}^2 A'_0}{A_0}\right), \tag{33}$$

$$P_{\perp 0} = \frac{1}{{B_0}^2}\left(\frac{A''_0}{A_0} - \frac{B'_0}{rB_0} - \frac{A'_0 B'_0}{A_0 B_0} + \frac{A'_0}{A_0 r}\right) + \frac{\alpha {G_0}^2}{2} - \frac{8\alpha A'_0 G''_0}{rA_0 {B_0}^4} + \frac{8}{rA_0 {B_0}^4}(3\alpha B'_0 A'_0 G'_0 - \alpha A''_0 B_0 G'_0), \tag{34}$$

and for perturbed configuration, from Eqs. (13)–(16), we have

$$\begin{aligned}\bar{\mu} = {} & -2\frac{T}{{B_0}^2}\left[\left(\frac{c}{r}\right)'' - \frac{1}{r}\left(\frac{b}{B_0}\right)' - \left(\frac{B'_0}{B_0} - \frac{3}{r}\right)\left(\frac{c}{r}\right)' - \left(\frac{B_0}{r}\right)^2\left(\frac{b}{B_0} - \frac{c}{r}\right)\right] - 2\mu_0 T\frac{b}{B_0} \\ & - T\alpha\Bigg[gG_0 + \frac{b{G_0}^2}{B_0} - \frac{4}{r^2 {B_0}^4}\left(4rG''_0\left(\frac{c}{r}\right) - \frac{bG''_0}{2B_0} + 4c\frac{{B_0}^2 G''_0}{r} + 2g''(1 - {B_0}^2)\right) - 4\frac{B'_0}{r^2 {B_0}^5}\Bigg(-12rG'_0\left(\frac{c}{r}\right)' + 18\frac{bG'_0}{B_0} \\ & - 2g'(3 - {B_0}^2) + 6\frac{b'G'_0}{B'_0} - 4c\frac{G'_0 {B_0}^2}{r} - 8bB_0 G'_0 + 2\frac{b'G'_0 {B_0}^2}{B'_0} + 4c''G'_0 {B_0}^3\Bigg)\Bigg],\end{aligned} \tag{35}$$

$$\begin{aligned}0 = {} & 2\frac{\dot{T}}{A_0 B_0}\left[\left(\frac{c}{r}\right)' - \frac{b}{rB_0} - \left(\frac{A'_0}{A_0} - \frac{1}{r}\right)\frac{c}{r}\right] + 8\alpha g\frac{\dot{T}}{{A_0}^4 {B_0}^4}\left[A_0 B_0 A'_0 A''_0 - \frac{{A_0}^2 B_0 A'_0}{r^2} A_0 {A'_0}^2 B'_0 + \frac{{A_0}^2 A'_0 {B_0}^3}{r^2}\right] \\ & + 8\alpha\frac{\dot{T} G'_0}{r^2 A_0^4 B_0^4}\left[-A_0^3 B_0 c' + A_0^2 A'_0 B_0 c + br^2 A_0^2 A''_0 - r^2 A_0^2 A'_0 \frac{bB'_0}{B_0} + bA_0^3 B_0^2\right] \\ & + 8\alpha g'\frac{\dot{T}}{r^2 A_0^3 B_0^3}\left[a_0^2 + r^2 A_0 A'_0\frac{B'_0}{B_0} - A_0^2 B_0^2 + r^2 A_0 A''_0\right],\end{aligned} \tag{36}$$

$$\bar{P}_r=-2\frac{\ddot{T}}{{A_0}^2}\frac{c}{r}+2\frac{T}{r{B_0}^2}\left[\left(\frac{c}{r}\right)'\left(1+r\frac{A_0'}{A_0}\right)+\left(\frac{a}{A_0}\right)'-\frac{B_0^2}{r}\left(\frac{b}{B_0}-\frac{c}{r}\right)\right]$$
$$-2P_{r0}T\frac{b}{B_0}+T\alpha\left[gG_0+b\frac{{G_0}^2}{B_0}\right]+8\alpha\frac{g\ddot{T}}{r^2A_0^3B_0^3}[2r^2A_0''B_0-2r^2A_0'B_0'$$
$$+A_0B_0-4rA_0B_0'-A_0B_0^3]+4\alpha\frac{A_0'T}{r^2B_0^4A_0}\left[-12rG_0'\left(\frac{c}{r}\right)'+12b\frac{G_0'}{B_0}-6g'\right.$$
$$\left.-6G_0'\left(\frac{A_0}{A_0'}\right)\left(\frac{a}{A_0}\right)'-4G_0'B_0^2\frac{c}{r}+2G_0'B_0^2\left(\frac{A_0}{A_0'}\right)\left(\frac{a}{A_0}\right)'+2g'B_0^2\right]+16\alpha\frac{G_0'\ddot{T}}{A_0^2B_0^2}\frac{c}{r^2}, \tag{37}$$

$$\bar{P_\perp}=-\frac{\ddot{T}}{A_0^2}\left(\frac{c}{r}+\frac{b}{B_0}\right)+\frac{T}{B_0^2}\left[\left(\frac{a}{A_0}\right)''+\left(\frac{c}{r}\right)''+\left(2\frac{A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{1}{r}\right)\left(\frac{a}{A_0'}\right)'-\left(\frac{A_0'}{A_0}+\frac{1}{r}\right)\left(\frac{b}{B_0}\right)'+\left(\frac{A_0'}{A_0}-\frac{B_0'}{B_0}+\frac{2}{r}\right)\left(\frac{c}{r}\right)\right]$$
$$-2P_{\perp0}T\frac{b}{B_0}+T\alpha\left(gG_0+b\frac{G_0^2}{B_0}\right)-8\alpha\frac{\ddot{T}}{rA_0^2B_0^3}(gB_0'-cB_0G_0''+cB_0'-b)-8\alpha\frac{TA_0'}{rA_0B_0^4}\left[c'G_0''+g''+G_0''\frac{A_0}{A_0'}\left(\frac{a}{A_0}\right)'\right]$$
$$+2T\alpha b\frac{A_0'G_0''}{rA_0B_0^5}+8\alpha\frac{T}{rA_0B_0^5}\left[A_0'B_0c''G_0'+3A_0'B_0'\left(g'-3b\frac{G_0'}{B_0}+c'G_0'+b'\frac{G_0'}{B_0'}+G_0'\frac{A_0}{A_0'}\left(\frac{a}{A_0}\right)\right)\right.$$
$$\left.-A_0''B_0\left(g'+a\frac{G_0'}{A_0}-3b\frac{G_0'}{A_0}+a''\frac{G_0'}{A_0}+b\frac{G_0'}{B_0}+c'G_0'\right)\right]. \tag{38}$$

The perturbation of the expansion scalar from Eq. (7) and shear from Eq. (9) yields

$$\bar{\Theta}=\frac{\dot{T}}{A_0}\left(\frac{b}{B_0}+2\frac{c}{r}\right), \tag{39}$$

$$\bar{\sigma}=\frac{\dot{T}}{A_0}\left(\frac{b}{B_0}-\frac{c}{r}\right), \tag{40}$$

while the static parts vanish.

IV. THE COLLAPSE EQUATION

The collapse equation will help to find the stability ranges for the anisotropic fluid. By making use of perturbation on the Bianchi identities given in Eqs. (19) and (20) with Eqs. (21)–(31), we obtain

$$0=\frac{1}{B_0}\left[P_{r0}'+(\mu_0+P_{r0})\frac{A_0'}{A_0}+\frac{2}{r}(P_{r0}-P_{\perp0})\right]+\left(\frac{A_0'}{A_0B_0^3}-2\frac{B_0'}{B_0^4}+\frac{2}{rB_0^3}\right)\left[-\frac{B_0^2}{2}\alpha G_0^2-8\alpha\frac{G_0'}{r^2A_0B_0^2}(-3A_0'+B_0^2A_0')\right]$$
$$+\frac{A_0'}{A_0^3B_0}\left[\alpha G_0^2\frac{A_0^2}{2}-8\alpha\frac{G_0''}{r^2B_0^4}\times(A_0^2-A_0^2B_0^2)-8\alpha\frac{G_0'}{r^2A_0B_0^5}(-3B_0'A_0^3+A_0^3B_0^2B_0')\right]+\frac{1}{B_0^3}(-\alpha B_0B_0'G_0^2-\alpha B_0^2G_0G_0')$$
$$-8\alpha\frac{G_0'}{(r^2A_0^3B_0^2)^2}\left[r^2\frac{A_0^3}{B_0}(-3A_0^2A_0''-6A_0{A_0'}^2+2{A_0}^2B_0A_0'B_0'+2A_0{A_0'}^2B_0^2+A_0^2A_0''B_0^2)\right.$$
$$\left.-\left(2r\frac{A_0^3}{B_0}+2r^2A_0^3\frac{B_0'}{B_0^2}+3r^2A_0'\frac{A_0^2}{B_0}\right)(-3A_0^2A_0'+A_0^2B_0^2A_0')\right]-8\alpha\frac{G_0''}{r^2A_0^3B_0^5}(-3A_0^2A_0'+A_0^2A_0'B_0^2)$$
$$+\frac{2}{r^3B_0}\left[r^2\alpha\frac{G_0^2}{2}-8rA_0^2A_0'\alpha\frac{G_0''}{A_0^3B_0^4}+8\alpha\frac{G_0'}{A_0^3B_0^5}(3rA_0^2A_0'B_0'-rA_0^2B_0A_0'')\right], \tag{41}$$

$$0=\frac{1}{A_0}\left[\dot{\bar{\mu}}+(\mu_0+P_{r0})\dot{T}\frac{b}{B_0}+2(\mu_0+P_{\perp0})\dot{T}\frac{c}{r}+D_1\dot{T}\right], \tag{42}$$

$$0=\frac{1}{B_0}\left[\bar{P}_r'+(\mu_0+P_{r0})T\left(\frac{a}{A_0}\right)'+(\bar{\mu}+\bar{P}_r)\frac{A_0'}{A_0}+2(P_{r0}-P_{\perp 0})T\left(\frac{c}{r}\right)'+\frac{2}{r}(P_r-P_\perp)\right]+D_2\ddot{T}+D_3T, \qquad (43)$$

where D_1, D_2, and D_3 are the dark source terms, appearing due to $f(G)$ gravity, and are defined in the Appendix. Integration of Eq. (42) with regard to t gives

$$\bar{\mu}=-\left[(\mu_0+P_{r0})\frac{b}{B_0}+2(\mu_0+P_{\perp 0})\frac{c}{r}+D_1\right]T. \qquad (44)$$

The linear perturbation on the mass function $m(t,r)$ from Eq. (18) leads to

$$m_0=\frac{r}{2}\left(1-\frac{1}{B_0^2}\right), \qquad (45)$$

$$\bar{m}=-\frac{T}{B_0^2}\left[r\left(\bar{c}-\frac{b}{B_0'}\right)+(1-B_0^2)\frac{c}{2}\right]. \qquad (46)$$

The perturbed configuration of the Ricci scalar given in Eq. (17) gives a second order differential equation as follows:

$$\ddot{T}-\omega T=0, \qquad (47)$$

here

$$\omega=\left(2\frac{c}{rA_0^2}+\frac{b}{A_0^2B_0}\right)^{-1}\left[2\frac{c''}{rB_0^2}+\frac{a''}{A_0B_0^2}-2\frac{A_0''}{A_0}\frac{b}{B_0^3}-a\frac{A_0''}{A_0^2B_0^2}+2\frac{c'}{r^2B_0^2}-2\frac{b}{r^2B_0^3}-2\frac{c}{r^3B_0^2}-2\frac{c'}{r}\frac{B_0'}{B_0^3}-2\frac{b'}{rB_0^3}+2\frac{c}{r^2}\frac{B_0'}{B_0^3}+6\frac{bB_0'}{r^2B_0^4}-\frac{b'}{b_0^3}\frac{A_0'}{A_0}-\frac{a'}{A_0}\frac{B_0'}{B_0^3}+a\frac{A_0'}{A_0^2}\frac{B_0'}{B_0^3}+3b\frac{A_0'}{A_0}\frac{B_0'}{B_0^4}+2\frac{c}{r^3}+2\frac{A_0'c'}{rA_0B_0^2}+2\frac{a'}{rA_0B_0^2}-2\frac{bA_0'}{rA_0B_0^3}-2\frac{cA_0'}{r^2A_0B_0^2}-2\frac{aA_0'}{rA_0^2B_0^2}\right]. \qquad (48)$$

The solution of the above equation contains oscillating and nonoscillating parts corresponding to stable and unstable configurations of the stellar interior. Since we are interested in determining the dynamical instability, we neglect one part so that the solution takes the form as

$$T(t)=-\exp(\sqrt{\omega}t), \qquad (49)$$

which corresponds to the unstable (nonoscillating) function. We consider the second law of thermodynamics which relates $\bar{P}$ and $\bar{\mu}$ via the adiabatic index as

$$\bar{P}_r=\Gamma\frac{P_{r0}}{\mu_0+P_{r0}}\bar{\mu}, \qquad \bar{P_\perp}=\Gamma\frac{P_{\perp 0}}{\mu_0+P_{\perp 0}}\bar{\mu}, \qquad (50)$$

where the adiabatic index, also known as Laplace's coefficient, is the ratio of the heat capacity at constant pressure to the heat capacity at constant volume. We can say that specific heat measures the stiffness of the fluid. Substituting the value of $\bar{\mu}$ in Eq. (50), we get

$$\bar{P_r}=-\Gamma\left[P_{r0}\frac{b}{B_0}-2P_{r0}\left(\frac{\mu_0+P_{\perp 0}}{\mu_0+P_{r0}}\right)\frac{c}{r}-\frac{P_{r0}}{\mu_0+P_{r0}}D_1\right]T, \qquad (51)$$

$$\bar{P_\perp}=-\Gamma\left[P_{\perp 0}\frac{b}{B_0}\left(\frac{\mu_0+P_{r0}}{\mu_0+P_{\perp 0}}\right)+2\frac{c}{r}P_{\perp 0}+\frac{P_{\perp 0}}{\mu_0+P_{\perp 0}}D_1\right]T. \qquad (52)$$

From Eq. (47), we obtain

$$\frac{\ddot{T}}{T}=\omega_\Sigma. \qquad (53)$$

Substituting the values of $\bar{\mu}, \bar{P}_r$ and $\bar{P_\perp}$ in Eq. (43), we get the collapse equation as

$$
\begin{aligned}
0 = {} & \frac{1}{B_0}\left[\Gamma\left(-P_{r0}\frac{b}{B_0}-2P_{r0}\left(\frac{\mu_0+P_{\perp 0}}{\mu_0+P_{r0}}\right)\frac{c}{r}-\frac{P_{r0}}{\mu_0+P_{r0}}\right)'+(\mu_0+P_{r0})\times\left(\frac{a}{A_0}\right)'\right]T \\
& -\frac{A_0'}{B_0A_0}\left[(\mu_0+P_{r0})\frac{b}{B_0}+2(\mu_0+P_{\perp 0})\frac{c}{r}+D_1+\Gamma\left[P_{r0}\frac{b}{B_0}+2P_{r0}\left(\frac{\mu_0+P_{\perp 0}}{\mu_0+P_{r0}}\right)\frac{c}{r}+\frac{P_{r0}}{\mu_0+P_{r0}}\right)\right] \\
& +\frac{1}{B}\left[2(P_{r0}-P_{\perp 0})\left(\frac{c}{r}\right)'-2\Gamma\frac{1}{r}\left(P_{r0}\frac{b}{B_0}+2P_{r0}\left(\frac{\mu_0+P_{\perp 0}}{\mu_0+P_{r0}}\right)\frac{c}{r}+\frac{P_{r0}}{\mu_0+P_{r0}}D_1\right)\right]T \\
& +2\Gamma\frac{T}{rB_0}\left[P_{\perp 0}\frac{b}{B_0}\left(\frac{\mu_0+P_{r0}}{\mu_0+P_{\perp 0}}\right)+2P_{\perp 0}\frac{c}{r}+\frac{P_{\perp 0}}{\mu_0+P_{\perp 0}}\right]+D_2\omega_\Sigma-D_3.
\end{aligned} \tag{54}
$$

This equation has a crucial significance in determining the stable/unstable regimes of the spherical star configuration.

V. STABILITY CONDITIONS UNDER NEWTONIAN AND POST-NEWTONIAN REGIMES

In this section, we will find the stability restraints from the collapse equation with the help of the adiabatic index at both N and pN regimes.

A. Newtonian limit

At the N approximation, we assume that geometry is defined with the flat background metric by which instability of the system is discussed in the framework of $f(G)$ gravity. The metric coefficients are constrained as $A_0 = 1$, $B_0 = 1$. Also, we consider that energy density is much greater than the pressure components. So we have $\mu_0 \gg P_{r0}, \mu_0 \gg P_{\perp 0}$. Then Eq. (55) takes the form as

$$
0=\Gamma\left[\left(P_{r0}\left(b+2\frac{c}{r}\right)\right)'+2\frac{P_{r0}}{r}\left(b+2\frac{c}{r}\right)-2\frac{P_{\perp 0}}{r}\left(b+2\frac{c}{r}\right)\right]-(\mu_0+P_{r0})a'-2(P_{r0}-P_{\perp 0})\left(\frac{c}{r}\right)'-D_2\omega_\Sigma-D_3, \tag{55}
$$

which is the condition for the hydrostatic equilibrium. From this equation, we can find a constraint on the adiabatic index for unstable regions as

$$
\Gamma<\frac{(\mu_0+P_{r0})a'+2(P_{r0}-P_{\perp 0})(\frac{c}{r})'+D_2\omega_\Sigma+D_3}{(P_{r0}(b+2\frac{c}{r}))'+\frac{2}{r}(b+2\frac{c}{r})(P_{r0}-P_{\perp 0})}. \tag{56}
$$

The system will be unstable if inequality (56) holds, i.e., if the effect of the numerator term is less than the term in the denominator. If the term in the numerator is greater than the term in the denominator then the system will be in dynamical stability. We found that Γ plays an integral role in describing the stability of the system.

B. Post-Newtonian limits

For post-Newtonian background, we choose $A_0 = 1 - m_0/r, B_0 = 1 + m_0/r$, and relativistic corrections up to order m_0/r. For this approximation, Eq. (54) becomes

$$
\begin{aligned}
0 = {} & -\Gamma\left(1-\frac{m_0}{r}\right)\left[P_{r0}'\left(b-\frac{bm_0}{r}+2\frac{c}{r}\right)+P_{r0}\left(b'+\frac{2}{r^2}(c'r-c)\right)\right] \\
& +\mu_0a'\left(1-\frac{m_0}{r}\right)+\frac{\mu_0m_0}{r^2}(a'r-a)+P_{r0}a'-b\frac{m_0\mu_0}{r^2}-2\mu_0m_0\frac{c}{r^3} \\
& -D_1\left(\frac{m_0}{r}+\frac{m_0^2}{r^3}\right)\left(1-\frac{m_0}{r}\right)+2\left(\frac{c}{r}\right)'P_{r0}-2\left(\frac{c}{r}\right)'P_{\perp 0}\left(1-\frac{m_0}{r}\right) \\
& -2\frac{\Gamma}{r}\left(b+2\frac{c}{r}\right)(P_{r0}-P_{\perp 0})+D_2\omega_\Sigma+D_3.
\end{aligned} \tag{57}
$$

From the above equation, we can extract the unstable constraint for our stellar interior as

$$\Gamma < \frac{\mu_0(a' - \frac{m_0 a}{r^2} - m_0\frac{b}{r^2} + 2c\frac{m_0}{r^3}) + P_{r0}a' + 2(\frac{c}{r})'(P_{r0} - P_{\perp 0})}{P'_{r0}L + P_{r0}M + \frac{2}{r}(b + 2\frac{c}{r})(P_{r0} - P_{\perp 0})} + \frac{D_2\omega_\Sigma + D_3 - D_1(\frac{m_0}{r} - \frac{m_0^2}{r^2} + \frac{m_0^2}{r^3})}{P'_{r0}L + P_{r0}M + \frac{2}{r}(b + 2\frac{c}{r})(P_{r0} - P_{\perp 0})}, \tag{58}$$

where L and M are defined as

$$L = \left[b\left(1 - 2\frac{m_0}{r}\right) + 2\frac{c}{r}\left(1 - \frac{m_0}{r}\right)\right],$$
$$M = \left[b'\left(1 - \frac{m_0}{r}\right) + 2\left(\frac{c}{r}\right)' + 2\frac{m_0}{r}\left(\frac{c}{r}\right)'\right].$$

The anisotropic spherical system will be unstable until inequality (58) holds. The system will be in hydrostatic equilibrium when both numerator and denominator terms have the same effects and the system will enter the state of stability when the numerator term has a greater effect than the denominator term.

VI. CONCLUSION

In this paper, we discussed a systematic analysis for a spherically symmetric, locally anisotropic fluid which collapses adiabatically, and we calculated the instability ranges for such a fluid in the background of $f(G)$ gravity. This objective is followed by acquiring the field equations and the dynamical equation. In order to witness the consequences of $f(G)$ gravity on the stability ranges, we have considered a particular class of $f(G)$ family, i.e., $f(G) = G + \alpha G^2$. Then the field equations and dynamical equations are perturbed, and, after some calculations, we get the collapse equation which helps to estimate limits of dynamical instability. Bekenstein [35] estimated the gravitational collapse of a charged fluid ball by generalizing the Oppenheimer-Volkoff equations of hydrostatic equilibrium and calculating a Christodoulou formula. Toyozawa and Shinozuka [36] analyzed the local and global stabilities of an electron within adiabatic approximation. Ori [37] studied the inner structure of a generic rotating black hole by applying the small perturbation approach. Moreover, stages of formation of the black hole and generalization of the Price's analysis are also deliberated.

Blondin *et al.* [38] deliberated the stability of the spherical accretion shocks occurring in star formation, core-collapse supernovae, etc. Feng *et al.* [39] investigated the aspects affecting stability of the shallow tunnel face. Mostly, dynamical instability of compact objects is described by the adiabatic index. We conclude that dynamical instability given by Γ depends upon anisotropic pressure, energy density, and the dark source terms of the $f(G)$ gravity. Moustakidis [40] analyzed stability criteria for the white dwarfs, neutron stars, and super-massive stars, like compact objects in a similar fashion. We concluded that

(i) The system will be dynamically unstable until inequalities (56) and (58) hold.
(ii) The compact objects will be in hydrostatic equilibrium when the inequalities (56) and (58) are violated.

It is noteworthy that if the dark source terms are eliminated then the constraints for dynamical instability in GR can be obtained.

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APPENDIX

The extra curvature ingredients indicated by χ_i appearing in Eq. (13) are given below:

$$\chi_1 = \frac{8}{AB^3R}(A\dot{B}R' - AB\dot{R}' + A'B\dot{R}),$$
$$\chi_2 = \frac{4}{B^4R^2}(A^2R'^2 - \dot{R}^2B^2 - A^2B^2),$$
$$\chi_3 = \frac{4}{AB^5R^2}(AB^2B'\dot{R}^2 - 3A^3B'R'^2 + B'B^2A^3 + 2R''R'BA^3 - 2AB^2\dot{B}\dot{R}R' - 2ABRR'\dot{B}^2 + 2ARB^2\dot{B}\dot{R}' - 2A'B^2\dot{B}R\dot{R}),$$
$$\chi_4 = \frac{4}{A^2B^3R^2}(3B^2\dot{B}\dot{R}^2 - A^2R'^2\dot{B} + A^2B^2\dot{B} - 2A^2B\dot{R}R'' + 2A^2B'R'\dot{R} - 2ARA'\dot{B}R' + 2ABRA'\dot{R}' - 2A'^2BR\dot{R}).$$

The values of ϕ_i arising in Eq. (15) are given as

$$\phi_1 = -\frac{8}{A^3BR}(A\dot{B}R' - AB\dot{R}' + A'B\dot{R}),$$
$$\phi_2 = \frac{4}{A^6B^3R^2}(2AB^4\dot{A}\,\dot{B}\,R^2 - 2A^2B^4\ddot{B}R^2 + 2A^3B^3R^2A'' - 2A^3B^2R^2A'B'$$
$$+ 4RA^4B^3R'' - \dot{R}^2A^2B^5 - 4A^2B^4R\dot{R}\,\dot{B} + A^4B^3R'^2 - 4A^4B^2RR'B' - A^4B^5),$$
$$\phi_3 = \frac{4}{A^5B^2R^2}(-2A^2B^2R\dot{A}R'' + \dot{A}B^4\dot{R}^2 - A^2B^2\dot{A}R'^2 - 2AB^4\dot{R}\,\ddot{R}$$
$$+ 2A^2B^2A'R'\dot{R} - A'B'RA^2B\dot{R} - 2ARB^2A'^2\dot{R} + 2A^2B^2RA'\dot{R}' - 2A^2BRA'\dot{B}R'),$$
$$\phi_4 = \frac{4}{A^3B^2R^2}(A'B^2\dot{R}^2 - 3A^2A'R'^2 + A^2B^2A' + 2AB^2R'\ddot{R} - 2\dot{A}\,\dot{R}\,B^2R' + 2AR\dot{B}^2R' - 2ABR\dot{B}\,\dot{R}' + 2A'BR\dot{B}\,\dot{R}).$$

The terms ψ_i occurring in Eq. (16) are

$$\psi_1 = \frac{4}{A^4B^3}(A^2BRR'' - B^2R\dot{B}\,\dot{R} - A^2RR'B'),$$
$$\psi_2 = \frac{4}{A^3B^4}(AB^2R\ddot{R} - RB^2\dot{A}\,\dot{R} - RA^2A'R'),$$
$$\psi_3 = \frac{4}{A^5B^3}(A^2BRA'\dot{R}' - ABRA'^2\dot{R} - A^2BR\dot{A}R'' + 3RB^2\dot{A}\,\dot{B}\,\dot{R}$$
$$+ A^2RB'R'\dot{A} - AR\dot{B}B^2\ddot{R} - AB^2R\dot{R}\,\ddot{B} + A^2BR\dot{R}A'' - A^2RA'B'\dot{R}),$$
$$\psi_4 = \frac{4}{A^3B^5}(-ABRR'\dot{B}^2 + AB^2R\dot{B}\,\dot{R}' - A^2BRA'R'' + 3RA^2A'B'R'$$
$$- ARB^2B'\ddot{R} + BRB'\dot{A}\,\dot{R} + ARB^2R'\ddot{B} - A^2BRR'A'' - B^2RR'\dot{A}\,\dot{B}).$$

The expressions of η_i which pop up in Eq. (14) are given as

$$\eta_1 = \frac{4}{A^4B^4R^2}(-A^2B^3\dot{B}\,\dot{R}\,R' + A^2B^4\dot{R}\,\dot{R}' - AB^3R^2A'\ddot{B} + A^2B^2R^2A'A''$$
$$- A^3B^2A'R'^2 + B^3R^2A'\dot{A}\,\dot{B} - A^2R^2BB'A'^2 + A^3B^4A'),$$
$$\eta_2 = \frac{4}{A^4B^4R^2}(-A^4B^2R'\dot{R}' + A^3B^2A'R'\dot{R} - A^2B^2R^2\dot{B}\,\ddot{B} + A^3BR^2\dot{B}A''$$
$$+ A^2B^3\dot{B}\dot{R}^2 + AB^2R^2\dot{A}\dot{B}^2 - A^3R^2\dot{B}B'A' + A^4B^3\dot{B}),$$
$$\eta_3 = \frac{4}{A^3B^3R^2}(AB^2R^2\ddot{B} - AB^3\dot{R}^2 + A^3BR'^2 - B^2R^2\dot{A}\,\dot{B} + A^2R^2A'B' - A^3B^3 - A^2BR^2A'').$$

The dark source terms D_1, D_2, and D_3 in Eqs. (42) and (43) are

$$D_1 = -\frac{1}{A_0}\left(-2\frac{a}{A_0} + \frac{b}{B_0} + 2\frac{c}{r}\right)\left[-8\alpha\frac{G_0''}{r^2B_0^4}(1 - B_0^2) - 8\alpha\frac{G_0'}{r^2B_0^5}(-3B_0'\right.$$
$$\left.+ B_0'B_0^2)\right] - \frac{1}{A_0}\left(\frac{b}{B_0} + 2\frac{c}{r}\right)\frac{\alpha G_0^2}{2} + \alpha g G_0\frac{1}{A_0} - 8\alpha\frac{G_0''}{A_0^2(r^2B_0^4)^2}[(2a$$
$$+ 2A_0^2c' - 2aB_0^2 - 2bA_0B_0)r^2B_0^4 - (A_0^2 - A_0^2B_0^2)(2rcB_0^4 - 4br^2B_0^3)]$$
$$- 8\alpha\frac{g''}{r^2A_0B_0^4}(1 - B_0^2) - 8\alpha\frac{G_0'}{A_0^3(r^2A_0B_0^5)^2}[r^2A_0B_0^5(-6A_0^3B_0'c' - 3b'A_0^3$$
$$- 9aA_0^2B_0' + b'A_0^3B_0^2 + 2bA_0^3B_0B_0' + 3aA_0^2B_0^2B_0' + 2c''A_0^3B_0)$$

$$
\begin{aligned}
&- (-3A_0^3B_0'A_0^3B_0^2B_0')(2rcA_0B_0^5 + r^2aB_0^5 + 5br^2a0B_0^5)]\\
&- 8\frac{\alpha}{A_0}\left(\frac{B_0'}{B_0^3} - 2\frac{1}{rB_0^2}\right)\left[\frac{g}{r^2A_0^4B_0^4}(r^2A_0^2B_0^2A_0'A_0'' - A_0'A_0^3B_0^2 - r^2A_0^2B_0A_0'^2B_0'\right.\\
&+ A_0^3B_0^4A_0') - \frac{G_0'}{r^2A_0^4B_0^4}(-c'A_0^4B_0^2 + cA_0^3B_0^2A_0' + br^2A_0^3B_0A_0'' + bA_0^4B_0^3\\
&\left. - br^2A_0^3A_0'B_0') + \frac{g'}{r^2A_0^3B_0^3}(A_0^3B_0 + r^2A_0^2A_0'B_0' - A_0^3B_0^3 - r^2A_0^2B_0A_0'')\right]\\
&- 8\frac{\alpha}{A_0B_0^2}\left[\frac{g}{(r^2A_0^4B_0^4)^2}[(r^2A_0^4B_0^4)(r^2A_0^2B_0^2A_0'A_0'' + 2r^2A_0^2B_0A_0'B_0'A_0''\right.\\
&+ 2rA_0'A_0^2B_0^2A_0'' + 2r^2A_0B_0^2A_0'^2A_0'' + r^2A_0^2B_0^2A_0''^2 - 2A_0^3B_0^2A_0'\\
&- 2A_0^3A_0'B_0B_0' - 3A_0^2B_0^2A_0'^2 - A_0^3B_0^2A_0'' - 2r^2A_0^2B_0A_0'^2B_0'\\
&- r^2A_0^2B_0A_0'^2B_0'' - 2rA_0^2B_0A_0'^2B_0' - 2r^2A_0B_0A_0'^3B_0' + r^2A_0^2A_0'^2B_0'^2\\
&+ 4A_0^4B_0^3B_0' + 3A_0^2B_0^4A_0'^2 + A_0^3B_0^4A_0'') + (2rA_0^4B_0^4 + 4r^2A_0^3B_0^4B_0'\\
&+ 4r^2A_0^4B_0^3B_0')(r^2A_0^2B_0^2A_0'A_0'' - A_0^3A_0'B_0^2 - r^2A_0^2B_0A_0'^2B_0' + A_0^3B_0^4A_0')]\\
&+ \frac{g'}{r^2A_0^4B_0^4}(r^2A_0^2B_0^2A_0'A_0'' - A_0^3B_0^2A_0' - r^2A_0^2B_0A_0'^2B_0' + A_0^3B_0^4A_0')\\
&+ \frac{G_0'}{(r^2A_0^4B_0^4)^2}[r^2A_0^4B_0^4(-c'B_0^2A_0^4 + cA_0'A_0^3B_0^2 + r^2bA_0^3B_0A_0'' - r^2bA_0^3A_0'B_0'\\
&+ bA_0^4B_0^3)' - (r^2A_0^4B_0^4)'(-c'B_0^2A_0^4 + cA_0'A_0^3B_0^2 + r^2bA_0^3B_0A_0'' + bA_0^4B_0^3\\
&- r^2bA_0^3A_0'B_0')] + \frac{G_0''}{r^2A_0^4B_0^4}(-c'B_0^2A_0^4 + cA_0'A_0^3B_0^2 + r^2bA_0^3B_0A_0''\\
&- r^2bA_0^3A_0'B_0' + bA_0^4B_0^3) + \frac{g'}{(r^2A_0^3B_0^3)^2}[r^2A_0^3B_0^3(A_0^3B_0 + r^2A_0^2A_0'B_0'\\
&- A_0^3B_0^3 - r^2A_0^2B_0A_0'')' - (r^2A_0^3B_0^3)'(A_0^3B_0 + r^2A_0^2A_0'B_0' - A_0^3B_0^3\\
&- r^2A_0^2B_0A_0'')] + \frac{g''}{r^2A_0^3B_0^3}(A_0^3B_0 + r^2A_0^2A_0'B_0' - A_0^3B_0^3 - r^2A_0^2B_0A_0'')\\
&\left. + \frac{b}{A_0B_0^3}\left(\frac{G_0'}{r^2A_0B_0^2}(-3A_0' + A_0'B_0^2)\right) + 16\alpha\frac{c}{r^2A_0}\left(-G_0''\frac{A_0'}{A_0B_0^4} + \frac{G_0'}{A_0B_0^5}(3A_0'B_0' - A_0''B_0)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
D_2 = {}& 8\frac{\alpha}{r^2A_0^2B_0}\left[\frac{g}{A_0^4B_0^4}(r^2A_0^2B_0^2A_0'A_0'' - A_0^3B_0^2A_0' - r^2A_0^2B_0B_0'A_0'^2\right.\\
&+ A_0^3A_0'B_0^4) + \frac{G_0'}{A_0^4B_0^4}(-c'B_0^2A_0^4 + cA_0'A_0^3B_0 + r^2bA_0^3B_0A_0'' - r^2bA_0^3A_0'B_0'\\
&\left. + bA_0^4B_0^3) + \frac{g'}{A_0^3B_0^3}(A_0^3B_0 + r^2A_0^2A_0'B_0' - A_0^3B_0^3 - r^2A_0^2B_0A_0'')\right]\\
&- 8\alpha\frac{g}{r^2A_0^3B_0^4}\left(-2\frac{B_0'}{B_0} + \frac{A_0'}{A_0} + \frac{2}{r}\right)(2r^2A_0''B_0 - 2r^2A_0'B_0' + A_0B_0\\
&- 4rA_0B_0' - A_0B_0^3) - 16\alpha c\frac{G_0'}{r^2A_0^2B_0^3}\left(-2\frac{B_0'}{B_0} + \frac{A_0'}{A_0} + \frac{2}{r}\right)\\
&- 8\alpha\frac{g}{(r^2A_0^5B_0^2)^2B_0^3}[(r^2A_0^5B_0^2)(2r^2A_0^3B_0^3A_0'' - 2r^2A_0^3B_0^2A_0'B_0' + A_0^4B_0^3\\
&- 4rA_0^4B_0^2B_0' - A_0^4B_0^5)' - (r^2A_0^5B_0^2)'(2r^2A_0^3B_0^3A_0'' - 2r^2A_0^3B_0^2A_0'B_0'
\end{aligned}
$$

$$+ A_0^4B_0^3 - 4rA_0^4B_0^2B_0' - A_0^4B_0^5)] - 8\alpha\frac{g'}{r^2A_0^5B_0^5}(2r^2A_0^3B_0^3A_0''$$
$$- 2r^2A_0^3B_0^2A_0'B_0' + A_0^4B_0^3 - 4rA_0^4B_0^2B_0' - A_0^4B_0^5) - 8\alpha\frac{G_0'}{(r^2A_0^3B_0^2)^2B_0^3}$$
$$\times [(r^2A_0^3B_0^2)(2c'A_0B_0^2 + 4cA_0B_0B_0' + 2cB_0^2A_0') - 2cA_0B_0^2] - 8\alpha\frac{G_0''}{r^2A_0^3B_0^5}$$
$$\times (2cA_0B_0^2) - 16\alpha\left[g\frac{B_0'}{r^2A_0^2B_0^4} - c\frac{G_0''}{r^2A_0^2B_0^3} - \frac{G_0'}{r^2A_0^2B_0^4}(-cB_0' + b)\right],$$

$$D_3 = -2\frac{B_0'}{B_0^4}\left[-\alpha G_0^2\frac{B_0^2}{2}\left(\frac{b'}{B_0'}\right) - \alpha gG_0B_0^2 - 8\alpha\frac{G_0'}{r^2A_0B_0^2}\left(-3A_0'(2r\left(\frac{c}{r}\right)'\right.\right.$$
$$\left.\left. + \frac{a'}{A_0'} - \frac{a}{A_0} + \frac{b'}{B_0'}\right) + A_0'B_0^2\left(-\frac{a}{A_0} - \frac{a'}{A_0'} - 2\frac{c}{r} + \frac{b'}{B_0'}\right)\right)\Big]$$
$$+ \frac{A_0'}{A_0B_0^3}\left[-\alpha G_0^2\frac{B_0^2}{2}\left(\frac{a'}{A_0'} - \frac{a}{A_0}\right) - \alpha gG_0B_0^2 - 8\alpha\frac{G_0'}{r^2A_0B_0^2}\left(-3A_0\left(2c' + 2\frac{a'}{A_0'}\right.\right.\right.$$
$$\left.\left.\left. - 2\frac{a}{A_0} - 2\frac{c}{r}\right) + A_0'B_0^2\left(-2\frac{a}{A_0} - 2\frac{c}{r}\right)\right) - 8\alpha\frac{g'}{r^2A_0B_0^2}(-3A_0' + A_0'B_0^2)\right]$$
$$+ \frac{2}{rB_0^3}\left[-\alpha G_0^2\frac{B_0^2}{2}\left(c' - \frac{c}{r}\right) - \alpha gG_0B_0^2 - 8\alpha\frac{G_0'}{r^2A_0B_0^2}\left(-3A_0'\left(3c' + \frac{a'}{A_0'}\right.\right.\right.$$
$$\left.\left. - \frac{a}{A_0} - 3\frac{c}{r}\right) + A_0'B_0^2\left(-\frac{a}{A_0} - \frac{a'}{A_0'} - 3\frac{c}{r} + c'\right)\right) - 8\alpha\frac{g'}{r^2A_0B_0^2}(-3A_0'$$
$$\left. + A_0'B_0^2)\right] + \frac{A_0'}{A_0^3B_0}\left[\alpha G_0^2\frac{A_0^2}{2}\left(-\frac{a}{A_0} + \frac{a'}{A_0'}\right) + \alpha gG_0A_0^2 - 8\alpha\frac{G_0''}{r^2B_0^4}\left(A_0^2\left(\right.\right.\right.$$
$$\left.\left. - \frac{a}{A_0} + 2c' - 2\frac{c}{r} + \frac{a'}{A_0'}\right) - A_0^2B_0^2\left(-\frac{a}{A_0} - 2\frac{c}{r} + \frac{a'}{A_0'}\right)\right) - 8\alpha\frac{g''}{r^2B_0^4}(A_0^2$$
$$- A_0^2B_0^2) - 8\alpha\frac{G_0'}{r^2B_0^5}\left[-3A_0^2B_0'\left(2c' + \frac{b'}{B_0'} - 2\frac{c}{r} - \frac{a}{A_0} + \frac{a'}{A_0'}\right)\right.$$
$$\left. + A_0^2B_0^2B_0'\left(\frac{b'}{B_0'} - \frac{a}{A_0} - 2\frac{c}{r} + \frac{a'}{A_0'}\right) + 2c''A_0^2B_0\right] - 8\alpha\frac{g'}{r^2B_0^5}(-3A_0^2B_0$$
$$\left. + A_0^2B_0^2B_0')\right] + \frac{1}{B_0^3}\left[-\alpha B_0B_0'G_0^2\frac{b'}{B_0'} - 2\alpha gG_0B_0B_0' - \alpha B_0^2(gG_0)'\right.$$
$$- 8\alpha\frac{G_0'}{(r^2A_0^3B_0^2)^2B_0^3}\left[r^2A_0^3B_0^2\left(-6c''A_0^2A_0' - 3A_0''A_0^2\left(2c' + \frac{a''}{A_0''} - \frac{a}{A_0}\right.\right.\right.$$
$$\left. - 2\frac{c}{r}\right) - 6A_0A_0'^2\left(2c' + 2\frac{a'}{A_0'} - 2\frac{a}{A_0} - 2\frac{c}{r}\right) + 2A_0^2A_0'B_0B_0'\left(\frac{b'}{B_0'} + \frac{a'}{A_0'}\right.$$
$$\left. - \frac{a}{A_0} - 2\frac{c}{r}\right) + 2A_0A_0'^2B_0^2\left(-2\frac{a}{A_0} + 2\frac{a'}{A_0'} - 2\frac{c}{r}\right) + A_0''A_0^2B_0^2\left(-\frac{a}{A_0}\right.$$
$$\left.\left. + \frac{a''}{A_0''} - 2\frac{c}{r}\right)\right) - \left(2rA_0^3B_0^2\left(-3\frac{c}{r} + c' - 3\frac{a}{A_0}\right) + 2r^2A_0^3B_0B_0'\left(-\frac{c}{r}\right.\right.$$
$$\left.\left.\left. + \frac{b'}{B_0'} - 3\frac{a}{A_0}\right) + 3r^2A_0^2A_0'B_0^2\left(-2\frac{c}{r} - 4\frac{a}{A_0} + \frac{a'}{A_0'}\right)\right)\right]$$
$$- 8\alpha\frac{g'}{(r^2A_0^3B_0^2)^2B_0^3}[(r^2A_0^3B_0^2)(-3A_0^2A_0' + A_0^2B_0^2A_0')'$$

$$
\begin{aligned}
&-(r^2A_0^3B_0^2)'(-3A_0^2A_0'+A_0^2B_0^2A_0')]-8\alpha\frac{G_0''}{r^2A_0^3B_0^5}\left[-3A_0^2A_0'\left(c'+\frac{a'}{A_0'}-\frac{a}{A_0}-2\frac{c}{r}\right)\right.\\
&\left.+A_0^2A_0'B_0^2\left(-\frac{a}{A_0}+\frac{a'}{A_0'}-2\frac{c}{r}\right)\right]-8\alpha\frac{g''}{r^2A_0^3B_0^5}(-3A_0'A_0^2+A_0^2A_0'B_0^2)\\
&+\frac{2}{r^3B_0}\left[r^2\alpha\frac{G_0^2}{2}\left(-\frac{c}{r}+c'\right)+r^2\alpha gG_0-8\alpha g\frac{rB_0'}{A_0^2B_0^3}-8r\alpha A_0'\frac{G_0''}{A_0B_0^4}\left(2c'\right.\right.\\
&\left.+3\frac{a'}{A_0'}-2\frac{c}{r}-\frac{a}{A_0}\right)-8\alpha r\frac{g''A_0'}{A_0B_0^4}+8\alpha\frac{G_0'}{A_0^3B_0^5}\left(-rc''A_0^2A_0'B_0\right.\\
&+3rA_0^2A_0'B_0'\left(2c'+\frac{a'}{A_0'}+\frac{b'}{B_0'}-2\frac{c}{r}-\frac{a}{A_0}\right)-rA_0^2B_0A_0''\left(\frac{a''}{A_0''}-2\frac{c}{r}\right.\\
&\left.\left.\left.-\frac{a}{A_0}+2c'\right)\right)+8\alpha\frac{g'}{A_0B_0^5}(3rA_0'B_0'-rA_0''B_0)\right]+\frac{b}{B_0}\left[\frac{1}{2}\alpha G_0^2\frac{A_0'}{A_0B_0}\right.\\
&\left.+\alpha\frac{G_0^2}{rB_0}\right]+\alpha\frac{b}{B_0}\left[240G_0'\frac{A_0'B_0'}{r^2A_0B_0^6}-120G_0'\frac{A_0'^2}{r^2A_0^2B_0^5}-240G_0'\frac{A_0'}{r^2A_0B_0^5}\right.\\
&-48G_0'\frac{A_0'B_0'}{r^2B_0^4}+24G_0'\frac{A_0^{2\prime}}{r^2A_0B_0^3}+48G_0'\frac{A_0'}{r^3B_0^3}+32G_0''\frac{A_0'}{r^2A_0B_0^5}\\
&-16G_0''\frac{A_0'}{r^2A_0B_0^3}-120G_0'\frac{A_0'B_0'}{r^2A_0B_0^6}+24G_0'\frac{A_0'B_0'}{r^2A_0B_0^4}-96G_0'\frac{A_0''}{r^2A_0B_0^5}\\
&-192G_0'\frac{A_0'^2}{r^2A_0^2B_0^5}+48G_0'\frac{A_0'B_0'}{r^2A_0B_0^4}+32G_0'\frac{A_0'^2}{r^2A_0^2B_0^3}+16G_0'\frac{A_0''}{r^2A_0B_0^3}\\
&+192G_0'\frac{A_0'}{r^3A_0B_0^5}+240G_0'\frac{A_0'B_0'}{r^2A_0B_0^6}+288G_0'\frac{A_0'^2}{r^2A_0^2B_0^5}-32G_0'\frac{A_0'}{r^3A_0B_0^3}\\
&-48G_0'\frac{A_0'B_0'}{r^2A_0B_0^4}-48G_0'\frac{A_0'^2}{r^2A_0^2B_0^3}-96G_0''\frac{A_0'}{rA_0B_0^2}+16G_0''\frac{A_0'}{r^2A_0B_0}\\
&\left.+64G_0''\frac{A_0'}{r^2A_0B_0^5}-240G_0'\frac{A_0'B_0'}{r^2A_0B_0^6}+64G_0'\frac{A_0''}{r^2A_0B_0^5}\right].
\end{aligned}
$$

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