Dirac sea and chiral anomaly in the quantum kinetic theory

Jian-Hua Gao,[1] Zuo-Tang Liang,[2] and Qun Wang[3]

[1]*Shandong Provincial Key Laboratory of Optical Astronomy and Solar-Terrestrial Environment, Institute of Space Sciences, Shandong University, Weihai, Shandong 264209, China*

[2]*Key Laboratory of Particle Physics and Particle Irradiation (MOE), Institute of Frontier and Interdisciplinary Science, Shandong University, Qingdao, Shandong 266237, China*

[3]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

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We revisit the chiral anomaly in the quantum kinetic theory in the Wigner function formalism under the background field approximation. Our results show that the chiral anomaly is actually from the Dirac sea or the vacuum contribution in the un-normal-ordered Wigner function. We also demonstrate that this contribution modifies the chiral kinetic equation for antiparticles.

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I. INTRODUCTION

The chiral anomaly is a novel and prominent quantum effect in particle physics and can only be understood at the quantum field level. The kinetic theory is a bridge to connect the macroscopic physical magnitude to the microscopic particle motion in classical phase space. Hence, it is a highly nontrivial task to incorporate the chiral anomaly into the kinetic approach in a consistent way so that the chiral kinetic theory is properly formulated. In recent years, a considerable amount of work on the chiral kinetic theory has been published where the chiral kinetic equation has been derived from various methods such as the semiclassical approach [1–7], the Wigner function formalism [8–13], the effective field theory [14–17], and the world-line approach [18–20].

In these publications, most works connect the chiral anomaly in the chiral kinetic theory with Berry's phase or Berry's curvature. In contrast, it has also been pointed out by Fujikawa and co-worker [21–23] that topological effects due to Berry's phase and the chiral anomaly are basically different from each other. The Berry phase arises only in the adiabatic limit, while the chiral anomaly is generic and independent of kinematic limits. This distinct difference has been further demonstrated by Mueller and Venugopalan in Refs. [18,19] where they found that the Berry phase arises from the real part of the world-line effective action in a particular adiabatic limit, while the chiral anomaly is from the imaginary part. In Ref. [24], Hidaka *et al.* performed the derivation of the chiral anomaly by using the nontrivial boundary condition of a distribution function. Recently, in our paper with other collaborators [12], we found that some singular boundary terms also result in a new source term contributing to the chiral anomaly, in contrast to the well-known scenario of the Berry phase term.

It is clear that such a controversial situation needs to be further clarified with a more fundamental approach. In a very recent paper by Yee and Yi [25], the authors try to clarify the relationship between the chiral anomaly and the Chern number of the Berry connection via the conventional Feynman diagram. In this paper, we try to clarify this situation from the quantum transport theory based on Wigner functions [26–29], a first principles approach from quantum field theory. To do this, we will start with the basic Wigner equations and derive the chiral anomaly step by step so that one can see clearly where the contribution comes from. The results obtained show that the Dirac sea or vacuum contribution that originated from the anticommutation relations between antiparticle field operators in the un-normal-ordered Wigner function cannot be dropped casually. This unique term that comes directly from the quantum field theory plays a central role in generating the right chiral anomaly in quantum kinetic theory both for massive and massless fermion systems. The coefficient of the chiral anomaly derived this way is universal and is independent of the phase space distribution function at zero momentum. We will also present the updated chiral kinetic equation for antiparticles with Dirac sea contribution.

The rest of the paper is organized as follows: In Sec. II, we give a very brief review of the Wigner function formalism of the relativistic quantum kinetic theory and present especially those equations that will be used in deriving the chiral anomaly. In Sec. III, we derive the chiral anomaly from Wigner equations and show in particular that it is generated from the Dirac sea contribution. In Sec. IV, we exhibit how the chiral kinetic equation is derived from Wigner function formalism at the chiral limit and give the updated chiral kinetic equation for antiparticles with Dirac sea contribution. Lastly, we summarize the paper in Sec. V.

II. THE WIGNER FUNCTION FORMALISM

We recall that the quantum kinetic theory based on Wigner functions is a first principles approach from quantum field theory. Here, the Wigner matrix $W(x,p)$ is the basic unit, and for spin-1/2 fermions, it is defined as the ensemble average of gauge invariant nonlocal bilinear Dirac spinor field,

$$W_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \langle \bar{\psi}_\beta(x_+) U(x_+, x_-) \psi_\alpha(x_-) \rangle, \quad (1)$$

where $x_\pm \equiv x \pm y/2$ are two space-time points centered at x with separation y, and U denotes the gauge link along the straight line between x_+ and x_-,

$$U(x_+, x_-) \equiv e^{-i\int_{x_-}^{x_+} dz^\mu A_\mu(z)}. \quad (2)$$

Here we did not define the ensemble average with normal ordering for the Dirac fields because the Wigner equation (3) below derived from the Dirac equation must be satisfied by the Wigner function without normal ordering instead of with normal ordering [30]. In addition, we did not include the path ordering in the definition of the gauge link above since we restrict ourselves to the background field approximation in this work. The electric charge has been absorbed into the gauge potential A_μ for brevity. Under the background field approximation, we obtain the equation satisfied by the Wigner matrix from the Dirac equation as given by [28]

$$\left[\gamma_\mu\left(\Pi^\mu + \frac{i}{2}G^\mu\right) - m\right] W(x,p) = 0, \quad (3)$$

where γ^μ's are Dirac matrices and m is the particle's mass. The operators G^μ and Π^μ denote the nonlocal generalizations of the space-time derivative ∂_x^μ and the momentum p^μ,

$$G^\mu \equiv \partial_x^\mu - j_0\left(\frac{1}{2}\hbar\Delta\right) F^{\mu\nu}\partial_\nu^p, \quad (4)$$

$$\Pi^\mu \equiv p^\mu - \frac{1}{2}\hbar j_1\left(\frac{1}{2}\hbar\Delta\right) F^{\mu\nu}\partial_\nu^p, \quad (5)$$

where j_0 and j_1 are the zeroth and first order spherical Bessel functions, respectively. The triangle operator is defined as $\Delta \equiv \partial^p \cdot \partial_x$, in which ∂_x acts only on the field strength tensor $F^{\mu\nu}$ but not on the Wigner function.

We emphasize once more that in the definition of the Wigner function given by Eq. (1) and the Wigner equation given by Eq. (3), there is no normal ordering in the Wigner matrix. We will illustrate that this point plays a central role in giving rise to the chiral anomaly in the quantum kinetic theory. The Hamiltonian derivation of the chiral anomaly from the physically correct normal ordering can be found in [31].

The Wigner equation given by Eq. (3) is a matrix equation that actually includes 32 equations for 16 independent components in the Wigner matrix. These 16 components are classified as scalar $\mathscr{F}$, pseudoscalar $\mathscr{P}$, vector $\mathscr{V}_\mu$, axial-vector $\mathscr{A}_\mu$, and tensor $\mathscr{S}_{\mu\nu}$ components according to the Lorentz transformation. They are all real functions of x and p defined by the Γ-matrix expansion of $W(x,p)$, i.e.,

$$W = \frac{1}{4}\left(\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right). \quad (6)$$

The chiral anomaly is a quantum effect that can be studied using the semiclassical expansion in terms of $\hbar^n$. It has been shown that [32], for massive particles, up to the first order of $\hbar$ in the expansion, we can choose $\mathscr{F}$ and $\mathscr{A}_\mu$ as independent dynamical Wigner functions and sort the 32 Wigner equations as follows: Eleven of them provide explicit expressions of other Wigner functions in terms of $\mathscr{F}$ and $\mathscr{A}_\mu$,

$$\mathscr{P} = -\frac{\hbar}{2m}\nabla^\mu\mathscr{A}_\mu, \quad (7)$$

$$\mathscr{V}_\mu = \frac{p_\mu}{m}\mathscr{F} - \frac{\hbar}{2m^2}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu p^\rho\mathscr{A}^\sigma, \quad (8)$$

$$\mathscr{S}_{\mu\nu} = -\frac{1}{m}\epsilon_{\mu\nu\rho\sigma}p^\rho\mathscr{A}^\sigma + \frac{\hbar}{2m^2}(\nabla_\mu p_\nu - \nabla_\nu p_\mu)\mathscr{F}, \quad (9)$$

five of them give transport equations,

$$p^\mu\nabla_\mu\mathscr{F} = \frac{1}{2m}p^\mu\Delta\tilde{F}_{\mu\nu}\mathscr{A}^\nu, \quad (10)$$

$$p^\nu\nabla_\nu\mathscr{A}_\mu = F_{\mu\nu}\mathscr{A}^\nu + \frac{1}{2m}p^\nu\Delta\tilde{F}_{\mu\nu}\mathscr{F}, \quad (11)$$

where $\nabla^\mu \equiv \partial_x^\mu - F^{\mu\nu}\partial_\nu^p$, $\epsilon^{0123} = 1$, and $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$, and another five provide mass shell equations with the following general solutions:

$$\mathscr{F} = \delta(p^2 - m^2)\mathcal{F} + \frac{\hbar}{m}\tilde{F}_{\mu\nu}p^\mu\mathcal{A}^\nu\delta'(p^2 - m^2), \quad (12)$$

$$\mathscr{A}_\mu = \delta(p^2 - m^2)\mathcal{A}_\mu + \frac{\hbar}{m}\tilde{F}_{\mu\nu}p^\nu\mathcal{F}\delta'(p^2 - m^2), \quad (13)$$

where $\mathcal{F}$ and $\mathcal{A}_\mu$ are arbitrary functions that are nonsingular for on-shell momentum $p^2 = m^2$. They can be taken as the fundamental functions replacing $\mathscr{F}$ and $\mathscr{A}_\mu$ in practice. Finally, there is another constraint equation for the axial vector component $\mathscr{A}_\mu$,

$$p^\mu \mathscr{A}_\mu = 0, \tag{14}$$

and the other ten Wigner equations are satisfied automatically, and thus are not needed to consider in practice.

We will now use these results to derive the chiral anomaly in the following. Other work on quantum kinetic theory for the massive fermion can be found in Refs. [33–38].

III. THE CHIRAL ANOMALY

Using the results presented in Sec. II, we can calculate the chiral anomaly to the first order in $\hbar$. In this section, we present the calculations step by step. To do this, we start with the expression of $\mathcal{P}$ given by Eq. (7). It should be mentioned that this expression can actually hold up to the second order of $\hbar$. By inserting the general solution of the mass shell equation for $\mathcal{A}_\mu$ given by Eq. (13) into Eq. (7) and integrating over the four-momentum p, we obtain the divergence of the axial current j^5_μ as

$$\hbar \partial^\mu_x j^5_\mu = -2m j_5 + \hbar X - \frac{\hbar^2}{8\pi^2} C F^{\mu\nu} \tilde{F}_{\mu\nu}, \tag{15}$$

where $j^5_\mu = \int d^4p \mathscr{A}_\mu$, $j_5 = \int d^4p \mathscr{P}$, and

$$X = F^{\mu\lambda} \int d^4p \partial^p_\lambda [\mathcal{A}_\mu \delta(p^2 - m^2)], \tag{16}$$

$$C = -\frac{2\pi^2}{m} \int d^4p \partial^\lambda_p [p_\lambda \mathcal{F} \delta'(p^2 - m^2)]. \tag{17}$$

We note that though there is an m in the denominator in Eqs. (7)–(11), the expression of Eqs. (15) and (16) is finite in the chiral limit $m = 0$ because $\mathcal{F} \propto m$. The result after taking the massless limit is consistent with the one directly from the chiral kinetic theory [8,9,12]. Therefore, Eq. (15) actually holds for both massive and massless fermions.

We now calculate these two coefficients X and C carefully to find the source of the chiral anomaly by using the solution of $\mathcal{A}_\mu$ and $\mathcal{F}$. First of all, it is easy to verify that X vanishes if $\mathcal{A}_\mu$ approaches zero rapidly at infinity in momentum space. Such a result is obvious since the chiral anomaly is a quantum effect and should not exist in the classical limit. This boundary condition seems reasonable for $\mathcal{A}_\mu$ from the result at zeroth order that can be calculated directly from the free quantum field theory, i.e.,

$$\mathcal{A}_\mu = \begin{cases} \frac{m}{4\pi^3} s_\mu (f^+ - f^-), & p_0 > 0, \\ \frac{m}{4\pi^3} s_\mu (\bar{f}^+ - \bar{f}^-), & p_0 < 0, \end{cases} \tag{18}$$

where s_μ is the spin polarization vector with $s^2 = -1$ and $s \cdot p = 0$, and $f^\pm$ and $\bar{f}^\pm$ are number densities in the phase space for particles and antiparticles with spins $\pm s_\mu$, respectively. They are defined as the ensemble averages of the normal-ordered number density operators and are expected to vanish at infinity in phase space. For $\mathcal{A}_\mu$, there is no nontrivial Dirac sea or vacuum contribution because for a free gas of fermions the energy levels are all degenerate with different spins, and the vacuum terms from different spins cancel each other. However, for fermions in strong magnetic field such as that discussed in [39], the ground states are nondegenerate with one specific spin. There is no cancellation and the vacuum term exists.

The situation is however different for the coefficient C where the solution of the scalar function $\mathcal{F}$ is needed. Here, at the same level as $\mathcal{A}_\mu$ given by Eq. (18), we have the result from the free quantum field theory as

$$\mathcal{F} = \begin{cases} \frac{m}{4\pi^3} (f^+ + f^-), & p_0 > 0, \\ \frac{m}{4\pi^3} (\bar{f}^+ + \bar{f}^- + 2\bar{f}_v), & p_0 < 0. \end{cases} \tag{19}$$

We see that, in contrast to the axial vector $\mathcal{A}_\mu$, there exists a Dirac sea or vacuum contribution $\bar{f}_v = -1$ to the scalar function $\mathcal{F}$ [39,40]. This contribution originates from the anticommutator of the antiparticle field in the definition of the Wigner function given by Eq. (1) without normal ordering. It takes the value $\bar{f}_v = -1$ that is universal and does not depend on the state of the system that we are considering.

Now we show how the universal $\bar{f}_v = -1$ gives rise to the universal coefficient C of the chiral anomaly in front of $F^{\mu\nu}\tilde{F}_{\mu\nu}$ in Eq. (15). We consider in Eq. (19) only this universal Dirac sea contribution $\bar{f}_v = -1$ and substitute it into Eq. (16) and obtain

$$\begin{aligned} C_v = &\int \frac{d^4p}{\pi} \partial_{p_0} \left[\frac{1}{4E_p} \delta'(p_0 + E_p)\right] \\ &- \int \frac{d^4p}{\pi} \partial_{\mathbf{p}} \cdot \left[\mathbf{p} \frac{1}{4E_p^2} \delta'(p_0 + E_p)\right] \\ &+ \int \frac{d^4p}{\pi} \partial_{\mathbf{p}} \cdot \left[\mathbf{p} \frac{1}{4E_p^3} \delta(p_0 + E_p)\right], \end{aligned} \tag{20}$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$, and we use the subscript "v" to denote the Dirac sea contribution only. It is easy to verify that the first and second terms vanish by direct calculation, while the last term survives and gives

$$C_{\mathrm{v}} = \int \frac{d^3\mathbf{p}}{2\pi} \partial_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{2E_p^3} \right) = 1. \tag{21}$$

This is just the right coefficient of the chiral anomaly. It is also obvious that this derivation does not depend on whether the system is for massive or massless fermions. At the chiral limit $m = 0$ and $E_p = |\mathbf{p}|$; hence, $\mathbf{p}/(2E_p^3)$ in the bracket in Eq. (21) is just the usual Berry curvature $\mathbf{\Omega} = \mathbf{p}/(2|\mathbf{p}|^3)$ with $\nabla \cdot \mathbf{\Omega} = 2\pi$, and this exhibits how the chiral anomaly comes from the Berry curvature. As we all know, we can carry out the integration above either directly in the three-dimensional momentum volume where the Berry monopole appears and only the infrared momentum contributes, or in the two-dimensional momentum boundary area by using Gauss's theorem in which only the ultraviolet momentum contributes. It should be noted that when we calculate the integral by using Gauss's theorem, we have to introduce an ultraviolet cutoff for momentum to regularize the sphere at infinity. After taking the limit of this cutoff at infinity, the integral is finite and independent on the cutoff. This illustrates how the ultraviolet and infrared regions in momentum space are connected. However, for the massive particle there is no infrared singularity or Berry monopole. When the Dirac sea contribution is included, the boundary conditions of distribution functions at infinity of momentum space for the particle and antiparticle are different and this will be compatible with the treatment given in [24].

Now we analyze the normal contributions associated with $f^{\pm}$ and $\bar{f}^{\pm}$. Totally, they give no contribution to the chiral anomaly because usually the normal distribution functions $f^{\pm}$ and $\bar{f}^{\pm}$ are all supposed to vanish rapidly at infinity in phase space so that they lead to no contribution after the integration in Eq. (16). The details of the calculations are given in the Appendix.

We note that although we relate the chiral anomaly with the Berry curvature through Eq. (21) at the chiral limit, it is different from the result given in Refs. [4,7,9] where the coefficient of the chiral anomaly is generated from

$$C = -\int \frac{d^3\mathbf{p}}{2\pi} \mathbf{\Omega} \cdot \partial_{\mathbf{p}} (f_{\mathbf{p}} + \bar{f}_{\mathbf{p}}). \tag{22}$$

After integrating by parts and using $\nabla \cdot \mathbf{\Omega} = 2\pi$, we obtain

$$C = f_{\mathbf{p}=0} + \bar{f}_{\mathbf{p}=0}. \tag{23}$$

Hence, it depends on the specific distribution function at infrared momentum. It is interesting that with the Fermi-Dirac distribution,

$$f^{\pm} = \frac{1}{e^{(|\mathbf{p}|-\mu_{\pm})/T} + 1}, \tag{24}$$

$$\bar{f}^{\pm} = \frac{1}{e^{(|\mathbf{p}|+\mu_{\pm})/T} + 1}, \tag{25}$$

Eq. (23) gives the right coefficient of the chiral anomaly. In our approach, a similar term to Eq. (22) indeed exists as well and is included in C_{n} where the subscript "n" indicates that only the normal distribution functions $f^{\pm}$ and $\bar{f}^{\pm}$ are involved; see the definition in Eq. (A2) in the Appendix. However, they always appear in the divergence form [see the first term $C_{\mathrm{n}}^{(1)}$ of C_{n} given by Eq. (A3)]

$$\begin{aligned} C_{\mathrm{n}}^{(1)} &= -\int \frac{d^3\mathbf{p}}{4\pi} \partial_{\mathbf{p}} \cdot \left[\frac{\mathbf{p}}{E_p^3} (f_{\mathbf{p}} + \bar{f}_{\mathbf{p}}) \right] \\ &= -\int \frac{d^3\mathbf{p}}{4\pi} \frac{\mathbf{p}}{E_p^3} \cdot \partial_{\mathbf{p}} (f_{\mathbf{p}} + \bar{f}_{\mathbf{p}}) \\ &\quad - \int \frac{d^3\mathbf{p}}{4\pi} (f_{\mathbf{p}} + \bar{f}_{\mathbf{p}}) \partial_{\mathbf{p}} \cdot \left(\frac{\mathbf{p}}{E_p^3} \right), \end{aligned} \tag{26}$$

where $f_{\mathbf{p}}$ and $\bar{f}_{\mathbf{p}}$ in our approach are defined by Eq. (A8). We see that $C_{\mathrm{n}}^{(1)}$ is divided into two terms, and the sum of them vanishes for $f_{\mathbf{p}}$ and $\bar{f}_{\mathbf{p}}$ that go to zero at infinity of momentum. At chiral limit $m = 0$ and with specific Fermi-Dirac distribution, the first term in $C_{\mathrm{n}}^{(1)}$ reproduces the result (23) and gives the coefficient of the chiral anomaly. However, the second term always cancels this term and the total contribution must vanish. Hence, in such a very special case, one might regard the Berry curvature as the effective source for the chiral anomaly because the Dirac sea contribution and the second term in Eq. (26) happen to cancel each other. However, such a case is a coincidence since it holds only for the chiral system with a very specific distribution constraint $f_{\mathbf{p}} + \bar{f}_{\mathbf{p}} = 1$ at the zero momentum point. It does not hold for massive fermions at all. In general, the Berry curvature contribution to the chiral anomaly is always canceled by the second term in Eq. (26), and the real contribution to the chiral anomaly comes actually from Dirac sea C_{v}.

Now let us consider another interesting term—the last term $C_{\mathrm{n}}^{(3)}$ in Eq. (A2). At the chiral limit and for the isotropic distribution function in momentum space, we have

$$C_{\mathrm{n}}^{(3)} = (f_{p_0=0} + \bar{f}_{p_0=0}) - (f_{p_0=0} + \bar{f}_{p_0=0}). \tag{27}$$

Once more, if we choose the Fermi-Dirac distribution, the first term gives the right coefficient "1" of the chiral anomaly, though it is always canceled by the second term.

To summarize, we see that for a massless Fermion with special distribution functions $f_{\mathbf{p}} + \bar{f}_{\mathbf{p}} = 1$ and $f_{p_0=0} + \bar{f}_{p_0=0} = 1$, we can rewrite Eq. (15) as

$$\begin{aligned} \partial_x^{\mu} j_{\mu}^{5(1)} &= -\frac{1}{8\pi^2} [C_{\mathrm{n}}^{(1)} + C_{\mathrm{n}}^{(3)} + C_{\mathrm{v}}] F^{\mu\nu} \tilde{F}_{\mu\nu} \\ &= -\frac{1}{8\pi^2} [1 - 1 + 1 - 1 + 1] F^{\mu\nu} \tilde{F}_{\mu\nu} \\ &= -\frac{1}{8\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu}. \end{aligned} \tag{28}$$

In our previous work in [12], the first, third, and fourth terms were obtained while the second term and last term were missing. Some other work kept only the third term corresponding to the Berry curvature term.

As we all know, the vector current must be conserved and does not have anomalous terms. At the end of this section, it would be valuable to verify this point as well. Making the time-space divergence of both sides and integrating over the 4-momentum, we obtain

$$\partial_x^\mu j_\mu = \int d^4p \partial_x^\mu \mathscr{V}_\mu = \int d^4p \left\{ \frac{p_\mu}{m} \partial_x^\mu \mathscr{F} + \frac{\hbar}{2m^2} \epsilon_{\mu\nu\rho\sigma} \partial_x^\mu [F^{\nu\lambda} \partial_\lambda^p (p^\rho \mathscr{A}^\sigma)] \right\}. \tag{29}$$

The last term must vanish because it is a total derivative term on momentum and there is no vacuum contribution for $\mathscr{A}^\sigma$. After dropping this term and using Eq. (10), we have

$$\partial_x^\mu j_\mu = \int d^4p \left[\frac{1}{m} F_{\mu\nu} \partial_p^\nu (p^\mu \mathscr{F}) + \frac{\hbar}{2m^2} \Delta \tilde{F}_{\mu\nu} (p^\mu \mathscr{A}^\nu) \right], \tag{30}$$

where we have used Maxwell's equation $\partial_x^\mu \tilde{F}_{\mu\nu} = 0$ when moving p^μ after the derivative operator Δ. Although the two terms above are both total derivatives, the first term could contribute from the vacuum term. We only keep the vacuum contribution and obtain

$$\begin{aligned} \partial_x^\mu j_\mu &= -\frac{1}{2\pi^3} F_{\mu\nu} \int d^4p \partial_p^\nu [p^\mu \delta(p^2 - m^2) \theta(-p_0)] \\ &= -\frac{1}{4\pi^3} F_{\mu\nu} \int d^4p \partial_p^\nu [p^\mu \delta(p^2 - m^2)] \\ &= 0. \end{aligned} \tag{31}$$

Hence, the Dirac sea or vacuum contribution does not influence the conservation law for electric charge as it should.

IV. THE CHIRAL KINETIC EQUATION

In this section, we show that the Dirac sea contribution modifies the chiral kinetic equation for the antiparticle. We now first recall how we obtain the chiral kinetic equation from the Wigner function approach. In the chiral limit, it is convenient to define the helicity basis,

$$\mathscr{J}_s^\mu = \frac{1}{2} (\mathscr{V}^\mu + s \mathscr{A}^\mu), \tag{32}$$

where $s = \pm$ is the chirality. In this chiral limit, the helicity bases are completely decoupled from each other and from all the other Wigner functions as well,

$$\begin{aligned} p^\mu \mathscr{J}_\mu &= 0, \\ \nabla^\mu \mathscr{J}_\mu &= 0, \\ 2s(p_\mu \mathscr{J}_\nu - p_\nu \mathscr{J}_\mu) &= -\hbar \epsilon_{\mu\nu\rho\sigma} \nabla^\rho \mathscr{J}^\sigma, \end{aligned} \tag{33}$$

where we have suppressed the lower index "s" for brevity and only kept the contribution up to the first order of $\hbar$.

As shown in [12], in the chiral limit, a disentanglement theorem of the Wigner function is valid. According to this theorem, only one of the four components of the Wigner function $\mathscr{J}_\mu$ is independent, and the other three can be determined completely from it. This independent Wigner function satisfies only one Wigner equation and the other equations are satisfied automatically. We especially have the freedom to choose which component is the independent one. In general, we can introduce a timelike 4-vector n^μ with normalization $n^2 = 1$ and choose $\mathscr{J}_n$ as the independent Wigner function. For simplicity, we assume n^μ is a constant vector. With the auxiliary vector n^μ, we can decompose any vector X^μ into the component parallel to n^μ and that perpendicular to n^μ,

$$X^\mu = X_n n^\mu + \bar{X}^\mu, \tag{34}$$

where $X_n = n \cdot X$ and $\bar{X} \cdot n = 0$. The electromagnetic tensor $F^{\mu\nu}$ can be decomposed into

$$F^{\mu\nu} = E^\mu n^\nu - E^\nu n^\mu + \epsilon^{\mu\nu\rho\sigma} n_\rho B_\sigma. \tag{35}$$

With $\mathscr{J}_n$ as the independent Wigner function, we obtain the other components

$$\bar{\mathscr{J}}_\mu = \bar{p}_\mu \frac{\mathscr{J}_n}{p_n} - \frac{s\hbar}{2p_n} \epsilon^{\mu\nu\rho\sigma} n_\nu \nabla_\sigma \left(p_\rho \frac{\mathscr{J}_n}{p_n} \right) \tag{36}$$

by contracting both sides of the last equation in Eq. (33) with n^ν. Substituting this relation into the first equation of Eq. (33), we obtain the general expression,

$$\frac{\mathscr{J}_n}{p_n} = f\delta(p^2) - \frac{s\hbar}{p_n} B \cdot p f \delta'(p^2). \tag{37}$$

It follows that,

$$\mathscr{J}_\mu \approx \left(g_{\mu\nu} + \frac{\hbar s}{2p_n} \epsilon_{\mu\nu\rho\sigma} n^\rho \nabla^\sigma \right) \times \left[p^\nu f \delta\left(p^2 - \hbar s \frac{B \cdot p}{p_n} \right) \right]. \tag{38}$$

Inserting this result into the second equation in Eq. (33), we obtain the covariant chiral kinetic equation up to the first order of $\hbar$ as

$$\nabla^\mu\left\{\left(g_{\mu\nu}+\frac{\hbar s}{2p_n}\epsilon_{\mu\nu\rho\sigma}n^\rho\nabla^\sigma\right)\times\left[p^\nu f\delta\left(p^2-\hbar s\frac{B\cdot p}{p_n}\right)\right]\right\}=0. \tag{39}$$

Integrating over p_n from 0 to $+\infty$ gives rise to the chiral kinetic equation for particles

$$\begin{aligned}&(1+s\hbar B\cdot\Omega)n\cdot\partial^x f_{\bar{p}}\\&\quad+[v^\mu+s\hbar(\hat{\bar{p}}\cdot\Omega)B^\mu+s\hbar\epsilon^{\mu\nu\rho\sigma}n_\rho E_\sigma\Omega_\nu]\bar{\partial}^x_\mu f_{\bar{p}}\\&\quad+(\tilde{E}^\mu+\epsilon^{\mu\nu\alpha\beta}v_\nu n_\alpha B_\beta+s\hbar E\cdot B\Omega^\mu)\bar{\partial}^p_\mu f_{\bar{p}}\\&\quad+s\hbar E\cdot B(\bar{\partial}^\mu_p\Omega_\mu)f_{\bar{p}}=0,\end{aligned} \tag{40}$$

where

$$f_{\bar{p}}=f(p_n=|\bar{p}|(1+\hbar sB\cdot\Omega)), \tag{41}$$

$$\Omega^\mu=\frac{\bar{p}^\mu}{2|\bar{p}|^3},\qquad |\bar{p}|=\sqrt{-\bar{p}^2},\qquad \hat{\bar{p}}_\mu=\frac{\bar{p}_\mu}{|\bar{p}|}, \tag{42}$$

$$v^\mu=\left(1+\frac{s\hbar B\cdot\bar{p}}{|\bar{p}|^3}\right)\hat{\bar{p}}^\mu+\frac{s\hbar B^\mu}{2|\bar{p}|^2}, \tag{43}$$

$$\tilde{E}^\mu=E^\mu-s\hbar|\bar{p}|(\bar{\partial}^\mu_x B^\lambda)\Omega_\lambda. \tag{44}$$

For finite momentum $\bar{p}^\mu$, the last term in Eq. (40) vanishes due to Berry's monopole in momentum space $\partial^p_\mu\Omega^\mu=2\pi\delta^3(\bar{p})$, and the chiral kinetic equation reduces to the usual form obtained in [4,7,9–12,14] and so on. However, once we leave the infrared region where the last term can be neglected, the place where the chiral anomaly comes from is also concealed, because this last term will always cancel the last term in the second line from below in Eq. (40) after integrating over the momentum.

By integrating over p_n from $-\infty$ to 0 and replacing $\bar{p}$ and s with $-\bar{p}$ and $-s$, respectively, we obtain the chiral kinetic equation for antiparticles,

$$\begin{aligned}&(1-s\hbar B\cdot\Omega)n\cdot\partial^x\bar{f}^{\rm t}_{\bar{p}}\\&\quad+[v^\mu-s\hbar(\hat{\bar{p}}\cdot\Omega)B^\mu-s\hbar\epsilon^{\mu\nu\rho\sigma}n_\rho E_\sigma\Omega_\nu]\bar{\partial}^x_\mu\bar{f}^{\rm t}_{\bar{p}}\\&\quad-(\tilde{E}^\mu+\epsilon^{\mu\nu\alpha\beta}v_\nu n^\alpha B^\beta-s\hbar E\cdot B\Omega_\mu)\bar{\partial}^\mu_p\bar{f}^{\rm t}_{\bar{p}}\\&\quad+s\hbar E\cdot B(\bar{\partial}^\mu_p\Omega_\mu)\bar{f}^{\rm t}_{\bar{p}}=0,\end{aligned} \tag{45}$$

where

$$\bar{f}^{\rm t}_{\bar{p}}=f(p_n=-|\bar{p}|(1-\hbar sB\cdot\Omega))|_{\bar{p}=-\bar{p},s=-s}, \tag{46}$$

$$v^\mu=\left[\left(1-\frac{s\hbar B\cdot\bar{p}}{|\bar{p}|^3}\right)\hat{\bar{p}}^\mu-\frac{s\hbar B^\mu}{2|\bar{p}|^2}\right]. \tag{47}$$

Here, $\bar{f}^t_{\bar{p}}=\bar{f}_{\bar{p}}+\bar{f}_{\rm v}$ denotes the total contribution of the normal $\bar{f}_{\bar{p}}$ and the vacuum distribution $\bar{f}_{\rm v}$. Using the free quantum Dirac field theory with $\bar{f}_{\rm v}=-1$, we obtain the kinetic equation of the normal distribution $\bar{f}_{\bar{p}}$ as

$$\begin{aligned}&(1-s\hbar B\cdot\Omega)n\cdot\partial^x\bar{f}_{\bar{p}}\\&\quad+[v^\mu-s\hbar(\hat{\bar{p}}\cdot\Omega)B^\mu-s\hbar\epsilon^{\mu\nu\rho\sigma}n_\rho E_\sigma\Omega_\nu]\bar{\partial}^x_\mu\bar{f}_{\bar{p}}\\&\quad-(\tilde{E}^\mu+\epsilon^{\mu\nu\alpha\beta}v_\nu n^\alpha B^\beta-s\hbar E\cdot B\Omega_\mu)\bar{\partial}^\mu_p\bar{f}_{\bar{p}}\\&\quad+s\hbar E\cdot B(\bar{\partial}^\mu_p\Omega_\mu)(\bar{f}_{\bar{p}}-1)=0.\end{aligned} \tag{48}$$

It should be noted that it is sufficient to use the result $\bar{f}_{\rm v}=-1$ from the free quantum field here in order to describe the chiral anomaly because the chiral anomaly term with $E\cdot B$ in Eq. (48) always arises at the first order of $\hbar$. The last term in Eq. (48) is different from that in Eq. (40) for the particle due to the Dirac sea contribution. There exists an inhomogeneous term $-s\hbar E\cdot B(\bar{\partial}^\mu_p\Omega_\mu)$ originated from the Dirac sea. As discussed in the last section, this term will eventually lead to the chiral anomaly. In the scenario of the Dirac sea, the difference between the particle and antiparticle is very natural.

V. SUMMARY

Starting from the quantum transport theory based on the quantum field theory, we find that the Wigner equation holds only for Wigner functions that are not normal ordered. It turns out that the Dirac sea or the vacuum contribution that originated from the anticommutation relation between antiparticle field operators in the un-normal-ordered Wigner function plays a central role in generating the right chiral anomaly for both massive fermions and massless fermions. Correspondingly, the chiral kinetic equation for antiparticles should include the contribution from the Dirac sea contribution.

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APPENDIX: INTEGRAL CALCULATION

In this appendix, we give the details on how to calculate the normal contribution from $f^\pm$ and $\bar{f}^\pm$ in Eq. (16).

Substituting the normal distribution function given by Eq. (19) into Eq. (16) and using the identities

$$\delta'(p^2-m^2) = \frac{1}{4p_0E_p}[\delta'(p_0-E_p)+\delta'(p_0+E_p)]$$
$$= \frac{1}{4E_p^3}[\delta(p_0-E_p)+\delta(p_0+E_p)]$$
$$+\frac{1}{4E_p^2}[\delta'(p_0-E_p)-\delta'(p_0+E_p)], \quad \text{(A1)}$$

we decompose Eq. (16) into the following three parts:

$$C_\mathrm{n} = C_\mathrm{n}^{(1)} + C_\mathrm{n}^{(2)} + C_\mathrm{n}^{(3)}, \quad \text{(A2)}$$

$$C_\mathrm{n}^{(1)} = -\int\frac{d^4p}{8\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^3}(f^+ + f^-)\delta(p_0-E_p)\right]$$
$$-\int\frac{d^4p}{8\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^3}(\bar{f}^+ + \bar{f}^-)\delta(p_0+E_p)\right], \quad \text{(A3)}$$

$$C_\mathrm{n}^{(2)} = -\int\frac{d^4p}{8\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^2}(f^+ + f^-)\delta'(p_0-E_p)\right]$$
$$+\int\frac{d^4p}{8\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^2}(\bar{f}^+ + \bar{f}^-)\delta'(p_0+E_p)\right], \quad \text{(A4)}$$

$$C_\mathrm{n}^{(3)} = -\int\frac{d^4p}{8\pi}\partial_{p_0}\left[\frac{1}{E_p}(f^+ + f^-)\delta'(p_0-E_p)\right]$$
$$-\int\frac{d^4p}{8\pi}\partial_{p_0}\left[\frac{1}{E_p}(\bar{f}^+ + \bar{f}^-)\delta'(p_0+E_p)\right]. \quad \text{(A5)}$$

For $C_\mathrm{n}^{(1)}$ and $C_\mathrm{n}^{(2)}$, we integrate p_0 over the delta function or derivative of delta function and obtain

$$C_\mathrm{n}^{(1)} = -\int\frac{d^3\mathbf{p}}{4\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^3}(f_\mathbf{p}+\bar{f}_\mathbf{p})\right], \quad \text{(A6)}$$

$$C_\mathrm{n}^{(2)} = -\int\frac{d^3\mathbf{p}}{4\pi}\partial_\mathbf{p}\cdot\left[\frac{\mathbf{p}}{E_p^2}(f'_\mathbf{p}+\bar{f}'_\mathbf{p})\right], \quad \text{(A7)}$$

where

$$f_\mathbf{p} = \frac{1}{2}(f^+ + f^-)|_{p_0=E_p},$$
$$\bar{f}_\mathbf{p} = \frac{1}{2}(\bar{f}^+ + \bar{f}^-)|_{p_0=-E_p},$$
$$f'_\mathbf{p} = -\frac{1}{2}\partial_{p_0}(f^+ + f^-)|_{p_0=E_p},$$
$$\bar{f}'_\mathbf{p} = \frac{1}{2}\partial_{p_0}(\bar{f}^+ + \bar{f}^-)|_{p_0=-E_p}. \quad \text{(A8)}$$

It is obvious that both $C_\mathrm{n}^{(1)}$ and $C_\mathrm{n}^{(2)}$ vanish for the normal distribution function that approaches zero rapidly at infinity in momentum space.

For $C_\mathrm{n}^{(3)}$, it is more convenient to integrate $|\mathbf{p}|$ over the derivative of delta function,

$$C_\mathrm{n}^{(3)} = -\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}^2\left[\sqrt{p_0^2-m^2}f_{p_0}\theta(p_0-m)\right]$$
$$-\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}^2\left[\sqrt{p_0^2-m^2}\bar{f}_{p_0}\theta(-p_0-m)\right]$$
$$+\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}\left[\sqrt{p_0^2-m^2}f'_{p_0}\theta(p_0-m)\right]$$
$$+\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}\left[\sqrt{p_0^2-m^2}\bar{f}'_{p_0}\theta(-p_0-m)\right], \quad \text{(A9)}$$

where

$$f_{p_0} = \frac{1}{2}(f^+ + f^-)|_{|\mathbf{p}|=\sqrt{p_0^2-m^2}}, \quad \text{(A10)}$$

$$\bar{f}_{p_0} = \frac{1}{2}(\bar{f}^+ + \bar{f}^-)|_{|\mathbf{p}|=\sqrt{p_0^2-m^2}}, \quad \text{(A11)}$$

$$f'_{p_0} = \frac{1}{2}\partial_{p_0}(f^+ + f^-)|_{|\mathbf{p}|=\sqrt{p_0^2-m^2}}, \quad \text{(A12)}$$

$$\bar{f}'_{p_0} = \frac{1}{2}\partial_{p_0}(\bar{f}^+ + \bar{f}^-)|_{|\mathbf{p}|=\sqrt{p_0^2-m^2}}, \quad \text{(A13)}$$

and $\theta(\pm p_0 - m)$ is the step function. It is obvious that the last two terms vanish because the function in square brackets vanishes at the boundary points $p_0 = \pm m, \pm\infty$. For the first two terms, when one of the derivative acts on f_{p_0} or the step function, the integral also vanishes. However, when the derivative acts on $\sqrt{p_0^2-m^2}$, the reciprocal of $\sqrt{p_0^2-m^2}$ will arise and leads to possible divergence at boundary point $p_0 = \pm m$,

$$C_\mathrm{n}^{(3)} = -\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}\left[\frac{p_0 f_{p_0}}{\sqrt{p_0^2-m^2}}\theta(p_0-m)\right]$$
$$-\int\frac{d\Omega dp_0}{4\pi}\partial_{p_0}\left[\frac{p_0 \bar{f}_{p_0}}{\sqrt{p_0^2-m^2}}\theta(-p_0-m)\right]. \quad \text{(A14)}$$

We expand each term above according to whether the derivative acts on the step function or not,

$$C_\mathrm{n}^{(3)} = -\int\frac{d\Omega}{4\pi}\int_m^{+\infty}dp_0\partial_{p_0}\left(\frac{p_0 f_{p_0}}{\sqrt{p_0^2-m^2}}\right)$$
$$-\int\frac{d\Omega dp_0}{4\pi}\frac{p_0 f_{p_0}}{\sqrt{p_0^2-m^2}}\delta(p_0-m)$$
$$-\int\frac{d\Omega}{4\pi}\int_{-\infty}^{-m}dp_0\partial_{p_0}\left(\frac{p_0 \bar{f}_{p_0}}{\sqrt{p_0^2-m^2}}\right)$$
$$+\int\frac{d\Omega dp_0}{4\pi}\frac{p_0 \bar{f}_{p_0}}{\sqrt{p_0^2-m^2}}\delta(p_0+m). \quad \text{(A15)}$$

In order to regularize the divergence at $p_0 = m$ and $p_0 = -m$, we set $p_0 = m + \epsilon$ and $p_0 = -m - \epsilon$, respectively. It follows that

$$C_{\rm n}^{(3)} = \frac{m+\epsilon}{\sqrt{(2m+\epsilon)\epsilon}} \int \frac{d\Omega}{4\pi} (f_{p_0=m+\epsilon} + \bar{f}_{p_0=-m-\epsilon}) - \frac{m+\epsilon}{\sqrt{(2m+\epsilon)\epsilon}} \int \frac{d\Omega}{4\pi} (f_{p_0=m+\epsilon} + \bar{f}_{p_0=-m-\epsilon}). \quad \text{(A16)}$$

We see that each term is divergent at the limit $\epsilon \to 0$ when $m \neq 0$, but every two terms cancel each other and give a null result. It is interesting that for the chiral limit $m = 0$ there is no divergence, and we obtain

$$C_{\rm n}^{(3)} = \int \frac{d\Omega}{4\pi} (f_{p_0=0} + \bar{f}_{p_0=0}) - \int \frac{d\Omega}{4\pi} (f_{p_0=0} + \bar{f}_{p_0=0}). \quad \text{(A17)}$$

As has been mentioned in Sec. III, the first term happens to be the right coefficient of the chiral anomaly if we choose the Fermi-Dirac distribution function given by Eq. (24).

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