Nonexistence of expansion-free dynamical stars with rotation and spatial twist

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Extending a previous work by the present authors, we investigate the existence of expansion-free dynamical stars with nonzero spatial twist and rotation and show that such stars cannot exist. First, it is shown that a rotating expansion-free dynamical star with zero twist cannot exist because such stars cannot radiate and are shear-free, in which case the energy density ρ is time independent. Second, we prove that a nonrotating expansion-free dynamical star with nonzero spatial twist also cannot exist, as either the strong energy condition must be violated, i.e., $\rho + 3p < 0$, or the star must be shear-free, in which case the star is static ($\Theta = \Omega = \Sigma = 0$). Finally, if we insist that the rotation and spatial twist are simultaneously nonzero, then the star cannot be shear-free, in which case we obtain a quadratic polynomial equation in ϕ and Σ with no real solutions. Therefore, such stars cannot exist.

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I. INTRODUCTION

Models of radiating stars in general relativity play a central role in the study of gravitational collapse and the astrophysics of gravitating bodies. Physically relevant exact models were obtained by Tewari and Charan [1], Tewari [2], and Ivanov [3–5]. These examples provide interesting insights into the processes involved during stellar evolution. It has also been found by Reddy *et al* [6] that anisotropy and dissipative effects during gravitational collapse have influence on the collapse rate and temperature profiles in radiating stars. Classes of exact solutions to Einstein's field equations (EFEs) have been obtained and are referred to as Euclidean stars, which, in the appropriate limit, have been shown to regain the solutions referred to as Newtonian stars [7–9]. In recent years, the method of Lie analysis of differential equations using symmetry invariance has proved an invaluable and systematic tool in obtaining general categories of exact solutions to the boundary condition of radiating stellar objects [10–12]. There is an important class of radiating stars, introduced by Herrera *et al.* [13], which are expansion-free. Expansion-free dynamical models imply the existence of a cavity or void. One important feature of expansion-free models is that matter distributions with a vanishing expansion scalar have to be inhomogeneous. These physical features should have important astrophysical consequences for spherically symmetric distributions. Also, such radiating astrophysical models might offer a plausible explanation for the existence of voids that have been observed on cosmological scales. Various authors have explored expansion-free dynamical models with different considerations. Studies containing descriptions of the physical properties of expansion-free dynamical radiating stars can be found in several works [14–16]. The peak in interest regarding these models is connected to the fact that such models have the possibility of helping to explain the existence of voids on cosmological scales. In 2008, Herrera and co-authors [13] studied such models with nonzero shear and showed that the appearance of a cavity (see Ref. [17] for more discussion) within an anisotropic and dissipative matter distribution that is undergoing an explosion, is inevitable. The same authors followed up on this result with a paper in 2009 [18] in which they ruled out the Skripkin expansion-free dynamical model (see Ref. [19]) with constant energy density and isotropic pressure. Another study in [20] involved the study of models collapsing adiabatically and showed that the instability was independent of the star's stiffness. In particular, it was shown that the instability was entirely governed by the pressure and the radial profile of the energy density. In a recent work by Sherif *et al.* [21], the authors employed, for the first time, the $1 + 1 + 2$ formalism (a semitetrad covariant method for analyzing the field equations) to study the properties of expansion-free models. With an emphasis on nonrotating and nontwisting stars, the authors found that a necessary condition for the existence of such stars is that they simultaneously accelerate and radiate. It was also shown in the same paper that these stars must possess a conformally flat geometry.

In this paper, we study the required geometric and thermodynamic properties for the existence of a relativistic

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expansion-free dynamical star with, at least, the rotation or spatial twist being nonzero. This analysis falls in the scope of stability analysis of self-gravitating systems (given in Refs. [22–26]). Our approach is to fix either the rotation or the spatial twist to zero and see whether, indeed, expansion-free dynamical models of such stars exist. In particular, we would like to know the restrictions that the addition of spatial twist and/or rotation induces on the geometric and matter quantities such as acceleration, heat flux, etc. As with our previous work [21], we make use of equivalent forms of the field equations from the $1+1+2$ semitetrad covariant formulation of general relativity [27–32]. The semitetrad formalism has proven to be an extremely useful approach in displaying geometrical features in a transparent fashion, which are generally very difficult to find using other approaches.

In Sec. II, we briefly introduce the $1+1+2$ semitetrad formalism and provide a definition for locally rotationally symmetric (LRS) spacetimes. In Sec. III, we present the results of the paper, a complete analysis of the expansion-free model with rotation and spatial twist. We conclude with a discussion of the results in Sec. IV.

II. LOCALLY ROTATIONALLY SYMMETRIC SPACETIMES AND THE 1+1+2 SEMITETRAD SPLITTING

We provide some background material in this section, covering the $1+1+2$ semitetrad covariant formalism as well as notes on, and calculations of, useful quantities utilized in this paper.

Stellar models that are rotating and twisting can be studied using the spacetime models known as locally rotationally symmetric spacetimes [33,34]. As such, we use this model to investigate expansion-free dynamical stars that are either rotating or possess spatial twist or both. We start by explicitly defining LRS spacetimes.

Definition.—A spacetime in which at each point, $p \in M$, there exists a continuous isotropy group generating a multiple transitive isometry group on M [34–39]. The general metric of LRS spacetimes is given by

$$\begin{aligned} ds^2 = &-A^2dt^2 + B^2d\chi^2 + F^2dy^2 \\ &+ [(F\bar{D})^2 + (Bh)^2 - (Ag)^2]dz^2 \\ &+ (A^2gdt - B^2hd\chi)dz, \end{aligned} \tag{1}$$

where A^2, B^2, F^2 are functions of t and χ, $\bar{D}^2$ is a function of y and k (k fixes the geometry of the 2-surfaces), and g, h are functions of y.

In the limiting case where $g = h = 0$, we recover the well-known spherically symmetric LRS II class of spacetimes which generalizes spherically symmetric solutions to EFEs. Such spacetimes with vanishing rotation and spatial twist were employed in [21] to study expansion-free and dynamic stellar models. LRS spacetimes, a generalization of LRS II spacetimes, on the other hand, include solutions with nonzero vorticity and nonzero spatial twist. Some of these solutions include the Gödel world model, the Kantowski-Sachs models, and the Bianchi models, invariant under the G_3 groups of types I, II, VIII, and IX (see, for example, Ref. [40]). In fact, the Gödel world model, a famous, albeit unphysical solution to the field equations [41], is an expansion-free model that rotates with zero spatial twist. Properties of such expansion-free dynamical stars will be investigated in Sec. III A, and properties necessary for their existence will be determined.

Next, we introduce the $1+1+2$ covariant splitting of spacetime and the resulting field equations for LRS spacetimes, as well as derivatives of the unit vector fields [29,32].

To start with, let (M, g_{ab}) be a spacetime manifold, with associated metric tensor g_{ab}. To any timelike congruence of an observer, we may associate a unit vector field u^a tangent to the congruence which satisfies $u^a u_a = -1$. One may then split M as follows: Given any 4-vector U^a in the spacetime, the projection tensor $h_a{}^b \equiv g_a{}^b + u_a u^b$ projects U^a onto the 3-space as

$$U^a = U u^a + U^{\langle a \rangle},$$

where U is the scalar along u^a and $U^{\langle a \rangle}$ is the projected 3-vector [42]. This splits g_{ab} into components associated with the u^a and spatial directions. This naturally gives rise to two derivatives:

(i) The covariant time derivative (or simply the dot derivative) along the observer's congruence. Given any tensor $S^{a..b}{}_{c..d}$, we have $\dot{S}^{a..b}{}_{c..d} \equiv u^e \nabla_e S^{a..b}{}_{c..d}$.
(ii) Fully orthogonally projected covariant derivative D with the tensor h_{ab} with the total projection carried out on all the free indices. Given any tensor $S^{a..b}{}_{c..d}$, we have $D_e S^{a..b}{}_{c..d} \equiv h^a{}_f h^p{}_c \ldots h^b{}_g h^q{}_d h^r{}_e \nabla_r S^{f..g}{}_{p..q}$.

This $1+3$ splitting of the spacetime irreducibly splits the covariant derivative of u^a as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab}. \tag{2}$$

In (2), the vector $A_a = \dot{u}_a$ is the acceleration vector, $\Theta \equiv D_a u^a$ (the trace of the fully orthogonally projected covariant derivative of u^a) is the expansion, and $\sigma_{ab} = D_{\langle b} u_{a \rangle}$ is the shear tensor (wherever used in this paper, angle brackets will denote the projected symmetric trace-free part of the tensor). In the particular case of LRS spacetimes, all vector and tensor quantities vanish identically (see Ref. [29] for details).

The splitting further allows for the energy momentum tensor to be decomposed as

$$T_{ab} = \rho u_a u_b + 2q_{(a} u_{b)} + p h_{ab} + \pi_{ab}, \tag{3}$$

where $\rho \equiv T_{ab} u^a u^b$ is the energy density, $q_a = -h_a{}^c T_{cd} u^d$ is the 3-vector defining the heat flux, $p \equiv (1/3) h^{ab} T_{ab}$ is the isotropic pressure, and π_{ab} is the anisotropic stress tensor.

If there is a preferred spatial direction along the unit normal vector field e^a, which is the case with LRS II spacetimes, the metric g_{ab} can be split into terms along the u^a and e^a directions (the vector field e^a splits the 3-space), as well as on the 2-surface, i.e.,

$$g_{ab} = N_{ab} - u_a u_b + e_a e_b, \tag{4}$$

where the projection tensor N_{ab} projects any 2-vector orthogonal to u^a and e^a onto the 2-surface defined by the sheet ($N_a{}^a = 2, u^a N_{ab} = 0, e^a N_{ab} = 0$), and e^a is defined such that $e^a e_a = 1$, and it is orthogonal to u^a, i.e., $u^a e_a = 0$. This is referred to as $1+1+2$ splitting. This splitting of the spacetime additionally gives rise to the splitting of the covariant derivatives along the e^a direction and on the 2-surface:

(i) The hat derivative is the spatial derivative along the vector field e^a: Given a 3-tensor $\psi_{a..b}{}^{c..d}$, we have $\hat{\psi}_{a..b}{}^{c..d} \equiv e^f D_f \psi_{a..b}{}^{c..d}$.

(ii) The delta derivative is the projected spatial derivative on the 2-sheet (projection by the tensor $N_a{}^b$), and the projection is carried out on all free indices: Given any 3-tensor $\psi_{a..b}{}^{c..d}$, we have $\delta_e \psi_{a..b}{}^{c..d} \equiv N_a{}^f..N_b{}^g N_h{}^c..N_i{}^d N_e{}^j D_j \psi_{f..g}{}^{h..i}$.

The complete set of $1+1+2$ covariant scalars fully describing the LRS class of spacetimes is [29]

$$\{A, \Theta, \phi, \Sigma, \mathcal{E}, \mathcal{H}, \rho, p, \Pi, Q, \Omega, \xi\}.$$

The quantity $\phi \equiv \delta_a e^a$ is the sheet expansion, $\Sigma \equiv \sigma_{ab} e^a e^b$ is the scalar associated with the shear tensor σ_{ab}, $\mathcal{E} \equiv E_{ab} e^a e^b$ is the scalar associated with the electric part of the Weyl tensor E_{ab}, $\mathcal{H} \equiv H_{ab} e^a e^b$ is the scalar associated with the magnetic part of the Weyl tensor $\mathcal{H}_{ab}$, $\Pi \equiv \pi_{ab} e^a e^b$ is the anisotropic stress scalar, and $Q \equiv -e^a T_{ab} u^b = q_a e^a$ is the scalar associated with the heat flux vector q_a. The quantities ξ and Ω are the spatial twist and rotation scalar, respectively, which are defined by $\xi = (1/2)\varepsilon^{ab}\delta_a e_b$ (where $\varepsilon_{ab} \equiv \varepsilon_{abc} e^c = u^d \eta_{\mathrm{dabcd}} e^c$ is the Levi-Civita 2-tensor, the volume element of the 2-surface) and $\Omega = e^a \omega_a$ (where $\omega^a = \Omega e^a + \Omega^a$ is the rotation vector, with Ω^a being the sheet component of ω^a).

The full covariant derivatives of the vector fields u^a and e^a are given by [29]

$$\nabla_a u_b = -A u_a e_b + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) \tag{5a}$$

$$+ N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right), \tag{5b}$$

$$\nabla_a e_b = -A u_a u_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a u_b + \frac{1}{2}\phi N_{ab}. \tag{5c}$$

We also note the useful expression

$$\hat{u}^a = \left(\frac{1}{3}\Theta + \Sigma\right) e^a. \tag{6}$$

Any given scalar ψ satisfies the commutation relation

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -A\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}. \tag{7}$$

We will utilize this relation throughout this work when seeking constraint equations. The field equations for LRS spacetimes are given as propagation and evolution of the covariant scalars [29]:

(i) *Evolution*:

$$\frac{2}{3}\dot{\Theta} - \dot{\Sigma} = A\phi - \frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)^2 - 2\Omega^2 + \mathcal{E} - \frac{1}{2}\Pi - \frac{1}{3}(\rho + 3p), \tag{8a}$$

$$\dot{\phi} = \left(\frac{2}{3}\Theta - \Sigma\right)\left(A - \frac{1}{2}\phi\right) + 2\xi\Omega + Q, \tag{8b}$$

$$\dot{\xi} = -\frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)\xi + \left(A - \frac{1}{2}\phi\right)\Omega, \tag{8c}$$

$$\dot{\Omega} = A\xi - \left(\frac{2}{3}\Theta - \Sigma\right)\Omega, \tag{8d}$$

$$\dot{\mathcal{H}} = -3\xi\mathcal{E} - \frac{3}{2}\left(\frac{2}{3}\Theta - \Sigma\right)\mathcal{H} + \Omega Q, \tag{8e}$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot{\rho} + \frac{1}{2}\dot{\Pi} = -\left(\frac{2}{3}\Theta - \Sigma\right)\left(\frac{3}{2}\mathcal{E} + \frac{1}{4}\Pi\right) + \frac{1}{2}\phi Q + 3\xi\mathcal{H} + \frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)(\rho + p), \tag{8f}$$

(ii) *Propagation*:

$$\frac{2}{3}\hat{\Theta} - \hat{\Sigma} = \frac{3}{2}\phi\Sigma + 2\xi\Omega + Q, \tag{9a}$$

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) + 2\xi^2 - \frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi, \tag{9b}$$

$$\hat{\xi} = -\phi\xi + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega, \tag{9c}$$

$$\hat{\Omega} = (A - \phi)\Omega, \tag{9d}$$

$$\hat{\mathcal{H}} = -\left(3\mathcal{E} + \rho + p - \frac{1}{2}\Pi\right)\Omega - 3\phi\mathcal{H} - Q\xi, \tag{9e}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\rho} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) - \frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)Q + 3\Omega\mathcal{H} \tag{9f}$$

(iii) *Evolution/Propagation*:

$$\hat{A} - \dot{\Theta} = -(A+\phi)A - \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 + \frac{1}{2}(\rho + 3p), \tag{10a}$$

$$\dot{\rho} + \hat{Q} = -\Theta(\rho + p) - (2A+\phi)Q - \frac{3}{2}\Sigma\Pi, \tag{10b}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\left(A + \frac{3}{2}\phi\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - (\rho + p)A, \tag{10c}$$

(iv) *Constraint:*

$$\mathcal{H} = 3\Sigma\xi - (2\mathcal{A} - \phi)\Omega. \tag{11}$$

Let us now analyze the expansion-free dynamical models with rotation and spatial twist.

III. RESULTS

In [21], we considered expansion-free dynamical stars that are nonrotating and nontwisting. It was shown that the existence of such models requires the star to simultaneously accelerate and radiate, in which case the star is necessarily conformally flat. Here, we consider the case in which at least one of Ω or ξ is nonvanishing. Thus, we consider the following three cases [34,37,38]:

(1) $\xi = 0; \Omega \neq 0$: These models fall under the class of spacetimes known as LRS I spacetimes, with e^a hypersurface orthogonal and u^a twisting. A well-known example is the Gödel solution.

(2) $\xi \neq 0; \Omega = 0$: These models fall under the class of spacetimes known as LRS III spacetimes, with e^a twisting and u^a hypersurface orthogonal.

(3) $\xi \neq 0; \Omega \neq 0$: These models, investigated in [38], have the property that the heat flux Q cannot be zero and specific energy conditions need to be satisfied, i.e.,

$$-\frac{1}{2}(\rho + p + \Pi) < Q < \frac{1}{2}(\rho + p + \Pi). \tag{12}$$

One therefore expects that an expansion-free dynamical model to exist in models with $\xi \neq 0$ and $\Omega \neq 0$. The star being dynamical implies that all of the thermodynamic quantities, including p, ρ, Π, Q, etc., are functions of time.

A. Case 1: $\xi = 0; \Omega \neq 0$

Let us start by considering the case of a rotating expansion-free star with no spatial twist. From (9c) we have

$$0 = \Sigma\Omega. \tag{13}$$

Since by assumption $\Omega \neq 0$, we must have $\Sigma = 0$. Furthermore, from (8c) we obtain

$$0 = \left(A - \frac{1}{2}\phi\right)\Omega, \tag{14}$$

which, from (11), gives $\mathcal{H} = 0$, so that for such stars the Weyl tensor is purely electric. Using (8e) one has

$$0 = \Omega Q, \tag{15}$$

from which we obtain $Q = 0$. Therefore, the star is not dynamical as the energy density is time independent, i.e., $\dot{\rho} = 0$ from (10b). It is also not difficult to show that such stars will necessarily accelerate. To see this, assume $A = 0$. Then, from (14), since by assumption $\Omega \neq 0$, we must have $\phi = 0$ as well. From (8a), (9b), (9e), and (10a), we obtain the constraints:

$$0 = -2\Omega^2 + \mathcal{E} - \frac{1}{3}(\rho + 3p) - \frac{1}{2}\Pi, \tag{16a}$$

$$0 = \mathcal{E} + \frac{2}{3}\rho + \frac{1}{2}\Pi, \tag{16b}$$

$$0 = 3\mathcal{E} + \rho + p - \frac{1}{2}\Pi, \tag{16c}$$

$$0 = -2\Omega^2 + \frac{1}{2}(\rho + 3p). \tag{16d}$$

Comparing (16b) and (16c), we obtain

$$0 = -\rho + p - 2\Pi. \tag{17}$$

Substituting (17) and (16b) into (16a), we obtain the constraint

$$0 = -2\Omega^2 - \frac{1}{2}(\rho + 3p), \tag{18}$$

which upon comparing to (16d) gives

$$0 = \rho + 3p. \tag{19}$$

Therefore, $\Omega = 0$ [from either (16d) or (19)], which contradicts the assumption that $\Omega \neq 0$. Hence, we have $A \neq 0$. In summary, we state the following theorems.

Theorem III.1: There cannot exist an expansion-free dynamical star with vanishing spatial twist and nonzero rotation.

Though these stars are not dynamical, we have enumerated several properties we expect such stars to have. In particular, the star is shear-free and accelerates without radiating.

B. Case 2: $\xi \neq 0; \Omega = 0$

Next, we consider nonrotating expansion-free dynamical stars with nonzero spatial twist. We state and prove the following:

Theorem III.2: There cannot exist an expansion-free dynamical star with vanishing rotation and nonzero spatial twist.

Proof: To prove this, we will show that for such a star to exist, the star will either be static or it will violate the strong energy condition (SEC). From (8d) we have

$$0 = A\xi, \tag{20}$$

so we must have $A = 0$ since by assumption $\xi \neq 0$. Using (10a) we have

$$\Sigma^2 = -\frac{1}{3}(\rho + 3p). \tag{21}$$

Thus, for such a star to exist, we must have $\rho + 3p < 0$, except in the instance where the star is shear-free, in which case the star is static ($\Omega = \Theta = \Sigma = 0$). ■

In fact, in this case we have shown that even the expansion-free condition cannot hold and, not only that, it is also not dynamical.

C. Case 3: $\xi \neq 0; \Omega \neq 0$

Finally, we consider the case of a simultaneously rotating and twisting expansion-free dynamical star. We start by taking the dot derivative of (8a) and the hat derivative of (9a) and obtain, respectively,

$$\begin{aligned}
-\hat{\dot{\Sigma}} &= \phi\hat{A} + A\hat{\phi} - \Sigma\hat{\sigma} - 4\Omega\hat{\Omega} + \hat{\mathcal{E}} - \frac{1}{2}\hat{\Pi} - \frac{1}{3}\hat{\rho} - \hat{p} \\
&= -A^2\phi - \frac{3}{2}A\phi^2 - A\left(\frac{2}{3}\rho + \mathcal{E} + \frac{1}{2}\Pi\right) + 3\phi\Sigma^2 \\
&\quad + 2\phi\Omega^2 + \frac{1}{2}\phi(\rho + 3p) + \frac{3}{2}\Sigma Q + 2\Omega\Sigma\xi + 3\Omega\mathcal{H} \\
&\quad - \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) - (\hat{p} + \hat{\Pi}),
\end{aligned} \tag{22a}$$

$$\begin{aligned}
-\dot{\hat{\Sigma}} &= \frac{3}{2}\Sigma\dot{\phi} + \frac{3}{2}\phi\dot{\Sigma} + 2\Omega\dot{\xi} + 2\xi\dot{\Omega} + \dot{Q} \\
&= -\frac{3}{2}A\Sigma^2 + \frac{3}{2}\phi\Sigma^2 + 6\Omega\Sigma\xi + \frac{3}{2}\Sigma Q - \frac{3}{2}A\phi^2 \\
&\quad + 2\phi\Omega^2 - \frac{3}{2}\phi\mathcal{E} + \frac{3}{4}\phi\Pi + \frac{1}{2}\phi(\rho + 3p) + 2A\Omega^2 \\
&\quad + 2A\xi^2 + \dot{Q}.
\end{aligned} \tag{22b}$$

Taking the difference of (22a) and (22b) and using (10c), we obtain

$$\begin{aligned}
-\hat{\dot{\Sigma}} + \dot{\hat{\Sigma}} &= -A^2\phi + \frac{1}{3}A(\rho + 3p) - A\left(\mathcal{E} - \frac{1}{2}\Pi\right) \\
&\quad + \frac{3}{2}\phi\Sigma^2 - 4\Omega\Sigma\xi - 6A\Omega^2 + \frac{1}{2}A\Sigma^2 + \Sigma Q \\
&\quad + 3\Omega\mathcal{H}.
\end{aligned} \tag{23}$$

Using the commutation relation in (7) on Σ, we have

$$\begin{aligned}
-\hat{\dot{\Sigma}} + \dot{\hat{\Sigma}} &= -A\dot{\Sigma} + \Sigma\hat{\Sigma} \\
&= -A^2\phi + \frac{1}{2}A\Sigma^2 + 2A\Omega^2 - A\left(\mathcal{E} - \frac{1}{2}\Pi\right) \\
&\quad + \frac{1}{3}A(\rho + 3p) + \frac{3}{2}\phi\Sigma^2 + 2\Omega\Sigma\xi + \Sigma Q.
\end{aligned} \tag{24}$$

Comparing (23) and (24) and using (11), we obtain the constraint

$$\left(\frac{14}{3}A - \phi\right)\Omega = \Sigma\xi. \tag{25}$$

Now, taking the dot derivative of (8b) and the hat derivative of (9b), we obtain, respectively,

$$\begin{aligned}
\hat{\dot{\phi}} &= -\Sigma\hat{A} - A\hat{\Sigma} + \frac{1}{2}\Sigma\hat{\phi} + \frac{1}{2}\phi\hat{\Sigma} + 2\Omega\hat{\xi} + 2\xi\hat{\Omega} + \hat{Q} \\
&= A^2\Sigma + \frac{5}{2}A\phi\Sigma - 2\Sigma^3 + 4\Sigma\Omega^2 - \frac{5}{6}\Sigma\rho - \frac{3}{2}\Sigma p \\
&\quad + 4A\Omega\xi - \Sigma\phi^2 + \Sigma\xi^2 + \frac{1}{2}(2A - \phi)Q - 5\phi\Omega\xi \\
&\quad - \frac{1}{2}\Sigma\mathcal{E} - \frac{1}{4}\Sigma\Pi + \hat{Q},
\end{aligned} \tag{26a}$$

$$\begin{aligned}
\dot{\hat{\phi}} &= -\phi\dot{\phi} - 2\Sigma\dot{\Sigma} + 4\xi\dot{\xi} - \frac{2}{3}\dot{\rho} - \left(\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi}\right) \\
&= 3A\phi\Sigma - \Sigma^3 - \frac{1}{2}\Sigma\phi^2 - 4\phi\Omega\xi - \frac{3}{2}\phi Q + \frac{1}{2}\Sigma\mathcal{E} \\
&\quad - 4\Sigma\Omega^2 + 2\Sigma\xi^2 - \frac{5}{4}\Sigma\Pi + 4A\Omega\xi - \frac{1}{6}\Sigma\rho - \frac{3}{2}\Sigma p \\
&\quad - 3\xi\mathcal{H} - \dot{\rho}.
\end{aligned} \tag{26b}$$

Taking the difference of (26a) and (26b) and using (10b), we obtain

$$\hat{\dot{\phi}} - \dot{\hat{\phi}} = A^2\Sigma - \frac{1}{2}A\phi\Sigma - \Sigma^3 + 8\Sigma\Omega^2 - \frac{2}{3}\Sigma\rho - \frac{1}{2}\Sigma\phi^2 - \Sigma\xi^2 - AQ - \phi\Omega\xi - \Sigma\mathcal{E} + \Sigma\Pi + 3\xi\mathcal{H}. \quad (27)$$

Using the commutation relation in (7) on ϕ, we have

$$\begin{aligned}\hat{\dot{\phi}} - \dot{\hat{\phi}} &= -A\dot{\phi} + \Sigma\hat{\phi} \\ &= A^2\Sigma - \frac{1}{2}A\phi\Sigma - 2A\Omega\xi - AQ - \frac{1}{2}\Sigma\phi^2 - \Sigma^3 \\ &\quad + 2\Sigma\xi^2 - \frac{2}{3}\Sigma\rho - \Sigma\mathcal{E} - \frac{1}{2}\Sigma\Pi. \end{aligned} \quad (28)$$

Comparing (27) and (28) and using (11), we obtain the constraint

$$(9\Sigma\xi + \phi\Omega - 5A\Omega)\xi + \left(\frac{3}{2}\Pi + 8\Omega^2\right)\Sigma = 0. \quad (29)$$

Next, taking the dot derivative of (8c) and the hat derivative of (9c), we obtain, respectively,

$$\begin{aligned}\hat{\dot{\xi}} &= \frac{1}{2}\xi\hat{\Sigma} + \frac{1}{2}\Sigma\hat{\xi} + \Omega\left(\hat{A} - \frac{1}{2}\hat{\phi}\right) + \left(A - \frac{1}{2}\phi\right)\hat{\Omega} \\ &= -\frac{5}{4}\phi\Sigma\xi - 2\Omega\xi^2 - \frac{1}{2}\xi Q + \frac{5}{2}\Omega\Sigma^2 - \frac{5}{2}A\phi\Omega \\ &\quad + \frac{3}{4}\Omega\phi^2 - 2\Omega^3 + \frac{5}{6}\Omega\rho + \frac{3}{2}\Omega p + \frac{1}{2}\Omega\mathcal{E} + \frac{1}{4}\Omega\Pi, \end{aligned} \quad (30a)$$

$$\begin{aligned}\dot{\hat{\xi}} &= -\xi\dot{\phi} - \phi\dot{\xi} + \Omega\dot{\Sigma} + \Sigma\dot{\Omega} \\ &= 2A\Sigma\xi - \phi\Sigma\xi - 2\Omega\xi^2 - \xi Q - 2A\phi\Omega + \frac{3}{2}\Omega\Sigma^2 \\ &\quad + \frac{1}{2}\Omega\phi^2 + 2\Omega^3 - \Omega\mathcal{E} + \frac{1}{2}\Omega\Pi + \frac{1}{3}\Omega(\rho + 3p). \end{aligned} \quad (30b)$$

Taking the difference of (30a) and (30b), we obtain

$$\hat{\dot{\xi}} - \dot{\hat{\xi}} = -\frac{1}{4}\phi\Sigma\xi + \frac{1}{2}\xi Q + \Omega\Sigma^2 - \frac{1}{2}A\phi\Omega + \frac{1}{4}\Omega\phi^2 - 4\Omega^3 + \frac{1}{2}\Omega(\rho + p). \quad (31)$$

Using the commutation relation in (7) on ξ, we have

$$\begin{aligned}\hat{\dot{\xi}} - \dot{\hat{\xi}} &= -A\dot{\xi} + \Sigma\hat{\xi} \\ &= -\frac{1}{2}\Sigma\xi - A^2\Omega + \frac{1}{2}A\phi\Omega - \phi\Sigma\xi + \Omega\Sigma^2. \end{aligned} \quad (32)$$

Comparing (31) and (32), we obtain the constraint

$$\left(A + \frac{3}{2}\phi\right)\Sigma\xi + \left(2A^2 + \frac{1}{2}\phi^2 + \rho + p\right)\Omega = \xi Q. \quad (33)$$

Let us now prove the following proposition:

Proposition III.3: An expansion-free dynamical star that is simultaneously rotating and twisting cannot be shear-free, if it exists.

Proof: Here we assume the existence of such stars and show that if $\Sigma = 0$, then the weak energy condition must be violated. We start by assuming that $\Sigma = 0$. Then, from (25) we obtain (taking into account that $\Omega \neq 0$)

$$A = \frac{3}{14}\phi, \quad (34)$$

and, therefore, from (29) we have

$$-\frac{1}{14}\phi\Omega\xi = 0. \quad (35)$$

Since by assumption $\xi \neq 0$, $\Omega \neq 0$, we must have $\phi = 0$, which implies $A = 0$ as well. Now, from (9) and (33) we have, respectively,

$$0 = 2\xi\Omega + Q, \quad (36a)$$

$$(\rho + p)\Omega = \xi Q. \quad (36b)$$

Substituting (36a) into (36b) and again noting that $\Omega \neq 0$, we obtain the energy condition

$$(\rho + p) = -2\xi^2, \quad (37)$$

which gives $(\rho + p) < 0$. ■

Finally, we state and prove the following:

Theorem III.4: There cannot exist an expansion-free dynamical star with both rotation and spatial twist nonvanishing.

Proof: As has been shown in [38], any scalar ψ in LRS spacetimes obtained via the $1+1+2$ decomposition satisfies the relation

$$\dot{\psi}\Omega = \hat{\psi}\xi. \quad (38)$$

Using (8c) and (9c) to substitute $\dot{\xi}$ and $\hat{\xi}$ for $\dot{\psi}$ and $\hat{\psi}$, respectively, in (38) we obtain

$$-(2A - \phi)\Omega = \Sigma\xi, \quad (39)$$

which, upon comparing to (25), gives

$$\phi = \frac{10}{3}A. \quad (40)$$

Using (8d) and (9d) to substitute $\dot{\Omega}$ and $\hat{\Omega}$ for $\dot{\psi}$ and $\hat{\psi}$, respectively, in (38) we obtain

$$-\phi\xi = \Omega\Sigma. \tag{41}$$

Substituting (40) into (39) [or equivalently (25)], we obtain

$$\frac{2}{5}\phi\Omega = \Sigma\xi. \tag{42}$$

It is clear that $\phi \neq 0$; otherwise, we would have $A = 0$ [from (40)], in which case from (25) [or alternatively (39)] we would have $\xi = 0$ (we have already shown that $\Sigma \neq 0$), contradicting the assumption that $\xi \neq 0$.

Now, multiplying both (41) and (42) by Ω, we obtain, respectively,

$$-\phi\Omega\xi = \Omega^2\Sigma, \tag{43a}$$

$$\frac{2}{5}\phi\Omega^2 = \Omega\xi\Sigma, \tag{43b}$$

which we can rewrite as

$$\Omega\xi = -\frac{\Omega^2\Sigma}{\phi}, \tag{44a}$$

$$\Omega\xi = \frac{2}{5}\frac{\phi\Omega^2}{\Sigma}, \tag{44b}$$

since $\phi \neq 0$, $\Sigma \neq 0$. Equating (44a) and (44b) and simplifying, we obtain

$$\left(\frac{2}{5}\phi^2 + \Sigma^2\right)\Omega^2 = 0. \tag{45}$$

Since by assumption $\Omega \neq 0$, we must have $(2/5)\phi^2 + \Sigma^2 = 0$, which is not possible over the set of real numbers $\mathbb{R}$ for nonzero ϕ and Σ. ■

IV. DISCUSSION

In a recent paper [21], expansion-free dynamical stars, for which the rotation and spatial twist are simultaneously zero, were investigated. It was shown that these stars exist under the particular conditions that the stars radiate, accelerate, and are conformally flat. As with the case of [21], we have utilized the $1 + 1 + 2$ semitetrad covariant formalism to study such stars. In this paper, we have shown that there cannot exist an expansion-free dynamical star with non-vanishing rotation or spatial twist. In the case where the spatial twist is zero and the star is rotating, the star can be expansion-free, but both the heat flux and the shear vanish, in which case the energy density is time independent. Thus, such expansion-free stars are not dynamical. If the rotation is zero and the spatial twist is nonvanishing, then the star cannot be expansion-free since, for this to happen, the star must be static (in which case the star is not dynamical) or the SEC must be violated. Lastly, it is shown that if we assume nonvanishing of both the rotation and the spatial twist, then the shear cannot be zero. Further analysis on the basis that the shear is nonzero, using both the commutation relation and a result relating the dot and hat derivatives of an arbitrary scalar [38], shows that this leads to a quadratic polynomial equation in ϕ and Σ with no real solution for nonzero ϕ and Σ. In our opinion, this result is a valuable contribution to the increasing literature on the expansion-free condition. This result has also severely restricted the prevalence of such stars.

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