Three-boson bound states in Minkowski space with contact interactions

E. Ydrefors,[1,2] J. H. Alvarenga Nogueira,[1,3] V. A. Karmanov,[4] and T. Frederico[1]

[1]*Instituto Tecnológico de Aeronáutica, DCTA, 12228-900 São José dos Campos, Brazil*
[2]*Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France*
[3]*Dipartimento di Fisica, Università di Roma "La Sapienza" INFN, Sezione di Roma "La Sapienza" Piazzale A. Moro 5—00187 Roma, Italy*
[4]*Lebedev Physical Institute, Leninsky Prospekt 53, 119991 Moscow, Russia*

(Received 2 March 2020; accepted 15 May 2020; published 26 May 2020)

The structure of the three-boson bound state in Minkowski space is studied for a model with contact interaction. The Faddeev-Bethe-Salpeter equation is solved both in Minkowski and Euclidean spaces. The results are in fair agreement for comparable quantities, like the transverse amplitude obtained when the longitudinal constituent momenta of the light-front valence wave function are integrated out. The Minkowski space solution is obtained numerically by using a recently proposed method based on the direct integration over the singularities of the propagators and interaction kernel of the four-dimensional integral equation. The complex singular structure of the Faddeev components of the Bethe-Salpeter vertex function for space and timelike momenta in an example of a Borromean system is investigated in detail. Furthermore, the transverse amplitude is studied as a mean to access the double-parton transverse momentum distribution. Following that, we show that the two-body short-range correlation contained in the valence wave function is evidenced when the pair has a large relative momentum in a back-to-back configuration, where one of the Faddeev components of the Bethe-Salpeter amplitude dominates over the others. In this situation a power-law behavior is derived and confirmed numerically.

DOI: 10.1103/PhysRevD.101.096018

I. INTRODUCTION

The Bethe-Salpeter (BS) approach is an important and efficient tool to investigate relativistic few-body systems. Solving the BS equation with a realistic interaction, especially for a three-body system, is technically a rather complicated problem. However, the principal qualitative properties can be understood through models that retain the main features of the physical system. One of these models, fundamental in nuclear physics and described (in the two-body case) in any textbook, is the zero-range interaction.

The three-body BS equation in Minkowski space with zero-range interaction was derived in Ref. [1] in 1992. Later on, in 2017, the equation was solved in Euclidean space [2] for the first time and then, quite recently, directly in Minkowski space [3]. Concerning the Euclidean space solution, the reasons of this time lag was due to the fact that, though the BS equation was given in Ref. [1] in a simple and transparent form, as it was there presented the equation did not allow to make the Wick rotation directly. Whether the rotating integration contour crosses the singularities or not, this depends on the point around which it is rotated. To determine the safe point one must make a shift of variables in the BS equation [1], as proposed in Ref. [2]. This was the key to success. As for the Minkowski space solution [3], the methods were absent until recently. In Ref. [3] the method developed in Ref. [4] was used.

The aforementioned method is based on the direct integration of the singularities of the propagators and interaction kernel [4]. It does not resort to the Nakanishi integral representation [5,6] and light-front (LF) projection. In the present paper we will follow this method and explore it for obtaining information on the structure of relativistic three-body systems.

However, the corresponding equation in the light-front dynamics (LFD) was derived and solved already in Ref. [1]. Then the stability of the solution was thoroughly explored in Ref. [7]. Finding the solution of the BS equation fully in Minkowski space is rather important for applications when used to calculate observables like parton distributions and electromagnetic form factors (see e.g., Ref. [8]). The Euclidean solution, though it provides the bound state spectrum, requires a careful analytical extension of the Euclidean BS amplitude, and for large enough momentum transfers overlapping cuts turns the task cumbersome, preventing to explore the whole range of momentum transfers. Besides that, the comparison between the BS and the LFD solutions provides valuable information about the structure of the system, i.e., regarding contribution of the higher Fock components etc.

As it was shown in Ref. [2], the effect coming from higher Fock components on the binding energy and transverse amplitude is huge, even for weakly bound states. This is different from the two-body case, where the truncation at the valence state does not present such a dramatic effect (see e.g., Refs. [9,10]). This difference in a three-body system is explained by the contribution of effective three-body forces of relativistic origin, as investigated in Ref. [11]. It is noteworthy that the BS equation for three bosons has a kernel analogous to the contribution provided by the quark exchange diagrams in quark-diquark models in the constituent quark picture [12], making even more appealing the outcomes of the Minkowski space approach to be presented as follows.

The conclusion that the effect coming from higher Fock components is sizable leads to raise doubts regarding the range of validity of valence inspired models, which are widely applied to hadron physics, as they might be inappropriate to describe certain features of the bound state dynamics, particularly for three-body systems. It is worth mentioning that even for two-boson bound states the contributions coming from higher Fock components, as shown by the calculations [9,13], can constitute more than 30% of the normalization. The BS equation and LFD approaches have already been used as a suitable framework in phenomenological applications. For instance, the calculation of the LF amplitudes in a simplified pion model with strongly bound constituent quarks was done through the solution of the BS equation directly in Minkowski space [14] and also the final state interaction in heavy meson decays was studied using a relativistic LF model [15,16].

As mentioned, the comparison of the binding energies calculated within LFD and BS equation for a one-boson exchange kernel presents a significant discordance [11], unlike what happens for two-body systems [17]. In Ref. [11] there was found an increasing effect of the three-body forces as the exchanged boson mass μ grows, what is relevant for the zero-range case, which corresponds effectively to $\mu \to \infty$. Although that work was quite instructive, the three-body forces were taken into account only perturbatively, producing a significant contribution to the bound state energy, what indicates the necessity to go beyond perturbation theory. It is essential to obtain the nonperturbative solution of the three-body BS and LFD equations, including three-body forces, in order to have a thorough understanding of the physical system.

In the nonrelativistic approach, within the Schrödinger equation, it is well known that the binding energy of a three-boson system with the two-body zero-range interaction is not bound from below, what is known as the Thomas collapse [18]. As shown in Ref. [1], and further explored numerically in Ref. [7], the relativistic effects result in an effective repulsion at small distances that prevents the Thomas collapse in the relativistic case. This result was found for the valence truncation, within the LFD framework. Therefore, exploring the complete amplitude by means of the BS equation, which includes higher Fock contributions, is necessary to describe such a relativistic three-body system.

Furthermore, the approach for three-boson systems allows one to explore within the relativistic context a wide and important field of research that is already very well established nonrelativistically, known as the Efimov physics [19,20]. The three-body approach developed here paves the way to explore many interesting relativistic phenomena and it is expected to bring more remarkable outcomes as further studies are done.

This paper is devoted to a detailed study of the Minkowski space solution of the three-boson Faddeev-BS equation in the case of the two-body zero-range interaction. As the goal is to address the zero-range interaction case, a major point is the influence of relativistic effects on the stability of the three-body system and the impact on its structure. To accomplish such a goal we focus on Borromean systems and the Faddeev-BS equation is solved both in Minkowski and Euclidean spaces. The choice made for Borromean states simplifies the computations in Minkowski space, as the bound state pole is absent in the two-body scattering amplitude, which is an input to the kernel of the Faddeev-BS equation.

The Minkowski space solution is obtained by the direct integration of the singularities of the propagators and interaction kernel [3], what allows us to explore in the space and timelike momenta regions the complex singular structure of the Faddeev components of the BS vertex function of such a Borromean state. We study in detail the numerical solutions by showing that both methods produce results in fair agreement for the Faddeev component of the transverse amplitude obtained from the corresponding component of the valence wave function, after integration over the longitudinal LF momentum fractions. In addition, the double-parton content of the transverse amplitude is studied, and we evidenced the two-body short-range correlation contained in the valence wave function. The kinematical condition to expose the pair short-range correlation was set for large relative momentum in a back-to-back configuration, in such situation the Faddeev component of the BS amplitude that brings the pair interaction is the dominant one. We also found, as expected for large relative momentum, a power-law behavior, that was confirmed numerically.

The paper is organized as follows. The theoretical formalism for the two-body scattering amplitude, 3-body BS equation and transverse amplitudes is outlined in Secs. II–VI. In Sec. VII the Wick rotation of the three-body BS amplitude is revisited to clarify that the transverse amplitude is independent of that. In Sec. VIII the numerical results are presented and discussed. The conclusions are then drawn in Sec. IX. Some of the more lengthy derivations, and also a brief summary of the numerical methods, are available in appendixes.

II. TWO-BODY SCATTERING AMPLITUDE

For the contact interaction (with the four-leg vertex $i\lambda$), the two-body amplitude $\mathcal{F}(M_{12}^2)$ is determined by the equation shown graphically in Fig. 1 (see also Ref. [1]). Iterating, we find that the first contribution is simply $i\lambda$, the second one is $(i\lambda)^2\mathcal{B}$, where $\mathcal{B}$ is the amputated from $(i\lambda)^2$ the bubble graph, etc. That is

$$i\mathcal{F}(M_{12}^2) = i\lambda + (i\lambda)^2\mathcal{B} + (i\lambda)^3\mathcal{B}^2 + \ldots = \frac{i\lambda}{1-(i\lambda)\mathcal{B}(M_{12}^2)} = \frac{1}{(i\lambda)^{-1} - \mathcal{B}(M_{12}^2)}, \quad (1)$$

or

$$\mathcal{F}(M_{12}^2) = \frac{1}{i[(i\lambda)^{-1} - \mathcal{B}(M_{12}^2)]}, \quad (2)$$

where

$$\mathcal{B}(M_{12}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2 - m^2 + i\epsilon)} \frac{i}{[(k-P)^2 - m^2 + i\epsilon]}. \quad (3)$$

Here m denotes the boson mass, and P is the total four-momentum of the two-body system, $P^2 \equiv M_{12}^2$, where M_{12}^2 denotes the squared effective off-shell mass.

The loop integration (3) has a log-type ultraviolet divergence that has to be regularized and renormalized by fixing the scattering amplitude at some physical value, which will be given by the bound state pole or scattering length. Although it is a well known and standard procedure, we will present it in the following for the sake of completeness as well as to fix our notation.

A. Normalization of the scattering amplitude

In the derivation of the two-body amplitude, we follow the definitions and normalization of Ref. [21]. According to it, the partial wave amplitude of angular momentum L is defined as

$$F_L(k_v) = \frac{1}{32\pi} \int_{-1}^{1} dz P_L(z) F(k_v, z), \quad (4)$$

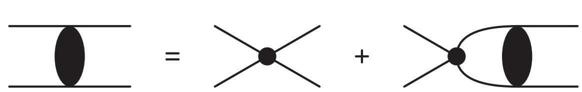

FIG. 1. Diagrammatic representation of the integral equation for the two-body scattering amplitude (1) of Ref. [1].

where $k_v = |\vec{k}|$ is the magnitude of the particle momentum in the rest-frame, z is the cosine of the center-of-mass (c.m.) scattering angle and $P_L(z)$ is the Legendre polynomial. For a z-independent amplitude (2):

$$F_0(k_v) = \frac{1}{16\pi} \mathcal{F}(M_{12}^2). \quad (5)$$

In the given normalization, the scattering amplitude is related to the phase shift by [4]

$$F_0(k_v) = \frac{\varepsilon_k}{k_v} \exp(i\delta_0) \sin\delta_0 = \varepsilon_k f_0(k_v), \quad (6)$$

with $\varepsilon_k = \sqrt{k_v^2 + m^2}$. The S-matrix is unitary if the phase-shift δ_0 is real and

$$f_0(k_v) = \frac{1}{k_v \cot\delta_0 - ik_v}, \quad (7)$$

is the standard nonrelativistic form of the scattering amplitude. The expansion in powers of k_v^2 in the low energy region, gives:

$$k_v \cot\delta_0 = -\frac{1}{a} + \frac{1}{2} r_0 k_v^2 + \cdots, \quad (8)$$

where a is the scattering length and r_0 the effective range. The relation to $\mathcal{F}(M_{12}^2)$ is

$$f_0(k_v) = \frac{1}{16\pi\varepsilon_k} \mathcal{F}(M_{12}^2), \quad (9)$$

and the two renormalization conditions to be used in the following, are based either on fixing the bound state pole or the scattering length. The latter condition reads

$$f_0(k_v = 0) = \frac{1}{16\pi m} \mathcal{F}(4m^2) = -a, \quad (10)$$

that fixes the scattering amplitude at the continuum branch point.

B. Renormalization via bound state pole

One way to calculate $\mathcal{B}(M_{12}^2)$ is to use the standard Feynman parametrization:

$$\frac{1}{ab} = \int_0^1 \frac{du}{[ua + (1-u)b]^2}, \quad (11)$$

with $a = k^2 - m^2 + i\epsilon$, $b = (k-P)^2 - m^2 + i\epsilon$, and then compute the 4D integral in the Euclidean space. However, for $M_{12}^2 \geq 4m^2$, the integrand of this integral becomes singular and this method is not so convenient.

Therefore, to calculate the amplitude (especially for $M_{12}^2 > 4m^2$) we use the initial integral (3) (written in the c.m. frame $\vec{P} = \vec{0}$) and start by performing the integration over k_0 by residues, i.e.,

$$\mathcal{B}(M_{12}^2) = -\int \frac{dk_0 d^3k}{(2\pi)^4} \frac{1}{(k_0^2 - k_v^2 - m^2 + i\epsilon)} \times \frac{1}{[(k_0 - M_{12})^2 - k_v^2 - m^2 + i\epsilon]} = 2\pi i (res_1(M_{12}) + res_2(M_{12})). \tag{12}$$

Here $res_{1,2}$ are the residues of the integrand in one of the two poles in the upper half plane of the complex variable k_0. The positions of the poles are

$$k_0^{(1)} = -\varepsilon_k + i\epsilon, \qquad k_0^{(2)} = M_{12} - \varepsilon_k + i\epsilon,$$

and the corresponding residues are given by

$$res_1(M_{12}^2) = \int_0^\Lambda \frac{k_v^2 dk_v}{(2\pi)^3} \frac{1}{\varepsilon_k} \frac{1}{[(\varepsilon_k + M_{12})^2 - \varepsilon_k^2 + i\epsilon]}, \tag{13}$$

$$res_2(M_{12}^2) = \int_0^\Lambda \frac{k_v^2 dk_v}{(2\pi)^3} \frac{1}{\varepsilon_k} \frac{1}{[M_{12}(M_{12} - 2\varepsilon_k) + i\epsilon]}, \tag{14}$$

where the integrals are regularized by the momentum cutoff Λ. If $-\infty < M_{12}^2 < 4m^2$, the integrals (13) and (14) are nonsingular ones. Contrary to this, if $M_{12} > 2m$, the second residue is represented as a sum of two contributions: the principal value of the integral over k_v and the delta-function contribution, i.e.,

$$\begin{aligned} res_2(M_{12}^2) &= res_{2a}(M_{12}^2) + res_{2b}(M_{12}^2) \\ &= PV \int_0^\Lambda \frac{k_v^2 dk_v}{(2\pi)^3} \frac{1}{\varepsilon_k} \frac{1}{M_{12}(M_{12} - 2\varepsilon_k)} \\ &\quad + \int_0^\Lambda \frac{k_v^2 dk_v}{(2\pi)^3} \frac{1}{\varepsilon_k} (-i\pi) \delta[M_{12}(M_{12} - 2\varepsilon_k)] \\ &= res_{2a}(M_{12}^2) + \frac{1}{2\pi i} \frac{y''}{16\pi}, \end{aligned} \tag{15}$$

where the delta function is integrated out by taking $\Lambda \to \infty$ and y'' is defined below, in Eq. (21).

As the contribution $res_{2a}(M_{12}^2)$ in (15) is divergent in the ultraviolet limit, it is necessary to perform a regularization process. By renormalizing we can express the bare parameters (in this work, the coupling constant λ) via observables (usually, in the field theory, via a "physical" coupling constant). From the condition that the two-body system has a bound state with the mass M_2 and the amplitude (2) has a pole at $M_{12} = M_2$, one finds for the coupling constant λ

$$(i\lambda)^{-1} = \mathcal{B}(M_2^2) = 2\pi i (res_1(M_2^2) + res_2(M_2^2)). \tag{16}$$

The denominator in (2) then becomes

$$\begin{aligned} & i[(i\lambda)^{-1} - \mathcal{B}(M_{12}^2)] \\ &= i[\mathcal{B}(M_{2B}^2) - \mathcal{B}(M_{12}^2)] \\ &= iPV \int_0^\infty \frac{(M_2^2 - M_{12}^2)}{32\pi^2 \varepsilon_k [k_v^2 - (\frac{1}{4}M_{12}^2 - m^2)]} \\ &\quad \times \frac{k_v^2 dk_v}{[k_v^2 + (m^2 - \frac{1}{4}M_2^2)]} - \frac{y''}{16\pi}. \end{aligned} \tag{17}$$

In Eq. (17), the principal value (PV) integral takes into account the singularity at $k_v = \sqrt{\frac{1}{4}M_{12}^2 - m^2}$ and the limit of $\Lambda \to \infty$ is taken since the integral is ultraviolet finite.

Now the integral (17), (in which the bare coupling constant λ is expressed via the two-body bound state mass M_2) is finite and its calculation in different domains of the variable M_{12} results for $\mathcal{F}(M_{12}^2)$ in:

(i) If $-\infty < M_{12}^2 \le 0 \quad (1 \ge y \ge 0)$, then:

$$\mathcal{F}(M_{12}^2) = \left[\frac{1}{16\pi^2 y} \log\frac{1+y}{1-y} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}}\right]^{-1}. \tag{18}$$

(ii) If $0 \le M_{12}^2 \le 4m^2 \quad (0 \le y' < \infty)$, then:

$$\mathcal{F}(M_{12}^2) = \left[\frac{\arctan y'}{8\pi^2 y'} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}}\right]^{-1}. \tag{19}$$

(iii) If $4m^2 \le M_{12}^2 < \infty \quad (0 \le y'' \le 1)$, then:

$$\mathcal{F}(M_{12}^2) = \left[\frac{y''}{16\pi^2} \log\frac{1+y''}{1-y''} - \frac{\arctan y'_{M_2}}{8\pi^2 y'_{M_2}} - i\frac{y''}{16\pi}\right]^{-1}. \tag{20}$$

Here $y'_{M_2} = \frac{M_2}{\sqrt{4m^2 - M_2^2}}$ and

$$y = \frac{\sqrt{-M_{12}^2}}{\sqrt{4m^2 - M_{12}^2}}, \qquad y' = \frac{M_{12}}{\sqrt{4m^2 - M_{12}^2}}, \qquad y'' = \frac{\sqrt{M_{12}^2 - 4m^2}}{M_{12}}. \tag{21}$$

We have not yet introduced the scattering length a in $\mathcal{F}(M_{12}^2)$ instead of the two-body bound state mass, which will allow to generalize the scattering amplitude for the case where no bound state exists. This will be discussed in detail in the next subsection. It should be also noticed that above the threshold $M_{12} > 2m$, the amplitude obtains in the denominator an imaginary part $-i\frac{y''}{16\pi}$.

C. Renormalization via scattering length

The two-body scattering amplitude (18)–(20) was obtained by fixing a bound state pole at $M_{12} = M_2$. However, it can happen that the two-body bound state is absent, hence, the two-body scattering amplitude has no any bound state pole. Besides that, the three-body bound state may exist in absence of the two-body one. In this situation a different condition will be used: the requirement that the scattering amplitude at zero energy is equal to $-a$, where a is the two-body scattering length. The (non-renormalized) two-body amplitude $\mathcal{F}(M_{12}^2)$ still has the form of Eq. (2). Its argument can be written as $M_{12}^2 = 4\varepsilon_k^2$. By using (2) and (10) we obtain

$$\frac{1}{16\pi m}\frac{1}{i[(i\lambda)^{-1} - \mathcal{B}(4m^2)]} = -a, \tag{22}$$

and therefore

$$(i\lambda)^{-1} = \mathcal{B}(4m^2) - \frac{1}{16i\pi ma}. \tag{23}$$

The two-body amplitude is then given by

$$\mathcal{F}(M_{12}^2) = \frac{1}{i[\mathcal{B}(4m^2) - \mathcal{B}(M_{12}^2)] - \frac{1}{16\pi ma}}. \tag{24}$$

The two-body scattering amplitude is obtained after substituting $2m$ for M_2 in Eq. (17), and in the different regions of M_{12}^2 it reads:

(i) If $-\infty < M_{12}^2 \leq 0 \quad (1 \geq y \geq 0)$, then:

$$\mathcal{F}(M_{12}^2) = 16\pi\left[\frac{1}{\pi y}\log\frac{1+y}{1-y} - \frac{1}{ma}\right]^{-1}. \tag{25}$$

(ii) If $0 \leq M_{12}^2 \leq 4m^2 \quad (0 \leq y' < \infty)$, then:

$$\mathcal{F}(M_{12}^2) = 16\pi\left[\frac{2}{\pi}\frac{\arctan y'}{y'} - \frac{1}{ma}\right]^{-1}. \tag{26}$$

(iii) If $4m^2 \leq M_{12}^2 < \infty \quad (0 \leq y'' \leq 1)$, then:

$$\mathcal{F}(M_{12}^2) = 16\pi\left[\frac{y''}{\pi}\log\frac{1+y''}{1-y''} - \frac{1}{ma} - iy''\right]^{-1}. \tag{27}$$

For negative scattering length a this amplitude has no poles.

The two-body amplitude can be written from Eq. (27) in terms of the c.m. frame momentum as:

$$\mathcal{F}(M_{12}^2) = 16\pi\left[\frac{k_v}{\varepsilon_k\pi}\log\frac{\varepsilon_k + k_v}{\varepsilon_k - k_v} - \frac{1}{ma} - i\frac{k_v}{\varepsilon_k}\right]^{-1}, \tag{28}$$

and then:

$$k_v \cot\delta_0 = \frac{k_v}{\pi}\log\frac{\varepsilon_k + k_v}{\varepsilon_k - k_v} - \frac{\varepsilon_k}{ma}, \tag{29}$$

which is real showing the unitarity of the model amplitude. The power expansion for small k_v and comparison with (8) allows to identify the effective range as

$$r_0 = \frac{4}{m\pi} - \frac{1}{m^2 a}, \tag{30}$$

which is determined by the terms $\propto 1/m, 1/m^2$, reflecting the relativistic origin of the model.

In the case when the bound state exists, one finds that

$$a = \frac{\pi y'_{M_2}}{2m\ \arctan(y'_{M_2})},$$
$$r_0 = \frac{2[2y'_{M_2} - \arctan(y'_{M_2})]}{\pi m y'_{M_2}}, \tag{31}$$

and Eqs. (25)–(27) then coincide with (18)–(20). Furthermore, for small binding energy $B \ll m$ the variable y'_{M_2} increases as $y'_{M_2} \sim \sqrt{m/B}$. The scattering length a also increases for $B \to 0$ with

$$a \to \frac{1}{\sqrt{mB}} \quad \text{and} \quad r_0 \to \frac{4}{m\pi}, \tag{32}$$

whereas the effective radius r_0 tends to a constant.

III. THREE-BODY BETHE-SALPETER EQUATION

The solution of the zero-range three-body BS equation for three identical spinless particles using the Faddeev decomposition of the full BS amplitude can be reduced to the solution of one single integral equation for the spectator vertex function $v_M(q,p)$ (external propagators are excluded). In the zero-range interaction case $v_M(q,p)$ depends upon both the total momentum p and on the four-momentum of the spectator particle q. The equation reads [1]:

$$v_M(q,p) = 2i\mathcal{F}(M_{12}^2)\int\frac{d^4k}{(2\pi)^4}\frac{i}{[k^2 - m^2 + i\epsilon]} \times \frac{i}{[(p-q-k)^2 - m^2 + i\epsilon]}v_M(k,p). \tag{33}$$

Notice that the momentum of the spectator particle, q, determines the effective mass of the two-boson subsystem, M_{12}, due to the four-momentum conservation (see below). Therefore, the vertex function does not depend on other momenta besides q and the total four-momentum p. The other two components of the integral equation (33) can be easily obtained through the cyclic permutation of the momentum of the constituent particles. The full BS amplitude in Minkowski space is recovered by multiplying the vertex function by the three external propagators and summing up the components, i.e.,

$$i\Phi_M(k_1,k_2,k_3;p) = i^3 \frac{v_M(k_1)+v_M(k_2)+v_M(k_3)}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)(k_3^2-m^2+i\epsilon)}, \quad (34)$$

where $v_M(k) \equiv v_M(k,p)$ (to simplify our notation) and the four-momenta obey the relation

$$k_1 + k_2 + k_3 = p. \quad (35)$$

The relativistic two-body zero-range scattering amplitude $\mathcal{F}(M_{12}^2)$ in (33) was derived in Sec. II and, being renormalized via scattering length, is given by Eqs. (25), (26) and (27). Its argument M_{12}^2 is expressed via three-body momenta as $M_{12}^2 = (p-q)^2$. One major simplification in Eq. (33) happens due to the fact that the amplitude $\mathcal{F}(M_{12}^2)$ does not depend on the loop integration variable k in the zero-range case. Due to that the two-body amplitude factors out in the integral equation, what does not happen for a finite-range interaction kernel like the one-boson exchange or the cross-ladder one.

Notice that in Refs. [1,7] the regime $M_{12}^2 > 4m^2$ of $\mathcal{F}(M_{12}^2)$ (27) was not presented, due to the range of the variables considered in those works. The amplitude in terms of the bound state mass, as presented in Refs. [1,7], is given in Eq. (19) (i.e., in the physical domain, $0 \le M_{12} \le 2m$).

Such a link is very important to understand the range covered by the results obtained previously, in Refs. [1,7], by considering only the situation where the two-body state is bound (i.e., $a > 0$) producing a real M_2 through Eq. (31), and the full support covered by Eqs. (25), (26) and (27), including also virtual two-body bound states (i.e., $a \in \mathbb{R}$). In other words, as mentioned, in the region for which $a < 0$ the amplitude $\mathcal{F}(M_{12}^2)$ has no pole in the physical domain and, therefore, the two-body bound state does not exist. The three-body system can still be formed though, as a Borromean bound state.

The goal now is to solve the scalar three-body BS equation (33), derived in Ref. [1], for the lowest angular momentum bound state, with zero-range interaction, fully in Minkowski space and retaining implicitly the Fock-space composition beyond the valence truncation. The adopted method is the direct integration of the singularities of the four-dimensional integral equation, developed recently for the two-body BS equation in Ref. [4]. The method does not rely on any ansatz, as e.g., the Nakanishi integral representation used for the solution of the two-body equation in Ref. [9]. No three-dimensional reduction of the covariant 4D equation, as the one done by performing the projection onto the LF plane, is adopted. Part of what is exposed here was published in Ref. [3]. Further comparisons with results obtained in the Euclidean space calculations will be provided to test the reliability of the method.

One interesting example of a calculation within the approach used here is the electromagnetic transition form factor [22], which quantifies the breakup of a two-body bound state. This highly complex calculation was performed through the direct integration method in Minkowski space, using as inputs the solutions, obtained by the same method [4], of the scattering and bound state BS equations. The transition form factor, including the final state interaction, was calculated in the whole kinematical region. It satisfied the nontrivial condition of current conservation explicitly verified numerically.

Equation (33) is a singular integral equation and solving it numerically is a very challenging task. For that reason, the equation requires a proper treatment to be rewritten in, at least, a less singular form before its numerical solution in the c.m. frame, $\vec{p} = \vec{0}$. The propagators, containing the strongest singularities of the BS equation kernel, are represented in the customary form [4]

$$\frac{1}{k^2 - m^2 + i\epsilon} = \frac{1}{k_0^2 - k_v^2 - m^2 + i\epsilon} = PV \frac{1}{k_0^2 - \varepsilon_k^2} - \frac{i\pi}{2\varepsilon_k}[\delta(k_0 - \varepsilon_k) + \delta(k_0 + \varepsilon_k)], \quad (36)$$

where PV denotes the principal value. The terms like $PV \int \ldots \frac{dk_0}{k_0^2 - \varepsilon_k^2}$ contain the singularities at $k_0 = \pm\varepsilon_k$ which are removed by subtracting integrals from the equation, with appropriate coefficients, in such a way that the final equation is not affected. For that, the following identities are used

$$PV \int_{-\infty}^{0} \frac{dk_0}{k_0^2 - \varepsilon_k^2} = PV \int_0^{\infty} \frac{dk_0}{k_0^2 - \varepsilon_k^2} = 0. \quad (37)$$

The second propagator in Eq. (33) can be integrated over the angles analytically. Denoting $z = \cos(\frac{\vec{k}\cdot\vec{q}}{k_v q_v})$ and recalling that $d^3k = k_v^2 dk_v dz d\varphi$, one can write that

$$\Pi(q_0, q_v, k_0, k_v) = \int \frac{i dz d\varphi}{[(p-q-k)^2 - m^2 + i\epsilon]} = \frac{i\pi}{q_v k_v}\left\{\log\left|\frac{(\eta+1)}{(\eta-1)}\right| - i\pi I(\eta)\right\}, \quad (38)$$

with

$$I(\eta) = \begin{cases} 1 & \text{if} \quad |\eta| \quad \le \quad 1 \\ 0 & \text{if} \quad |\eta| \quad > \quad 1 \end{cases}, \quad (39)$$

and

$$\eta = \frac{(M_3 - q_0 - k_0)^2 - k_v^2 - q_v^2 - m^2}{2 q_v k_v}, \quad (40)$$

where M_3 denotes the bound state mass of the three-body system. The BS equation (33) turns, after integration over the angles, into an integral equation with the kernel (38), that is still singular. However, these singularities are weakened by integration and become logarithmic ones and discontinuities, that will be treated numerically.

Once the propagators are expressed as in Eq. (36), the principal value singularities are subtracted and the angular integrations are performed, Eq. (33) acquires the following form:

$$\begin{aligned} v_M(q_0,q_v) = \frac{\mathcal{F}(M_{12}^2)}{(2\pi)^4}\int_0^\infty k_v^2 dk_v \Bigg\{ & \frac{2\pi i}{2\varepsilon_k}[\Pi(q_0,q_v;\varepsilon_k,k_v)v_M(\varepsilon_k,k_v) + \Pi(q_0,q_v;-\varepsilon_k,k_v)v_M(-\varepsilon_k,k_v)] \\ & - 2\int_{-\infty}^0 dk_0 \left[\frac{\Pi(q_0,q_v;k_0,k_v)v_M(k_0,k_v) - \Pi(q_0,q_v;-\varepsilon_k,k_v)v_M(-\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right] \\ & - 2\int_0^\infty dk_0 \left[\frac{\Pi(q_0,q_v;k_0,k_v)v_M(k_0,k_v) - \Pi(q_0,q_v;\varepsilon_k,k_v)v_M(\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right]\Bigg\}. \end{aligned} \tag{41}$$

This equation has now, besides the unknown analytical behavior of $v_M(q_0,q_v)$ that will be discovered numerically, only weak singularities and discontinuities, but unlike (33) the singularities in $k_0 = \pm\varepsilon_k$ no longer exist.

The logarithmic singularities of the kernel (38), $\Pi(q_0,q_v,k_0,k_v)$, at $\eta = \pm 1$ can be found for fixed values of q_0, q_v and k_v, what makes the numerical treatment in k_0 easier. Their positions with respect to the variable k_0 are

$$\begin{aligned} k_0 &= (M_3-q_0) + \sqrt{m^2+(k_v\pm q_v)^2}, \\ k_0 &= (M_3-q_0) - \sqrt{m^2+(k_v\pm q_v)^2}. \end{aligned} \tag{42}$$

Analogously, the position of the singularities can be found for the variable k_v, so that the integration over this variable can be optimized numerically. The positions of the singularities of $\Pi(q_0,q_v,\pm\varepsilon_k,k_v)$ as a function of k_v are given by

$$k_v = \frac{\pm\sqrt{M_{12}^2(M_{12}^2+q_v^2)(M_{12}^2-4m^2)} \pm q_v M_{12}^2}{2M_{12}^2}, \tag{43}$$

where $M_{12}^2 = (M_3-q_0)^2 - q_v^2$. The expression under the square root is non-negative if

$$M_{12}^2 \geq 4m^2 \quad \text{or} \quad M_{12}^2 \leq 0, \tag{44}$$

and, therefore, for existing real singularities in k_v one needs to ensure one of the following conditions for q_0: $q_0 < M_3 - \sqrt{q_v^2+4m^2}$ or $M_3 - q_v < q_0 < M_3 + q_v$ or $q_0 > M_3 + \sqrt{q_v^2+4m^2}$. This means that the branching points that need to be considered while fixing the mesh numerically to separate the regions with and without singularities in k_v are

$$\begin{aligned} q_0^{(1)} &= M_3 - \sqrt{q_v^2+4m^2}, \\ q_0^{(2)} &= M_3 - q_v, \\ q_0^{(3)} &= M_3 + q_v, \\ q_0^{(4)} &= M_3 + \sqrt{q_v^2+4m^2}, \end{aligned} \tag{45}$$

with $q_0^{(1)} < q_0^{(2)} < q_0^{(3)} < q_0^{(4)}$. As it can be seen from Eqs. (18)–(20), these branching points are also present in the two-body amplitude $\mathcal{F}(M_{12}^2)$. For more details on the behavior of the $\mathcal{F}(M_{12}^2)$ amplitude, see Sec. C 2.

IV. RELATION BETWEEN THE BS AMPLITUDE AND LF WAVE FUNCTION

In this section and in Appendix A, we establish the relation between the three-body BS amplitude and the three-body LF wave function (LFWF). It generalizes to the three-body system the relation (3.57) or (3.58) from Ref. [23] for the two-scalar system and can be easily generalized to the arbitrary n-body case.

The three-body BS amplitude is defined analogously to the two-body one, namely:

$$\Phi_M(x_1,x_2,x_3;p) = \langle 0|T(\varphi(x_1)\varphi(x_2)\varphi(x_3))|p\rangle. \tag{46}$$

As is shown in Appendix A, the three-body LFWF can be related to the BS amplitude through

$$\begin{aligned} \psi(\vec{k}_{1\perp},\xi_1;\vec{k}_{2\perp},\xi_2;\vec{k}_{3\perp},\xi_3) \\ = \frac{(p^+)^2}{\sqrt{2\pi}}\xi_1\xi_2\xi_3\int dk_1^- dk_2^- \Phi_M(k_1,k_2,k_3;p), \end{aligned} \tag{47}$$

where $\xi_1+\xi_2+\xi_3 = 1$ and the plus and minus momentum components are given by $p^\pm = p^0 \pm p^3$, with analogous definitions for the other momenta.

Similarly, one has for the two-body case that

$$\psi(\vec{k}_{1\perp},\xi_1;\vec{k}_{2\perp},\xi_2) = \frac{p^+}{\sqrt{2\pi}}\xi_1\xi_2\int dk_1^-\Phi_M(k_1,k_2;p). \tag{48}$$

By introducing the relative variable $k=\frac{1}{2}(k_1-k_2)$, $\Phi_M\equiv\Phi_M(k,p)$, Eq. (48) can be written on the form:

$$\psi(\vec{k}_\perp,\xi) = \frac{p^+}{\sqrt{2\pi}}\xi(1-\xi)\int dk^-\Phi_M(\vec{k}_\perp,k^+,k^-;p). \tag{49}$$

It differs from Eq. (3.57) of Ref. [23] by the degrees of π due to the presence of the factor $(2\pi)^{-3/2}$ in Eq. (3.53) of Ref. [23] which we did not introduce above.

Noteworthy that the support of the function $\psi(\vec{k}_\perp,\xi)$ in the variable ξ, or of the integral

$$\int dk^-\Phi_M(\vec{k}_\perp,k^+,k^-;p),$$

is $0<\xi<1$, as it should be. This follows from the fact that Φ_M is not an arbitrary function, but it is defined, in the coordinate space, by

$$\Phi_M(x_1,x_2;p) = \langle 0|T(\varphi(x_1)\varphi(x_2))|p\rangle,$$

analogously to the three-boson BS amplitude defined in Eq. (46). This support is also automatically obtained if one represents the BS amplitude in the Nakanishi form (see Ref. [24], Appendix D). The same conclusion is also valid for the three-body case, when using Eq. (47).

V. NONRELATIVISTIC LIMIT

In this section, the nonrelativistic limits of the three-body Euclidean BS and valence LF equations, i.e., Eqs. (7) and (10) of Ref. [2], are considered. The Euclidean BS equation will be first analysed. Representing the three-body mass M_3 as $M_3=3m-B_3$, with B_3 denoting the three-body binding energy, and truncating the denominator in Eq. (7) of Ref. [2], and the terms in the fraction of the argument of the log in Eq. (8) of the aforementioned reference, to the leading terms of momenta and the binding energies, one gets

$$\begin{aligned} K &= \frac{\Pi_E(q_4,q_v,k_4,k_v)}{(k_4-\frac{i}{3}M_3)^2+k_v^2+m^2} \\ &= \frac{\frac{1}{2}\log\frac{(k_4+q_4+\frac{i}{3}M_3)^2+(q_v+k_v)^2+m^2}{(k_4+q_4+\frac{i}{3}M_3)^2+(q_v-k_v)^2+m^2}}{(k_4-\frac{i}{3}M_3)^2+k_v^2+m^2} \\ \Rightarrow K_{nr} &\approx \frac{\frac{1}{2}\log\frac{\frac{2}{3}B_3+\frac{(k_v+q_v)^2}{2m}+i(k_4+q_4)}{\frac{2}{3}B_3+\frac{(k_v-q_v)^2}{2m}+i(k_4+q_4)}}{2m(\frac{1}{3}B_3-ik_4)}. \end{aligned} \tag{50}$$

At first glance, one could neglect the terms $\frac{(k_v\pm q_v)^2}{2m}$ in comparison to (k_4+q_4), however, this would result in $K_{nr}\equiv 0$, and therefore it is necessary to keep them.

Following Ref. [1], one can write E_2 through $M_{12}^2=(2m-E_2)^2$, the two-body bound state mass as $M_2=2m-B_2$ and introduce them in the two-body amplitude $\mathcal{F}(M_{12}^2)$ in the physical domain $(0\le M_{12}^2\le 4m^2)$ [see Eq. (19)] to elaborate the nonrelativistic limit. In such case, $m\to\infty$, and the $\mathcal{F}(M_{12}^2)$ amplitude becomes

$$\mathcal{F}(M_{12}^2) = \frac{16\pi\sqrt{m}}{\sqrt{E_2}-\sqrt{B_2}}, \tag{51}$$

or, alternatively,

$$\mathcal{F}(-M_{12}^2) = \frac{16\pi\sqrt{m}}{\sqrt{2m-\sqrt{-M_{12}^2}}-\sqrt{B_2}}. \tag{52}$$

Since $M_{12}^2=(\frac{2}{3}p-iq_4)^2-q_v^2=-(\frac{2}{3}iM_3+q_4)^2-q_v^2$ in the limit $m\to\infty$ one gets

$$\begin{aligned} E_2 &= 2m-\sqrt{-\left[\frac{2}{3}i(3m-B_3)+q_4\right]^2-q_v^2} \\ &\approx \frac{2}{3}B_3+iq_4+\frac{q_v^2}{4m}. \end{aligned} \tag{53}$$

Substituting it in (51), one finds for the scattering amplitude

$$\mathcal{F}(M_{12}^2) = \frac{16\pi\sqrt{m}}{\sqrt{\frac{2}{3}B_3+iq_4+\frac{q_v^2}{4m}}-\sqrt{B_2}}. \tag{54}$$

After these manipulations, Eq. (7) of Ref. [2] obtains the form

$$\begin{aligned} \tilde{v}'_E(q_4,q_v) &= \frac{1}{\pi^2\sqrt{m}}\frac{1}{\sqrt{\frac{2}{3}B_3+iq_4+\frac{q_v^2}{4m}}-\sqrt{B_2}} \\ &\quad\times\int_0^\Lambda dk_v\int_{-\infty}^\infty\frac{dk_4}{(\frac{1}{3}B_3-ik_4)} \\ &\quad\times\log\frac{\frac{2}{3}B_3+\frac{(k_v+q_v)^2}{2m}+i(k_4+q_4)}{\frac{2}{3}B_3+\frac{(k_v-q_v)^2}{2m}+i(k_4+q_4)}\tilde{v}'_E(k_4,k_v), \end{aligned} \tag{55}$$

where it was introduced a cutoff Λ to prevent the Thomas collapse [18]. In order to obtain the time independent equation, the integration over k_4 needs to be performed. Since this is a lengthy derivation, it will not be done here explicitly.

For the three-body LF equation given by Eq. (10) of Ref. [2], the nonrelativistic limit, obtained by following the same steps as before, reads

$$\Gamma_{nr}(\vec{q}) = \frac{1}{\pi^2 m^{3/2}} \frac{1}{\sqrt{E_2} - \sqrt{B_2}} \times \int \frac{\Gamma_{nr}(\vec{k}) d^3k}{B_3 + \frac{q_v^2}{2m} + \frac{k_v^2}{2m} + \frac{(\vec{q}+\vec{k})^2}{2m}}, \quad (56)$$

where

$$E_2 = 2m - M_{12} \approx B_3 + \frac{3}{4}\frac{q_v^2}{m}. \quad (57)$$

Here the factor $\frac{1}{\sqrt{E_2}-\sqrt{B_2}}$ is originating from the two-body amplitude (51) when $m \to \infty$.

Equation (56) is the same as Eq. (18) of Ref. [1]. This equation is known as the Skorniakov-Ter-Martirosyan equation [25]. The nonrelativistic equation can be also written in the form

$$\Gamma_{nr}(\vec{q}) = \frac{1}{\pi^2 \sqrt{m}} \frac{1}{\sqrt{B_3 + \frac{3}{4}\frac{q_v^2}{m}} - \sqrt{B_2}} \times \int \frac{\Gamma_{nr}(\vec{k}) d^3k}{k_v^2 + \vec{k}\cdot\vec{q} + q_v^2 + mB_3}, \quad (58)$$

and, for the s-wave, after integrating over the angles, it reads

$$\Gamma_{nr}(q_v) = \frac{2}{\pi\sqrt{m}} \frac{1}{\sqrt{B_3 + \frac{3}{4}\frac{q_v^2}{m}} - \sqrt{B_2}} \times \int_0^\Lambda \log\left(\frac{k_v^2 + k_v q_v + q_v^2 + mB_3}{k_v^2 - k_v q_v + q_v^2 + mB_3}\right) \Gamma_{nr}(k_v) \times \frac{k_v dk_v}{q_v}. \quad (59)$$

The above equation, like (55), and in contrast to (33), requires a cutoff in order to allow a physical solution avoiding the Thomas collapse [18].

VI. TRANSVERSE AMPLITUDES

The vertex function $v(q_0, q_v)$ is fundamentally dependent on the metric (Euclidean or Minkowski one) adopted to define the integral equation. The transverse amplitude is, instead, an useful quantity for comparison between calculations performed in Euclidean and Minkowski spaces. Furthermore, it gives information on the valence wave function integrated in the longitudinal momenta in the present model, where an infinite number of Fock-components are taken into account implicitly by the Bethe-Salpeter framework. The rich structure of the three-boson bound state transverse amplitude is investigated numerically in Sec. VIII B, as it is a mean to exploit the double parton momentum dependence of the valence wave function, and also gives access to the dynamical correlation between the constituents.

The derivation of the expressions for the Minkowski transverse amplitude is presented below in Sec. VI A. The final amplitude, computed with the BS amplitude obtained from the solution of the BS equation in Minkowski space (41), is expected to coincide with the one defined in Euclidean space. The expressions for the latter one will be derived in Sec. VI B.

A. Minkowski space

As mentioned, the BS amplitude Φ_M can be written in terms of the three vertex components by introducing the external propagators. It is given above by Eq. (34).

The transverse amplitude can be defined via Φ_M as

$$\begin{aligned} L(\vec{k}_{1\perp}, \vec{k}_{2\perp}) &= L_1(\vec{k}_{1\perp}, \vec{k}_{2\perp}) + L_2(\vec{k}_{1\perp}, \vec{k}_{2\perp}) + L_3(\vec{k}_{1\perp}, \vec{k}_{2\perp}) \\ &= \int_{-\infty}^{\infty} dk_{10} \int_{-\infty}^{\infty} dk_{1z} \int_{-\infty}^{\infty} dk_{20} \int_{-\infty}^{\infty} dk_{2z} \\ &\quad \times i\Phi_M(k_{10}, k_{1z}, k_{20}, k_{2z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}). \end{aligned} \quad (60)$$

However, in the three identical boson case, only one of its Faddeev components L_i is enough for the comparison with the transverse amplitude derived from the Euclidean BS solution. The first Faddeev component is given by

$$L_1(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = i \int_{-\infty}^{\infty} dk_{10} \int_{-\infty}^{\infty} dk_{1z} \frac{v_M(k_{10}, k_{1v})}{k_1^2 - m_1^2 + i\epsilon} \times \chi(k_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}), \quad (61)$$

with

$$\chi(k_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp}) = i^2 \int \frac{d^2k_2}{(k_2^2 - m_2^2 + i\epsilon)[(p' - k_2)^2 - m_3^2 + i\epsilon]}, \quad (62)$$

where the following quantities enter: $k_i = (k_{i0}, k_{iz})$ and $d^2k_i = dk_{i0}dk_{iz}$ with $i = 1, 2$. Moreover,

$$m_2^2 = m^2 + |\vec{k}_{2\perp}|^2, \qquad m_3^2 = m^2 + (\vec{p}_\perp - \vec{k}_{1\perp} - \vec{k}_{2\perp})^2, \quad (63)$$

and $p' = (p'_0, p'_z) = p - k_1 = (p_0 - k_{10}, p_z - k_{1z})$.

The two-dimensional integral in (62) can be performed by first introducing the Feynman parametrization (11) and then making the transformation $k_2 \to k_2 + (1-u)p'$. After integration over k_2, using a Wick rotation as $k_0 = ik_4$, we find

$$\chi(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) = i^2\int_0^1 du\int\frac{d^2k_2}{(k_2^2+D+i\epsilon)^2} = -\pi i^3\int_0^1\frac{du}{D+i\epsilon}, \tag{64}$$

with

$$D = u(1-u)p'^2 - m_2^2 u - (1-u)m_3^2. \tag{65}$$

The denominator D is zero at

$$u_\mp = \frac{1}{2p'^2}\Big[p'^2 - m_2^2 + m_3^2 \mp\sqrt{((m_2-m_3)^2-p'^2)((m_2+m_3)^2-p'^2)}\Big], \tag{66}$$

but for $p'^2<(m_2+m_3)^2$, the equality $D=0$ is never satisfied in the interval $0<u<1$, so the term $i\epsilon$ can be dropped out in Eq. (64) and the integral over the Feynman parameter u can be performed safely analytically, giving the following

$$\begin{aligned}\chi(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) &= \frac{\pi i^3}{p'^2(u_- - u_+)}\int_0^1 du\left[\frac{1}{u-u_-}-\frac{1}{1-u_+}\right]\\ &= -\frac{i\pi}{p'^2(u_- - u_+)}[\log(1-u_-)\\ &\quad -\log(-u_-) - \log(-1+u_+) + \log(u_+)],\end{aligned} \tag{67}$$

with $u_\pm$ defined in (66).

In the situation where $p'^2>(m_2+m_3)^2$, the zeroes of the denominator, $u_\pm$, are placed on the real axis for the interval $u\in[0,1]$. For that reason, one can separate χ in two terms, analogously to what was done in (36), i.e.,

$$\chi(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) = \chi'(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) + \chi''(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}), \tag{68}$$

where

$$\chi'(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) = \frac{\pi i^3}{p'^2(u_- - u_+)}\left[\mathrm{PV}\int_0^1\frac{du}{u-u_-} - \mathrm{PV}\int_0^1\frac{du}{u-u_+}\right], \tag{69}$$

and

$$\begin{aligned}\chi''(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) &= \frac{\pi i^3}{p'^2(u_- - u_+)}\\ &\quad\times\left[-i\pi\int_0^1 du\,\delta(u-u_-) - i\pi\int_0^1 du\,\delta(u-u_+)\right]\\ &= \frac{2\pi^2}{\sqrt{[p'^2-(m_2-m_3)^2][p'^2-(m_2+m_3)^2]}}.\end{aligned} \tag{70}$$

The principal value integrals in Eq. (69) can be carried out analytically and one obtains for χ' the following expression

$$\chi'(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp}) = i\pi\frac{\log\frac{m_2^2+m_3^2-p'^2-\sqrt{[p'^2-(m_2-m_3)^2][p'^2-(m_2+m_3)^2]}}{m_2^2+m_3^2-p'^2+\sqrt{[p'^2-(m_2-m_3)^2][p'^2-(m_2+m_3)^2]}}}{\sqrt{[p'^2-(m_2-m_3)^2][p'^2-(m_2+m_3)^2]}}. \tag{71}$$

The contribution $L_1(\vec{k}_{1\perp},\vec{k}_{2\perp})$ can subsequently be written in the form

$$\begin{aligned}L_1(\vec{k}_{1\perp},\vec{k}_{2\perp}) = -i\int_{-\infty}^{\infty}dk_{1z}\Bigg\{&\frac{i\pi}{2\tilde{k}_{10}}[\chi(\tilde{k}_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(\tilde{k}_{10},k_{1v})+\chi(-\tilde{k}_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(-\tilde{k}_{10},k_{1v})]\\ &-\int_0^\infty dk_{10}\left[\frac{\chi(-k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(-k_{10},k_{1v})}{k_{10}^2-\tilde{k}_{10}^2} - \frac{\chi(-\tilde{k}_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(-\tilde{k}_{10},k_{1v})}{k_{10}^2-\tilde{k}_{10}^2}\right]\\ &-\int_0^\infty dk_{10}\left[\frac{\chi(k_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(k_{10},k_{1v})}{k_{10}^2-\tilde{k}_{10}^2} - \frac{\chi(\tilde{k}_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})v_M(\tilde{k}_{10},k_{1v})}{k_{10}^2-\tilde{k}_{10}^2}\right]\Bigg\},\end{aligned} \tag{72}$$

where

$$\tilde{k}_{10} = \sqrt{k_{1z}^2 + |\vec{k}_{1\perp}|^2 + m^2}. \tag{73}$$

Similarly to the treatment of the BS equation in Sec. III, propagators like $[k_1^2-m_1^2+i\epsilon]^{-1}$ were expressed in the form (36) and subtractions were made to eliminate the principal value singularities at $k_0=\pm\tilde{k}_{10}$.

It should be noticed that the function χ in Eq. (72) has square-root singularities at $p'^2=(m_2\pm m_3)^2$. The functions $\chi(\pm\tilde{k}_{10},k_{1z};\vec{k}_{1\perp},\vec{k}_{2\perp})$ are thus singular at

$$k_{1z} = \pm\frac{2}{M_3}\sqrt{(M_3+m_1)^2-(m_2+m_3)^2}\times\sqrt{(M_3-m_1)^2-(m_2+m_3)^2}. \tag{74}$$

Furthermore, for fixed k_{1z}, the positions of the singular points of the functions $\chi(-k_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp})$ and $\chi(k_{10}, k_{1z}; \vec{k}_{1\perp}, \vec{k}_{2\perp})$ are given by

$$k_{10} = -M_3 + \sqrt{k_{1z}^2 + (m_2 + m_3)^2}, \tag{75}$$

and

$$k_{10} = M_3 \pm \sqrt{k_{1z}^2 + (m_2 + m_3)^2}, \tag{76}$$

respectively. In this case only the singular points located on the positive k_0 axis need to be considered [see Eq. (72)]. In fact, it turns out that the integrands in Eq. (72) are symmetric with respect to $k_{1z} \to -k_{1z}$. Therefore, one needs to consider only the region where $k_{1z} > 0$ and multiply the equation by a factor 2. Furthermore, only the positive solutions of Eq. (74) are needed.

B. Euclidean space

The expressions for the transverse amplitudes in the Euclidean space, presented in Ref. [2], will be derived in detail in this section and in Appendix B. As mentioned in that paper, the following change of variables in the original equation (33) was performed,

$$k_i = k'_i + \frac{p}{3}, \quad (i = 1, 2, 3), \tag{77}$$

in order to allow the Wick rotation without crossing any singularities. The primed momenta satisfy the relation

$$k'_1 + k'_2 + k'_3 = 0. \tag{78}$$

The BS amplitude in Minkowski space can be written as

$$\begin{aligned} i\tilde{\Phi}_M(k'_1, k'_2, k'_3; M_3) &= i^3 \frac{\tilde{v}_M(k'_1) + \tilde{v}_M(k'_2) + \tilde{v}_M(k'_3)}{(k'_1 + \frac{p}{3})^2 - m^2 + i\epsilon} \\ &\quad \times \frac{1}{[(k'_2 + \frac{p}{3})^2 - m^2 + i\epsilon][(k'_3 + \frac{p}{3})^2 - m^2 + i\epsilon]} \\ &= i^3 \frac{\tilde{v}_M(k'_1) + \tilde{v}_M(k'_2) + \tilde{v}_M(-k'_1 - k'_2)}{[(k'_1 + \frac{p}{3})^2 - m^2 + i\epsilon][(k'_2 + \frac{p}{3})^2 - m^2 + i\epsilon]} \\ &\quad \times \frac{1}{(k'_1 + k'_2 - \frac{p}{3})^2 - m^2 + i\epsilon}, \end{aligned} \tag{79}$$

where

$$\tilde{\Phi}_M(k'_1, k'_2, k'_3; p) = \Phi_M\left(k'_1 + \frac{p}{3}, k'_2 + \frac{p}{3}, k'_3 + \frac{p}{3}; p\right), \tag{80}$$

and

$$\tilde{v}_M(k'_i) = v_M\left(k'_i + \frac{p}{3}\right). \tag{81}$$

Now, in new (shifted) integration variable one can perform the Wick rotation in the Eq. (33) and transform this equation into the Euclidean space. For the full Euclidean BS amplitude one gets:

$$\begin{aligned} &i\tilde{\Phi}_E(k'_{14}, k'_{1z}, \vec{k}'_{1\perp}; k'_{24}, k'_{2z}, \vec{k}'_{2\perp}) \\ &\quad = -i^3 \frac{\tilde{v}_E(k'_{14}, k'_{1v}) + \tilde{v}_E(k'_{24}, k'_{2v}) + \tilde{v}_E(k'_{34}, k'_{3v})}{(k'_{14} - i\frac{M_3}{3})^2 + k'^2_{1z} + m_1^2} \\ &\qquad \times \frac{1}{(k'_{24} - i\frac{M_3}{3})^2 + k'^2_{2z} + m_2^2} \\ &\qquad \times \frac{1}{(k'_{14} + k'_{24} + i\frac{M_3}{3})^2 + (k'_{1z} + k'_{2z})^2 + m_3^2}, \end{aligned} \tag{82}$$

where

$$\begin{aligned} k'_{iv} &= \sqrt{|\vec{k}'_{i\perp}|^2 + k'^2_{iz}}, \\ m_i^2 &= |\vec{k}'_{i\perp}|^2 + m^2, \quad (i = 1, 2, 3) \\ \vec{k}'_{3\perp} &= -(\vec{k}'_{1\perp} + \vec{k}'_{2\perp}). \end{aligned} \tag{83}$$

The full Euclidean transverse amplitude, corresponding to the Minkowski one given by (60), reads

$$\begin{aligned} &L(\vec{k}'_{1\perp}, \vec{k}'_{2\perp}) \\ &L_1(\vec{k}'_{1\perp}, \vec{k}'_{2\perp}) + L_2(\vec{k}'_{1\perp}, \vec{k}'_{2\perp}) + L_3(\vec{k}'_{1\perp}, \vec{k}'_{2\perp}) \\ &\quad = -\int_{-\infty}^{\infty} dk'_{14} \int_{-\infty}^{\infty} dk'_{1z} \int_{-\infty}^{\infty} dk'_{24} \int_{-\infty}^{\infty} dk'_{2z} \\ &\qquad \times i\tilde{\Phi}_E(k'_{14}, k'_{1z}, k'_{24}, k'_{2z}; \vec{k}'_{1\perp}, \vec{k}'_{2\perp}). \end{aligned} \tag{84}$$

By insertion of Eq. (82) in (84), it is found that one of the contributions to the transverse amplitude is given by

$$\begin{aligned} L_1(\vec{k}'_{1\perp}, \vec{k}'_{2\perp}) = &-\int_{-\infty}^{\infty} dk'_{1z} \int_{-\infty}^{\infty} dk'_{14} \\ &\times \chi(k'_{14}, k'_{1z}; \vec{k}'_{1\perp}, \vec{k}'_{2\perp}) \tilde{v}(k'_{1v}, k'_{14}) \\ &\times \frac{i}{(k'_{14} - i\frac{M_3}{3})^2 + k'^2_{1z} + m_1^2}, \end{aligned} \tag{85}$$

where the function χ is derived in Appendix B and reads

$$\chi(k'_{14}, k'_{1z}; \vec{k}'_{1\perp}, \vec{k}'_{2\perp}) = -\pi \int_0^1 \frac{du}{A}. \tag{86}$$

Here the denominator A is given by

$$A = au^2 + bu + c, \tag{87}$$

with

$$a = -k_{1z}'^2 - \left(k_{14}' + \frac{2}{3} i M_3\right)^2,$$
$$b = k_{1z}'^2 + \left(k_{14}' + \frac{2}{3} i M_3\right)^2 + m_2^2 - m_3^2,$$
$$c = m_3^2, \tag{88}$$

where the m_i's are defined by Eqs. (63) and (83).

VII. WICK ROTATION IN THE THREE-BODY BS EQUATION

It was shown in Ref. [2] that the three-body BS equation (33), after the introduction of the shifted variables (77) allows the Wick rotation without crossing any singularities. The validity of this rotation in the complex plane is a key for the equivalence between the Minkowski- and Euclidean-space transverse amplitudes. Therefore, we outline in this section some of the main points related to the Wick rotation of (33).

The BS equation (33) enclosing the shifted variables takes the form

$$\tilde{v}_M(q', p) = 2i\mathcal{F}(M_{12}'^2) \int \frac{d^4k'}{(2\pi)^4} \frac{i}{[(k' + \frac{p}{3})^2 - m^2 + i\epsilon]} \times \frac{i}{[(\frac{p}{3} - q' - k')^2 - m^2 + i\epsilon]} \tilde{v}_M(k', p), \tag{89}$$

where $M_{12}'^2 = (\frac{2}{3}p - q')^2$ and $\tilde{v}_M$ is defined by (81).

In the center-of-mass frame the pole in the upper half of the k_0' complex plane of the first propagator is located at

$$k_{01}'^{(+)} = -\frac{M_3}{3} - \sqrt{k_v'^2 + m^2} + i\epsilon, \tag{90}$$

and for the second one the position of the pole is

$$k_{02}'^{(+)} = \eta' - q_0' + i\epsilon, \tag{91}$$

with

$$\eta' = \frac{M_3}{3} - \sqrt{(\vec{k}' + \vec{q}')^2 + m^2}. \tag{92}$$

Consequently, as seen from (90), the first propagator does not have any poles in the first quadrant. Moreover, since $M_3 < 3m$ one has that $\eta' < 0$. The Wick rotation of (89) can thus be done safely without crossing any singularities.

Contrary to this, for the BS equation in the form (33), written in terms of the unshifted variables (i.e., k and q), the second propagator has a pole at

$$k_{02}^{(+)} = \eta - q_0 + i\epsilon, \tag{93}$$

where

$$\eta = M_3 - \sqrt{(\vec{k} + \vec{q})^2 + m^2}, \tag{94}$$

and if $M_3 > m$, there exist $\vec{k}$ and $\vec{q}$ such that $\eta > 0$, and the Wick rotation is thus not permitted.

Each of the three Faddeev components of the transverse amplitude, given by (60), is like the right-hand side of the BS equation (33), an integral over a vertex function times a product of propagators. It is therefore clear that the Wick rotation (after introduction of the shifted variables) also should hold for the transverse amplitude. The propagators entering the definition of the BS amplitude (79) have poles of the form (90), and should therefore not cause any additional problems in this respect. The expected equivalence between the transverse amplitudes computed in Minkowski and Euclidean spaces respectively, is confirmed by the numerical results to be presented in Sec. VIII B. Though, in this paper the transverse amplitudes, depending on k_x, k_y, are compared, we want to stress that the aforementioned arguments are also applicable to the more general amplitudes, not integrated over k_z, i.e., depending on k_x, k_y, k_z.

VIII. RESULTS AND DISCUSSION

A. Vertex function in Minkowski space

The Faddeev component of the vertex function is quantitatively studied in Minkowski space, as it carries the dynamical content of the relativistic three-body model. From it the full BS amplitude of the system can be constructed. In Minkowski space it has a nontrivial analytic structure since several branch points given by (45) are present in the kernel of Eq. (41), and reflected in the cusps appearing in the vertex function, as will be presented in what follows. Evidently, the Euclidean equation, obtained after the Wick rotation (see Ref. [2]) does not present in its integration path any singularity as well as the corresponding solution. Despite this, both solutions can be compared as we are going to present.

We solved Eq. (41) adopting a spline decomposition of the vertex function $v_M(q, p)$ used in Ref. [3], see Appendix C. The inputs are the scattering length a and the three-body binding energy B_3, the same as in the Euclidean calculations performed in Ref. [2]. In the numerical solution of (41), we multiplied the right-hand side of it by a parameter (eigenvalue) λ. A consistent solution then corresponds to an eigenvalue of $\lambda = 1.0$.

Three results for the eigenvalue are given in Table I, for the following values of the two-body scattering length: $am = -1.280$, $am = -1.500$ and $am = -1.705$. The results for the eigenvalue λ, expected to be real and equal

TABLE I. Eigenvalues of the three-body ground state for three scattering lengths, a, computed in [3] by using the Euclidean three-body binding energies.

B_3/m	am	λ
0.006	−1.280	$0.999 - 0.054i$
0.395	−1.500	$1.000 + 0.002i$
1.001	−1.705	$0.997 + 0.106i$

to one, present small deviations from the unity and also an imaginary part.

Nevertheless, it is important to mention another potential source of error: cutoffs were introduced to constrain the domains of the variables q_v and q_0. It is very difficult to reach a reasonable convergence considering the full domains, as the size of the region where the singularities [given by Eqs. (42) and (43)] appear is enlarged along the axes. Moreover, the asymptotic regions start at larger momenta.

The actual values used to truncate the variables were $q_v^{\max}/m = 6.0$ and $q_0^{\max}/m = 13.0$, for the two smallest binding energies, or $q_0^{\max}/m = 15.0$, for the case where $B_3/m = 1.001$. Regardless, the convergence was reached within about 10% for the worst case. On the other hand, in the Euclidean calculations it is possible to take into account the whole range of the involved variables q_v, q_4, k_v and k_4 through a mapping procedure. The fact that in the Minkowski approach cutoffs were applied while the whole domains were used in the Euclidean calculations makes the results not fully comparable. This might be one of the reasons why in λ small nonzero imaginary parts appear and for the deviations from 1 obtained in the real part.

The rest of this section will be devoted to a detailed study of the representative case with $B/m = 0.395$ and $am = -1.5$. Though, important to point out that many of the stated conclusions are valid also for the other cases in Table I.

In Fig. 2 it is shown the calculated real and imaginary parts of the vertex function $v_M(q_0, q_v)$ versus q_0 for three fixed values of q_v. For all three cases there is a quite good agreement between the numerical peak positions and the analytical ones, given by Eq. (45). Though, worth to mention that, due to the scale of the figure, some of the peaks for the case $q_v/m = 2.5$ are not clearly visible. The peaks in Fig. 2 appear as branching points of the kernel $\Pi(q_0, q_v, \pm\varepsilon_k, k_v)$, defining its singularities, as discussed in Sec. III. Interestingly, the aforementioned positions correspond to $M_{12}^2 = 0$ and $M_{12}^2 = 4m^2$, which give the branching points of the two-body scattering amplitude $\mathcal{F}(M_{12}^2)$. In Fig. 2 it is seen that for small values of q_v a singularity appears at $q_0 \approx M_3$. The distance between the external peaks, corresponding to $M_{12}^2 = 4m^2$, is equal to $2\sqrt{q_v^2 + 4m^2}$, an increasing function with respect to q_v. This fact makes things more complicated from the numerical point of view, as for large values of q_v a very wide region of q_0 has to be covered. This imposes the need of cutoffs for the variables.

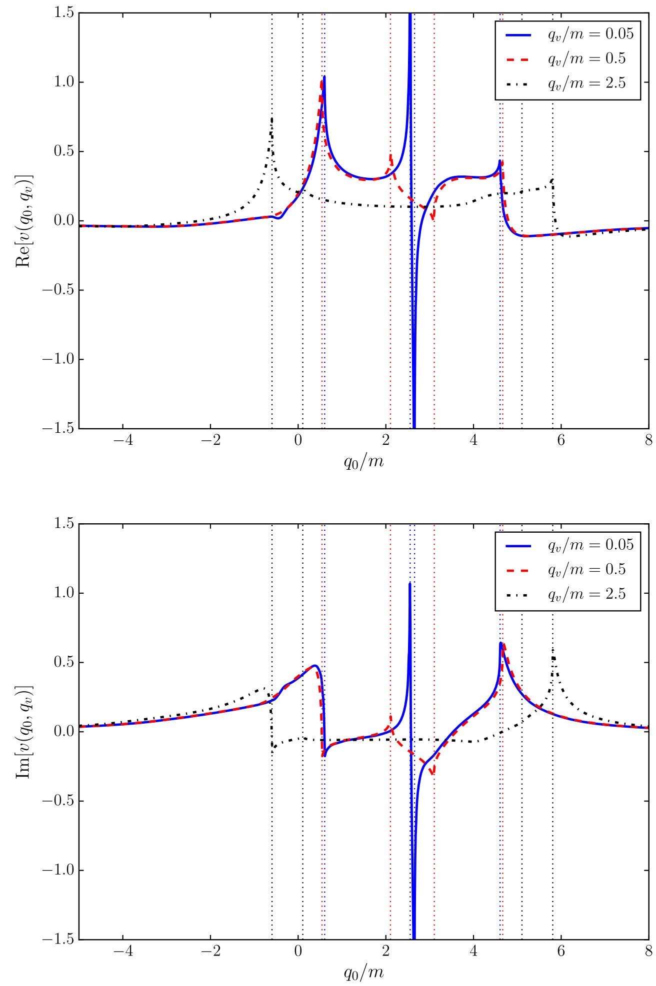

FIG. 2. Real (upper panel) and imaginary (lower panel) parts of the vertex function, $v_M(q_0, q_v)$ with respect to q_0. For each value of q_v the analytical positions of the peaks, given in Eq. (45), are shown with the vertical dotted lines.

Similarly, in Fig. 3 we present the real and imaginary parts of $v_M(q_0, q_v)$ with respect to q_v, for $q_0 = m$. In the figure is seen a peak (both in the real and imaginary part) which corresponds to the branching point $q_v = M_3 - q_0$. However, for the other points given by Eq. (45), no solution exists such that $q_0 = m$ and $q_v \geq 0$. It can be seen in the figure that the amplitude asymptotically goes to zero for large values of q_v, as expected.

Furthermore, in Fig. 4, we compare for $q_0 = M_3/3$ the calculated real and imaginary parts of $v_M(q_0, q_v)$ versus q_v with the corresponding Euclidean results for $\tilde{v}_E(q', p) = v_E(q' + p/3, p)$, i.e., $q'_4 = 0$. It is seen that the results for both real and imaginary parts are practically the same for small values of q_v. However, the Minkowskian amplitude has a peak at the branching point $q_v = 2M_3/3$ and the amplitudes also differ significantly at larger values

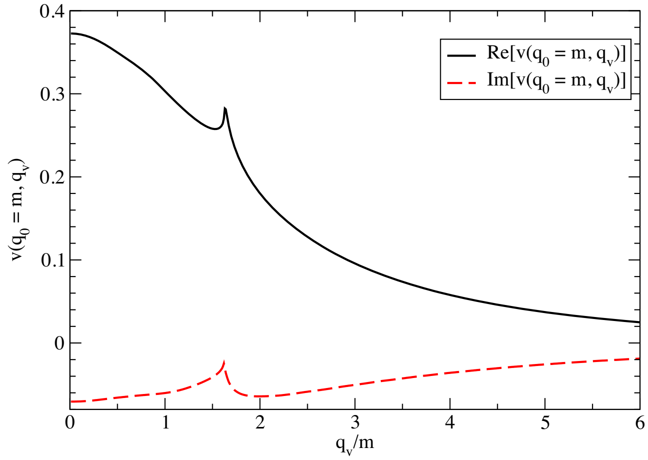

FIG. 3. Real and imaginary parts of the vertex function, $v_M(q_0 = m, q_v)$ with respect to q_v.

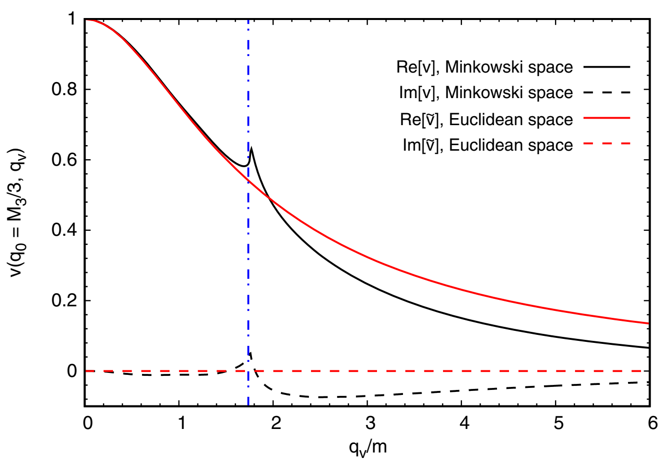

FIG. 4. Real and imaginary parts of the Minkowski space vertex function, $v_M(q_0 = M_3/3, q_v)$ versus q_v, compared with the corresponding Euclidean results for $\tilde{v}_E(q_4', q_v')$.

of q_v, a difference coming from the contribution of the cut necessary to be taken into account to match the limit of $q_4' \to 0$ with the Minkowski point. For the sake of comparison both amplitudes are normalized to 1 for $q_4' = 0$ and $q_v = 0$.

B. Transverse amplitude

Although the transverse amplitude has a different meaning with respect to the standard distribution amplitude (see Refs. [26,27]), it shares in common with the distribution amplitude the valence wave function content of the full LFWF in Fock space. However, the transverse amplitude has a distinctive feature with respect to the distribution amplitude as it can be obtained as well from the BS amplitude in Euclidean space, making it a useful quantity to compare with the corresponding Minkowski space quantity. The demonstration of equivalence between the transverse amplitude obtained in Minkowski and Euclidean spaces has been given in Ref. [28] resorting to the Nakanishi integral representation of the BS amplitude [5,6] in a two-boson system. Furthermore, as discussed in Sec. VII, the Wick rotation of the BS equation (33) can be performed without crossing any singularities and should also hold for the transverse amplitude. Consequently, the transverse amplitudes computed in Minkowski and Euclidean spaces should agree with each other also for the three-body system, which is confirmed by the results presented in this section.

In what follows we will exploit the structure of the transverse amplitude associated with one Faddeev component. The reader has to keep in mind that the full transverse amplitude from the three-body wave function is a coherent sum of the three components. However, to expose the consequences of our dynamical assumptions it is simpler to look individually to each Faddeev component, as the sum of $L_i(|\vec{k}_{i\perp}|, |\vec{k}_{j\perp}|, \theta_{ij})$ where θ_{ij} denotes the angle between $\vec{k}_{i\perp}$ and $\vec{k}_{j\perp}$, will present a much more complex 3D landscape as a function of the two independent transverse momenta.

In Fig. 5 it is displayed the modulus of the contribution $L_1(|\vec{k}_{1\perp}|, |\vec{k}_{2\perp}|, \theta)$ to the transverse amplitude versus $|\vec{k}_{1\perp}|$, calculated from Eq. (72). It is also shown, for comparison, the corresponding Euclidean results calculated through Eq. (85). It is seen that the Minkowski and Euclidean results are in fair agreement with each other. The nonsmooth behavior of the BS solution in Minkowski space, shown in Fig. 2, is washed out and makes the agreement between the Euclidean and Minkowski space transverse amplitudes even more remarkable.

The dependence of the modulus of the transverse amplitude $|L_1|$ on the angle θ between $\vec{k}_{1\perp}$ and $\vec{k}_{2\perp}$ is also displayed, in Fig. 6. The modulus of L_1 is a slowly

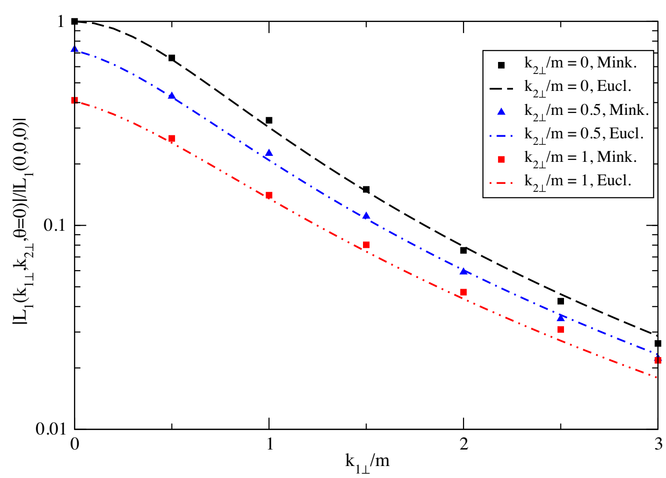

FIG. 5. Transverse amplitude modulus, $|L_1|$ for $\theta = 0$ versus $|\vec{k}_{1\perp}|/m$ for $|\vec{k}_{2\perp}|/m = 0.0,\ 0.5,\ 1.0$, obtained in Minkowski space compared with the one calculated in Euclidean space.

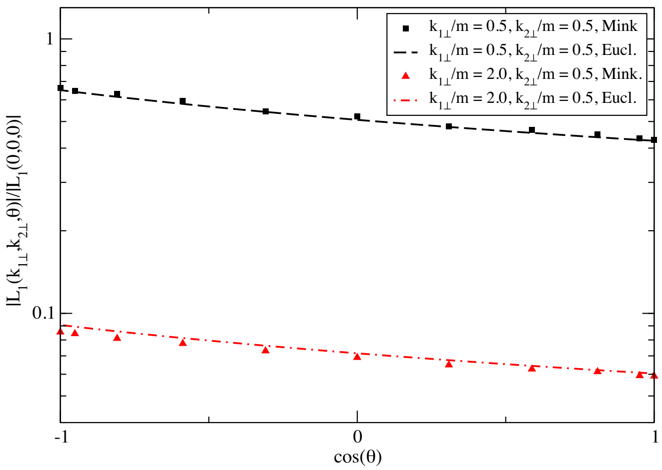

FIG. 6. Transverse amplitude modulus, $|L_1|$ as a function of $\cos(\theta)$ for $(|\vec{k}_{1\perp}|/m, |\vec{k}_{2\perp}|/m) = (0.5, 0.5), (2.0, 0.5)$, obtained in Minkowski space compared with the one computed in Euclidean space.

decreasing function with respect to $\cos\theta$. As seen in the figure, a satisfactory agreement is again found between the Euclidean (lines) and Minkowski (symbols) calculations. Although the interaction is active in the s-wave and the vertex function is dependent only on the time component and the modulus of the spatial momentum in the rest-frame, the Faddeev component of the BS amplitude acquired a weak angular dependence, as we illustrated, due to the presence of the individual propagators and momentum conservation, essentially containing both s- and p-wave dependencies. The mixing of higher waves appears as the binding energy increases, which is expected as the bound state mass becomes smaller increasing the sensitivity of the denominator in Eq. (86) to variations of the relative angle, as it is transparent in the case of the transverse amplitude in Euclidean space.

The three-dimensional structure of the transverse amplitude is further exposed in Fig. 7, where we plot the absolute value of the amplitude $L_1(|\vec{k}_{1\perp}|, |\vec{k}_{2\perp}|, \theta)$ with respect to $|\vec{k}_{1\perp}|$ and $|\vec{k}_{2\perp}|$ for different fixed values of θ. The computations were for simplicity performed in Euclidean space, since it has already been shown above that the Euclidean and Minkowski calculations give the same results for the transverse amplitude. One can conclude from the figure that the transverse amplitude becomes wider when the value of is θ is increased, due to correlations proportional to $\vec{k}_{1\perp} \cdot \vec{k}_{2\perp}$. This effect is clearly visible, in particular, for the antialigned case ($\theta = \pi$), i.e., when the transverse momenta obey the relation $|\vec{k}_{3\perp}|^2 = (|\vec{k}_{1\perp}| - |\vec{k}_{2\perp}|)^2$. For $\theta = \pi$ and along $|\vec{k}_{1\perp}| = |\vec{k}_{2\perp}|$ there is a clear enhancement of the transverse amplitude which exhibits a bump due to the vanishing value of $\vec{k}_{3\perp}$. The development of this pattern is seen by inspecting the evolution of the amplitude with θ and

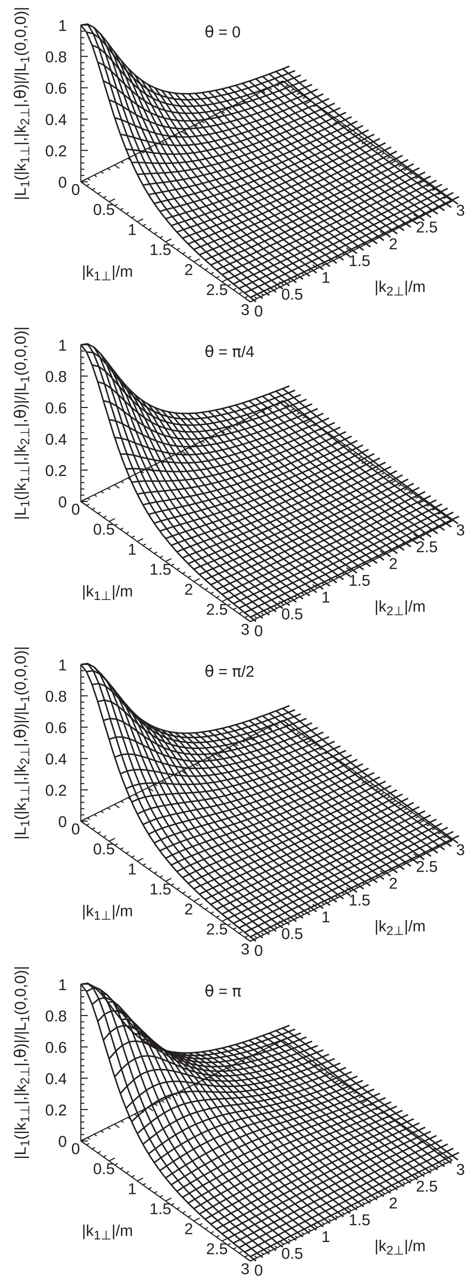

FIG. 7. Transverse contribution, $L_1(|\vec{k}_{1\perp}|, |\vec{k}_{2\perp}|, \theta)$ with respect to $|\vec{k}_{1\perp}|$ and $|\vec{k}_{2\perp}|$, for $\theta = 0, \pi/4, \pi/2, \pi$ calculated in Euclidean space.

comparing the results for the $\pi/2$ and π angles. This feature is further enhanced when the three-body bound state mass decreases for a strongly bound system.

C. Short-range correlation

The two-body short-range correlation (in the context of nonrelativistic nuclear physics, see e.g., Ref. [29]) is exhibited by the model for large relative momentum, $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|$, and a back-to-back momentum configuration of particles 2 and 3, where the transverse amplitude is dominated by the component L_1. Note that this amplitude is also symmetric by the exchange of $\vec{k}_{2\perp}$ and $\vec{k}_{3\perp}$ due to the bosonic nature of the system with equal masses. In the relativistic context the concept of the short-range two-body correlation has not yet been developed, but its emergent imprints are observed in the transverse amplitude.

The plot for L_1 as a function of $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|$ for the back-to-back configuration ($|\vec{k}_{1\perp}| = 0$) is shown in Fig. 8. In the situation when $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|$ is large, only L_1 dominates in the full three-body transverse amplitude, being the counterpart of the factorization of the nonrelativistic wave function as a two-body term depending on the relative distance between the two particles times a function of the relative coordinates of the other $N-2$ particles [29]. In the model with contact interaction, the denominator in Eq. (86) provides the momentum dependence of the "wave function" of the short-range correlated pair in the valence LFWF of the three-body system. The relative momentum behavior shown in the figure reflects the free propagators of the bosons, as the contact interaction does not bring any momentum dependence. This property is shared by the nonrelativistic three-body model with a zero-range potential. Furthermore, for $|\vec{k}_{1\perp}| = 0$ the denominator of Eq. (86) provides the large momentum behavior as $L_1 \sim |\vec{k}_{3\perp}|^{-2}$ that leads to $L_1 \sim |\vec{k}_{2\perp} - \vec{k}_{3\perp}|^{-2}$ in the back-to-back configuration, as confirmed by our results. The scale for the asymptotic behavior is naturally fixed by the individual boson mass, as it can be seen in Fig. 8, where L_1 shows an accentuated drop in this momentum range. In the inset plot the asymptotic behavior for large relative momentum is shown.

The asymptotic property of L_1 also follows from the structure of the valence LFWF, which has the three-body LF propagator as the dominant factor for the contact interaction and at large momentum is just the inverse of the free three-body mass, i.e.,

$$M_0^2 = \sum_i \frac{\vec{k}_{i\perp}^2}{\xi_i}.$$

Note that $M_0^{-2} \sim |\vec{k}_{2\perp} - \vec{k}_{3\perp}|^{-2}$ for $|\vec{k}_{1\perp}| = 0$, which is the power-law behavior seen in Fig. 8. Such asymptotic form is expected to change for models with finite range interactions.

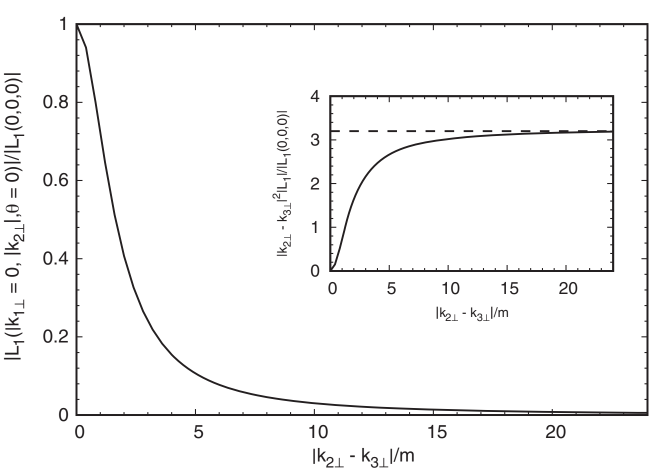

FIG. 8. Transverse amplitude modulus, $|L_1|$ as a function of $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|$ for $|\vec{k}_{1\perp}| = 0$. In the inset of the figure is shown $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|^2 |L_1|$ versus $|\vec{k}_{2\perp} - \vec{k}_{3\perp}|$ with a dashed line indicating the asymptotic behavior.

IX. CONCLUSIONS

The three-body BS equation has been solved, directly in Minkowski space, by standard analytical and numerical methods, where (i) no ansatz or assumption has been introduced to represent the BS amplitude and (ii) the singularities from the kernel are treated analytically and numerically directly in the four-dimensional equation. The application of the direct integration method to the three-body Faddeev-BS equation was already presented in Ref. [3]. However, in the present paper the Minkowski-space structure of the three-body system has been analyzed in far greater detail and brings up the structure of the short-range correlated pair in the Borromean system.

The computed amplitude turns out to be highly peaked, indicating the presence of a singular behavior, as shown in Sec. VI. This is very different from what was found for the amplitude computed through the Wick-rotated equation in Ref. [2], due to the presence of branching points and the associated cuts. In one example we expose the cut contribution to the amplitude comparing results from Minkowski and Euclidean calculations.

Although the BS amplitudes obtained from the solution in Euclidean and Minkowski spaces are fundamentally different, they can be compared by means of the transverse amplitude. The comparison shows a notable agreement, giving more confidence on the reliability of the direct integration method. Furthermore, the transverse amplitude reveals the structure of the short-range correlated pair in the valence wave function, which was found when the pair has large relative momentum in a back-to-back configuration. We found that the Faddeev component of the BS amplitude defined with the pair interaction dominates over the others, and a power-law behavior of the type $\sim|\vec{k}_{i\perp} - \vec{k}_{j\perp}|^{-2}$ is found for the associated transverse amplitude and confirmed by our numerical results.

In this work, we show that the results obtained by the direct integration in Minkowski space agree well with the Euclidean results for comparable quantities, however this method is quite demanding from the numerical point of view. One possible way to improve on this and additionally be able to treat more realistic kernels and/or the spin degree of freedom is to transform the BS equation into a nonsingular form by using the Nakanishi integral representation [5,6]. The formulation of the BS approach to the three-body problem via Nakanishi integral representation is already in progress and computations based on this method will be undertaken in the near future. Once the BS amplitude of the three-body state is known in Minkowski space it can be used to investigate electromagnetic form factors, the diversity of parton longitudinal and transverse momentum distributions, as well as the space-time structure of the pair short-range correlation.

Furthermore, one interesting direction for future explorations of the three-body system is to consider particles with nonequal masses, as a framework, for example, to study baryons with a heavy-light content. Of course still many steps have to be considered to include the subtle physics of quantum chromodynamics (QCD) in a continuum model (see e.g., Refs. [12,30]), but now in Minkowski space. We expect that the formulation will also allow to explore excited states and, through it, in the low energy region, the Efimov phenomena relativistically.

ACKNOWLEDGMENTS

We are grateful to Jaume Carbonell for stimulating discussions. We thank for the support from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) of Brazil. J. H. A. N. acknowledges the support of the Grant No. 2014/19094-8 and V. A. K. of the Grant No. 2015/22701-6 from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP). E. Y. thanks for the financial support of the Grants No. 2016/25143-7 and No. 2018/21758-2 from FAPESP. T. F. thanks the FAPESP Thematic Project Grant no. 17/05660-0. V. A. K. is also sincerely grateful to group of theoretical nuclear physics of ITA, São José dos Campos, Brazil, for kind hospitality during his visits.

APPENDIX A: DERIVATION OF THE RELATION BETWEEN THE BS AMPLITUDE AND LFWF

In this Appendix, we derive in detail the relation between the three-body BS amplitude and the three-body LFWF.

The three-body BS amplitude is defined by Eq. (46). Let us define the integral

$$I_3 = \int d^4x_1 d^4x_2 d^4x_3 \delta(\omega\cdot x_1)\delta(\omega\cdot x_2)\delta(\omega\cdot x_3) \times \Phi_M(x_1,x_2,x_3;p)\exp(ik_1\cdot x_1 + ik_2\cdot x_2 + ik_3\cdot x_3), \tag{A1}$$

where $\omega=(\omega_0,\vec{\omega})$, $\omega^2=0$ and $k_{1,2,3}$ are the on-shell momenta, i.e., $k_1^2=k_2^2=k_3^2=m^2$. The delta functions in (A1) restrict the variation of the arguments of the coordinate space BS amplitude to the LF hyperplane $\omega\cdot x=0$.

We now represent the δ-functions in (A1) in the integral form

$$\delta(\omega\cdot x_i) = \frac{1}{2\pi}\int \exp(-i\omega\cdot x_i\tau_i)d\tau_i. \tag{A2}$$

Due to translation invariance, when all x_i's are shifted by $a\colon x\to x+a$, the BS amplitude obtains a factor $\exp(-ia\cdot p)$:

$$\Phi_M(x_1+a,x_2+a,x_3+a;p) = \exp(-ia\cdot p)\Phi_M(x_1,x_2,x_3;p), \tag{A3}$$

like the nonrelativistic wave function.

We introduce then the BS amplitude (46) in momentum space. We define it, extracting the delta function, responsible for conservation of momenta:

$$\Phi_M(x_1,x_2,x_3;p) = \frac{(2\pi)^4}{(2\pi)^{12}}\int d^4k_1' d^4k_2' d^4k_3' \times \exp(-ik_1'\cdot x_1 - ik_2'\cdot x_2 - ik_3'\cdot x_3) \times \delta^{(4)}(k_1'+k_2'+k_3'-p)\Phi_M(k_1',k_2',k_3';p). \tag{A4}$$

Because of the delta-function in (A4) the amplitude $\Phi_M(x_1,x_2,x_3;p)$ satisfies the relation (A3). We emphasize that here, in contrast to Eq. (A1), all the arguments $k'_{1,2,3}$ of the BS amplitude are the off-mass-shell momenta.

We substitute (A4) and (A2) in (A1) and integrate over $x_{1,2,3}$ and $k'_{1,2,3}$. The result reads:

$$I_3 = \frac{1}{(2\pi)^3}\int d\tau_1 d\tau_2 d\tau_3 (2\pi)^4 \times \delta(k_1+k_2+k_3-p-\omega\tau_1-\omega\tau_2-\omega\tau_3) \times \Phi_M(k_1-\omega\tau_1,k_2-\omega\tau_2,k_3-\omega\tau_3;p), \tag{A5}$$

where the variables $\tau_{1,2,3}\in]-\infty,\infty[$. To avoid confusion, we emphasize again: the four-momenta $k_{1,2,3}$ in this formula are the on-shell momenta, according to the definition of (A1), whereas, the arguments of the BS amplitude are the off-shell momenta, like $k'_{1,2,3}$ in (A4); namely: $(k_1-\omega\tau_1)^2 = m^2-2(\omega\cdot k_1)\tau_1 \neq m^2$ since $\tau_1\neq 0$ and similarly for other arguments except for $p^2=M_3^2$.

We subsequently introduce the variable $\tau=\tau_1+\tau_2+\tau_3$ and represent the integral (A5) in the following three equivalent forms:

$$I_{3a} = \frac{(2\pi)^4}{(2\pi)^3}\int d\tau\,\delta(k_1+k_2+k_3-p-\omega\tau)\int d\tau_1 d\tau_2 \times \Phi_M(k_1-\omega\tau_1, k_2-\omega\tau_2, k_3-\omega(\tau-\tau_1-\tau_2); p),$$
$$I_{3b} = \frac{(2\pi)^4}{(2\pi)^3}\int d\tau\,\delta(k_1+k_2+k_3-p-\omega\tau)\int d\tau_2 d\tau_3 \times \Phi_M(k_1-\omega(\tau-\tau_2-\tau_3), k_2-\omega\tau_2, k_3-\omega\tau_3; p),$$
$$I_{3c} = \frac{(2\pi)^4}{(2\pi)^3}\int d\tau\,\delta(k_1+k_2+k_3-p-\omega\tau)\int d\tau_1 d\tau_3 \times \Phi_M(k_1-\omega\tau_1, k_2-\omega(\tau-\tau_1-\tau_3), k_3-\omega\tau_3; p). \tag{A6}$$

It can be also represented as:

$$I_3 = \frac{(2\pi)^4}{(2\pi)^3}\int d\tau\delta(k_1+k_2+k_3-p-\omega\tau) \times \int d\tau_1 d\tau_2 d\tau_3 \delta(\tau_1+\tau_2+\tau_3-\tau) \times \Phi_M(k_1-\omega\tau_1, k_2-\omega\tau_2, k_3-\omega\tau_3; p), \tag{A7}$$

where by means of the delta function $\delta(\tau_1+\tau_2+\tau_3-\tau)$ one can exclude any τ_i and obtain Eq. (A6). Depending on the convenience, one can chose any of the forms in (A6) to calculate the double integral over τ_i, τ_j. With the standard choice $\omega^\mu = (1,0,0,-1)$, i.e., in the LF coordinates, $\vec{\omega}_\perp = 0, \omega^+ = 0, \omega^- = 2$, these integrals are reduced to the integrals over the k_i^- components. The value of τ is determined from the conservation law $k_1+k_2+k_3 = p+\omega\tau$. For example, squaring this equation, we find

$$\tau = \frac{(k_1+k_2+k_3)^2 - M_3^2}{2(\omega\cdot p)} = \frac{1}{2(\omega\cdot p)}\left(\frac{k_{1\perp}^2+m^2}{x_1}+\frac{k_{2\perp}^2+m^2}{x_2}+\frac{k_{3\perp}^2+m^2}{x_3} - M_3^2\right) \equiv \frac{1}{2(\omega\cdot p)}(M_0^2 - M_3^2). \tag{A8}$$

On the other hand, the integral (A1) can be expressed in terms of the three-body LFWF. We assume that the LF plane is the limit of a spacelike plane, therefore the operators $\varphi(x_1), \varphi(x_2), \varphi(x_3)$, commute with each other, and, hence, the symbol of the T product in (46) can be omitted. In the considered representation, the Heisenberg operators $\varphi(x)$ in (46) are identical on the light front $\omega\cdot x = 0$ to the Schrödinger ones (just as in the ordinary formulation of field theory the Heisenberg and Schrödinger operators are identical for $t=0$). The Schrödinger operator $\varphi(x)$ (for the spinless case, for simplicity), which for $\omega\cdot x = 0$ is the free field operator, is given by:

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\varepsilon_k}} \times [a(\vec{k})\exp(-ik\cdot x) + a^\dagger(\vec{k})\exp(ik\cdot x)]. \tag{A9}$$

We represent the state vector $|p\rangle \equiv \phi(p)$ in (A1) in the form of the expansion via the Fock states:

$$|p\rangle = (2\pi)^{3/2}\int \psi(k_1,k_2,k_3,p,\omega\tau) \times \delta^{(4)}(k_1+k_2+k_2-p-\omega\tau)2(\omega\cdot p)d\tau \times \frac{d^3k_1}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_1}}}\frac{d^3k_2}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_2}}}\frac{d^3k_3}{(2\pi)^{3/2}\sqrt{2\varepsilon_{k_3}}} \times a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)a^\dagger(\vec{k}_3)|0\rangle + \cdots. \tag{A10}$$

In (A9) and (A10) the four-momenta $k_{1,2,3}$ are on mass-shells. We substitute this expression in $\Phi_M(x_1,x_2,x_3;p)$, Eq. (46). Since the vacuum state on the light front is always "bare," the creation operator, applied to the vacuum state $\langle 0|$ gives zero, and in the operators $\varphi(x)$ the part containing the annihilation operators only survives. This cuts out the three-body Fock component in the state vector. We thus obtain:

$$\Phi_M(x_1,x_2,x_3;p) = (2\pi)^{3/2}\int \psi(k_1,k_2,k_3,p,\omega\tau) \times \delta^{(4)}(k_1'+k_2'+k_2'-p-\omega\tau)2(\omega\cdot p)d\tau \times \exp(-ik_1'x_1 - ik_2'x_2 - ik_3'x_3) \times \frac{d^3k_1'}{(2\pi)^3 2\varepsilon_{k_1'}}\frac{d^3k_2'}{(2\pi)^3 2\varepsilon_{k_2'}}\frac{d^3k_3'}{(2\pi)^3 2\varepsilon_{k_3'}}. \tag{A11}$$

Then we substitute this $\Phi_M(x_1,x_2,x_3;p)$ in (A1) and integrate over $x_{1,2,3}$. The integration, for example, over x_1 and then over k_1' is fulfilled as follows

$$\int \frac{d^4x_1 d^3k_1'}{(2\pi)^4 2\varepsilon_{k_1'}}\exp(-ik_1'x_1)\exp(-i\omega\cdot x_1\tau_1)d\tau_1\exp(ik_1x_1) = \int \delta^{(4)}(k_1-k_1'-\omega\tau_1)d\tau_1\frac{d^3k_1'}{2\varepsilon_{k_1'}} = \int \delta^{(4)}(k_1-k_1'-\omega\tau_1)d\tau_1 d^4k_1'\theta(\omega\cdot k_1')\delta(k_1'^2-m^2) = \int \theta(\omega\cdot k_1)\delta((k_1+\omega\tau_1)^2-m^2) = \int d\tau_1\delta(2(\omega\cdot k_1)\tau_1) = \frac{1}{2(\omega\cdot k_1)}, \tag{A12}$$

and similarly for the integrations over x_2, k_2' and x_3, k_3'. We used here that $\theta(k_{10}') = \theta(\omega\cdot k_1')$ for $k_{10}' > 0$. Then for I_3 we get (cf. Eq. (3.56) from Ref. [23]):

$$I_3 = \frac{(2\pi)^{3/2}2(\omega\cdot p)}{2(\omega\cdot k_1)2(\omega\cdot k_2)2(\omega\cdot k_3)}\int d\tau\psi(k_1,k_2,k_3;p,\omega\tau) \times \delta^{(4)}(k_1+k_2+k_3-p-\omega\tau). \tag{A13}$$

Comparing (A13) and (A7), we find:

$$\psi(k_1,k_2,k_3,p,\omega\tau) = \frac{1}{\sqrt{2\pi}}\frac{2(\omega\cdot k_1)2(\omega\cdot k_2)2(\omega\cdot k_3)}{2(\omega\cdot p)} \times\int d\tau_1 d\tau_2 d\tau_3\delta(\tau_1+\tau_2+\tau_3-\tau) \times\Phi_M(k_1-\omega\tau_1,k_2-\omega\tau_2,k_3-\omega\tau_3;p). \tag{A14}$$

As mentioned, in ordinary LFD, Eq. (A14) corresponds to the integration over k^-. This equation makes the link between the three-body BS amplitude Φ_M and the wave function ψ defined on the light front specified by ω. As it is seen from the above derivation, it is generalizable (with the same coefficient $1/\sqrt{2\pi}$) for arbitrary number of particles. Simply the number of the factors $2(\omega\cdot k_i)$ and of the arguments increases. In the LF coordinates Eq. (A14) obtains the form:

$$\psi(\vec{k}_{1\perp},\xi_1;\vec{k}_{2\perp},\xi_2;\vec{k}_{3\perp},\xi_3) = \frac{1}{\sqrt{2\pi}}\frac{(p^+)^2}{2}2\xi_1 2\xi_2 2\xi_3\int d\tau_1 d\tau_2 d\tau_3\delta(\tau_1+\tau_2 +\tau_3-\tau)\Phi_M(\tilde{k}_1;\tilde{k}_2;\tilde{k}_3;p), \tag{A15}$$

where $\tilde{k}_i\equiv\{\vec{k}_{i\perp},k_i^+,k_i^--2\tau_i\}$ and $0<\xi_i<1$ $(\xi_1+\xi_2+\xi_3=1)$ denotes the longitudinal momentum fraction of particle i.

We introduce now new integration variables: $k_1'^-=k_1^- - 2\tau_1$, etc, and then

$$\psi(\vec{k}_{1\perp},\xi_1;\vec{k}_{2\perp},\xi_2;\vec{k}_{3\perp},\xi_3) = \frac{(p^+)^2}{\sqrt{2\pi}}\xi_1\xi_2\xi_3\int dk_1^- dk_2^-\Phi_M(k_1,k_2,k_3;p) = \frac{(p^+)^2}{\sqrt{2\pi}}\xi_1\xi_2\xi_3\int dk_1^- dk_2^-\Phi_M(k_1,k_2,k_3;p). \tag{A16}$$

In the last line, we omitted the integration over the 3rd argument since it is not independent. In the above formula it is understood that $k_3=p-k_1-k_2$. One can chose any pair of arguments: (12), (13) or (23), depending on the convenience.

APPENDIX B: CALCULATING THE EUCLIDEAN TRANSVERSE AMPLITUDE

In this Appendix we derive in detail the function χ occurring in the expression for the Euclidean transverse amplitude, Eq. (85).

From Eqs. (82) and (84), we define χ as the integral

$$\chi(k'_{14},k'_{1z};\vec{k}'_{1\perp},\vec{k}'_{2\perp}) = \int_{-\infty}^{\infty}dk'_{20}\int_{-\infty}^{\infty}dk'_{2z}\frac{i}{(k'_{24}-i\frac{M_3}{3})^2+k'^2_{2z}+m_2^2} \times\frac{i}{(k'_{14}+k'_{24}+i\frac{M_3}{3})^2+(k'_{1z}+k'_{2z})^2+m_3^2}. \tag{B1}$$

The two propagators in (B1) can then be put together by using the Feynman parametrization (11) leading to the result

$$\frac{i}{(k'_{24}-i\frac{M_3}{3})^2+k'^2_{2z}+m_2^2} \times\frac{i}{(k'_{14}+k'_{24}+i\frac{M_3}{3})^2+(k'_{1z}+k'_{2z})^2+m_3^2} = -\int_0^1\frac{du}{D^2}, \tag{B2}$$

where the denominator reads

$$\begin{aligned} D &= k'^2_{24}+k'^2_{2z}+(1-u)[k'^2_{14}+k'^2_{1z}]+\frac{2}{3}iM_3k'_{24}+2(1-u)k'_{1z}k'_{2z}+\frac{2}{3}(1-u)k'_{14}(3k'_{24}+iM_3) \\ &\quad -\frac{4}{3}iuM_3k'_{24}+(1-u)m_3^2+um_2^2-\frac{M_3^2}{9} \\ &= k'^2_{24}+k'^2_{2z}+(1-u)[k'^2_{14}+k'^2_{1z}]+2\left[(1-u)k'_{14}-\frac{iM_3}{3}(-1+2u)\right]k'_{24} \\ &\quad +2(1-u)k'_{1z}k'_{2z}+\frac{2}{3}iM_3(1-u)k_{14}+(1-u)m_3^2+u\,m_2^2-\frac{M_3^2}{9}. \end{aligned} \tag{B3}$$

We subsequently eliminate the terms linear in k'_{24} and k'_{2z}, by performing in Eqs. (B1), (B2) and (B3) the transformations

$$k'_{24} \to k'_{24} - \alpha, \quad k'_{2z} \to k'_{2z} - \beta, \tag{B4}$$

with

$$\alpha = (1-u)k'_{14} - \frac{iM_3}{3}(-1+2u), \tag{B5}$$

and

$$\beta = (1-u)k'_{1z}. \tag{B6}$$

By these transformations the denominator (B3) is changed into

$$D \to \tilde{D} = k'^2_{24} + k'^2_{2z} + A, \tag{B7}$$

where

$$A = u(1-u)[k'^2_{14} + k'^2_{1z}] + (1-u)m_3^2 + um_2^2 + \frac{4}{3}iM_3u(1-u)k'_{14} - \frac{4}{9}M_3^2u(1-u). \tag{B8}$$

The integrals over k'_{24} and k'_{2z} in (B1) can now be performed analytically, and the result is

$$\chi(k'_{14}, k'_{1z}; \vec{k}'_{1\perp}, \vec{k}'_{2\perp}) = -\int_0^1 du \int_{-\infty}^{\infty} dk'_{20} \int_{-\infty}^{\infty} \frac{dk'_{2z}}{(k'^2_{24} + k'^2_{2v} + A)^2} = -2\pi \int_0^1 du \int_0^{\infty} \frac{k'dk'}{(k'^2 + A)^2} = -\pi \int_0^1 \frac{du}{A}. \tag{B9}$$

Alternatively, one can write the quantity A in the form

$$A = au^2 + bu + c, \tag{B10}$$

with

$$a = -k'^2_{1z} - \left(k'_{14} + \frac{2}{3}iM_3\right)^2, \quad b = k'^2_{1z} + \left(k'_{14} + \frac{2}{3}iM_3\right)^2 + m_2^2 - m_3^2, \quad c = m_3^2. \tag{B11}$$

APPENDIX C: NUMERICAL METHODS

We solve in this work Eq. (41) by expanding the amplitude $v_M(q_0, q_v)$ in a bicubic spline basis, on a finite domain $\Omega = I_{q_0} \times I_{q_v} = [-q_0^{\max}, q_0^{\max}] \times [0, q_v^{\max}]$, i.e.,

$$v_M(q_0, q_v) = \sum_{k=0}^{2N_{q_0}+1} \sum_{l=0}^{2N_{q_v}+1} A_{ij} S_k(q_0) S_l(q_v), \tag{C1}$$

where the unknown coefficients A_{ij} are to determined. In the numerical implementation, the interval $I_x (x = q_0, q_v)$ is partitioned into N_x subintervals, so that good convergence was reached. The adopted spline functions, $S_j(x)$ are given by [8]

$$S_{2i}(x) = \begin{cases} 3\left(\frac{x-x_{i-1}}{h_i}\right)^2 - 2\left(\frac{x-x_{i-1}}{h_i}\right)^3, \\ \text{if } x \in [x_{i-1}, x_i] \\ 3\left(\frac{x_{i+1}-x}{h_{i+1}}\right)^2 - 2\left(\frac{x_{i+1}-x}{h_{i+1}}\right)^3, \\ \text{if } x \in [x_i, x_{i+1}] \\ 0, \quad \text{if } x \notin [x_{i-1}, x_{i+1}] \end{cases}$$

$$S_{2i+1}(x) = \begin{cases} \left[-\left(\frac{x-x_{i-1}}{h_i}\right)^2 + \left(\frac{x-x_{i-1}}{h_i}\right)^3\right] h_i, \\ \text{if } x \in [x_{i-1}, x_i] \\ \left[\left(\frac{x_{i+1}-x}{h_{i+1}}\right)^2 - \left(\frac{x_{i+1}-x}{h_{i+1}}\right)^3\right] h_{i+1}, \\ \text{if } x \in [x_i, x_{i+1}] \\ 0, \quad \text{if } x \notin [x_{i-1}, x_{i+1}] \end{cases} \tag{C2}$$

with $h_i = x_i - x_{i-1}$.

By using (C1), Eq. (41) can be transformed to a generalized eigenvalue problem of the form

$$\sum_{i'j'} F_{iji'j'} A_{i'j'} = \lambda(M_3) \sum_{i'j'} V_{iji'j'} A_{i'j'}, \tag{C3}$$

where

$$F_{iji'j'} = S_{i'}(q_0^{(i)}) S_{j'}(q_v^{(j)}), \tag{C4}$$

and the array $V_{iji'j'}$ is the right-hand side of (41) with v_M replaced by $S_{i'}(q_0^{(i)}) S_{j'}(q_v^{(j)})$. The variable q_0 (q_v) has here been discretized on a mesh consisting of $2N_{q_0} + 2$ ($2N_{q_v} + 2$) points. The three-body mass M_3, or equivalently the three-body binding energy B_3, can subsequently be obtained from the condition

$$\lambda(M_3) = 1. \tag{C5}$$

Equation (C5) constitutes a nonlinear equation relative to M_3 and is rather time-consuming to solve. For simplicity, we use thus instead as inputs in the calculations the scattering length a and the M_3, obtained from the solution of the Euclidean BS equation. Eq. (C3) is then solved for the eigenvalue λ and the coefficients A_{ij}.

The kernel $\Pi(q_0, q_v, k_0, k_v)$ [see Eq. (38)], which enters Eq. (41) has logarithmic singularities, and the analytic expressions for the singular points are given by Eqs. (42)

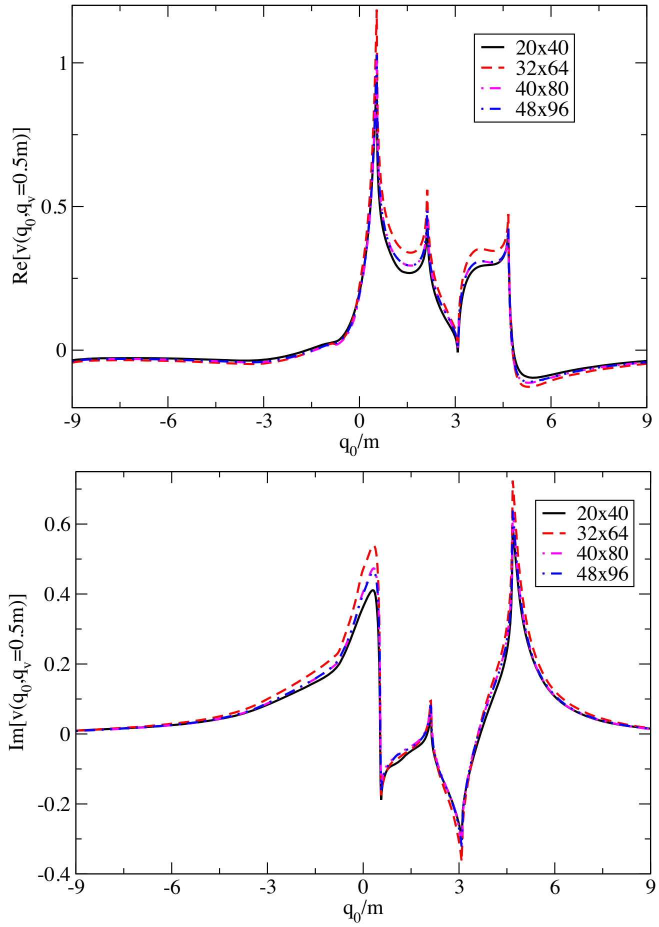

FIG. 9. Convergence of the real (left panel) and imaginary (right panel) parts of the vertex function $v_M(q_0, q_v = 0.5m)$ with respect to the size of the basis, $N_{q_v} \times N_{q_0}$. In the calculations we used $B_3/m = 0.395$.

and (43). In the present work, the integrals over k_0 and k_v are computed by dividing a given integration interval into subintervals $I_i = [a_i, b_i]$, so that each subinterval contains at most one singular point which is just one of the end points of the subinterval. For each subinterval, the integrand singularity is subsequently weakened by adopting a change of variables of the form

$$\int_{a_i}^{b_i} f(x)dx = \int_0^{\sqrt{b_i - a_i}} 2tf(a_i + t^2)dt, \qquad \text{(C6)}$$

for a subinterval with a singularity at a_i, and

$$\int_{a_i}^{b_i} f(x)dx = \int_0^{\sqrt{b_i - a_i}} 2tf(b_i - t^2)dt, \qquad \text{(C7)}$$

if the singularity is at the end point b_i. The resulting integrals involving smooth functions can then be performed by Gauss-Legendre integration.

1. Numerical convergence

As mentioned, in this work the three-body BS equation is solved by using an expansion of the amplitude $v_M(q_0, q_v)$ in terms of a finite number of spline functions. Evidently, it is important to check that the adopted number basis functions is enough.

For this purpose, we show in Fig. 9 the real and imaginary parts of $v_M(q_0, q_v = 0.5m)$, computed by using different number of subintervals N_{q_v} and N_{q_0}, corresponding to the variables q_v and q_0. In the calculations we used the parameters $am = -1.5$ and $B_3/m = 0.395$. It is seen in the figure that for $N_{q_v} \geq 40$ and $N_{q_0} \geq 80$, the solution is well converged.

2. Behavior of $\mathcal{F}(M_{12}^2)$

For negative a, the function $\mathcal{F}(M_{12}^2)$ is nonsingular and continuous. However, the function may change rapidly in the neighborhood of the transition points $M_{12}^2 = 0$ and $M_{12}^2 = 4m^2$. In terms of q_0 (for a given q_v) these are

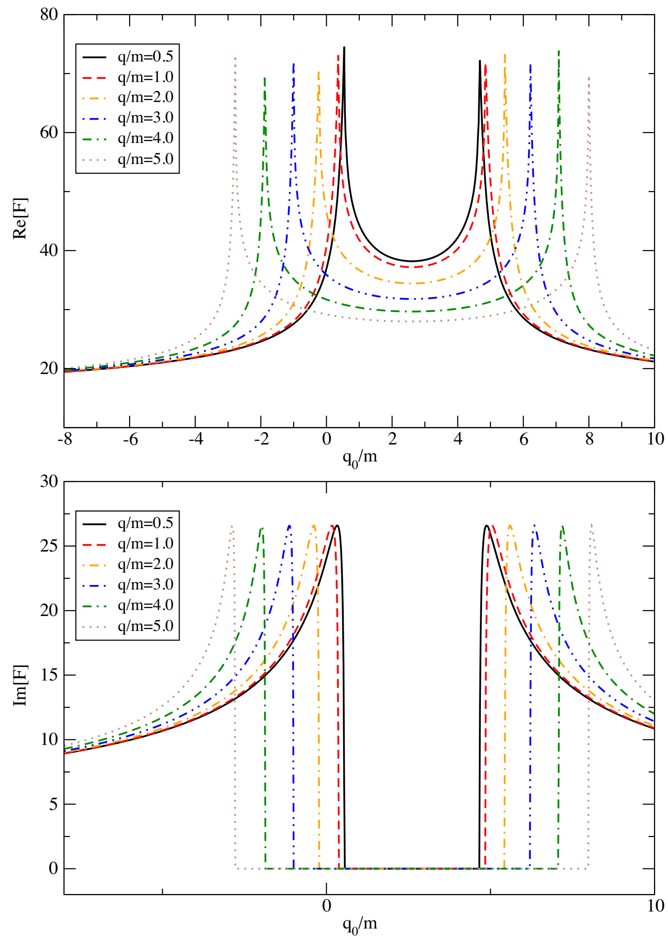

FIG. 10. Real and imaginary parts of $\mathcal{F}(M_{12}^2)$ with respect to q_0 for different fixed values of q_v.

$$q_0 = M_3 \pm q_v, \qquad q_0 = M_3 \pm \sqrt{q_v^2 + 4m^2}. \tag{C8}$$

In the Fig. 10 the real and imaginary parts of $\mathcal{F}$ are shown as functions of q_0 (for selected values of q_v) in the case of $M_3/m = 2.605$ corresponding to $am = -1.5$. It is seen in the figures that close to $q_0 = M_3 \pm \sqrt{q_v^2 + 4m^2}$ (i.e., $M_{12}^2 = 4m^2$), the amplitude has a nonsmooth behavior.

Although the nonsmoothness exists, this was shown to not be problematic in solving the equation. To show that, we tested solving the problem proposing a factorization of the form

$$v_M(q_0, q_v) = \mathcal{F}(M_{12}^2(q_0, q_v))\tilde{v}_M(q_0, q_v), \tag{C9}$$

by introducing

$$\tilde{\Pi}(q_0, q_v; k_0, k_v) = \mathcal{F}(M_{12}^2(q_0, q_v))\Pi(q_0, q_v; k_0, k_v), \tag{C10}$$

and obtaining an integral equation in terms of the function $\tilde{v}_M$ instead of v_M. The resulting equation was solved by expanding $\tilde{v}_M$ in splines. The result showed no significant difference between the solutions with and without the decomposition, with the convergence being achieved with a similar set of basis functions and integration points.

[1] T. Frederico, Phys. Lett. B **282**, 409 (1992).
[2] E. Ydrefors, J. H. Alvarenga Nogueira, V. Gigante, T. Frederico, and V. A. Karmanov, Phys. Lett. B **770**, 131 (2017).
[3] E. Ydrefors, J. H. Alvarenga Nogueira, V. A. Karmanov, and T. Frederico, Phys. Lett. B **791**, 276 (2019).
[4] J. Carbonell and V. A. Karmanov, Phys. Rev. D **90**, 056002 (2014).
[5] N. Nakanishi, Phys. Rev. **130**, 1230 (1963).
[6] N. Nakanishi, Prog. Theor. Phys. Suppl. **43**, 1 (1969).
[7] J. Carbonell and V. A. Karmanov, Phys. Rev. C **67**, 037001 (2003).
[8] J. Carbonell and V. A. Karmanov, Few Body Syst. **49**, 205 (2011).
[9] T. Frederico, G. Salmè, and M. Viviani, Phys. Rev. D **89**, 016010 (2014).
[10] C.-R. Ji and Y. Tokunaga, Phys. Rev. D **86**, 054011 (2012).
[11] V. A. Karmanov and P. Maris, Few Body Syst. **46**, 95 (2009).
[12] G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, and C. S. Fischer, Prog. Part. Nucl. Phys. **91**, 1 (2016).
[13] D. S. Hwang and V. A. Karmanov, Nucl. Phys. **B696**, 413 (2004).
[14] W. de Paula, T. Frederico, G. Salmè, and M. Viviani, Phys. Rev. D **94**, 071901 (2016); W. de Paula, T. Frederico, G. Salmè, M. Viviani, and R. Pimentel, Eur. Phys. J. C **77**, 764 (2017).
[15] J. H. Alvarenga Nogueira, T. Frederico, and O. Lourenço, Few Body Syst. **58**, 98 (2017).
[16] K. S. F. F. Guimarães, O. Lourenço, W. de Paula, T. Frederico, and A. C. dos Reis, J. High Energy Phys. 08 (2014) 135.
[17] M. Mangin-Brinet and J. Carbonell, Phys. Lett. B **474**, 237 (2000).
[18] L. H. Thomas, Phys. Rev. **47**, 903 (1935).
[19] T. Frederico, A. Delfino, L. Tomio, and M. T. Yamashita, Prog. Part. Nucl. Phys. **67**, 939 (2012).
[20] V. Efimov, Phys. Lett. B **33**, 563 (1970); Nucl. Phys. **A362**, 45 (1981).
[21] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (Dover Publications, New York, 1980).
[22] J. Carbonell and V. A. Karmanov, Phys. Rev. D **91**, 076010 (2015).
[23] J. Carbonell, B. Desplanques, V. A. Karmanov, and J.-F. Mathiot, Phys. Rep. **300**, 215 (1998).
[24] V. A. Karmanov and J. Carbonell, Eur. Phys. J. A **27**, 1 (2006).
[25] G. V. Skorniakov and K. A. Ter-Martirosyan, Sov. Phys. JETP **4**, 648 (1957).
[26] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).
[27] S. J. Brodsky, H.-C. Pauli, and S. S. Pinsky, Phys. Rep. **301**, 299 (1998).
[28] C. Gutierrez, V. Gigante, T. Frederico, G. Salmè, M. Viviani, and L. Tomio, Phys. Lett. B **759**, 131 (2016).
[29] O. Hen, G. A. Miller, E. Piasetzky, and L. B. Weinstein, Rev. Mod. Phys. **89**, 045002 (2017).
[30] A. Bashir, L. Chang, I. C. Cloet, B. El-Bennich, Y. X. Liu, C. D. Roberts, and P. C. Tandy, Commun. Theor. Phys. **58**, 79 (2012).