

Hamiltonian formulation of dust cloud collapse

Nick Kwidzinski,^{1,*} Daniele Malafarina,^{2,†} Jan J. Ostrowski^{3,‡}, Włodzimierz Piechocki^{3,§} and Tim Schmitz^{1,||}

¹*Institute for Theoretical Physics, University of Cologne, Zùlpicher Strasse 77, 50937 Köln, Germany*

²*Department of Physics, Nazarbayev University, 53 Kabanbay Batyr, 010000 Nur-Sultan, Kazakhstan*

³*Department of Fundamental Research, National Centre for Nuclear Research, Pasteura 7, 02-093 Warszawa, Poland*



(Received 20 January 2020; accepted 13 April 2020; published 8 May 2020)

We consider the gravitational collapse of a self-gravitating spherical dust cloud in the Hamiltonian formalism. We address both homogeneous and inhomogeneous cases. Our novel derivation of the Hamiltonian of the system is based on an improved variational principle. It differs from usual treatments due to the presence of an extra boundary term added to the Hilbert action. As expected, the standard equations of motion are retrieved. However, differently from other treatments, the total Hamiltonian obtained with our procedure in the Schwarzschild time gauge is identical to the total mass of the system as measured from infinity, as it would be expected. Implications for the quantization of the system are suggested.

DOI: [10.1103/PhysRevD.101.104017](https://doi.org/10.1103/PhysRevD.101.104017)

I. INTRODUCTION

Analytical toy models for gravitational collapse in General Relativity (GR) are of great importance from several perspectives. First, they provide dynamical scenarios that lead to the formation of black holes from regular initial data [1]. Second, they can be used to investigate the mathematical properties of the theory, for example, in the Hamiltonian formalism [2]. Third, they provide a simple test bed to study possible pathways toward the quantization of the gravitational field, since we expect that the singularities that generically arise in GR should not be present in a fully quantum-gravitational model (see, e.g., Refs. [3,4]).

Einstein's equations for collapse of perfect fluids are formally identical to the equations describing the Universe expansion in the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models. The simplest and most studied solution of Einstein's equations describing collapse is the so-called Oppenheimer-Snyder-Datt (OSD) model, which describes the complete collapse of a spherical cloud composed of homogeneous pressureless matter (i.e., dust) [5]. It is well known that OSD collapse in comoving coordinates leads to the formation of a static black hole in a finite comoving time, even though far-away observers never see the boundary of the cloud crossing the horizon.

The simplest extension of the OSD solution is the so-called Lemaître-Tolman-Bondi (LTB) collapse model,

which describes a spherical cloud composed of inhomogeneous dust [6]. The interest in the LTB solution as a theoretical toy model for collapse resides in the fact that, depending on the radial dependence of the energy density at the initial time, the complete collapse may produce a naked singularity; i.e., at the instant of formation, the central singularity may not be covered by the horizon [7].

Obtaining the Hilbert action and a global Hamiltonian for an asymptotically flat solution of Einstein's equations is not a trivial matter since typically integrals diverge at spatial infinity. However, one would expect the Hamiltonian of the system to be related to the total energy as measured by observers at spatial infinity. In Ref. [8], it was shown that by performing the Legendre transformations on an appropriate finite surface one obtains a quasilocal Hamiltonian, which in turn leads to the correct global Hamiltonian once the surface is shifted to infinity.

In the present article, we apply the above idea to the OSD and LTB cases. The same procedure was used in the case of collapse of a thin shell in Ref. [9], in which the Hamiltonian for an observer at infinity was found to be equal to the total mass of the collapsing shell regardless of the equation of state of the matter content. As expected, the equations of motion obtained from the improved variational principle coincide with the equations of motion obtained by different procedures. For a different derivation of this equality, see Ref. [10] and references therein.

The importance of finding the Hamiltonian that correctly describes the energy of the system, however, appears when one attempts to quantize the system. It is well known that different quantization procedures, based on different Hamiltonian formulations, provide different results,

*nk@thp.uni-koeln.de

†daniele.malafarina@nu.edu.kz

‡Jan.Jakub.Ostrowski@ncbj.gov.pl

§wlodzimierz.piechocki@ncbj.gov.pl

||tschmitz@thp.uni-koeln.de

making nontrivial the issue of quantizing even the simplest gravitating systems [3]. The hope is that the results obtained for collapse with the improved variational principle may provide a path toward a viable quantization of gravitational systems.

The paper is organized as follows. In Sec. II, we review the matching between the collapsing dust cloud and the exterior Schwarzschild spacetime. Section III is devoted to the application of the variational principle developed in Ref. [8] to the case of dust collapse. In Sec. IV, we obtain the Hamiltonian for dust collapse using the above variational principle. Finally, in Sec. V, we summarize the results of this article and mention the possible implications for quantum gravity.

Throughout the paper, we make use of natural units setting $c = G = 1$.

II. MATCHING DUST COLLAPSE TO SCHWARZSCHILD EXTERIOR

The general theory for matching two manifolds across a hypersurface was developed by Darmois [11], and it was applied to spherical symmetry by many authors [12]. In the following, we will consider the matching between the collapsing dust and an exterior Schwarzschild metric across a spherical surface. Cosmological models can be used, to some extent, for describing collapse to a black hole after implementing the condition that one deals with an isolated object. In our case, this reduces to the problem of matching a finite region of FLRW or LTB spacetime with the Schwarzschild geometry. Therefore, in what follows, the interior region is described by a portion of FLRW or LTB that extends up to a finite boundary radius, whereas the exterior region is described by the Schwarzschild line element. The interface between the matter field in the interior and the vacuum exterior is used to match both spacetimes. In what follows, we apply the standard method of matching, thus requiring continuity of the first and second fundamental forms across the hypersurface separating the two manifolds (see, e.g., Ref. [13]).

A. Matching conditions for FLRW

For the interior, we consider the homogeneous collapse of dust with the line element in comoving hyperspherical coordinates $\{\tau, \chi, \theta, \phi\}$ given by

$$ds_-^2 = -d\tau^2 + a(\tau)^2[d\chi^2 + h^2(\chi)d\Omega^2]. \quad (1)$$

We consider the marginally bound (corresponding to flat cosmological models) and bound (corresponding to closed cosmological models) cases simultaneously. The unbound (i.e., open) case is somehow less relevant for collapse as it describes a matter cloud with positive initial velocity at spatial infinity. The function $h(\chi)$ is given by

$$h(\chi) = \begin{cases} \chi, & \text{in the flat case}(k = 0) \\ \sin(\chi), & \text{in the closed case}(k = +1). \\ \sinh(\chi), & \text{in the open case}(k = -1) \end{cases} \quad (2)$$

The interior is filled with a homogeneous field of comoving dust particles; that is, the energy-momentum tensor is given by

$$T^{\mu\nu} = \epsilon u^\mu u^\nu, \quad \text{where } \mathbf{u} = u^\mu \partial_\mu = \partial_\tau. \quad (3)$$

The conservation of energy momentum implies that $\epsilon = \epsilon_0/a^3$ with $\epsilon_0 > 0$. For the exterior, we have the Schwarzschild geometry in Schwarzschild coordinates $\{t, r, \theta, \phi\}$,

$$ds_+^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2d\Omega^2. \quad (4)$$

The matching will be performed on the hypersurface Σ defined in parametric form by $\Phi_-(\tau, \chi) = \chi - \chi_b = 0$ in the interior, and $\Phi_+(t, r) = r - \psi(t) = 0$ in the exterior. Note that $\chi_b > 0$ in the flat case while $0 < \chi_b < \pi/2$ in the closed case. On Σ , we can consider $\tau = f(t)$. We choose the coordinates $\{t, \theta, \phi\}$ to parameterize the hypersurface. The metrics on Σ are then given by

$$ds_+^2|_\Sigma = -\left(1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}\right)dt^2 + \psi^2d\Omega^2, \quad (5)$$

$$ds_-^2|_\Sigma = -\dot{f}^2dt + a^2h^2(\chi_b)d\Omega^2. \quad (6)$$

Then, the first matching condition gives (see Appendix),

$$\dot{f}^2 = \left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}, \quad (7)$$

$$\psi(t) = h(\chi_b)a(\tau(t)), \quad (8)$$

where with a dot we denote partial differentiation with respect to the Schwarzschild time t . We can eliminate ψ from (7) by using (8) to obtain

$$\dot{f}^2 = \frac{\left(1 - \frac{2M}{h(\chi_b)a}\right)^2}{1 - \frac{2M}{h(\chi_b)a} + (h(\chi_b)a')^2}, \quad (9)$$

where the prime denotes differentiation with respect to τ .

The normal covectors $\mathbf{n}^\pm = n_\mu^\pm dx_\pm^\mu$ on both sides read

$$\mathbf{n}^+ = |\dot{f}|^{-1}(dr - \dot{\psi}dt), \quad \mathbf{n}^- = a d\chi, \quad (10)$$

where we made use of Eq. (7) for brevity. The nonvanishing components of the second fundamental form and their traces on Σ are given by

$$\begin{aligned}
 K_{tt}^+ &= -\frac{\dot{f}^2}{\dot{\psi}} \partial_t \left[\frac{1}{|\dot{f}|} \left(1 - \frac{2M}{\psi} \right) \right], \\
 K_{\theta\theta}^+ &= |\dot{f}|^{-1} (\psi - 2M), & K_{\phi\phi}^+ &= |\dot{f}|^{-1} (\psi - 2M) \sin^2 \theta, \\
 K_{\theta\theta}^- &= ah \partial_\chi h|_{\chi=\chi_b}, & K_{\phi\phi}^- &= a \sin^2(\theta) h \partial_\chi h|_{\chi=\chi_b} \\
 K^- &= \frac{2}{ah} \partial_\chi h|_{\chi=\chi_b}.
 \end{aligned} \tag{11}$$

The matching condition

$$[K_{\theta\theta}] = 0 = [K_{\phi\phi}] \tag{12}$$

then yields that

$$(a')^2 + k = \frac{2M}{h^3(\chi_b)a}, \tag{13}$$

where we used the first matching condition, that is, Eqs. (7) and (8). The remaining matching condition $K_{tt}^+ = 0$ can be shown to be satisfied if the other matching conditions (7), (8), and (12) are already imposed.

So far, we have three equations that might be used to determine the unknown functions a , ψ , and f . What remains undetermined is the mass parameter M in the exterior region. This can be fixed by imposing the Friedmann equation

$$(a')^2 + k = \frac{8\pi}{3} \epsilon a^2. \tag{14}$$

Comparing this to equation (13) yields

$$M = \frac{4\pi}{3} h^3(\chi_b) \epsilon_0. \tag{15}$$

Let us now investigate the hyperbolic angle μ between surfaces $\tau = \text{const}$ on one side and surfaces $t = \text{const}$ on the other. The unit vector orthogonal to the surfaces $\tau = \text{const}$ on the OSD side is

$$m_-^\mu = \{1, 0, 0, 0\}, \tag{16}$$

whereas the unit vector orthogonal to the surfaces $t = \text{const}$ on the Schwarzschild side reads

$$m_+^\mu = \left\{ \frac{1}{\sqrt{1 - 2M/r}}, 0, 0, 0 \right\}. \tag{17}$$

The hyperbolic angle μ is defined to be

$$|\mu| := \text{arcosh} |g_{\mu\nu} m_+^\mu m_-^\nu|. \tag{18}$$

Consider the normalized 4-velocity of the dust particles on the boundary Σ . In the FLRW coordinates, it is given by

$$\mathbf{u} = \partial_\tau = m_-^\mu \partial_\mu = \mathbf{m}_-, \tag{19}$$

while in the Schwarzschild ones, it is given by

$$\mathbf{u} = \frac{\partial_t + \dot{\psi} \partial_r}{\|\partial_t + \dot{\psi} \partial_r\|}. \tag{20}$$

Since $\mathbf{m}_- = \mathbf{u}$, it follows that

$$\cosh \mu = |\langle \mathbf{u}, \mathbf{n} \rangle| = \frac{1 - \frac{2M}{\psi}}{\sqrt{(1 - \frac{2M}{\psi})^2 - \dot{\psi}^2}}. \tag{21}$$

Furthermore, we will choose the sign of μ in such a way that it coincides with the sign of $\dot{\psi}$. One can then write

$$\dot{\psi} = \left(1 - \frac{2M}{\psi} \right) \tanh \mu. \tag{22}$$

Using the above equation, the matching condition (12), which is simply the second Friedmann equation, can then be written as

$$1 - \frac{2M}{\psi} = \frac{1 - h^2(\chi_b)k}{\cosh^2 \mu}. \tag{23}$$

B. Matching conditions for LTB

The above derivation can be extended in a straightforward way to the case of inhomogeneous dust. Again, we use comoving coordinates $\{\tau, \rho, \theta, \phi\}$ on the LTB side and Schwarzschild coordinates $\{t, r, \theta, \phi\}$ on the Schwarzschild side.

The Schwarzschild metric is given by (4), while the LTB metric reads

$$ds_-^2 = -d\tau^2 + \frac{(\partial_\rho R)^2}{1 + 2E} d\rho^2 + R^2 d\Omega^2, \tag{24}$$

where $R = R(\rho, \tau)$ is the aerial radius and $E = E(\rho)$ is one arbitrary function resulting from the integration of Einstein's equations. Another function that is useful to consider is the Misner-Sharp mass $F(\rho)$, which defines the amount of matter contained within the comoving shell labeled by ρ [14]. $F(\rho)$ is related to the dust density by

$$4\pi\epsilon = \frac{\partial_\rho F}{R^2 \partial_\rho R}, \tag{25}$$

from which it is easy to see that it represents the active gravitational mass. Notice that for dust collapse F does not depend on τ , meaning that the amount of matter contained within the comoving radius ρ is conserved throughout

collapse. Then, the system of Einstein's equations is fully solved once one integrates the equation of motion for $R(\tau, \rho)$, which can be given in the form

$$(\partial_\tau R)^2 = \frac{2F}{R} + 2E. \quad (26)$$

Solutions to (26) read (in parametric form)

$$E < 0, \quad R = -\frac{F}{2E}(1 - \cos \alpha),$$

$$\alpha - \sin \alpha = \frac{(-2E)^{3/2}}{F}(\tau - t_B); \quad (27)$$

$$E = 0, \quad R = \left(\frac{9}{2}F(\tau - t_B)^2\right)^{1/3}; \quad (28)$$

$$E > 0, \quad R = \frac{F}{2E}(\cosh \alpha - 1),$$

$$\sinh \alpha - \alpha = \frac{(2E)^{3/2}}{F}(\tau - t_B), \quad (29)$$

where α is an auxiliary angle and t_B is another function of ρ , which in cosmology is called the big bang time. Notice that, in general, Schwarzschild exterior and LTB interior may define one LTB spacetime as the Schwarzschild metric belongs to the LTB family.

The matching hypersurface Σ has the topology of $\mathbb{S}^2 \times \mathbb{R}$. Similarly to the OSD case, we parametrize this hypersurface in the following way: $\Phi_-(\tau, \rho) = \rho - \rho_b = 0$ from the interior, and $\Phi_+(t, r) = r - \psi(t) = 0$ from the exterior. It follows that

$$dr = \dot{\psi} dt, \quad (30)$$

and by choosing $\{t, \theta, \phi\}$ with $\tau = f(t)$ as a coordinates on the hypersurface, we get the line elements as

$$ds_+|_\Sigma = -\left(1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}\right)dt^2 + \psi^2 d\Omega^2,$$

$$ds_-|_\Sigma = -\dot{f}^2 dt^2 + R_b^2 d\Omega^2, \quad R_b = R(\rho_b, t). \quad (31)$$

Again, we have the following conditions for continuity of the metric, i.e., the first matching conditions:

$$\dot{f}^2 = 1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}, \quad (32)$$

$$\psi = R_b, \quad \dot{\psi}(t) = \frac{\partial \tau}{\partial t} \partial_\tau R_b = \dot{f} \partial_\tau R_b. \quad (33)$$

In analogy with Eq. (9), combining the above equations gives

$$\dot{f}^2 = \frac{(1 - \frac{2M}{R_b})^2}{1 - \frac{2M}{R_b} + (\partial_\tau R_b)^2}. \quad (34)$$

Normal vectors to the boundary hypersurface read

$$\mathbf{n}^+ = |\dot{f}|^{-1}(dr - \dot{\psi} dt), \quad \mathbf{n}^- = |\partial_\rho R_b / \sqrt{1 + 2E_b}| d\rho. \quad (35)$$

We can calculate the extrinsic curvature of the boundary surface as

$$K_{tt}^+ = -\frac{\dot{f}^2}{\dot{\psi}} \partial_t \left[\frac{1}{|\dot{f}|} \left(1 - \frac{2M}{\psi} \right) \right];$$

$$K_{\theta\theta}^+ = |\dot{f}|^{-1}(\psi - 2M), \quad K_{\theta\theta}^- = R_b \sqrt{1 + 2E_b};$$

$$K_{\phi\phi}^+ = K_{\theta\theta}^+ \sin^2(\theta), \quad K_{\phi\phi}^- = K_{\theta\theta}^- \sin^2(\theta). \quad (36)$$

In addition, the trace of the extrinsic curvature reads

$$K^- = \frac{2\sqrt{1 + 2E_b}}{R_b}. \quad (37)$$

The second matching conditions then reduce to

$$K_{\theta\theta}^+ = K_{\theta\theta}^-, \quad |\dot{f}|^{-1}(\psi - 2M) = R_b \sqrt{1 + 2E_b}, \quad (38)$$

and using (34), we obtain

$$(\partial_\tau R_b)^2 = \frac{2M}{R_b} + 2E_b, \quad (39)$$

which is equivalent to (26) provided that $M = F_b$.

Finally, performing calculations analogously to (16)–(21), we can express the field equation (26) in terms of hyperbolic angle (with no explicit time derivative) as follows:

$$1 - \frac{2M}{\psi} = \frac{1 + 2E_b}{\cosh^2 \mu}. \quad (40)$$

III. VARIATIONAL PRINCIPLE

Following Ref. [9], the full action for our model contains the following contributions:

$$S = \int_{\mathcal{D}_-} \mathcal{L}_{\text{grav}} + \int_{\mathcal{D}_-} \mathcal{L}_{\text{dust}} + \int_{\Sigma} \mathcal{L}_{\text{grav}}$$

$$+ \int_{\mathcal{D}_+} \mathcal{L}_{\text{grav}} + \int_{\partial \mathcal{D}} \mathcal{L}_{\text{boundary}}. \quad (41)$$

The domains are sketched in Fig. 1. The matching between Friedmann or LTB interior \mathcal{D}_- and Schwarzschild exterior \mathcal{D}_+ is implemented here by a boundary term on the

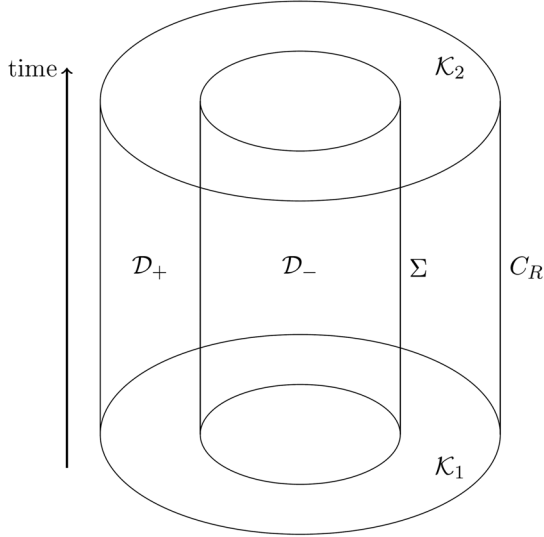


FIG. 1. Total domain of integration \mathcal{D} , consisting of Friedmann (and LTB) interior \mathcal{D}_- and Schwarzschild exterior \mathcal{D}_+ , with Σ as the matching surface. For boundaries, we have a surface of constant Schwarzschild radius C_R , where we take $R \rightarrow \infty$ at the end, and two surfaces of constant time $\mathcal{K}_{1/2}$. Note that $\mathcal{K}_{1/2}$ are not smooth where they overlap Σ , since in the interior we use dust proper time and in the exterior we use Schwarzschild Killing time.

matching hypersurface Σ , given by the jump in the trace of the Arnowitt-Deser-Misner (ADM) momentum on Σ . The mass of the Schwarzschild exterior is taken to be dynamical, $M = M(t)$. All the necessary Gibbons-Hawking-York (GHY) terms are included in the boundary term $\int_{\partial\mathcal{D}} \mathcal{L}_{\text{boundary}}$. In addition to the GHY terms on the surfaces of constant time \mathcal{K}_1 and \mathcal{K}_2 , we require edge terms on the intersection $\Sigma \cap \mathcal{K}$, where $\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_2$. For a discussion of such terms, see Ref. [15]. Lastly, we require a GHY term on the spacelike boundary of infinite curvature radius $C_R|_{R \rightarrow \infty}$.

When we have the Lagrangian in place, the dynamical quantities to be varied will be the degrees of freedom left open for the FLRW or LTB interior and Schwarzschild exterior: the rescaled scale factor $\psi(t)$ and the Schwarzschild mass $M(t)$. As usual, when applying the variational principle, these dynamical quantities are kept fixed at the timelike boundaries $\mathcal{K}_{1/2}$.

We will further work in a gauge fixed picture where we choose the coordinate frames in the interior and exterior as we have done in the last section. The final result will thus be an actual physical Hamiltonian generating evolution in Schwarzschild Killing time t , and not a constraint.

A. Schwarzschild exterior

Let us first take care of all exterior terms in the action. We start with the boundary terms in $\int_{\partial\mathcal{D}} \mathcal{L}_{\text{boundary}}$, excluding the one for $\mathcal{D}_+ \cap \mathcal{K}_{1/2}$. The latter will be discussed with

the respective interiors. First, we have the GHY term on C_R . For this, we need the extrinsic curvature on that surface. The spacelike normal vector on C_R is given by

$$\mathbf{n} = \sqrt{1 - \frac{2M}{R}} \partial_r. \quad (42)$$

The only nonvanishing component of n_μ is $n_r = 1/\sqrt{1 - 2M/r}$. Therefore, the extrinsic curvature tensor is given by

$$K_{ij} = -\Gamma_{ij}^r n_r, \quad \text{where } i, j = t, \theta, \phi. \quad (43)$$

The nonvanishing components of the Christoffel symbols Γ_{ij}^r are

$$\begin{aligned} \Gamma_{tt}^r &= \frac{(R - 2M)M}{R^3}, & \Gamma_{\theta\theta}^r &= -(R - 2M), \\ \Gamma_{\phi\phi}^r &= -(R - 2M)\sin^2\theta. \end{aligned} \quad (44)$$

Finally, we obtain the trace of the ADM momentum as defined in Appendix,

$$Q = 2(3M - 2R) \sin\theta. \quad (45)$$

This gives for the corresponding GHY boundary term

$$-\frac{1}{16\pi} \int_{C_R} d^3y Q = R(t_2 - t_1) - \frac{3}{2} \int_{t_1}^{t_2} dt M. \quad (46)$$

To obtain the final expression for $\int_{C_R} \mathcal{L}_{\text{boundary}}$, we need take the term from above and subtract the same surface term which we would obtain by embedding the surface into flat spacetime, e.g., by using the metric

$$ds_0^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + dr^2 + r^2 d\Omega^2 \quad (47)$$

and considering the surface $r = R$. The trace of the ADM momentum for this surface is given by

$$Q_0 = -2(R - M) \sin\theta + \mathcal{O}(1/R^2). \quad (48)$$

Then,

$$\begin{aligned} \int_{C_R} \mathcal{L}_{\text{boundary}} &= -\frac{1}{16\pi} \lim_{R \rightarrow \infty} \int_{\partial\mathcal{D}} d^3y (Q - Q_0) \\ &= -\frac{1}{2} \int_{t_1}^{t_2} dt M. \end{aligned} \quad (49)$$

Next, we compute the GHY term on surfaces of constant $t = t_{1/2}$ (i.e., $\mathcal{D}_+ \cap \mathcal{K}_{1/2}$). The timelike normal vector on that surface reads

$$\mathbf{n} = \pm \frac{1}{\sqrt{1 - \frac{2M}{r}}} \partial_t, \quad (50)$$

where we take the upper sign for the surface with constant t_2 and the lower sign for the surface with constant t_1 . The only nonvanishing component of the covector n_μ is $n_t = \mp \sqrt{1 - \frac{2M}{r}}$. Therefore, the extrinsic curvature on $\mathcal{D}_+ \cap \mathcal{K}_{1/2}$ is given by

$$K_{ij} = -\Gamma_{ij}^t n_t, \quad \text{where } i, j = r, \theta, \phi. \quad (51)$$

The only nonvanishing component of the connection is $\Gamma_{rr}^t = \dot{M}r^2/(r-2M)^3$, and therefore the only nonvanishing component of the extrinsic curvature is

$$K_{rr} = \pm \frac{\dot{M}r^{3/2}}{(r-2M)^{5/2}} \Big|_{t=t_{1/2}}. \quad (52)$$

This gives

$$Q = \pm 2 \frac{\dot{M}r^3}{(r-2M)^2} \sin \theta. \quad (53)$$

The corresponding boundary term is therefore

$$\begin{aligned} & \int_{\mathcal{D}_+ \cap \mathcal{K}_1} \mathcal{L}_{\text{boundary}} + \int_{\mathcal{D}_+ \cap \mathcal{K}_2} \mathcal{L}_{\text{boundary}} \\ &= \frac{1}{2} \left[\int_{\psi(t)}^R dr \frac{\dot{M}r^3}{(r-2M)^2} \right]_{t_1}^{t_2}. \end{aligned} \quad (54)$$

There are joint terms on the intersections between Σ with \mathcal{K}_1 and \mathcal{K}_2 . Following Ref. [15], they are given by

$$\int_{\Sigma \cap \mathcal{K}_{1/2}} \mathcal{L}_{\text{boundary}} = \frac{1}{8\pi} \int \sqrt{\gamma} d^2y \operatorname{arsinh}(n_\mu m^\mu), \quad (55)$$

where $\mathbf{n} = 1/\sqrt{1 - 2M/\psi} dt$ is the normal covector to $\mathcal{K}_{1/2}$ on the Schwarzschild side and $\mathbf{m} = -(1/\dot{f}) dx$ is the normal covector to Σ . We obtain that

$$\int_{\Sigma \cap \mathcal{K}} \mathcal{L}_{\text{boundary}} = \frac{1}{2} [\psi^2 \mu]_{t_1}^{t_2}. \quad (56)$$

Next, we want to compute the Schwarzschild bulk term

$$\int_{\mathcal{D}_+} \mathcal{L}_{\text{grav}} = \frac{1}{16\pi} \int_{\mathcal{D}_+} d^4x \sqrt{-g_+} {}^{(4)}\mathcal{R}_+. \quad (57)$$

The Ricci scalar for the Schwarzschild spacetime with a time-dependent mass term $M = M(t)$ is given by

$${}^{(4)}\mathcal{R}_+ = \frac{\partial}{\partial t} \frac{2\dot{M}r}{(r-2M)^2}. \quad (58)$$

Using Leibniz's integral rule, we find that

$$\begin{aligned} \int_{\mathcal{D}_+} d^4x \sqrt{-g_+} {}^{(4)}\mathcal{R}_+ &= 4\pi \int_{t_1}^{t_2} dt \int_{\psi(t)}^R dr r^2 {}^{(4)}\mathcal{R}_+ \\ &= 8\pi \left[\int_{\psi(t)}^R dr \frac{\dot{M}r^3}{(r-2M)^2} \right]_{t_1}^{t_2} \\ &\quad + 8\pi \int_{t_1}^{t_2} dt \frac{\dot{\psi} \dot{M} \psi^3}{(\psi-2M)^2}. \end{aligned} \quad (59)$$

Hence,

$$\int_{\mathcal{D}_+} \mathcal{L}_{\text{grav}} = \frac{1}{2} \left[\int_{\psi(t)}^R dr \frac{\dot{M}r^3}{(r-2M)^2} \right]_{t_1}^{t_2} + \frac{1}{2} \int_{t_1}^{t_2} dt \frac{\dot{\psi} \dot{M} \psi^3}{(\psi-2M)^2}. \quad (60)$$

The first term on the right-hand side cancels with the GHY term on $\mathcal{K} \cap \mathcal{D}_+$ given by Eq. (54).

B. FLRW interior

Next, we want to consider the gravitational and matter terms on the Friedmann side, that is, on \mathcal{D}_- including the GHY terms on $\mathcal{D}_- \cap \mathcal{K}$,

$$S_{\text{interior}} = \int_{\mathcal{D}_-} \mathcal{L}_{\text{grav}} + \int_{\mathcal{D}_-} \mathcal{L}_{\text{dust}} + \int_{\mathcal{D}_- \cap \mathcal{K}} \mathcal{L}_{\text{boundary}}. \quad (61)$$

The Gibbons-Hawking-York term S_{GHY} provides a contribution on the spacelike boundary at constant times on the Friedmann side. The Ricci scalar on the Friedmann side is given by

$${}^{(4)}\mathcal{R}_- = \frac{6}{a^2} (a'^2 + aa'' + k), \quad (62)$$

and the trace for the extrinsic curvature on $\mathcal{D}_- \cap \mathcal{K}$ is $K = 3a'/a$.

According to Brown and Kuchař [16], we can model the matter action by

$$\mathcal{L}_{\text{dust}} = -\frac{1}{2} \sqrt{-g} \epsilon (g_{\mu\nu} u^\mu u^\nu + 1), \quad (63)$$

with ϵ being the dust density and $\mathbf{u} = dT = T'd\tau$ being the 4-velocity covector of the dust field. The variable $T(\tau)$ is the proper time of the dust particles. The dust density ϵ is not dynamical and serves a Lagrange multiplier, which ensures the normalization of \mathbf{u} .

The full action now reads

$$S_{\text{interior}} = \frac{C}{2} \int_{\tau_1}^{\tau_2} d\tau a^3 \left[-\frac{a'^2}{a^2} + \frac{k}{a^2} + \frac{4\pi}{3} \epsilon (T'^2 - 1) \right], \quad (64)$$

where we defined the constant $C := 3 \int_0^{\chi_b} d\chi h^2(\chi)$.

For the sake of simplicity, we first perform the Legendre transform in the matter sector. The momentum conjugate to the dust proper time reads

$$p_T = \frac{4\pi C a^3 \epsilon}{3} T'. \quad (65)$$

Variation with respect to ϵ yields the primary constraint $p_\epsilon = \partial L / \partial \epsilon' = 0$. The secondary constraint $p'_\epsilon = 0$ yields that $\epsilon = 3p_T / (4\pi C a^3)$. We immediately implement this

constraint equation after performing the Legendre transform in T . The action then becomes

$$S_{\text{interior}} = \frac{C}{2} \int_{\tau_1}^{\tau_2} d\tau \left[a^3 \left(-\frac{a'^2}{a^2} + \frac{k}{a^2} - \frac{2p_T}{Ca^3} \right) \right]. \quad (66)$$

Since p_T is a constant of motion, we can keep it in the action not as a dynamical quantity but as an external parameter, controlling the dust density in the interior. To this end, we have dropped in the above the Liouville term $p_T T'$. Note that matching exterior and interior as done in the last section leads to the identification $M = h^3(\chi_b) p_T / C$, but we will not implement this before variation.

We now switch to the variable $\psi = a/h(\chi_b)$ and to the Schwarzschild time t to get

$$\begin{aligned} S_{\text{interior}} &= \frac{1}{2} \int_{t_1}^{t_2} dt \sqrt{1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}} \left(-2p_T + \frac{Ck\psi}{h(\chi_b)} - \frac{C\psi}{h^3(\chi_b)} \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}} \right) \\ &= \frac{1}{2} \int_{t_1}^{t_2} dt \sqrt{1 - \frac{2M}{\psi}} \left[\frac{-2p_T + \frac{C}{h(\chi_b)} k\psi}{\cosh \mu} - \frac{C}{h^3(\chi_b)} (\psi - 2M) \frac{\sinh^2 \mu}{\cosh \mu} \right]. \end{aligned} \quad (67)$$

Next, we wish to compute the term on the matching surface Σ . We need to compute the jump $[Q]$ in the trace of the ADM momentum on the surface Σ . We already have the extrinsic curvature on the Friedmann side. The result on the Schwarzschild side in (11) changes due to the time dependence of M . For the computation, we switch to the coordinate $x = r - \psi$. The normal covector is then

$$\mathbf{n} = \frac{1}{f} dx. \quad (68)$$

$K_{\theta\theta}^+$ and $K_{\phi\phi}^+$ stay the same as in (11), but

$$K_{tt}^+ = -\frac{\dot{f}^2}{\dot{\psi}} \partial_t \left[\frac{1}{|\dot{f}|} \left(1 - \frac{2M}{\psi} \right) \right] - \frac{\dot{M} \dot{f} [\psi^2 \dot{\psi}^2 + (\psi - 2M)^2]}{\psi \dot{\psi} (\psi - 2M)^2}. \quad (69)$$

From this, we get

$$[Q] = 4 \sin \theta \left[\frac{\psi^2}{2f} K_{tt}^+ - \psi + 2M + \partial_x h(\chi_b) \psi \dot{f} \right]. \quad (70)$$

Finally,

$$\begin{aligned} \int_{\Sigma} \mathcal{L}_{\text{grav}} &= -\frac{1}{16\pi} \int_{\Sigma} [Q] d^3y \\ &= \int_{t_1}^{t_2} dt \left(-\frac{1}{2} \frac{\dot{\psi} \dot{M} \psi^3}{(\psi - 2M)^2} - \frac{1}{2} \psi^2 \dot{\mu} \right. \\ &\quad \left. + \frac{3M}{2} - \psi + \partial_x h(\chi_b) f \dot{\psi} \right), \end{aligned} \quad (71)$$

where we have used that

$$\frac{\psi^2 \ddot{\psi}}{2\dot{f}^2} = \frac{1}{2} \psi^2 \dot{\mu} + \frac{\dot{\psi} \psi}{\dot{f}^2 (1 - \frac{2M}{\psi})} \left(-\dot{M} + \frac{M \dot{\psi}}{\psi} \right). \quad (72)$$

The first term cancels with the one from the Schwarzschild bulk (60), and the second term can be partially integrated. The resulting boundary term on the $t = \text{const.}$ surfaces cancels with the edge (56).

Now, combining everything, we see that what remains is the Friedmann term (67) and the boundary term on C_R and Σ apart from the first term. Finally, the Lagrangian is given by

$$\begin{aligned} L_{\text{tot}} &= -\frac{\psi \sqrt{1 - \frac{2M}{\psi}}}{\cosh \mu} \left[\frac{C}{2h^3(\chi_b)} \left(1 - \frac{2M}{\psi} \right) \sinh^2 \mu \right. \\ &\quad \left. - \frac{C}{2h(\chi_b)} k + \frac{p_T}{\psi} - \partial_x h(\chi_b) \right] \\ &\quad + M - \psi + \psi \dot{\psi} \mu. \end{aligned} \quad (73)$$

Since this Lagrangian does not depend on \dot{M} , one can express the mass as a function of ψ and $\dot{\psi}$ by enforcing the equation of motion for M ,

$$\frac{\partial L_{\text{tot}}}{\partial M} = 0. \quad (74)$$

Using

$$\frac{\partial \mu}{\partial M} = \frac{2 \sinh \mu \cosh \mu}{\psi \left(1 - \frac{2M}{\psi}\right)}, \quad (75)$$

we then find

$$-\sqrt{1 - \frac{2M}{\psi}} \cosh \mu + \frac{C}{2h^3(\chi_b)} \left(1 - \frac{2M}{\psi}\right) \sinh^2 \mu = -\frac{C}{2h(\chi_b)} k + \frac{p_T}{\psi} - \partial_\chi h(\chi_b). \quad (76)$$

This expression will now define M as a function of ψ and μ . We can then simplify (73) and get

$$L_{\text{tot}} = -\frac{C}{h^3(\chi_b)} \psi \left(1 - \frac{2M}{\psi}\right)^{\frac{3}{2}} \frac{\sinh^2 \mu}{\cosh \mu} - M + \psi \dot{\psi} \mu. \quad (77)$$

C. LTB interior with $E = \text{const}$

First, we calculate the action for LTB interior and $\tau = \text{const}$ boundaries. The Ricci scalar built from LTB metric is

$${}^{(4)}\mathcal{R}\sqrt{-g} = \frac{\partial_\rho(R + R(\partial_\rho R)^2 - (1 + 2E)R)}{\sqrt{1 + 2E}} \sin \theta. \quad (78)$$

To perform the integration along the ρ coordinate, we will impose the condition $E = \text{const}$, which reduces the class of LTB models but still includes inhomogeneous matter and curvature distribution cases. Thus, the gravitational part of LTB interior and boundary action reads

$$\mathcal{S}_{LTB} = \frac{1}{4} \int_{t_1}^{t_2} \frac{(R_b(\partial_\rho R_b)^2 - 2ER_b)}{\sqrt{1 + 2E}} d\tau - \frac{1}{2} \int_0^{\rho_0} \frac{R^2 \partial_\rho R}{\sqrt{1 + 2E}} \left(\frac{\partial_\tau \partial_\rho R}{\partial_\rho R} + 2 \frac{\partial_\tau R_b}{R} \right) d\rho. \quad (79)$$

The boundary term can be written as

$$\mathcal{S}_{\text{boundary}} = \frac{3}{4} \int_{t_1}^{t_2} d\tau \frac{R_b(\partial_\rho R_b)^2}{\sqrt{1 + 2E}}, \quad (80)$$

which leads to the total action

$$\mathcal{S}_{\text{grav}} = -\frac{1}{2} \int_{t_1}^{t_2} d\tau \left(\frac{R_b(\partial_\rho R_b)^2 + ER_b}{\sqrt{1 + 2E}} \right). \quad (81)$$

Changing variables to t and ψ gives

$$\mathcal{S}_{\text{grav}} = -\frac{1}{2\sqrt{1 + 2E}} \int_{t_1}^{t_2} dt \sqrt{1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}} \times \left(\frac{\dot{\psi}^2 \psi}{1 - \frac{2M}{\psi} - \frac{\dot{\psi}^2}{1 - \frac{2M}{\psi}}} + E\psi \right), \quad (82)$$

which, using the hyperbolic angle, can be rewritten as

$$\mathcal{S}_{\text{grav}} = \frac{1}{2} \int_{t_1}^{t_2} dt \sqrt{1 - \frac{2M}{\psi}} \left(\frac{-E\psi}{\cosh \mu \sqrt{1 + 2E}} - \frac{(\psi - 2M) \sinh^2 \mu}{\sqrt{1 + 2E} \cosh \mu} \right). \quad (83)$$

The jump in the extrinsic curvature on the shell reads

$$[Q] = 4 \sin \theta \left(\frac{\psi^2}{2f} K_{tt}^+ - \psi + 2M + \sqrt{1 + 2E} \psi f \right). \quad (84)$$

Lastly, we consider the matter. The Lagrangian density for dust is

$$\mathcal{L}_{\text{dust}} = -\frac{1}{2} \sqrt{-g} \epsilon (g_{\mu\nu} u^\mu u^\nu + 1). \quad (85)$$

The action can be written as

$$\mathcal{S}_{\text{dust}} = 2\pi \int d\tau \int d\rho \frac{R^2 \partial_\rho R}{\sqrt{1 + 2E}} \epsilon(\rho) (T^2 - 1). \quad (86)$$

In the following, we will assume that the density has an integral form given by the equation of motion (25) for LTB. When putting $F(0) = 0$, this allows us to write

$$\mathcal{S}_{\text{dust}} = \frac{1}{2} \int d\tau \frac{F_b}{\sqrt{1 + 2E}} (T^2 - 1). \quad (87)$$

Comparing with (64), we see that, in analogy to the FLRW case, the momentum p_T is given by

$$p_T = \frac{F_b}{\sqrt{1 + 2E}} T', \quad (88)$$

and as a secondary constraint, we get the identification

$$p_T = \frac{F_b}{\sqrt{1 + 2E}}. \quad (89)$$

Recall that matching interior and exterior is done by setting $F_b = M$. To implement this here, we will set $M = \sqrt{1 + 2E} p_T$ after variation.

In total, this leads to the Lagrangian

$$L_{\text{tot}} = -\frac{\psi \sqrt{1 - \frac{2M}{\psi}}}{\cosh \mu} \left[\frac{1}{2\sqrt{1 + 2E}} \left(1 - \frac{2M}{\psi} \right) \sinh^2 \mu + \frac{E}{2\sqrt{1 + 2E}} + \frac{p_T}{\psi} - \sqrt{1 + 2E} \right] + M - \psi + \psi \dot{\psi} \mu. \quad (90)$$

Note that this Lagrangian can be obtained from the Lagrangian for an FLRW interior (73) using the following identifications:

$$\begin{aligned} \frac{h(\chi_b)}{C^3} &\rightarrow \sqrt{1 + 2E}, & h^2(\chi_b)k &\rightarrow -E, \\ \partial \chi h(\chi_b) &\rightarrow \sqrt{1 + 2E}. \end{aligned} \quad (91)$$

It follows that the reduction undertaken for FLRW leading to (77) can also be done here. The result will be (77) with the replacement (91).

IV. HAMILTONIAN FORMULATION

We can now discuss the Hamiltonian formulation for both the FLRW and LTB interior at the same time. We will work with the Lagrangian (77), keeping in mind that all results apply to the LTB model when identifying the constants involved according to (91). In a first step to find

the corresponding Hamiltonian, we want to compute the momentum conjugate to ψ . As $L_{\text{tot}} = L_{\text{tot}}(\dot{\psi}, \mu(\dot{\psi}), M(\mu))$, we get

$$p_\psi = \frac{dL_{\text{tot}}}{d\dot{\psi}} = \frac{\partial L_{\text{tot}}}{\partial \dot{\psi}} + \left(\frac{\partial L_{\text{tot}}}{\partial \mu} + \frac{\partial L_{\text{tot}}}{\partial M} \frac{\partial M}{\partial \mu} \right) \frac{\partial \mu}{\partial \dot{\psi}}. \quad (92)$$

From (77) and (76), we find

$$\frac{\partial L_{\text{tot}}}{\partial \dot{\psi}} = \psi \mu, \quad (93)$$

$$\begin{aligned} \frac{\partial L_{\text{tot}}}{\partial \mu} &= -\psi \left(1 - \frac{2M}{\psi} \right) \\ &\times \tanh \mu \left(\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \frac{\cosh^2 \mu + 1}{\cosh \mu}} - 1 \right), \end{aligned} \quad (94)$$

$$\frac{\partial L_{\text{tot}}}{\partial M} = \frac{3C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \frac{\sinh^2 \mu}{\cosh \mu}} - 1, \quad (95)$$

$$\frac{\partial M}{\partial \mu} = \psi \left(1 - \frac{2M}{\psi} \right) \sinh \mu \frac{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \cosh \mu} - 1}{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \sinh^2 \mu} - \cosh \mu}, \quad (96)$$

and from (22), we find

$$\frac{\partial \mu}{\partial \dot{\psi}} = \frac{\cosh^2 \mu}{\left(1 - \frac{2M}{\psi} \right) - \frac{2}{\psi} \frac{\partial M}{\partial \mu} \sinh \mu \cosh \mu} \quad (97)$$

$$= \frac{\cosh^2 \mu}{1 - \frac{2M}{\psi} \frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \sinh^2 \mu} - \cosh \mu} \frac{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \sinh^2 \mu} - \cosh \mu}{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi} \sinh^2 \mu} (1 - 2\cosh^2 \mu) - \cosh \mu (1 - 2\sinh^2 \mu)}. \quad (98)$$

The momentum is then given by

$$p_\psi = \psi \mu - \frac{C}{h^3(\chi_b)} \psi \sqrt{1 - \frac{2M}{\psi}} \sinh \mu. \quad (99)$$

Finally, the Hamiltonian can be expressed as

$$H = p_\psi \dot{\psi} - L_{\text{tot}} = M, \quad (100)$$

as was the case in Ref. [9].

We will now demonstrate that the Hamiltonian (100) really gives the correct equations of motion for the Oppenheimer-Snyder model. We have to keep in mind that $M(\mu, \psi)$ is given by (76) and $\mu(\psi, p_\psi)$ in turn is implicitly given by (99). This gives us the equations of motion as

$$\dot{\psi} = \frac{\partial M}{\partial \mu} \frac{\partial \mu}{\partial p_\psi}, \quad (101)$$

$$\dot{p}_\psi = -\frac{\partial M}{\partial \psi} - \frac{\partial M}{\partial \mu} \frac{\partial \mu}{\partial \psi}. \quad (102)$$

As it turns out, it is more useful to consider $\dot{\mu}$ instead of \dot{p}_ψ , which is then given by

$$\dot{\mu} = -\frac{\partial M}{\partial \psi} \frac{\partial \mu}{\partial p_\psi}. \quad (103)$$

From (76), we get (96) as well as

$$\frac{\partial M}{\partial \psi} = \frac{M}{\psi} + \frac{p_T}{\psi} \frac{\sqrt{1 - \frac{2M}{\psi}}}{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi}} \sinh^2 \mu - \cosh \mu}, \quad (104)$$

and from (99),

$$\frac{\partial \mu}{\partial p_\psi} = \frac{1}{\psi} \frac{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi}} \sinh^2 \mu - \cosh \mu}{\frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi}} \cosh^2 \mu - \cosh \mu}, \quad (105)$$

where we have used (96).

In total, this gives us the equations of motion as

$$\dot{\psi} = \left(1 - \frac{2M}{\psi}\right) \tanh \mu, \quad (106)$$

$$\dot{\mu} = -\frac{M \frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi}} (\sinh^2 \mu + \frac{h^3(\chi_b) p_T}{CM}) - \cosh \mu}{\psi^2 \frac{C}{h^3(\chi_b)} \sqrt{1 - \frac{2M}{\psi}} \cosh^2 \mu - \cosh \mu}. \quad (107)$$

We can see that the first equation simply gives us back the definition of the hyperbolic angle μ from the canonical formalism.

To bring the equation for $\dot{\mu}$ into a recognizable form, we have to do a bit more work. We want to demonstrate that the above is equivalent to the second Friedmann equation in the form (23). Recall that for dust the first Friedmann equation follows directly from the second one, meaning that this is sufficient to demonstrate that our Hamiltonian gives the correct dynamics.

To this end, we solve (76) for $\cosh \mu$,

$$\cosh \mu = \frac{1}{\sqrt{1 - \frac{2M}{\psi}}} \left[\frac{h^3(\chi_b)}{C} \pm \sqrt{\left(\partial_\chi h(\chi_b) - \frac{h^3(\chi_b)}{C}\right)^2 + \frac{2}{\psi} \left(\frac{h^3(\chi_b)}{C} p_T - M\right)} \right]. \quad (108)$$

In principle, the sign should be fixed such that $\cosh \mu$ is always positive, but since it will not influence the result, we will leave it open. Inserting this into the right-hand side of (107) allows us to integrate the equation with regard to t , giving

$$\cosh \mu = \frac{A}{\sqrt{1 - \frac{2M}{\psi}}} \left[\frac{h^3(\chi_b)}{C} \pm \sqrt{\left(\partial_\chi h(\chi_b) - \frac{h^3(\chi_b)}{C}\right)^2 + \frac{2}{\psi} \left(\frac{h^3(\chi_b)}{C} p_T - M\right)} \right], \quad (109)$$

where A is a positive real constant of integration. Note that we used here that the Hamiltonian M is a constant of motion. This expression is, aside from a constant, identical to (108), demonstrating that (107) is simply the time derivative of (108) and in turn (76).

Imposing $p_T = CM/h^3(\chi_b)$, we see that (108) gives us

$$\cosh \mu = \frac{1}{\sqrt{1 - \frac{2M}{\psi}}} \left[\frac{h^3(\chi_b)}{C} \pm \left| \partial_\chi h(\chi_b) - \frac{h^3(\chi_b)}{C} \right| \right], \quad (110)$$

where the sign has to be chosen such that $\cosh \mu$ is positive. For $k = 0$, this does not make a difference since there $h'(\chi_b) = h^3(\chi_b)/C = 1$, giving (23), using that $h'(\chi_b) = \sqrt{1 - h^2(\chi_b)} \tilde{k}$. For $k = \pm 1$, both signs give different positive results, in addition to (23) also giving the solution

$$\cosh \mu = \frac{2 \frac{h^3(\chi_b)}{C} - \partial_\chi h(\chi_b)}{\sqrt{1 - \frac{2M}{\psi}}}. \quad (111)$$

In summary, we see that the Hamilton equations of motion simply give us the definition of μ (or alternatively express p_ψ in terms of $\dot{\psi}$) and the time derivative of the equation $H = M = \text{const}$, while this equation itself already gives the dynamics of the Oppenheimer-Snyder model, provided we choose $p_T = CM/h^3(\chi_b)$.

It is for both $k = \pm 1$ possible to identify an effective curvature \tilde{k} with $0 < \pm \tilde{k} < 1$ as $2h^3(\chi_b)/C - \partial_\chi h(\chi_b) = \sqrt{1 - h^2(\chi_b)} \tilde{k}$. One can then rescale all other quantities here, normalizing \tilde{k} , such that (111) can be seen as a rescaled Friedmann equation.

For an LTB interior with $E = \text{const}$, the situation is a bit more simple. Applying (110) to the LTB case by using (91) only gives the single equation of motion (40).

V. CONCLUSIONS

In this article, we have developed the Hamiltonian formulation of dust collapse with the use of a quasilocal

variational principle. The main result is that the global Hamiltonian H of the OSD or LTB gravitational system, for an observer at spatial infinity, is positive definite. This is consistent with the theorem in general relativity on positivity of the total energy of an isolated gravitational system with non-negative local mass density (see, e.g., Refs. [17–19] and references therein). Furthermore, the Hamiltonian obtained is equal to the total mass of the system, $H = M$, as measured by observers at infinity. This result, which resembles what is usually obtained in Newtonian systems, albeit intuitive, is not readily obtained from standard Hamiltonian treatments in general relativity.

Our results are obtained by making use of the improved variational principle [8] that was later applied to the dynamics of a gravitational dust shell [9]. Roughly speaking, it consists in considering an extra boundary term added to the Hilbert action and assuming that M , treated as a dynamical variable, is a function of an evolution parameter at the level of the variational procedure. Then, the fact that M is a constant becomes a consequence of the equations of motion in Hamilton's dynamics. We can see that the variational principle used for the simple shell model [9] extends to the more realistic cases of the OSD and LTB systems and encourages applying to still more realistic models of collapsing massive stars.

In the Schwarzschild black hole, M is just a parameter that determines the position of the event horizon. In our case, like a true Hamiltonian, it has an additional dynamical structure as it depends on configuration variables and matter fields and turns out to be constant only as a consequence of the equations of motion.

In this article, we have restricted our analysis to a single outermost dust shell of the LTB system, and we consider only the case with $E = \text{const}$, which reduces the original field theory LTB model to a mechanical system. This allows us to reduce the dynamics to a simple one-dimensional system where the configuration variables are position and velocity of the outermost shell. In the homogeneous OSD case, all shells obey the same dynamics as the outermost, while in the inhomogeneous LTB case with $E = \text{const}$, the radial position of the inner shells can be treated as a parameter, rather than a true degree of freedom. Because of this simplification, both gravitational systems, the LTB and OSD, can be described within one formalism. However, it should be noted that the LTB case is particularly interesting as it may admit the existence of naked singularities (see, e.g., Refs. [20–22]), which are appealing both from the observational and theoretical perspectives as they may provide the valuable keys to the construction of a viable theory of quantum gravity. More specifically, assuming that the dust density has radial dependence with a quadratic term, i.e., $\epsilon(\rho) = \epsilon_0 + \epsilon_2\rho^2$, it can be shown that there exist values for (ϵ_2, E) for which the comoving time of formation of the singularity coincides with the comoving time of formation of trapped

surfaces [23]. Then, for a set of values of the boundary radius of the matter cloud, there exist null geodesics originating at the singularity that reach far-away observers [24]. This seemingly nonphysical result can be understood if one treats the singularity as the limit of a regime where quantum gravity effects dominate. Then, a naked singularity merely describes a system where quantum gravity effects may be observable for far-away observers.

Quantization of the interior of both the LTB and OSD black holes has already been done in Refs. [25,26]. Robustness of these results will be analyzed in the near future. In fact, if a different quantization procedure for dust collapse were to provide the same results, this could be taken as an indication of the general validity of the results.

ACKNOWLEDGMENTS

We would like to thank Claus Kiefer and Jerzy Kijowski for helpful discussions and Dejan Stojkovic for feedback on our paper. This work was partially supported by the German-Polish bilateral project DAAD and MNiSW, Grant No. 57391638, and by Nazarbayev University Faculty Development Competitive Research Grant No. 090118FD5348.

APPENDIX: THEORY OF MATCHING

Consider two four-dimensional manifolds \mathcal{M}^+ and \mathcal{M}^- separated by a three-dimensional timelike hypersurface Σ . In general, we may express the line element on both sides in terms of the coordinates $\{x_{\pm}^0, x_{\pm}^1, x_{\pm}^2, x_{\pm}^3\}$ as

$$ds_{\pm}^2 = g_{\mu\nu}^{\pm} dx_{\pm}^{\mu} dx_{\pm}^{\nu}, \quad (\text{A1})$$

while the line element on the hypersurface, in terms of the coordinates $\{y^1, y^2, y^3\}$, is

$$ds_{\Sigma}^2 = \gamma_{ab} dy^a dy^b \quad (\text{A2})$$

where $a, b = 1, 2, 3$. The hypersurface may be expressed in parametric form on either side of the matching as

$$\Phi^{\pm}(x_{\pm}^{\mu}(y^a)) = 0 \quad (\text{A3})$$

so that the line element on the surface in terms of the coordinates on \mathcal{M}^{\pm} is

$$\gamma_{ab}^{\pm} = \frac{\partial x_{\pm}^{\mu}}{\partial y^a} \frac{\partial x_{\pm}^{\nu}}{\partial y^b} g_{\mu\nu}^{\pm}. \quad (\text{A4})$$

The induced metric is the same on both sides if we can find a set of coordinates for which

$$\gamma_{ab}^{\pm} = \gamma_{ab}. \quad (\text{A5})$$

Let us now define the unit vector normal to Σ on both sides as

$$n_{\mu}^{\pm} = \frac{1}{\sqrt{|\frac{\partial\Phi^{\pm}}{\partial x^{\alpha}} \frac{\partial\Phi^{\pm}}{\partial x^{\beta}} g_{\pm}^{\alpha\beta}|}} \frac{\partial\Phi^{\pm}}{\partial x^{\mu}}. \quad (\text{A6})$$

The second fundamental form (or extrinsic curvature) is defined by

$$\begin{aligned} K_{ab}^{\pm} &= \frac{\partial x_{\pm}^{\mu}}{\partial y^a} \frac{\partial x_{\pm}^{\nu}}{\partial y^b} \nabla_{\mu} n_{\nu}^{\pm} \\ &= -n_{\sigma}^{\pm} \left(\frac{\partial^2 x_{\pm}^{\sigma}}{\partial y^a \partial y^b} + \Gamma_{\mu\nu}^{\sigma} \frac{\partial x_{\pm}^{\mu}}{\partial y^a} \frac{\partial x_{\pm}^{\nu}}{\partial y^b} \right). \end{aligned} \quad (\text{A7})$$

The boundary surface does not carry any energy-momentum tensor, and therefore the matching is smooth, if

$$[\gamma_{ab}] = 0 \quad (\text{A8})$$

$$[K_{ab}] = 0, \quad (\text{A9})$$

where we have used the notation

$$[A] = A^{+} - A^{-}$$

for a generic quantity A . The ADM momentum density of the hypersurface is given by

$$Q_{ab} = \sqrt{|\det\gamma_{cd}|} (\gamma_{ab} K - K_{ab}), \quad (\text{A10})$$

and its trace reads

$$Q = \gamma^{ab} Q_{ab} = -2\sqrt{|\det\gamma_{cd}|} \gamma^{ab} K_{ab}. \quad (\text{A11})$$

For more details, see Ref. [13].

-
- [1] P. S. Joshi and D. Malafarina, *Int. J. Mod. Phys. D* **20**, 2641 (2011).
- [2] P. Hájíček and J. Kijowski, *Phys. Rev. D* **62**, 044025 (2000).
- [3] P. Hájíček and C. Kiefer, *Int. J. Mod. Phys. D* **10**, 775 (2001).
- [4] A. Gózdź, W. Piechocki, and G. Plewa, *Eur. Phys. J. C* **79**, 45 (2019).
- [5] J. R. Oppenheimer and H. Snyder, *Phys. Rev.* **56**, 455 (1939); S. Datt, *Z. Phys.* **108**, 314 (1938).
- [6] G. Lemaître, *Ann. Soc. Sci. Bruxelles I A* **53**, 51 (1933); R. C. Tolman, *Proc. Natl. Acad. Sci. U.S.A.* **20**, 169 (1934); H. Bondi, *Mon. Not. R. Astron. Soc.* **107**, 410 (1947).
- [7] P. S. Joshi and I. H. Dwivedi, *Commun. Math. Phys.* **146**, 333 (1992).
- [8] J. Kijowski, *Gen. Relativ. Gravit.* **29**, 307 (1997).
- [9] J. Kijowski, G. Magli, and D. Malafarina, *Phys. Rev. D* **74**, 084017 (2006).
- [10] A. Saini and D. Stojkovic, *Phys. Rev.* **89**, 044003 (2014).
- [11] G. Darmois, *Memorial de Sciences Mathematiques, XXV* (Gauthier-Villars, 1927); W. Israel, *Nouvo Cimento* **44B**, 1 (1966); **44B** 463E, (1966).
- [12] F. Fayos, X. Jaen, E. Llanta, and J. M. M. Senovilla, *Classical Quantum Gravity* **8**, 2057 (1991); F. Fayos, X. Jaen, E. Llanta, and J. M. M. Senovilla, *Phys. Rev. D* **45**, 2732 (1992); F. Fayos, J. M. M. Senovilla, and R. Torres, *Phys. Rev. D* **54**, 4862 (1996).
- [13] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, England, 2004).
- [14] C. Misner and D. Sharp, *Phys. Rev.* **136**, B571 (1964).
- [15] G. Hayward, *Phys. Rev. D* **47**, 3275 (1993).
- [16] J. D. Brown and K. V. Kuchař, *Phys. Rev. D* **51**, 5600 (1995).
- [17] R. Schön and S. T. Yau, *Commun. Math. Phys.* **65**, 45 (1979); **79**, 47 (1981); **79**, 231 (1981).
- [18] E. Witten, *Commun. Math. Phys.* **80**, 381 (1981).
- [19] J. Jezierski and J. Kijowski, *Phys. Rev. D* **36**, 1041 (1987).
- [20] R. Giambò and G. Magli, *Differential Geometry and its Applications* **18**, 285 (2003).
- [21] P. S. Joshi and I. H. Dwivedi, *Phys. Rev. D* **47**, 5357 (1993).
- [22] D. Malafarina, *Universe* **3**, 48 (2017).
- [23] P. S. Joshi and D. Malafarina, *Int. J. Mod. Phys. D* **20**, 2641 (2011); P. S. Joshi, D. Malafarina, and R. V. Saraykar, *Int. J. Mod. Phys. D* **21**, 1250066 (2012).
- [24] P. S. Joshi, *Gravitational Collapse and Spacetime Singularities* (Cambridge University Press, Cambridge, England, 2008).
- [25] C. Kiefer and T. Schmitz, *Phys. Rev. D* **99**, 126010 (2019).
- [26] T. Schmitz, *Phys. Rev. D* **101**, 026016 (2020).