Stability of the isotropic pressure condition

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(Received 29 April 2020; accepted 4 May 2020; published 13 May 2020)

We investigate the conditions for the (in)stability of the isotropic pressure condition in collapsing spherically symmetric, dissipative fluid distributions. It is found that dissipative fluxes, and/or energy density inhomogeneities and/or the appearance of shear in the fluid flow, force any initially isotropic configuration to abandon such a condition, generating anisotropy in the pressure. To reinforce this conclusion we also present some arguments concerning the axially symmetric case. The consequences of our results are analyzed.

DOI: 10.1103/PhysRevD.101.104024

I. INTRODUCTION

In theoretical physics it is usual to resort to different kinds of assumptions in order to solve (almost) any specific problem. Assumptions are restrictions imposed to simplify the problem under consideration, reflecting some of the essential aspects of the systems. Since all physical systems are subject to fluctuations, those essential aspects are as well. Accordingly, the following questions naturally arise in the study of almost any physical problem:

(i) Is any result obtained under assumption A similar to that obtained under the "quasiassumption" $A + \epsilon$ (where $\epsilon \ll 1$)? This question concerns the stability of the result.

(ii) Under which conditions does assumption A remain valid all along the evolution of the system? This question concerns the stability of the assumption itself.

In this paper we endeavor to answer the questions above, in relation to the isotropic pressure condition.

For many years, both in the Newtonian and the relativistic regime, the isotropy of the pressure (the Pascal principle) has been a common (and a fundamental) assumption in the study of stellar structure and evolution. Therefore, the two questions above deserve to be answered for the isotropic pressure condition.

The first question, concerning the stability of the result, has a known answer. Indeed, let us recall that even a small pressure anisotropy may lead to results drastically different from the ones obtained by assuming isotropic pressure, due to the possible appearance of crackings in the fluid distributions produced by the presence of arbitrarily small pressure anisotropy [1]. Thus, the stability of a specific result against small deviations from the isotropic pressure condition is not assured in general, and should be checked in each case.

Here, we focus on the question concerning the stability of the isotropic pressure condition, i.e., under which conditions such an assumption remains valid all along the evolution. More specifically, we endeavor to answer the following (related) questions:

(i) What physical properties of the fluid distribution are related (and how) to the appearance of pressure anisotropy in an initially isotropic fluid?

(ii) Under which conditions does an initially isotropic configuration remain isotropic all along its evolution (stability problem)?

The relevance of the problem under consideration is illustrated, on the one hand, by the fact that many important results concerning relativistic fluids rely on the Pascal principle, and on the other hand, by the fact that pressure anisotropy is expected to be produced by physical processes usually present in very compact objects. This in turn explains the renewed interest in self-gravitating systems with anisotropic pressure observed in recent years. Indeed, the number of papers devoted to this issue is so large that we ask for the indulgence of the reader for not being exhaustive with the corresponding bibliography. Just as a small partial sample, let us mention the review paper [2] with a comprehensive bibliography until 1997, and some of the recent works that have appeared so far in the current (2020) year [3–38].

Our approach heavily relies on a differential equation relating the Weyl tensor to different physical variables. It is an evolution equation containing time derivatives of those variables. This equation was first derived in [39–41] for configurations without any specific symmetry; afterwards it was reobtained and used in different contexts (see for example [42–44]).

We consider general fluid distributions endowed with anisotropic pressure and dissipating energy during its

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evolution. The specific physical (microscopic) phenomena behind these fluid characteristics are not discussed here; instead we are concerned only by the macroscopic (hydrodynamic) manifestations of those phenomena.

As we see, only a highly unlikely cancellation of terms containing the heat flux, the energy-density inhomogeneity and the shear of the fluid could ensure that the pressure isotropy condition remains valid all along the evolution. To complement our discussion, we also consider the axially symmetric case.

The manuscript is organized as follows: In the next section we introduce all the variables and conventions used throughout the paper, for the spherically symmetric case. In Secs. III we briefly present the basic differential equation our study is based upon. Then with all these elements we tackle in Sec. IV the problem of identifying the conditions required for the pressure isotropy assumption to remain valid all along the evolution of the system, and of identifying the physical causes of the departure from such a condition. In order to strengthen further our case, we expose some arguments concerning the axially symmetric case in Sec. V. A summary of the obtained results and a discussion on their potential consequences are presented in Sec. VI. Finally an appendix with the expressions of Einstein equations and conservation equations for the spherically symmetric case is included.

II. ENERGY-MOMENTUM TENSOR, RELEVANT VARIABLES AND FIELD EQUATIONS

Let us consider a spherically symmetric distribution of collapsing fluid, non-necessarily bounded. The fluid is assumed to be locally anisotropic (principal stresses unequal) and undergoing dissipation in the form of heat flow (to model dissipation in the diffusion approximation).

Choosing comoving coordinates, the general metric can be written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where A, B and R are functions of t and r and are assumed positive. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$.

The matter energy-momentum tensor $T_{\alpha\beta}$ has the form

$$T_{\alpha\beta} = (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta + q_\alpha V_\beta + V_\alpha q_\beta, \quad (2)$$

where μ is the energy density, P_r the radial pressure, $P_\perp$ the tangential pressure, q^α the heat flux describing dissipation in the diffusion approximation, V^α the four velocity of the fluid, and χ^α a unit four vector along the radial direction. These quantities satisfy

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \chi^\alpha\chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0. \quad (3)$$

We do not explicitly add dissipation in the free streaming approximation, bulk viscosity and/or shear viscosity to the system because they can be absorbed into the energy density μ, and the radial and tangential pressures, P_r and $P_\perp$, of the collapsing fluid.

Alternatively, we may write the energy-momentum tensor in its canonical form,

$$T_{\alpha\beta} = \mu V_\alpha V_\beta + P h_{\alpha\beta} + \Pi_{\alpha\beta} + q(V_\alpha\chi_\beta + \chi_\alpha V_\beta) \quad (4)$$

with

$$P = \frac{P_r + 2P_\perp}{3}, \qquad h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta,$$

$$\Pi_{\alpha\beta} = \Pi\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right), \qquad \Pi = P_r - P_\perp.$$

Since we assume that our observer is comoving with the fluid,

$$V^\alpha = A^{-1}\delta_0^\alpha, \qquad q^\alpha = qB^{-1}\delta_1^\alpha, \qquad \chi^\alpha = B^{-1}\delta_1^\alpha, \quad (5)$$

where q is a function of t and r.

The four acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta}V^\beta, \qquad \Theta = V^\alpha{}_{;\alpha}, \quad (6)$$

and its shear $\sigma_{\alpha\beta}$ by

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta h_{\alpha\beta}. \quad (7)$$

From (6) with (5) we have for the four acceleration and its scalar a

$$a_1 = \frac{A'}{A}, \qquad a^2 = a^\alpha a_\alpha = \left(\frac{A'}{AB}\right)^2, \quad (8)$$

where $a^\alpha = a\chi^\alpha$, and for the expansion

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \quad (9)$$

where the prime stands for differentiation with respect to r and the dot stands for differentiation with respect to t.

With (5) we obtain for the shear (7) its nonzero components

$$\sigma_{11} = \frac{2}{3}B^2\sigma, \qquad \sigma_{22} = \frac{\sigma_{33}}{\sin^2\theta} = -\frac{1}{3}R^2\sigma, \quad (10)$$

and its scalar

$$\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \frac{2}{3}\sigma^2, \quad (11)$$

where

$$\sigma = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R}\right). \quad (12)$$

Then, the shear tensor can be written as

$$\sigma_{\alpha\beta} = \sigma\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right). \quad (13)$$

We can define the velocity U of the collapsing fluid as the variation of the areal radius R as measured from its area, with respect to proper time, i.e.,

$$U = \frac{\dot{R}}{A}. \quad (14)$$

A. Weyl tensor

In general the Weyl tensor $C^{\rho}_{\alpha\beta\mu}$ may be defined through its electric and magnetic parts. However in the spherically symmetric case the magnetic part vanishes identically, and the electric part of the Weyl tensor is defined by

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu}V^\mu V^\nu, \quad (15)$$

with the following nonvanishing components,

$$E_{11} = \frac{2}{3}B^2\mathcal{E},$$
$$E_{22} = -\frac{1}{3}R^2\mathcal{E},$$
$$E_{33} = E_{22}\sin^2\theta, \quad (16)$$

where

$$\mathcal{E} = \frac{1}{2A^2}\left[\frac{\ddot{R}}{R} - \frac{\ddot{B}}{B} - \left(\frac{\dot{R}}{R} - \frac{\dot{B}}{B}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{R}}{R}\right)\right] + \frac{1}{2B^2}\left[\frac{A''}{A} - \frac{R''}{R} + \left(\frac{B'}{B} + \frac{R'}{R}\right)\left(\frac{R'}{R} - \frac{A'}{A}\right)\right] - \frac{1}{2R^2}. \quad (17)$$

Observe that we may also write $E_{\alpha\beta}$ as

$$E_{\alpha\beta} = \mathcal{E}\left(\chi_\alpha\chi_\beta - \frac{1}{3}h_{\alpha\beta}\right). \quad (18)$$

Finally, using the field equations the following expression may be obtained for $\mathcal{E}$ (see [43] or [44] for details):

$$\mathcal{E} = -4\pi\Pi + \frac{4\pi}{R^3}\int_0^r R^3\mu' d\tilde{r} - \frac{12\pi}{R^3}\int_0^r qUBR^2 d\tilde{r}. \quad (19)$$

III. AN EVOLUTION EQUATION FOR $\mathcal{E}$

As mentioned in the introduction a differential equation for the Weyl tensor plays a central role in our work; this equation, which follows from the Bianchi identities, was originally found in [39,40] and was reobtained in [42]. Here we use the notation used in [44]; it reads

$$\frac{\partial}{\partial t}[\mathcal{E} - 4\pi(\mu - \Pi)] = \frac{3\dot{R}}{R}[4\pi(\mu + P_\perp) - \mathcal{E}] + 12\pi q\frac{AR'}{BR}. \quad (20)$$

In the next section we elaborate on this equation, rewriting it in such a way that it may be regarded as an evolution equation for the anisotropy Π, thereby providing the conditions ensuring the propagation in time of pressure isotropy.

IV. THE EVOLUTION OF THE PRESSURE ISOTROPY CONDITION

Let us start our discussion by noticing a fact which is seldom mentioned in the study of relativistic hydrodynamics; we have in mind the "asymmetry" in the role played by the radial and tangential pressure in the context of general relativity. Indeed, in the static case the Tolman-Oppenheimer-Volkoff equation may be written at once from (A7) as

$$P_r' + (\mu + P_r)\frac{A'}{A} + 2(P_r - P_\perp)\frac{R'}{R} = 0. \quad (21)$$

The above equation is the hydrostatic equilibrium equation and the physical meaning of its different terms is well known: the first term is just the gradient of pressure opposing gravity, the second term describes the gravitational "force" and finally the third term describes the effect of the pressure anisotropy, whose sign depends on the difference between principal stresses.

The remarkable fact is that while the radial pressure enters into the gravitational force term, the tangential pressure does not. In other words there is not a "self-regenerative effect" of the tangential pressure, which explains why anisotropic spheres may be more compact than isotropic ones (if $P_\perp > P_r$). This is a purely relativistic effect, since in the Newtonian limit the radial pressure in the second term of (21) vanishes and both principal stresses appear symmetrically in the hydrostatic equilibrium equation. In other words, in relativistic hydrodynamics there seems to be an *intrinsic* anisotropy, in the sense that the role played by principal stresses is different.

A hint about the origin of the departure from the pressure isotropy condition during the evolution is provided by the following "qualitative" analysis of a system leaving the equilibrium from a static fluid distribution with isotropic pressure.

Thus, let us assume that our system is forced to abandon the state of equilibrium, and we take a "snapshot" of the system immediately after that, at a time scale shorter than the thermal relaxation time, the thermal adjustment time and the hydrostatic time. Therefore, at this time scale we have

$$q \approx U \approx \Theta \approx \sigma \approx 0 \Rightarrow \dot{R} \approx \dot{B} \approx 0; \tag{22}$$

obviously the time derivatives of the above quantities are small but nonvanishing.

Then, evaluating the anisotropic scalar Π at this time scale, we obtain from (A4) and (A5)

$$8\pi\Pi \approx \frac{1}{A}\left(\frac{\ddot{B}}{B} - \frac{\ddot{R}}{R}\right) \approx \dot{\sigma}, \tag{23}$$

where the fact has been assumed that the fluid is initially isotropic in the pressure.

Thus it appears that unless we assume that the fluid evolves shear free, at least within the time scale under consideration, it will depart from the initial isotropic pressure condition. It might be argued that for some unknown physical reasons, some "isotropization" process brings the system back to the isotropic pressure condition. However this is a highly speculative assumption and the fact remains that the expected tendency of the system is to develop pressure anisotropy.

This result, although valid only for the time scale under consideration, should not be underestimated. Indeed, once the system is removed from equilibrium, it faces two possible scenarios: (a) the fluid is stable and gets back to a static regime within a time scale of the order of hydrostatic time, or (b) it is unstable, and enters into a dynamic regime until eventually reaching a final equilibrium state. In the former case (a), there is no reason to think that the acquired anisotropy given by (23) would disappear in the new equilibrium state, and therefore the resulting configuration, unlike the initial one, even if it is static should in principle exhibit pressure anisotropy.

In the latter case (b), we see next that the departure from the isotropic pressure condition is the rule, for any time scale, even if we assume that the evolution proceeds shear free.

To do so we elaborate on (20) as follows.

Using (A6) and (12) we may write (20) in the form

$$\frac{\partial}{\partial t}(\mathcal{E} + 4\pi\Pi) + \frac{\dot{R}}{R}(3\mathcal{E} + 4\pi\Pi)$$
$$= -4\pi(\mu + P_r)A\sigma - \frac{4\pi q}{B}\left(2A' - \frac{AR'}{R}\right) - \frac{4\pi q' A}{B}, \tag{24}$$

or introducing for simplicity the dissipative factor ($\Psi_{\rm diss}$),

$$\Psi_{\rm diss} \equiv -\frac{4\pi}{B}\left[\left(2A' - \frac{AR'}{R}\right)q + q'A,\right], \tag{25}$$

we may rewrite (24) as an evolution equation for the anisotropy Π as

$$\dot{\Pi} + \frac{\dot{R}}{R}\Pi + \frac{1}{4\pi}\left(\dot{\mathcal{E}} + \frac{3\mathcal{E}\dot{R}}{R}\right) = -(\mu + P_r)A\sigma + \frac{1}{4\pi}\Psi_{\rm diss}. \tag{26}$$

The above equation may be integrated, producing

$$\Pi = -\frac{1}{4\pi R}\int_0^t R\left(\dot{\mathcal{E}} + \frac{3\mathcal{E}\dot{R}}{R}\right)d\tilde{t} - \frac{1}{R}\int_0^t (\mu + P_r)A\sigma R d\tilde{t}$$
$$+ \frac{1}{4\pi R}\int_0^t R\Psi_{\rm diss} d\tilde{t}, \tag{27}$$

where the initial condition $\Pi(t = 0) = 0$ has been imposed.

At this stage, we may identify in the equation above three different factors forcing the system to abandon the pressure isotropy condition. The first integral provides the contribution from the Weyl tensor, the second one depends on the shear of the flow and the last one describes the role played by the dissipative processes through the dissipative factor.

We next transform the equation above by expressing the Weyl tensor terms in the first integral, through its expression (19).

Thus using (19) in (27), we obtain after some simple calculations

$$\Pi\dot{R} = \frac{R}{2}(\mu + P_r)A\sigma - \frac{R\Psi_{\rm diss}}{8\pi} - \frac{3}{2R^2}\frac{\partial}{\partial t}\left(\int_0^r qUBR^2 d\tilde{r}\right)$$
$$+ \frac{1}{2R^2}\frac{\partial}{\partial t}\left(\int_0^r R^3\mu' d\tilde{r}\right). \tag{28}$$

We see from the above equation that unless a highly unlikely cancellation of the four terms on the right occurs, the system will abandon the pressure isotropic condition.

We next analyze the axially symmetric dissipative case.

V. THE AXIALLY SYMMETRIC CASE

A general approach to analyze axially and reflection symmetric fluids was developed in [45] based on the $1 + 3$ formalism [39–41]. Thus it is not difficult to realize that the analysis presented in the previous section could be extended to the axially symmetric case, by using Eqs. (B10)–(B13) in [45]. However such analysis would involve extremely long expressions, making it difficult to extract useful information. Instead, still using the results of [45], we present in this section a more qualitative approach, which however provides enough arguments as to consider the departure from the pressure isotropy as the rule instead of the exception, in this case too.

More specifically, as we already did at the beginning of the previous section, we analyze the behavior of the system immediately after its departure from equilibrium. By immediately we mean at the smallest time scale at which we can observe the first signs of dynamical evolution. Such a time scale is assumed to be smaller than the thermal relaxation time, the hydrostatic time, and the thermal adjustment time.

Thus, we consider axially (and reflection) symmetric sources. For such systems the line element may be written as

$$ds^2 = -A^2dt^2 + B^2(dr^2 + r^2d\theta^2) + C^2d\phi^2 + 2Gd\theta dt, \tag{29}$$

where A, B, C, G are positive functions of t, r and θ. We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

We assume that our source is filled with an anisotropic and dissipative fluid.

The energy-momentum tensor may be written in the canonical form, as

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + Pg_{\alpha\beta} + \Pi_{\alpha\beta} + q_\alpha V_\beta + q_\beta V_\alpha. \tag{30}$$

Choosing the fluid to be comoving in our coordinates,

$$V^\alpha = \left(\frac{1}{A}, 0, 0, 0\right); \qquad V_\alpha = \left(-A, 0, \frac{G}{A}, 0\right). \tag{31}$$

We next define a canonical orthonormal tetrad (say $e^{(a)}_\alpha$), by adding to the four velocity vector $e^{(0)}_\alpha = V_\alpha$, three spacelike unitary vectors (these correspond to the vectors $\mathbf{K}, \mathbf{L}, \mathbf{S}$ in [45])

$$e^{(1)}_\alpha = (0, B, 0, 0); \quad e^{(2)}_\alpha = \left(0, 0, \frac{\sqrt{A^2B^2r^2 + G^2}}{A}, 0\right), \tag{32}$$

$$e^{(3)}_\alpha (0, 0, 0, C), \tag{33}$$

with $a = 0, 1, 2, 3$ (latin indices labeling different vectors of the tetrad).

Then the anisotropic tensor may be expressed through three scalar functions defined as (see [46] for details)

$$\Pi_{(2)(1)} = e^\alpha_{(2)} e^\beta_{(1)} T_{\alpha\beta}, \tag{34}$$

$$\Pi_{(1)(1)} = \frac{1}{3}(2e^\alpha_{(1)}e^\beta_{(1)} - e^\alpha_{(2)}e^\beta_{(2)} - e^\alpha_{(3)}e^\beta_{(3)})T_{\alpha\beta}, \tag{35}$$

$$\Pi_{(2)(2)} = \frac{1}{3}(2e^\alpha_{(2)}e^\beta_{(2)} - e^\alpha_{(3)}e^\beta_{(3)} - e^\alpha_{(1)}e^\beta_{(1)})T_{\alpha\beta}. \tag{36}$$

The heat flux vector may be defined in terms of the two tetrad components $q_{(1)}$ and $q_{(2)}$, as

$$q_\mu = q_{(1)}e^{(1)}_\mu + q_{(2)}e^{(2)}_\mu \tag{37}$$

or, in coordinate components (see [45])

$$q^\mu = \left(\frac{q_{(2)}G}{A\sqrt{A^2B^2r^2 + G^2}}, \frac{q_{(1)}}{B}, \frac{Aq_{(2)}}{\sqrt{A^2B^2r^2 + G^2}}, 0\right), \tag{38}$$

$$q_\mu = \left(0, Bq_{(1)}, \frac{\sqrt{A^2B^2r^2 + G^2}q_{(2)}}{A}, 0\right). \tag{39}$$

The four acceleration may be expressed through two scalar functions

$$a_\alpha = V^\beta V_{\alpha;\beta} = a_{(1)}e^{(1)}_\mu + a_{(2)}e^{(2)}_\mu, \tag{40}$$

with

$$a_{(1)} = \frac{A'}{AB}; \quad a_{(2)} = \frac{A}{\sqrt{A^2B^2r^2 + G^2}}\left[\frac{A_{,\theta}}{A} + \frac{G}{A^2}\left(\frac{\dot{G}}{G} - \frac{\dot{A}}{A}\right)\right], \tag{41}$$

where the dot and the prime denote derivatives with respect to t and r respectively.

For the expansion scalar we obtain

$$\Theta = V^\alpha_{;\alpha} = \frac{1}{A}\left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \frac{G^2}{A(A^2B^2r^2 + G^2)}\left(-\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G}\right), \tag{42}$$

whereas the shear tensor is defined in terms of two scalar functions $\sigma_{(1)(1)}$ and $\sigma_{(2)(2)}$, which may be written in terms of the metric functions and their derivatives as (see [45])

$$\sigma_{(1)(1)} = \frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) + \frac{G^2}{3A(A^2B^2r^2 + G^2)}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{G}}{G}\right), \tag{43}$$

$$\sigma_{(2)(2)} = \frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) + \frac{2G^2}{3A(A^2B^2r^2 + G^2)}\left(-\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{G}}{G}\right). \tag{44}$$

Finally, for the vorticity tensor

$$\Omega_{\beta\mu} = \Omega_{(a)(b)}e^{(a)}_\beta e^{(b)}_\mu, \tag{45}$$

we find that it is determined by a single basis component,

$$\Omega_{(1)(2)} = -\Omega_{(2)(1)} = -\Omega = -\frac{G(\frac{G'}{G} - \frac{2A'}{A})}{2B\sqrt{A^2B^2r^2 + G^2}}. \tag{46}$$

It is important to recall that we have to impose regularity conditions, necessary to ensure elementary flatness in the vicinity of the axis of symmetry, and in particular at the center (see [47–49]); thus as $r \approx 0$

$$\Omega = \sum_{n\geq 1}\Omega^{(n)}(t, \theta)r^n, \tag{47}$$

implying, because of (46), that in the neighborhood of the center

$$G = \sum_{n \geq 3} G^{(n)}(t,\theta) r^n. \quad (48)$$

Next, we need a transport equation; here we use the Müller-Israel-Stewart second order phenomenological theory for dissipative fluids [50–53]. However, the main conclusions generated by our analysis are not dependent on the transport equation chosen, as far as it is a causal one, i.e., that it leads to a Cattaneo-type equation [54], leading thereby to a hyperbolic equation for the propagation of thermal perturbations.

Thus, the transport equation for the heat flux reads [51–53]

$$\tau h^{\mu}_{\nu} q^{\nu}_{;\beta} V^{\beta} + q^{\mu} = -\kappa h^{\mu\nu}(T_{,\nu} + T a_{\nu}) - \frac{1}{2}\kappa T^2 \left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha} q^{\mu}, \quad (49)$$

where τ, κ, T denote the relaxation time, the thermal conductivity and the temperature, respectively. Contracting (49) with $e^{(2)}_{\mu}$ we obtain

$$\frac{\tau}{A}(\dot{q}_{(2)} + A q_{(1)}\Omega) + q_{(2)} = -\frac{\kappa}{A}\left(\frac{G\dot{T} + A^2 T_{,\theta}}{\sqrt{A^2B^2r^2 + G^2}} + A T a_{(2)}\right) - \frac{\kappa T^2 q_{(2)}}{2}\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}, \quad (50)$$

whereas the contraction of (49) with $e^{(1)}_{\mu}$ produces

$$\frac{\tau}{A}(\dot{q}_{(1)} - A q_{(2)}\Omega) + q_{(1)} = -\frac{\kappa}{B}(T' + B T a_{(1)}) - \frac{\kappa T^2 q_{(1)}}{2}\left(\frac{\tau V^{\alpha}}{\kappa T^2}\right)_{;\alpha}. \quad (51)$$

Let us now take a snapshot of the system, just after it has abandoned the equilibrium. As mentioned before, by "just after" we mean on the smallest time scale, at which we can detect the first signs of dynamical evolution. The following results follow from this evaluation (see [46] for details).

- (i) At the time scale at which we are observing the system, which is smaller than the hydrostatic time scale, the kinematical quantities $\Omega(G)$, Θ, $\sigma_{(1)(1)}$, $\sigma_{(2)(2)}$ keep the same values they have in equilibrium; i.e., they are neglected [of course not so for their time derivatives, which are assumed to be small, say of order $O(\epsilon)$ (where $\epsilon \ll 1$)] but nonvanishing.
- (ii) The heat flux vector should also be neglected (once again, not so for its time derivative).
- (iii) From the above conditions it follows at once that first order time derivatives of the metric variables A, B, C can be neglected.

Then, we have for the four acceleration

$$a_{(1)} = \frac{A'}{AB}; \qquad a_{(2)} = \frac{1}{\mathrm{Br}}\left(\frac{A_{,\theta}}{A} + \frac{\dot{G}}{A^2}\right), \quad (52)$$

and from the remaining kinematical variables

$$\dot{\Theta} = \frac{1}{A}\left(\frac{2\ddot{B}}{B} + \frac{\ddot{C}}{C}\right), \qquad \dot{\sigma}_{(1)(1)} = \dot{\sigma}_{(2)(2)} \equiv \dot{\bar{\sigma}} = \frac{1}{3A}\left(\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C}\right), \quad (53)$$

$$\dot{\Omega} = \frac{1}{AB^2 r}\left(\frac{\dot{G}'}{2} - \frac{\dot{G}A'}{A}\right). \quad (54)$$

Now, at thermal equilibrium, when the heat flux vanishes, the Tolman conditions for thermal equilibrium [55]

$$(TA)' = (TA)_{,\theta} = 0 \quad (55)$$

are valid.

Thus, the evaluation of (51) and (50) just after leaving the equilibrium produces respectively

$$\dot{q}_{(1)} = 0, \quad (56)$$

and

$$\tau \dot{q}_{(2)} = -\frac{\kappa A T_{,\theta}}{\mathrm{Br}} - \kappa A T a_{(2)}, \quad (57)$$

or, using (55),

$$\tau \dot{q}_{(2)} = -\frac{\kappa T \dot{G}}{ABr}. \quad (58)$$

Therefore, at the very beginning of the evolution, the dissipative process starts with contributions along the $e^{(2)}_{\mu}$ (meridional) direction.

With the information above we may calculate the components of the Einstein tensor $G_{\alpha\beta}$ and evaluate them just after the system leaves the equilibrium. At this time scale, this tensor has three types of terms: On the one hand, there are terms with first time derivatives of the metric functions A, B, C, which are set to 0, next, there are terms that neither contain G nor first time derivatives of A, B, C (these correspond to the expressions in equilibrium), and finally, there are terms with first time derivatives of G and/or second time derivatives of A, B, C, which of course are not neglected. Then it follows from the Einstein equations (see [46] for details)

$$8\pi\mu = 8\pi\mu_{(\mathrm{eq})}, \quad (59)$$

$$8\pi P = 8\pi P_{(\mathrm{eq})} - \frac{2}{3A}\dot{\Theta} + \frac{2}{3A^2B^2r^2}\left(\dot{G}_{,\theta} + \dot{G}\frac{C_{,\theta}}{C}\right), \quad (60)$$

$$8\pi\Pi_{(1)(1)} = 8\pi\Pi_{(1)(1)(\mathrm{eq})} + \frac{\dot{\sigma}}{A} + \frac{1}{3A^2B^2r^2}\left[\dot{G}_{,\theta} - \dot{G}\left(\frac{3B_{,\theta}}{B} - \frac{C_{,\theta}}{C}\right)\right], \quad (61)$$

$$8\pi\Pi_{(2)(2)} = 8\pi\Pi_{(2)(2)(\mathrm{eq})} + \frac{\dot{\sigma}}{A} + \frac{1}{3A^2B^2r^2}\left[-2\dot{G}_{,\theta} + \dot{G}\left(\frac{3B_{,\theta}}{B} + \frac{C_{,\theta}}{C}\right)\right], \quad (62)$$

$$8\pi\Pi_{(2)(1)} = 8\pi\Pi_{(2)(1)(\mathrm{eq})} - \frac{\dot{\Omega}}{A} + \frac{\dot{G}}{A^2B^2r}\left[\frac{(Br)'}{Br} - \frac{A'}{A}\right], \quad (63)$$

where (eq) stands for the value of the quantity at equilibrium.

Now, let us assume that initially the pressure of the system is isotropic, i.e., $\Pi_{(1)(1)(\mathrm{eq})} = \Pi_{(2)(2)(\mathrm{eq})}$ and $\Pi_{(2)(1)(\mathrm{eq})} = 0$. The fundamental question we have to answer is the following: may these conditions propagate in time? To simplify the discussion, let us assume that out of equilibrium, at the time scale considered here, we still have $\Pi_{(1)(1)} = \Pi_{(2)(2)}$; then it follows from (60), (61) that

$$\dot{G} = B^2 f(t, r), \quad (64)$$

which by an appropriate choice of the arbitrary function f (referred to as the fluid news function in [46]) satisfies regularity conditions and is not in contradiction with any of the equations describing the system. Thus in principle we may assume that once the system abandons the equilibrium, it may keep (at the time scale under consideration) the condition $\Pi_{(1)(1)} = \Pi_{(2)(2)}$. However, the situation is quite different for the scalar $\Pi_{(2)(1)}$. In fact, if we impose the condition $\Pi_{(2)(1)} = 0$, then because of (63), we have

$$\frac{\dot{\Omega}}{A} = \frac{\dot{G}}{A^2B^2r}\left[\frac{(Br)'}{Br} - \frac{A'}{A}\right], \quad (65)$$

which together with (46) produces

$$\dot{G} = B^2r^2 g(t, \theta), \quad (66)$$

where g is an arbitrary function of its arguments. But (66) clearly violates the regularity condition (48), close to the center. Accordingly, at the time scale under consideration we must have $\Pi_{(2)(1)} \neq 0$, more precisely

$$8\pi\Pi_{(2)(1)} = \frac{f(t, r)}{2A^2r}\left(\ln\frac{r^2}{f}\right)'. \quad (67)$$

Thus we see that, after leaving the equilibrium, at the time scale under consideration, the condition $\Pi_{(1)(1)} = \Pi_{(2)(2)}$ may be assumed to hold. However for the off diagonal tension $\Pi_{(2)(1)}$ the situation is quite different, at our time scale. Indeed, the function f controls the evolution of the system as it abandons the equilibrium; accordingly it must be different from 0, and so should $\Pi_{(2)(1)}$, even if we assume it to vanish initially.

It is worth emphasizing that a nonvanishing function f triggers the onset of dissipative processes as it follows from (58), and the appearance of shear, according to Eq. (62) in [46]. Additionally, as shown in [45] (Sec. X), the dissipative processes are responsible (among other factors) for the appearance of energy-density inhomogeneities. Therefore, as in the spherically symmetric case, here too the above-mentioned physical factors bring on the onset of pressure anisotropy as the system exits from the initial state with isotropic pressure.

Although the above argument is valid only for the time scale under consideration, it nevertheless brings out the tendency of the system to develop pressure anisotropy during the evolution. More so, it should be stressed, as we did in the spherically symmetric case, that if the system returns to a static regime within a time scale of the order of hydrostatic time scale, it will do so keeping the nonvanishing value of $\Pi_{(2)(1)}$ acquired after leaving the equilibrium; i.e., in the new static regime the fluid would be anisotropic.

VI. DISCUSSION

Fundamental results have been obtained during the last decades concerning Newtonian and relativistic fluids, under the assumption that the pressure is isotropic. However we know that the appearance of small amounts of pressure anisotropy may be enough to produce quite different results, under otherwise the same general conditions. Also, we know that many physical processes producing pressure anisotropy are expected to be present in very compact objects. Based on these comments we felt compelled to pose the following question: under which conditions would an initial fluid configuration with isotropic pressure remain so during its evolution?

For the spherically symmetric case the qualitative analysis presented at the beginning of Sec. IV points out the tendency of the system to abandon the isotropic pressure condition (at least for a specific time scale). Furthermore, this result strongly suggests that if the system is stable and gets back to equilibrium after having been removed from it, in this new state of equilibrium the fluid would be anisotropic. The more rigorous analysis presented next confirmed this tendency for an arbitrary scale time, and allows us to identify the physical factors inducing the appearance of pressure anisotropy. According to (28) these factors are the shear, the heat flux vector through the dissipative factor and the first integral in (28), and the energy-density inhomogeneity. This point deserves a deeper analysis.

Indeed, as is apparent from (28), in order for an initial fluid configuration with isotropic pressure to remain isotropic all along the evolution, we must require the fluid to be nondissipative, shear free and homogeneous in the energy density, unless we admit the highly unlikely cancellation of the four terms on the right of (28). Alternatively, as can be seen from (27), the conditions ensuring the stability of the pressure isotropic condition are conformal flatness, vanishing shear and absence of dissipation, unless, again, we assume the exceptional cancellation of the three terms on the right of (27).

Now, it has been shown in [44] that the shear and/or local anisotropy of pressure and/or dissipative fluxes entail the formation of energy-density inhomogeneities. On the other hand it has been shown in [43] that the departure from the shear-free condition is controlled by a single scalar function defined in terms of the anisotropy of the pressure, the dissipative variables and the energy-density inhomogeneity. Thus even if we assume that initially not only the pressure anisotropy but also the shear and the energy-density inhomogeneity vanishes, the dissipative flux would enhance the departure of the isotropic pressure condition through two different channels: on the one hand by its contribution as described by (28), and on the other hand by inducing departures from the shear-free condition and energy-density homogeneity. Thus only imposing the conformal flatness, the nondissipation, and the shear-free conditions all along the evolution can we ensure that the fluid evolves keeping the isotropic pressure condition at all times. However, it is worth recalling that dissipation due to the emission of massless particles (photons and/or neutrinos) is the only plausible mechanism to carry away the bulk of the huge binding energy of the collapsing star, leading to a neutron star or black hole. In other words, the adiabatic condition imposed by a collapsing scenario is very unrealistic and dissipation has to be taken into account in any physically meaningful description of stellar evolution, thereby entailing the departure from the isotropic pressure condition.

In Sec. VI we analyzed the same question for axially symmetric fluid systems. However, for simplicity, we did not deduce the explicit equation of evolution for the anisotropic scalars, but, instead, considered the system at the shortest time scale at which the first signs of dynamical evolution can be observed. It was shown that starting from an initially isotropic fluid, at the time scale under consideration, the evolution leads to an anisotropic fluid. Indeed, it appears that even if we assume that the two anisotropic scalar functions $\Pi_{(1)(1)}$ and $\Pi_{(2)(2)}$ remain equal after the departure from equilibrium, the third anisotropic scalar $\Pi_{(2)(1)}$ must be necessarily different from 0 after leaving the initial state. Also, as in the spherically symmetric case, shear, dissipative processes and energy-density inhomogeneities are related to the onset of pressure anisotropy. Once again, since we expect that the final stages of stellar evolution should be accompanied by intense dissipative processes, we should expect some degree of pressure anisotropy to appear in the nonspherical collapse too.

To summarize, what we have learned so far is that an initial fluid configuration with isotropic pressure would tend to develop pressure anisotropy as it evolves, under conditions expected in stellar evolution. Of course the magnitude of the acquired pressure anisotropy would depend on the specific data of the system. However the obtained result allows us to conclude that as well as it is wise to check the stability of any specific result obtained under the assumption of the isotropic pressure condition against fluctuations of this latter condition, it would also be wise to check the stability of the condition itself, in each case.

ACKNOWLEDGMENTS

This work was partially supported by Ministerio de Ciencia, Innovacion y Universidades, Grant No. PGC2018-096038-B-I00.

APPENDIX: EINSTEIN EQUATIONS AND CONSERVATION EQUATIONS FOR THE SPHERICALLY SYMMETRIC CASE

Einstein's field equations for the interior spacetime (1) are given by

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \tag{A1}$$

and its nonzero components with (1) and (2) become

$$\begin{aligned} 8\pi T_{00} &= 8\pi\mu A^2 \\ &= \left(2\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R} \\ &\quad -\left(\frac{A}{B}\right)^2\left[2\frac{R''}{R}+\left(\frac{R'}{R}\right)^2-2\frac{B'}{B}\frac{R'}{R}-\left(\frac{B}{R}\right)^2\right], \end{aligned} \tag{A2}$$

$$8\pi T_{01} = -8\pi qAB = -2\left(\frac{\dot{R}'}{R}-\frac{\dot{B}}{B}\frac{R'}{R}-\frac{\dot{R}}{R}\frac{A'}{A}\right), \tag{A3}$$

$$\begin{aligned} 8\pi T_{11} &= 8\pi P_r B^2 \\ &= -\left(\frac{B}{A}\right)^2\left[2\frac{\ddot{R}}{R}-\left(2\frac{\dot{A}}{A}-\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right]+\left(2\frac{A'}{A}+\frac{R'}{R}\right)\frac{R'}{R} \\ &\quad -\left(\frac{B}{R}\right)^2, \end{aligned} \tag{A4}$$

$$\begin{aligned} 8\pi T_{22} &= \frac{8\pi}{\sin^2\theta}T_{33} \\ &= 8\pi P_\perp R^2 \\ &= -\left(\frac{R}{A}\right)^2\left[\frac{\ddot{B}}{B}+\frac{\ddot{R}}{R}-\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{R}}{R}\right)+\frac{\dot{B}}{B}\frac{\dot{R}}{R}\right] \\ &\quad +\left(\frac{R}{B}\right)^2\left[\frac{A''}{A}+\frac{R''}{R}-\frac{A'}{A}\frac{B'}{B}+\left(\frac{A'}{A}-\frac{B'}{B}\right)\frac{R'}{R}\right]. \end{aligned} \tag{A5}$$

The nontrivial components of the Bianchi identities, $T^{\alpha\beta}_{;\beta} = 0$, from (A1) yield

$$T^{\alpha\beta}_{;\beta}V_\alpha = -\frac{1}{A}\left[\dot{\mu} + (\mu + P_r)\frac{\dot{B}}{B} + 2(\mu + P_\perp)\frac{\dot{R}}{R}\right] - \frac{1}{B}\left[q' + 2q\frac{(AR)'}{AR}\right] = 0, \quad \text{(A6)}$$

$$T^{\alpha\beta}_{;\beta}\chi_\alpha = \frac{1}{A}\left[\dot{q} + 2q\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\right] + \frac{1}{B}\left[P'_r + (\mu + P_r)\frac{A'}{A} + 2(P_r - P_\perp)\frac{R'}{R}\right] = 0. \quad \text{(A7)}$$

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