$\mathbb{Z}_n$ modified XY and Goldstone models and vortex confinement transition

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The modified XY model is a modification of the XY model by the addition of a half-periodic term. The modified Goldstone model is a regular and continuum version of the modified XY model. The former admits a vortex molecule, that is, two half-quantized vortices connected by a domain wall, as a regular topological soliton solution to the equation of motion, while the latter admits it as a singular configuration. Here, we define the $\mathbb{Z}_n$ modified XY and Goldstone models as the $n=2$ case to be the modified XY and Goldstone models, respectively. We exhaust all stable and metastable vortex solutions for $n=2$, 3 and find a vortex confinement transition from an integer vortex to a vortex molecule of n $1/n$-quantized vortices, depending on the ratio between the term of the XY model and the modified term. We find that, for the case of $n=3$, a rod-shaped molecule is the most stable, while a Y-shaped molecule is metastable. We also construct some solutions for the case of $n=4$. The vortex confinement transition can be understood in terms of the $\mathbb{C}/\mathbb{Z}_n$ orbifold geometry.

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I. INTRODUCTION

The XY model is a lattice model describing a lot of physical systems such as superconductors and superfluids. Its Hamiltonian is given by $\mathcal{H}_{\rm XY} = -J\sum_{\langle i,j\rangle}\cos(\vartheta_i - \vartheta_j)$, where i and j label the lattice sites and $\langle i,j\rangle$ implies a pair of nearest neighbors. In the continuum limit, it becomes just a free U(1) scalar field theory or nonlinear O(2) model. One of the most nontrivial features of this model is to exhibit a topological phase transition called the Berezinskii-Kosterlitz-Thouless (BKT) transition [1–4] in $2+1$ dimensions, which separates bound vortices at a low temperature and liberated pairs of vortex and antivortex at a high temperature. The BKT transition yields quasi-long-range order with algebraically decaying correlations, although long-range order with continuous symmetry is forbidden by the Coleman-Mermin-Wagner theorem [5–7]. The BKT transition has been confirmed experimentally in various condensed matter systems such as ^{4}He films [8], thin superconductors [9–13], Josephson-junction arrays [14,15], colloidal crystals [16–19], and ultracold atomic Bose gases [20]. One of the drawbacks of the XY model may be the fact that vortices are described as discontinuous configurations, becoming a singular configuration in the continuum limit. To overcome this problem, one can introduce a Higgs (amplitude) degree of freedom together with a potential term along the Higgs direction, and then the model becomes the Goldstone or linear O(2) model, allowing vortices as regular solutions to the equation of motion. The vortex core singularity is resolved by the Higgs field, while the large-distance behavior can be capture by the XY model.

The modified XY model is a modification of the XY model by the addition of a half-periodic term [21–24]:

$$\mathcal{H}_{\rm mXY} = -J\sum_{\langle i,j\rangle}\cos(\vartheta_i - \vartheta_j) - J'\sum_{\langle i,j\rangle}\cos[2(\vartheta_i - \vartheta_j)], \quad (1)$$

where the second term is the half-periodic term. This model admits a vortex molecule, that is, two half-quantized vortices connected by a domain wall, as a singular configuration, and its existence is crucial in the phase diagram, as is so for the XY model. When the coupling J' of the modified term is large enough compared with the coupling J, there exists an Ising-type phase transition [22–24] as a consequence of the presence of domain walls. The modified model in Eq. (1) and its various modifications [25–33] are of great importance and interest because of the applicability to various systems such as superfluidity in atomic Bose gases [34], arrays of unconventional Josephson junctions [35], or high-temperature superconductivity [36]. The modified Goldstone or modified linear O(2) model is a regular (complemented by the

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Higgs mode) and continuum version of the modified XY model [37]:

$$\mathcal{H}_{\mathrm{mGoldstone}} = \int d^d x \left[a|\nabla\phi|^2 + b|\nabla\phi^2|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right], \tag{2}$$

where $\phi = r\exp(i\vartheta)$ is a complex scalar field (r is the Higgs field in the same sense with that of the Goldstone or Abelian-Higgs model) and λ, a, and b are positive coupling constants determined from the lattice model. Here, we denote spatial dimensions by d, but we focus on $d = 2$ in the following sections. This model admits a vortex molecule of half-quantized vortices connected by a domain wall as a regular topological soliton solution to the equation of motion when b is large enough [37], while for small b the molecule collapses to an integer vortex. The phase diagram is quite rich, and there is a two-step phase transition of BKT type and of Ising type [37].

In this paper, as a generalization of the modified XY and Goldstone models, we define the $\mathbb{Z}_n$ modified XY and Goldstone models:

$$\mathcal{H}_{\mathbb{Z}_n\mathrm{mXY}} = -J\sum_{\langle i,j\rangle}\cos(\vartheta_i - \vartheta_j) - J'\sum_{\langle i,j\rangle}\cos[n(\vartheta_i - \vartheta_j)] \tag{3}$$

and

$$\mathcal{H}_{\mathbb{Z}_n\mathrm{mGoldstone}} = \int d^d x \left[a|\nabla\phi|^2 + b|\nabla\phi^n|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \right], \tag{4}$$

respectively. The case of $n = 2$ corresponds to the usual modified XY and Goldstone models. We study vortex solutions in this model with particular attention to the cases of $n = 2$, 3. We exhaust stable and metastable vortex solutions for these cases and find a vortex confinement transition from an integer vortex to a vortex molecule, depending on the ratio between a and b. We find for the case of $n = 3$ that a rod-shaped molecule is the most stable, while a Y-shaped molecule is metastable. We also give some examples of (meta)stable vortices in the case of $n = 4$. This transition can be understood in terms of the $\mathbb{C}/\mathbb{Z}_n$ orbifold geometry; the model can be written in the form of a nonlinear sigma model with the target space $\mathbb{C}/\mathbb{Z}_n$ with a possible orbifold singularity resolved. If vacua are far from the origin in the target space, a vortex becomes a molecule of $1/n$ fractional vortices, while if the vacua are close to the origin, the vortex becomes an integer vortex.

This paper is organized as follows. In Sec. II, we introduce our model. In Sec. III, we construct vortex solutions. Section IV is devoted to a summary and discussion.

II. THE MODEL AND GEOMETRY

In this section, we formulate our model and discuss geometric properties. The Lagrangian of the $\mathbb{Z}_n$ modified Goldstone model is given by

$$\begin{aligned}\mathcal{L} &= a\partial_\mu\phi^*\partial^\mu\phi + \frac{b}{n}\partial_\mu\phi^{*n}\partial^\mu\phi^n - \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \\ &= (a + bn|\phi^{n-1}|^2)\partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda}{2}(|\phi|^2 - v^2)^2.\end{aligned} \tag{5}$$

The vacua are S^1 defined by $|\phi|^2 = v^2$. This model is just a nonlinear sigma model with the target space metric

$$g(\phi, \phi^*) = a + bn|\phi^{n-1}|^2. \tag{6}$$

In the limit of $\lambda \to \infty$, the model reduces to an $O(2)$ nonlinear sigma model (or the XY model) with the Lagrangian $\mathcal{L} = (a + bnv^{2n-2})\partial_\mu\phi^*\partial^\mu\phi$ with a constraint $|\phi|^2 = v^2$. It is sometimes useful to rewrite the Lagrangian by a new field $\Phi = \phi^n$ as

$$\begin{aligned}\mathcal{L} &= a\partial_\mu\Phi^{*1/n}\partial^\mu\Phi^{1/n} + \frac{b}{n}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{\lambda}{2}(|\Phi^{1/n}|^2 - v^2)^2 \\ &= \frac{1}{n}\left(\frac{a}{n}|\Phi^{-[(n-1)/n]}|^2 + b\right)\partial_\mu\Phi^*\partial^\mu\Phi - \frac{\lambda}{2}(|\Phi^{1/n}|^2 - v^2)^2.\end{aligned} \tag{7}$$

Let us discuss the asymptotic behavior of the target space geometry. Writing $\phi = re^{i\theta}$ or $\Phi = Re^{i\Theta}$ ($R = r^n$ and $\Theta = n\theta$), the geometry behaves in two different ways separated by the critical radius $r = r_c$ defined by

$$r_c = \left(\frac{a}{bn}\right)^{1/(2n-2)}, \qquad R_c = \left(\frac{a}{bn}\right)^{n/(2n-2)}. \tag{8}$$

Then, we can see that the metric behaves differently at large and short distances as follows:

(i) For the large distance $r \gg r_c$ ($R \gg R_c$), the first term in the metric in Eq. (6) is negligible, and the Lagrangian reduces to

$$\begin{aligned}\mathcal{L}_{\mathrm{large}} &= \frac{b}{n}\partial_\mu\Phi^*\partial^\mu\Phi - \frac{\lambda}{2}(|\Phi^{1/n}|^2 - v^2)^2 \\ &= \frac{b}{n}\partial_\mu\phi^{*n}\partial^\mu\phi^n - \frac{\lambda}{2}(|\phi|^2 - v^2)^2.\end{aligned} \tag{9}$$

One observes that Φ is a good coordinate rather than ϕ. The kinetic term of the Lagrangian in Eq. (9) is just a free scalar field in terms of Φ, but the target space is rather an orbifold:

$$\mathcal{M} \simeq \mathbb{C}/\mathbb{Z}_n. \tag{10}$$

This is because all $\phi\omega^a$ with $a = 0, 1, 2, \ldots n-1$ yield the same Φ, where $\omega^n = 1, \omega = \exp(2\pi i/n)$.

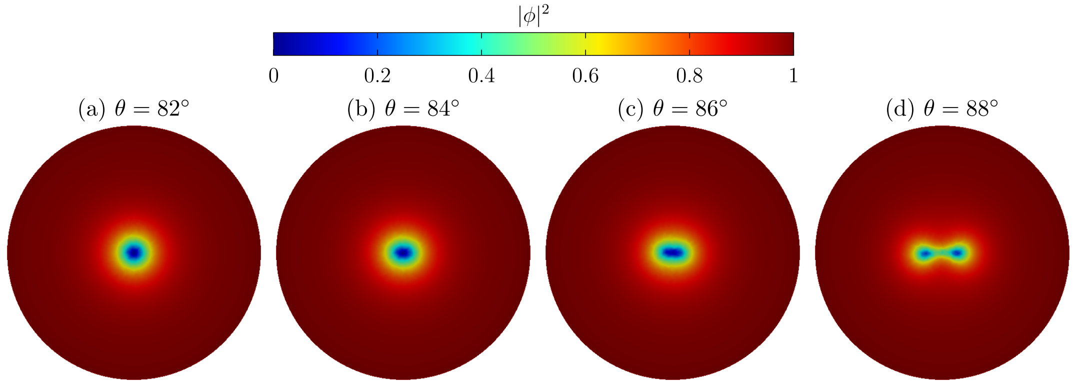

FIG. 1. Spatial configuration of $|\phi|^2$ for an integer vortex with $n = 2$. The radius of the system is 20.

This metric has an orbifold singularity in the origin, but it is not the case for the whole metric.

(ii) In fact, the short-distance behavior $[r \ll r_c$ $(R \ll R_c)]$ is dominated by the first term in the metric, and the Lagrangian reduces to

$$\mathcal{L}_{\text{short}} = a\partial_\mu\phi^*\partial^\mu\phi - \frac{\lambda}{2}(|\phi|^2 - v^2)^2$$
$$= a\partial_\mu\Phi^{*1/n}\partial^\mu\Phi^{1/n} - \frac{\lambda}{2}(|\Phi^{1/n}|^2 - v^2)^2. \quad (11)$$

The Lagrangian is nothing but the usual Goldstone model in terms of ϕ. In this case, ϕ is a good coordinate in which the metric is smooth at the origin $\phi = 0$. Therefore, we have seen that a possible singularity in the orbifold $\mathbb{C}/\mathbb{Z}_n$ is resolved in the full metric, and the whole target space is smooth.

III. VORTICES

Comparing the vacua $r = v$ and the critical radius $r = r_c$ around which the geometry behaves differently, we find two different scheme of the structure of vacua and, consequently, that of vortices. When the vacua exist inside the critical radius $r = r_c$ from the origin in the target space, that is, $v \ll r_c$, we do not need the outside geometry, in which case the Lagrangian reduces to the usual Goldstone model of ϕ admitting the S^1 vacua and integer global vortices. A single vortex configuration is of the form of $\phi = f(\rho)\exp(i\varphi)$ with the polar coordinates (ρ, φ).

On the other hand, when $r_c \ll v$, the Lagrangian is well described by Φ in Eq. (7), which is asymptotically reducing the Lagrangian Eq. (9) of a Goldstone model in terms of Φ. The vacua are $\Phi^{1/n} = ve^{i\alpha}$. Note that $\phi \sim \phi\omega^a$ yield the same Φ with $\omega^n = 1$, $\omega = \exp(2\pi i/n)$. If $a = 0$, Φ is always a good coordinate, and the model admits $1/n$ quantized (fractional) global vortices $\Phi = g^n(\rho)\exp(i\varphi)$, $[\phi = g(\rho)\exp(i\varphi/n)]$. However, if $a \neq 0$, these fractional vortices cannot exist alone, since only ϕ is a good coordinate in the vicinity of the origin of the target space. Instead, n of them must be confined to one integer vortex.

In summary, we have the following two cases:

(i) $r_c \gg v$, $R_c \gg v^n$.—integer vortex scheme;

(ii) $r_c \ll v$, $R_c \ll v^n$.—fractional vortex scheme.

We have numerically obtained the stationary solution with one integer vortex in two-dimensional space by minimizing the energy

$$\mathcal{E} = \int_\Omega dx^2\left\{a|\nabla\phi|^2 + \frac{b}{n}|\nabla\phi^n|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2\right\}. \quad (12)$$

The solution can be calculated by solving the equation

$$0 = \frac{\delta\mathcal{E}}{\delta\phi^*} = -a\nabla^2\phi - b(\nabla^2\phi^n)\phi^{*n-1} + \lambda(|\phi|^2 - v^2)\phi, \quad (13)$$

under the boundary condition $\phi = ve^{i\varphi}$ (on $\partial\Omega$). As numerical parameters, we have chosen $\lambda = v = 1$. To find the solution of Eq. (13), we have used the conjugate-gradient method on the discretized space by using FreeFem++ for the finite-element method [38]. a and b are parametrized by θ as $a = \cos\theta$ and $b = \sin\theta$, respectively. Figures 1 and 2 show the spatial configuration of $|\phi|^2$ for an integer vortex with $n = 2$ and $n = 3$, respectively. For small θ, the vortex has the circular structure which is qualitatively the same as that for the usual Goldstone model with $\theta = 0$. On the other hand, at larger θ, the circular integer vortex becomes energetically unstable and splits into n fractional vortices connected with line defects. We call this structure a vortex molecule. For $n = 3$, furthermore, there are several metastable solutions for Eq. (13) when θ is large. The triangular-shaped molecule in Fig. 2(d) is metastable and has the higher relative energy $\bar{\mathcal{E}} \equiv \mathcal{E} - \mathcal{E}_{\text{sym}} \sim -1.184$ than the rod-shaped molecule in Fig. 2(c) having the lowest relative energy $\mathcal{E} - \mathcal{E}_{\text{sym}} \sim -1.837$, where $\mathcal{E}_{\text{sym}}$ denotes the energy for the symmetric solution satisfying $\phi = g(\rho)e^{i\varphi}$.

Figure 3 shows the dependence of the relative energy $\bar{\mathcal{E}}$ on θ. The zero relative energy $\bar{\mathcal{E}} = 0$ shows that the circular

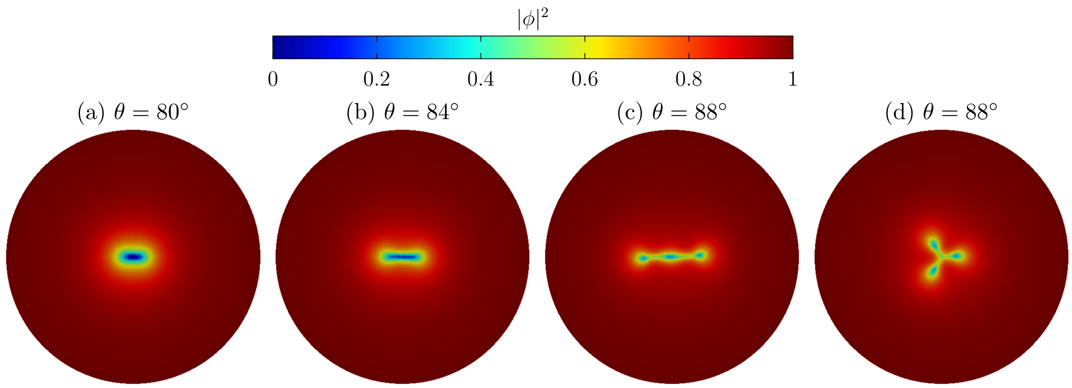

FIG. 2. Spatial configuration of $|\phi|^2$ for an integer vortex with $n = 3$. The radius of the system is 30. While the solution in (c) is the stable ground state, the solution in (d) is the metastable state having higher energy than that in (c).

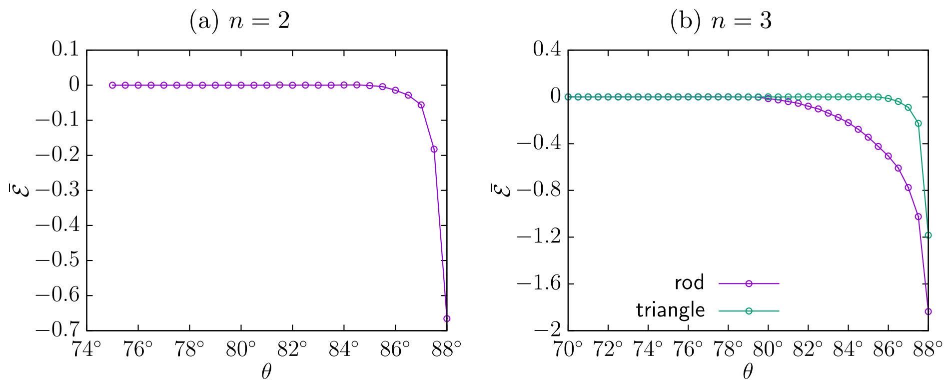

FIG. 3. Dependence of the relative energy $\bar{\mathcal{E}}$ on θ.

integer vortex solution is the stable solution. In the case of $n = 2$, the circular integer vortex solution changes to unstable against the vortex molecule solution at $\theta \sim 85°$. In the case of $n = 3$, the circular integer vortex solution changed into the rod-shaped vortex molecule solution at $\theta \sim 80°$. While the only rod-shaped vortex molecule appears as the stable solution at $80° \lesssim \theta \lesssim 85°$, the triangular-shaped vortex molecule solution appears as the metastable solution at $\theta \gtrsim 85°$.

In the case of $n = 4$, we have found three symmetric solutions as shown in Figs. 4(a)–4(c), i.e., rod-shaped, cross-shaped, and triangular-shaped molecules for Figs. 4(a), 4(b), and 4(c), respectively. Being different from the $n = 2$ and $n = 3$ cases, the rod-shaped solution in Fig. 4(a) is not the most stable but metastable, having higher energy than that for the triangular-shaped solution in Fig. 4(c), and we expect that the triangular-shaped solution is the most stable (minimum energy) solution. However, there is a large

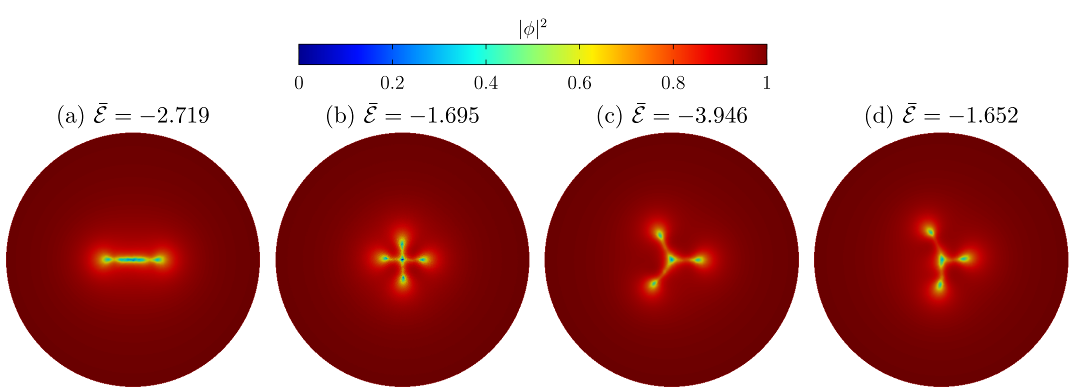

FIG. 4. Spatial configuration of $|\phi|^2$ for an integer vortex with $n = 4$ and $\theta = 88°$. The radius of the system is 40.

number of nonsymmetric solutions as shown in Fig. 4(d), and we have not exhausted all of them. Although all nonsymmetric solutions that we have found thus far have higher energies than those of symmetric solutions, we cannot give a clear conclusion that all nonsymmetric solutions truly have higher energies than those of symmetric solutions, and the triangular-shaped solution is the most stable solution. We will report the detailed analysis for $n \geq 4$ elsewhere.

IV. SUMMARY AND DISCUSSION

As a generalization of the modified XY and Goldstone models, we have defined the $\mathbb{Z}_n$ modified XY and Goldstone models, having a $2\pi/n$ periodic term in addition to the usual XY (2π periodic) term. We have pointed out that the modified Goldstone model can be regarded as a nonlinear sigma model with the target space of the orbifold geometry $\mathbb{C}/\mathbb{Z}_n$ with the orbifold singularity resolved. Depending on the vacua, we have found two different schemes: the integer vortex scheme, in which the XY term is dominant, and the fractional vortex scheme, in which the modified term is dominant. We have exhausted vortex solutions for $n = 2$, 3 and have found a vortex confinement transition from an integer vortex in the integer vortex scheme to a vortex molecule, i.e., n $1/n$-quantized vortices connected by a domain wall (or walls) in the fractional vortex scheme. In the case of $n = 3$, we have found a Y-shaped molecule as a metastable solution, and the most stable solution is a rod-shaped molecule for the fractional vortex regime.

As a related topic, two (or n) complex scalar fields coupled by a Josephson term(s) also admit a vortex molecule of half-quantized ($1/n$ quantized) vortices; see Refs. [39–47] for two-component Bose-Einstein condensates (BECs), Refs. [48–52] for n-component BECs, and Ref. [53] for a spinor BEC. In this case, in contrast to the case of the modified model, there is no two-step phase transition [54,55], but it is unclear what the crucial difference is between the two cases, although both admit similar solutions.

If we gauge the U(1) symmetry of the Goldstone model, we have an Abelian-Higgs model. While the former admits a global vortex as we have discussed in this paper, the latter admits a local Abrikosov-Nielsen-Olesen vortex [56,57]. It is an interesting question whether a modified Abelian-Higgs model admits local vortices of the molecule type. There are several questions such as whether there is a critical Bogomol'nyi-Prasad-Sommerfield coupling and whether it admits a supersymmetric extension; see, e.g., Ref. [58]. Whether there is any superconductor described by such models is also an interesting question.

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