Reissner-Nordström black holes supporting nonminimally coupled massive scalar field configurations

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It has recently been demonstrated that static spatially regular scalar fields, which are nonminimally coupled to the electromagnetic field of a charged central black hole, can be supported in the exterior regions of the black-hole spacetime. In the present paper, we use *analytical* techniques in order to study the physical and mathematical properties of the externally supported linearized scalar field configurations (scalar "clouds") in the dimensionless large-mass regime $\mu r_+ \gg 1$ (here, μ and r_+ are respectively the proper mass of the supported scalar field and the outer horizon radius of the central supporting black hole). In particular, we derive a remarkably compact analytical formula for the discrete resonant spectrum $\{\alpha_n(\mu; Q/M)\}_{n=0}^{n=\infty}$ which characterizes the dimensionless coupling parameter of the composed black hole–nonminimally coupled linearized massive scalar field configurations. The physical significance of this resonant spectrum stems from the fact that, for a given value of the dimensionless black-hole electric charge Q/M, the fundamental (smallest) eigenvalue $\alpha_0(\mu)$ determines the critical existence line of the composed black hole–massive field system, a boundary line which separates nonlinearly coupled hairy charged black hole–massive scalar field configurations from bald Reissner-Nordström black holes. The analytical results derived in this paper are confirmed by direct numerical computations.

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I. INTRODUCTION

The canonical no-hair theorems presented in Refs. [1–3] have revealed the physically interesting fact that, in the composed Einstein-Maxwell-scalar theory, static scalar field configurations cannot be supported in the exterior regions of asymptotically flat black-hole spacetimes with spatially regular horizons. This physically intriguing property also characterizes black-hole spacetimes in which the scalar fields are nonminimally coupled to the Ricci curvature scalar of the spacetime [4,5].

Intriguingly, it has recently been demonstrated in the physically interesting works [6,7] that spatially regular massless scalar field configurations which are characterized by a nonminimal coupling of the form $f(\phi)F_{\mu\nu}F^{\mu\nu}$ to the electromagnetic Maxwell tensor [see Eq. (4) below] *can* be supported in spherically symmetric asymptotically flat charged black-hole spacetimes. This phenomenon, which is known by the name black-hole spontaneous scalarization, has also been studied in Refs. [8,9] in the physically interesting context of massive scalar field configurations which are nonminimally coupled to the electromagnetic tensor of the charged black-hole spacetime.

As explicitly proved in Refs. [6,7], if the nontrivial scalar field–electromagnetic field coupling function is characterized by the weak-field functional behavior $f(\phi) = 1 + \alpha\phi^2 + O(\phi^4)$, then the bald (scalarless) charged Reissner-Nordström black-hole spacetime is a valid solution of the field equations in the trivial $\phi \equiv 0$ limit. This is a physically desirable property of the nontrivially coupled Einstein-Maxwell-scalar theory. Here, $\alpha > 0$ is a dimensionless physical parameter which determines the strength of the nonminimal coupling between the supported scalar field and the electromagnetic field of the charged black-hole spacetime.

The numerical results presented in the physically interesting works [6–9] have demonstrated that, for a given value of the dimensionless charge-to-mass ratio Q/M [10] of the black-hole spacetime, the nontrivially coupled Einstein-Maxwell-scalar system is characterized by the existence of a critical *existence line* $\alpha = \alpha(\mu; Q/M)$ which marks the boundary between hairy charged black hole–nonminimally coupled massive scalar field configurations and bald (scalarless) Reissner-Nordström black-hole spacetimes (here, μ is the proper mass of the nonminimally coupled scalar field). In particular, the critical existence line of the composed system corresponds to spatially regular linearized field configurations which are supported by a central charged Reissner-Nordström black hole (the term "scalar clouds" is usually used in the physics literature [11,12] in order to describe these linearized scalar field configurations which sit on the critical existence line of the system).

It is important to emphasize the fact that, as nicely demonstrated in Refs. [6,7], the critical existence line of the black hole–field system is universal in the sense that different scalar field–electromagnetic field coupling functions $\{f(\phi)\}$ with the same weak-field behavior, $f(\phi) = 1 + \alpha\phi^2 + O(\phi^4)$, are characterized by the same functional behavior $\alpha = \alpha(\mu; Q/M)$ of the critical existence line.

The main goal of the present paper is to study, using *analytical* techniques, the physical and mathematical properties of the composed Einstein-Maxwell–nonminimally coupled scalar field theory in the dimensionless regime $M\mu \gg 1$ of large field masses. In particular, we shall derive a remarkably compact analytical formula for the critical existence line $\alpha = \alpha(\mu; Q/M)$ which characterizes the Reissner-Nordström black hole–linearized massive scalar field cloudy configurations. Interestingly, the analytically derived resonance formula [see Eq. (31) below] for the composed black-hole-field system would provide a simple analytical explanation for the *numerically* discovered [8,9] monotonic functional behavior of the relation $\alpha = \alpha(\mu; Q/M)$ along the critical existence line of the system.

II. DESCRIPTION OF THE SYSTEM

We shall study the physical and mathematical properties of linearized massive scalar field configurations (scalar clouds) which are nontrivially coupled to the electromagnetic field of a charged Reissner-Nordström black hole. The line element of the spherically symmetric charged black-hole spacetime can be expressed in the form [13]

$$ds^2 = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1)$$

where

$$h(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \qquad (2)$$

Here, M and Q are respectively the black-hole mass and its electric charge. The black-hole horizon radii $\{r_+, r_-\}$ are determined by the polynomial equation $h(r = r_\pm) = 0$, which yields

$$r_\pm = M + (M^2 - Q^2)^{1/2}. \qquad (3)$$

The composed charged black hole–nonminimally coupled massive scalar field system is characterized by the action [6–9]

$$S = \int d^4x\sqrt{-g}[R - 2\nabla_\alpha\phi\nabla^\alpha\phi - 2\mu^2\phi^2 - f(\phi)\mathcal{I}], \qquad (4)$$

where the nontrivial coupling between the scalar field ϕ and the electromagnetic Maxwell tensor $F_{\mu\nu}$ of the central charged black hole is induced by the source term

$$\mathcal{I} = F_{\mu\nu}F^{\mu\nu}. \qquad (5)$$

The coupling function $f(\phi)$ of the supported massive scalar field configurations is characterized by the universal quadratic behavior [6–9]

$$f(\phi) = 1 + \alpha\phi^2 \qquad (6)$$

in the weak-field regime, where the dimensionless expansion constant α is the physical coupling parameter of the composed black-hole-field theory. We shall henceforth assume $\alpha > 0$.

The action (4), when varied with respect to the wave function of the massive scalar field, yields the differential equation [6–9]

$$\nabla^\nu\nabla_\nu\phi = \frac{1}{4}f_{,\phi}\mathcal{I}. \qquad (7)$$

Substituting into (7) the line element (1) of the curved black-hole spacetime and using the field decomposition [14]

$$\phi(r,\theta,\phi) = \sum_{lm}\frac{\psi_{lm}(r)}{r}Y_{lm}(\theta)e^{im\phi}, \qquad (8)$$

one finds that the spatial behavior of the static nonminimally coupled massive scalar field configurations, which are supported by the central charged Reissner-Nordström black hole, is determined by the ordinary differential equation

$$\frac{d^2\psi}{dy^2} - V\psi = 0, \qquad (9)$$

where the tortoise coordinate y in the Schrödinger-like equation (9) is related to the radial coordinate r by the compact differential relation [15]

$$\frac{dr}{dy} = h(r). \qquad (10)$$

Here [6–9],

$$V(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \times \left[\mu^2 + \frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} - \frac{\alpha Q^2}{r^4}\right] \qquad (11)$$

is the effective potential of the composed black hole–nonminimally coupled massive scalar field system.

In the next section, we shall use analytical techniques in order to determine the discrete resonant spectrum $\{\alpha_n(\mu, l, M, Q)\}_{n=0}^{n=\infty}$ of the dimensionless physical parameter α. This resonant spectrum is determined by the Schrödinger-like radial differential equation (9) with the following boundary conditions [6–9]:

$$\psi(r=r_+)<\infty;\qquad \psi(r\to\infty)\to\frac{1}{r}e^{-\mu r}. \tag{12}$$

These physically motivated boundary conditions at the outer black-hole horizon and at spatial infinity correspond to spatially regular bound-state massive scalar field configurations which are supported by the central charged black hole.

III. DISCRETE RESONANT SPECTRUM OF THE COMPOSED CHARGED BLACK HOLE–LINEARIZED MASSIVE SCALAR FIELD SYSTEM: WKB ANALYSIS

In the present section, we shall derive a remarkably compact analytical formula for the discrete resonant spectrum $\{\alpha_n(\mu,l,M,Q)\}_{n=0}^{n=\infty}$ which characterizes the composed charged black hole–linearized massive scalar field configurations in the dimensionless large-mass regime

$$M\mu\gg\max\{1,l\}. \tag{13}$$

As we shall now show explicitly, the Schrödinger-like equation (9), which determines the radial functional behavior of the spatially bounded nonminimally coupled massive scalar field configurations in the charged black-hole spacetime (1), is amenable to a Wentzel-Kramers-Brillouin (WKB) analysis in the large-mass regime (13). In particular, a standard second-order WKB analysis of the Schrödinger-like radial equation (9) yields the well-known discrete quantization condition [16–18]

$$\int_{y_{t-}}^{y_{t+}}dy\sqrt{-V(y;M,Q,l,\mu,\alpha)}=\left(n+\frac{1}{2}\right)\cdot\pi;\qquad n=0,1,2,\ldots. \tag{14}$$

The two integration boundaries $\{y_{t-},y_{t+}\}$ of the WKB formula (14) are the classical turning points [with $V(y_{t-})=V(y_{t+})=0$] of the composed charged black hole–massive field binding potential (11). The resonant parameter n (with $n\in\{0,1,2,\ldots\}$) characterizes the infinitely large discrete resonant spectrum $\{\alpha_n(\mu,l,M,Q)\}_{n=0}^{n=\infty}$ of the black hole–field system.

Using the relation (10) between the radial coordinates y and r, one can express the WKB resonance equation (14) in the form

$$\int_{r_{t-}}^{r_{t+}}dr\frac{\sqrt{-V(r;M,Q,l,\mu,\alpha)}}{h(r)}=\left(n+\frac{1}{2}\right)\cdot\pi;\qquad n=0,1,2,\ldots, \tag{15}$$

where the two polynomial relations [see Eq. (11)]

$$1-\frac{2M}{r_{t-}}+\frac{Q^2}{r_{t-}^2}=0 \tag{16}$$

and

$$\frac{l(l+1)}{r_{t+}^2}+\frac{2M}{r_{t+}^3}-\frac{2Q^2}{r_{t+}^4}-\frac{\alpha Q^2}{r_{t+}^4}=0 \tag{17}$$

determine the radial turning points $\{r_{t-},r_{t+}\}$ of the composed black-hole-field binding potential (11).

We shall now prove that the WKB resonance equation (15) can be studied *analytically* in the regime (13) of large field masses. To this end, it proves useful to define the dimensionless physical quantities

$$x\equiv\frac{r-r_+}{r_+};\qquad \tau\equiv\frac{r_+-r_-}{r_+}, \tag{18}$$

in terms of which the composed black hole–massive field interaction term (11) has the form of a binding potential well,

$$V[x(r)]=-\tau\left(\frac{\alpha Q^2}{r_+^4}-\mu^2\right)\cdot x+\left[\frac{\alpha Q^2(5r_+-6r_-)}{r_+^5}-\mu^2\left(1-\frac{2r_-}{r_+}\right)\right]\cdot x^2+O(x^3), \tag{19}$$

in the near-horizon region

$$x\ll\tau. \tag{20}$$

From the near-horizon expression (19) of the black-hole-field binding potential, one obtains the dimensionless expressions

$$x_{t-}=0 \tag{21}$$

and

$$x_{t+}=\tau\cdot\frac{\frac{\alpha Q^2}{r_+^4}-\mu^2}{\frac{\alpha Q^2(5r_+-6r_-)}{r_+^5}-\mu^2(1-\frac{2r_-}{r_+})} \tag{22}$$

for the classical turning points of the WKB integral relation (15).

Taking cognizance of Eqs. (20) and (22), one finds that our analysis is valid in the regime [see Eq. (31) below]

$$\alpha\simeq\frac{\mu^2r_+^4}{Q^2}, \tag{23}$$

in which case the near-horizon binding potential and its outer turning point can be approximated by the remarkably compact expressions

$$V(x) = -\tau\left[\left(\frac{\alpha Q^2}{r_+^4} - \mu^2\right)\cdot x - 4\mu^2\cdot x^2\right] + O(x^3) \tag{24}$$

and

$$x_{t+} = \frac{1}{4}\left(\frac{\alpha Q^2}{\mu^2 r_+^4} - 1\right). \tag{25}$$

In addition, from Eqs. (2) and (18), one finds the near-horizon relation

$$h(x) = \tau\cdot x + (1-2\tau)\cdot x^2 + O(x^3). \tag{26}$$

Substituting Eqs. (18), (24), (25), and (26) into Eq. (15), one obtains the integral relation

$$\frac{1}{\sqrt{\tau}}\int_0^{x_{t+}} dx\sqrt{\frac{\frac{\alpha Q^2}{r_+^2} - \mu^2 r_+^2}{x} - 4\mu^2 r_+^2} = \left(n+\frac{1}{2}\right)\cdot\pi; \quad n = 0,1,2,\ldots. \tag{27}$$

Defining the dimensionless radial coordinate

$$z \equiv \frac{x}{x_{t+}}, \tag{28}$$

one can express the WKB resonance equation (27) in the mathematically compact form

$$\frac{2\mu r_+ x_{t+}}{\sqrt{\tau}}\int_0^1 dz\sqrt{\frac{1}{z} - 1} = \left(n+\frac{1}{2}\right)\cdot\pi; \quad n = 0,1,2,\ldots, \tag{29}$$

which yields the relation [19]

$$\frac{\mu r_+ x_{t+}}{\sqrt{\tau}} = n + \frac{1}{2}; \quad n = 0,1,2,\ldots. \tag{30}$$

From Eqs. (25) and (30), one finds the discrete resonant spectrum

$$\alpha_n = \frac{\mu^2 r_+^4}{Q^2}\left[1 + \frac{4\sqrt{\tau}}{\mu r_+}\left(n+\frac{1}{2}\right)\right]; \quad n = 0,1,2,\ldots \tag{31}$$

for the dimensionless coupling parameter of the composed charged black hole–nonminimally coupled linearized massive scalar field configurations in the regime $\mu r_+ \gg 1$ [see (13)] of large field masses. The analytically derived relation (31) can also be written as the discrete resonant formula [20]

$$(\mu r_+)_n = \sqrt{\alpha\frac{r_-}{r_+}} - \sqrt{\frac{r_+ - r_-}{r_+}}\cdot(2n+1) \quad \text{for}\ \alpha \gg 1 \tag{32}$$

for the dimensionless mass parameter which characterizes the nonminimally coupled massive scalar field clouds in the large-coupling $\alpha \gg 1$ regime.

IV. NUMERICAL CONFIRMATION

It is of physical interest to test the accuracy of the analytically derived resonant spectrum (32) in the large-coupling (large-α) regime against the corresponding exact (numerically computed [8,9]) resonant spectrum.

In Table I, we present the analytically calculated [see Eq. (32)] dimensionless mass parameter $[\mu r_+(\alpha)]^{\text{analytical}}$, which characterizes the composed charged black hole–nonminimally coupled massive scalar field cloudy configurations in the large-coupling $\alpha \gg 1$ regime, for various values of the dimensionless coupling parameter α of the theory. We also present the corresponding exact (numerically computed [8]) values of the dimensionless mass parameter $[\mu r_+(\alpha)]^{\text{numerical}}$ which characterizes the composed black-hole-field system. The data presented in Table I reveal the fact that the agreement between the

TABLE I. Composed Reissner-Nordström black hole–nonminimally coupled massive scalar field cloudy configurations. We display, for various values of the dimensionless coupling parameter α of the theory and for various values of the discrete resonant parameter n, the analytically calculated values of the dimensionless mass parameter $[\mu r_+(\alpha)]^{\text{analytical}}$ which characterizes the composed black-hole-field system [see the analytically derived resonant spectrum (32)]. We also display the corresponding exact (numerically computed [8]) values $[\mu r_+(\alpha)]^{\text{numerical}}$ of the dimensionless mass parameter of the composed black-hole-field system. The data presented are for the case of a central supporting charged black hole with the dimensionless charge-to-mass ratio $Q/M = 0.7$. One finds that the agreement between the analytically derived resonant spectrum (32) and the corresponding numerically computed spectrum [8] becomes extremely good in the dimensionless large-coupling $\alpha \gg 1$ regime of the composed black-hole-field system. Interestingly, the data presented reveal the fact that the agreement between the analytically derived resonant spectrum (32) and the numerical results of Ref. [8] is quite good already in the dimensionless $\mu r_+ = O(1)$ regime.

$\alpha(n)$	82.52(0)	256.6(0)	469.8(0)	675.5(1)	2194(1)	4074(1)
$(\mu r_+)^{\text{analytical}}$	2.80	5.63	7.94	7.88	16.39	23.33
$(\mu r_+)^{\text{numerical}}$	2.75	5.60	7.92	7.88	16.39	23.33

approximated (analytically derived) resonant spectrum (32) and the corresponding exact (numerically computed) resonant spectrum becomes extremely good in the dimensionless large-coupling $\alpha \gg 1$ regime of the theory. In fact, the agreement between the analytically derived resonant spectrum (32) and the numerical results of Ref. [8] is found to be quite good already in the $\mu r_+ = O(1)$ regime.

V. $\alpha/\mu^2 r_+^2 \to r_+^2/Q^2$ LIMIT

Interestingly, it has been demonstrated numerically in Refs. [8,9] (see, in particular, Fig. 4 of Ref. [8]) that the dimensionless physical parameter α *diverges* in the $\beta \to \beta_c$ limit, where the physical parameter β is defined by the dimensionless relation

$$\frac{\alpha}{\beta} \equiv \mu^2 r_+^2. \tag{33}$$

Here, the critical parameter β_c is given by the simple relation [8]

$$\beta_c \equiv \frac{r_+^2}{Q^2}. \tag{34}$$

Remarkably, our results provide a simple analytical explanation for the numerically observed [8,9] intriguing divergent functional behavior of the coupling parameter α in the $\beta \to \beta_c$ limit. In particular, from the resonant spectrum (31), one finds the simple relation

$$\alpha_n = \frac{16\beta_c \tau (n+\frac{1}{2})^2}{(\frac{\beta}{\beta_c} - 1)^2} \quad \text{for } \beta \to \beta_c^+. \tag{35}$$

We have therefore proved analytically that, in agreement with the numerical results presented in Refs. [8,9], the physical coupling parameter α of the composed black hole–massive field system diverges quadratically in the $\beta \to \beta_c$ limit.

VI. SUMMARY AND CONCLUSIONS

The recently published highly important works [6–9] have explicitly proved, using numerical techniques, that asymptotically flat charged black holes with spatially regular horizons can support external static matter configurations which are made of (massless as well as massive) scalar fields. This physically intriguing phenomenon owes its existence to a nonminimal coupling between the supported scalar fields and the electromagnetic field of the charged black-hole spacetime [see Eq. (4)].

The interesting numerical results presented in Refs. [6–9] have revealed the fact that, in the nonrivial field theory (4), the boundary between hairy charged black hole–nonminimally coupled scalar field configurations and bald (scalarless) Reissner-Nordström black holes is determined by a critical existence line $\alpha = \alpha_0(\mu; Q/M)$, where the physical parameter α determines the strength of the nontrivial coupling between the supported scalar configurations and the electromagnetic field of the central charged black hole. In particular, the critical existence line of the system is composed of *linearized* scalar field configurations which are supported by central charged Reissner-Nordström black holes. Interestingly, it has been demonstrated in Refs. [6–9] that the linearized external scalar configurations (scalar clouds), which are supported in the charged black-hole spacetime, are characterized by a discrete resonant spectrum $\{\alpha_n(\mu; Q/M)\}_{n=0}^{n=\infty}$ of the nontrivial scalar field–electromagnetic field coupling parameter α.

In the present paper, we have used analytical techniques in order to explore the physical and mathematical properties of the composed Reissner-Nordström black hole–nonminimally coupled linearized massive scalar field system in the dimensionless large-coupling regime $\alpha \gg 1$. In particular, we have derived the discrete WKB resonant spectrum (32) for the dimensionless mass parameter μr_+ which characterizes the composed black hole–massive field cloudy configurations. Furthermore, we have explicitly demonstrated that the analytically derived resonant spectrum (32) for the dimensionless mass parameter of the composed black-hole-field system agrees remarkably well (see the data presented in Table I) with the corresponding numerically computed resonant spectrum of Ref. [8].

Interestingly, the analytically derived resonant formula (32) yields the remarkably compact expression [21]

$$(\mu r_+)_{\max} = \sqrt{\alpha \frac{r_-}{r_+}} - \sqrt{\frac{r_+ - r_-}{r_+}} \quad \text{for } \alpha \gg 1 \tag{36}$$

for the critical existence line which characterizes the charged black hole–massive scalar field configurations.

The α-dependent critical line (36) is physically important in the composed Einstein-Maxwell–massive scalar theory (4) since it marks, in the large-mass regime, the boundary between the hairy charged black hole–massive scalar field configurations and the bald (scalarless) Reissner-Nordström black-hole spacetimes. In particular, for given parameters $\{M, Q\}$ of the central supporting black hole and for a given value of the nontrivial coupling parameter α, the hairy charged black hole–massive scalar field configurations are characterized by the critical inequality $\mu(\alpha; Q/M) \leq \mu_{\max}(\alpha; Q/M)$.

It is physically interesting to point out that the analytically derived formula (36) for the critical existence line of the composed charged black hole–massive scalar field system implies, in agreement with the recently published important numerical results of Refs. [8,9], that, for given values of the black-hole physical parameters $\{M, Q\}$, the dimensionless mass parameter μr_+ of the nonminimally coupled linearized scalar field configurations is a monotonically increasing function of the dimensionless coupling parameter α of the theory.

In addition, our analysis provides a simple analytical explanation for the numerically observed [8,9] divergent functional behavior of the coupling parameter α in the $\beta \to \beta_c \equiv r_+^2/Q^2$ limit, where $\beta \equiv \alpha/\mu^2 r_+^2$. In particular, using analytical techniques, we have explicitly proved that the dimensionless physical parameter α diverges *quadratically* [see Eq. (35)] in the $\beta \to \beta_c$ limit.

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[20] Here, we have used the identity $Q^2 = r_+ r_-$ for the central supporting charged Reissner-Nordström black hole.
[21] Note that the critical existence line of the system corresponds to the fundamental ($n = 0$) resonant mode of (32).