Some extensions for the energy conditions inspired by vacuum nonlinear electrodynamics

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In this paper, we discuss some specific features for the stress-energy tensor of vacuum nonlinear electrodynamics and based on these features we propose the inequalities which in form are close to the energy conditions widely used in general relativity and which can be interpreted as their possible nonlinear extension. The modified energy conditions were verified for the perfect fluid and real scalar field with an arbitrary coupling to the kinetic term.

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I. INTRODUCTION

Contemporary field theory considers a lot of theoretical models, verification of the experimental status for which at present is difficult or impossible with the various reasons. For this reason the paramount importance is in search and investigation of criteria, violation of which for the tested model will lead to the contradiction of fundamental physical principles. Of course, violation of such criteria does not negate the need for a comprehensive experimental study of the particular model predictions; however, this allows to classify such a model as an exotic.

Among the many fundamental principles the special place is given to the criteria of causality, unitarity, and to the energy conditions. The causality criteria guarantee that the group velocity for the elementary field excitations do not exceed the speed of light in vacuum, while the unitarity criteria provides the positive definiteness of the norm of every elementary excitation of the vacuum. These requirements provide a significant restriction on the Lagrangian of the model under consideration. Such restrictions on theoretical models of vacuum electrodynamics, coming from the noted principles, was obtained in [1] as a set of inequalities on the Lagrangian and its derivatives.

In turn, the energy conditions have the form of various inequalities imposed on the matter stress-energy tensor components T_{ik} and originating from general requirement, that the field energy density should be non-negative, when measured by any observer. The formulation of energy conditions can be quite varied. From the set of different types for these conditions [2,3], one should especially note the weak energy condition (WEC), the null energy condition (NEC), and the dominant energy condition (DEC). The weak energy condition guarantees positive definiteness of the matter energy for an observer traversing any timelike curve, which manifests in the form of inequality:

$$T_{ik}a^i a^k \geq 0, \tag{1}$$

where a^k is any timelike vector. This condition is satisfied for most of the known types of matter; however, there are predictions about possible violation of it in the inflating space-times [4].

The null energy condition in form is similar to WEC; however a^k now is a null vector. In general relativity this condition ensures the absence of repulsion for null geodesics, focused by the matter. It also is a clue assumption for area [5] and singularity theorems [2,6]. The condition is satisfied by most reasonable classical fields; however, its violation is expected in some models of scalar-tensor gravity, for instance, in Horndesky theory [7,8].

Both the WEC and the NEC conditions do not preclude superluminal speed for the energy propagation; however the WEC supplemented by the requirement that the energy flux be a future-pointing causal vector

$$T_{ik}T^{km}a^i a_k \geq 0, \tag{2}$$

leads to the dominant energy condition which, widely believed, is deprived of the pointed disadvantage. Nevertheless, it should be noted that there are some assumptions [3,9,10] indicating that the DEC is not enough to solve the superluminal problem. These assumptions in the particular cases seem to be self-consistent; however, it is very difficult to generalize them to an arbitrary stress-energy tensor.

The energy conditions listed above are heuristic in nature and their fulfillment should be verified in each particular case for the stress-energy tensor of the matter in question. In this paper, we establish some new relations arising for the stress-energy tensor of an arbitrary Lorentz-invariant

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nonlinear electrodynamics with the most general dependence on the electromagnetic field tensor invariants. In order to find out whether the established relations are universal, or are valid only for the electromagnetic field, we will check their fulfillment for other types of matter.

The paper is organized as follows: In Sec. II, we discuss vacuum nonlinear electrodynamics models and obtain some specific features for their stress-energy tensor. In this section we will formulate a statement for the deviator of the stress-energy tensor which will be verified for other types of matter on the subsequent sections. In Sec. III we perform verification of the statement for the perfect fluid and in Sec. IV we will check it for scalar field with an arbitrary coupling to kinetic term in the Lagrangian. In the last section we summarize our results. For more convenience, we will use geometerized units ($G = c = \hbar = 1$) and the metric signature $\{+,-,-,-\}$.

II. VACUUM NONLINEAR ELECTRODYNAMICS AND ITS SPECIFIC FEATURES

Vacuum nonlinear electrodynamics (NED) models arise from the assumption about the nonlinear dependence of the model Lagrangian on the invariants of the electromagnetic field tensor. The specific form of the Lagrangian depends on the model choice. Such nonlinear generalization can solve some problems inherent to the Maxwell electrodynamics and also gives the predictions for new effects.

For instance, the Born-Infeld model [11] solves the problem of the infinite energy of a pointlike charge by bounding the field strength in the charge center. Heisenberg-Euler theory [12] considers quantum radiative corrections caused by electron-positron vacuum polarization in a strong electromagnetic field and predicts birefringence for electromagnetic waves in vacuum. There are various modifications and extensions of the NED models [13–15]. In this paper, we will not specify the particular form of the Lagrangian $\mathcal{L}$, assuming only that it is an arbitrary function of the electromagnetic field invariants $J_2 = F_{ik}F^{ki}$ and $J_4 = F_{ik}F^{kl}F_{lm}F^{mi}$. In this case the action for a vacuum nonlinear electrodynamics, in the space-time with the metric tensor g_{ik} can be represented in the form

$$S = \int \sqrt{-g}\mathcal{L}(J_2, J_4)d^4x, \tag{3}$$

where g is the metric determinant. It is easy to derive the symmetric stress-energy tensor for the action (3):

$$T_{ik} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ik}} = 4\left[\frac{\partial\mathcal{L}}{\partial J_2} + J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]F^{(2)}_{ik} + \left[(2J_4 - J_2^2)\frac{\partial\mathcal{L}}{\partial J_4} - \mathcal{L}\right]g_{ik}, \tag{4}$$

with the trace

$$T = T^i_i = 4\left[\frac{\partial\mathcal{L}}{\partial J_2}J_2 + 2J_4\frac{\partial\mathcal{L}}{\partial J_4} - \mathcal{L}\right], \tag{5}$$

and where the second power of the electromagnetic field tensor is introduced as $F^{(2)}_{ik} = F_{im}F^{m\cdot}_{\cdot k}$. In general, there are no reasons to assert that the weak energy condition (1) takes place for an arbitrary vacuum nonlinear electrodynamics. Fulfillment of this condition depends on the choice of Lagrangian and on specific model phenomenology. Nevertheless, some general relations inherent to an arbitrary nonlinear vacuum electrodynamics may be established. To obtain such relations, we construct the deviator of the stress-energy tensor (4)

$$\mathcal{D}_{ik} = T_{ik} - \frac{T}{4}g_{ik} = 4\left[\frac{\partial\mathcal{L}}{\partial J_2} + J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]\left\{F^{(2)}_{ik} - \frac{J_2}{4}g_{ik}\right\}. \tag{6}$$

It is easy to see that the deviator $\mathcal{D}_{ik}$ is conformal to the Maxwell electrodynamics stress-energy tensor, which up to a constant coefficient corresponds to the expression in the curly brackets. Moreover, as it was noted earlier, the Lagrangian $\mathcal{L}$ must fulfill unitarity and causality conditions, which were considered for an arbitrary NED in assumption that $(\mathbf{E}, \mathbf{B}) = 0$ in the paper [1]. These restrictions lead to the set of inequalities, one of which has exactly the same form as the first multiplier in (6)

$$\frac{\partial\mathcal{L}}{\partial J_2} + J_2\frac{\partial\mathcal{L}}{\partial J_4} \geq 0. \tag{7}$$

Because the weak energy condition (1) is satisfied for the Maxwell electrodynamics, the quadratic form based on the deviator of the stress-energy tensor will be non-negatively defined on the set of causal vectors for an arbitrary NED: $\mathcal{D}_{ik}a^i a^k \geq 0$.

This statement can be extended and generalized. To make this, we derive an arbitrary power p of the stress-energy tensor deviator $\mathcal{D}^{(p)}_{ik} = \overbrace{\mathcal{D}_{im_1}\mathcal{D}^{m_1m_2}\cdots\mathcal{D}_{m_{p-1}k}}^{p}$. After some cumbersome calculations we obtain the general expression for even and odd powers of the deviator:

$$\mathcal{D}^{(2m)}_{ik} = 2^m\left[\frac{\partial\mathcal{L}}{\partial J_2} + J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]^{2m}\left[2J_4 - \frac{J_2^2}{2}\right]^m g_{ik}, \tag{8}$$

$$\mathcal{D}^{(2m+1)}_{ik} = 2^{m+2}\left[\frac{\partial\mathcal{L}}{\partial J_2} + J_2\frac{\partial\mathcal{L}}{\partial J_4}\right]^{2m+1}\left[2J_4 - \frac{J_2^2}{2}\right]^m \times\left\{F^{(2)}_{ik} - \frac{J_2}{4}g_{ik}\right\}, \tag{9}$$

where m is an arbitrary unsigned integer.

The multiplier (7) for all the expressions is non-negative, as well as the multiplier $2J_4 - J_2^2/2$. Therefore, in consequence with the causality principle (7), the quadratic form for an arbitrary power p of the deviator, defined on the set of causal vectors a^k, will be not negative for an arbitrary NED:

$$\mathcal{D}^{(p)}_{ik} a^i a^k \geq 0, \qquad a_k a^k \geq 0. \tag{10}$$

The last expression can be rewritten in terms of the Ricci tensor deviator. However, for such a representation, it will be necessary to take a certain form of gravity equations, for instance, to use the Einstein equations for the connection between the components of the Ricci tensor and the energy-momentum tensor. Such an approach involves the selection of a specific theoretical model of gravity and therefore limits the generality, so we will carry out all further considerations in terms of the stress-energy tensor.

For the special class of NED with the zero trace of the stress-energy tensor [15], the statement (10) will be valid not only for the deviator but also for an arbitrary power of the stress-energy tensor, and in the case when $p = 1$ it leads to the weak energy condition. In general, the statement (10) can be interpreted as a nonlinear energy condition. Nonlinear energy conditions constructed in the form of power-law combinations of the stress-energy tensor components already have been mentioned in the literature. For example, in [16], the so-called "flux energy condition" (FEC) was proposed as a quadratic form based on the second power of vacuum averaged stress energy-tensor. The paper [17] also considers a quadratic energy condition in the form

$$(T_{ai} - \varepsilon\eta_{ai})\eta^{ik}(T_{bk} - \varepsilon\eta_{bk})V^a V^b \geq \varepsilon^2, \tag{11}$$

where $\eta_{ia} = \mathrm{diag}\{1, -1, -1, -1\}$ is the Minkowski metric, ε is an arbitrary positive coefficient, and V^a are the components of the timelike vector. The deviator condition (10) under the specific assumptions can be rewritten in the form close to (11). Other nonlinear energy conditions, which are in form significantly different from (10) are also mentioned in Ref. [17]. For instance, the determinant energy condition assumes that $\det T_{ik} \geq 0$, and the trace-of-square energy condition impose the restriction $T_{ik}T^{ik} \geq 0$.

All of the listed above nonlinear energy conditions, including (10), are empirical in nature, and as already noted in [18], unfortunately, do not allow a vivid physical interpretation. Primarily, this is due to the fact that the linear energy conditions obtained their physical meaning, due to the coincidence of the terms in some well-studied equations with the expressions for these energy conditions. For instance, in the Raychaudhuri equation, which describes the expansion of congruences of geodesic rays, one of the terms coincides by the form with the WEC. Nonlinear equations are still poorly studied, and so far, the equations which include the nonlinear power-law energy conditions are unknown. Moreover, if such equations exists, the interpretation of the energy conditions based on them would be rather complicated due to nonlinearity. In view of the described difficulties, we propose to consider nonlinear energy conditions only as algebraic constraints for the energy-momentum tensor established for a certain type of matter, which can be extended on the other types of matter with or without certain restrictions. The study of the possibility of such an extension and search of the limitations for this seems as the main aim of further consideration.

In order to find out whether the property (10) of the stress-energy tensor deviator is specific only for nonlinear electrodynamics, or if it is not a unique feature, in the next sections we will test this property for the other types of matter.

III. VERIFICATION FOR A PERFECT FLUID

Let us check whether the inequality (10) is valid for an arbitrary dynamics of the perfect fluid, for which the stress-energy tensor and its trace have a form:

$$T_{ik} = (w + p)u_i u_k - p g_{ik}, \qquad T = w - 3p, \tag{12}$$

where w is the rest energy density, p is isotropic pressure and u_k is a four-velocity with the norm $u_k u^k = 1$. It is easy to derive the deviator for the stress-energy tensor:

$$\mathcal{D}_{ik} = (w + p)\left[u_i u_k - \frac{g_{ik}}{4}\right]. \tag{13}$$

As previously, after some calculations, we can obtain the expression for an even and odd power of the deviator, substitution of which to the inequality (10) gives

$$\mathcal{D}^{(2m+1)}_{ik} a^i a^k = \left(\frac{w+p}{4}\right)^{2m+1} [(3^{2m+1}+1)(u_k a^k)^2 - a_k a^k],$$
$$\mathcal{D}^{(2m)}_{ik} a^i a^k = \left(\frac{w+p}{4}\right)^{2m} [(3^{2m}-1)(u_k a^k)^2 + a_k a^k]. \tag{14}$$

To find out the sign of the last expressions we will resort to a technique close to the Newman-Penrose formalism [19–21]. Let us introduce two isotropic four-vectors l^k and n^k. We take into account that any causal future-pointing vector a^k can be represented as a superposition of two noncollinear isotropic vectors, also pointing to the future $a^k = \alpha l^k + \beta n^k$, where α,β are real constants. Without loss of generality, one can take that $l^k n_k = 1$, so the causality $a_k a^k \geq 0$ sets the restriction on the coefficients $\alpha\beta \geq 0$.

Now we supplement the set of vectors l^k and n^k with the two other complex isotropic vectors m^k and $\bar{m}^k$, so that altogether these vectors form a tetrade with the following scalar products:

$$l_k l^k = n_k n^k = m_k m^k = n_k m^k = l_k m^k = 0,$$
$$l_k n^k = -\bar{m}_k m^k = 1, \tag{15}$$

where the bar denotes complex conjugation. The tetrade represents a basis, which can be used for four-velocity decomposition:

$$u_k = f_1 l_k + f_2 n_k + f_3(m_k + \bar{m}_k) + i f_4(\bar{m}_k - \bar{m}_k), \tag{16}$$

where the coefficients f are the fluid dynamics dependent functions. From the realness of four-velocity $u_k = \bar{u}_k$ one can conclude that all these coefficients are real. Another property of the decomposition follows from the norm $u_k u^k = 1$ of four-velocity:

$$2f_1 f_2 = 1 + 2(f_3^2 + f_4^2) \geq 1. \tag{17}$$

Finally, it is helpful to obtain some auxiliary expressions:

$$(u_k a^k)^2 = (\alpha f_2 + \beta f_1)^2, \qquad a_k a^k = 2\alpha\beta \geq 0, \tag{18}$$

substitution of which into (14) gives

$$\mathcal{D}_{ik}^{(2m+1)} a^i a^k = \left(\frac{w+p}{4}\right)^{2m+1} \times [(3^{2m+1}+1)(\alpha^2 f_2^2 + \beta^2 f_1^2) + 2\alpha\beta\{(3^{2m+1}+1) f_1 f_2 - 1\}],$$
$$\mathcal{D}_{ik}^{(2m)} a^i a^k = \left(\frac{w+p}{4}\right)^{2m} [(3^{2m}-1)(\alpha f_2 + \beta f_1)^2 + 2\alpha\beta]. \tag{19}$$

Since the sum of the pressure and the rest energy density is non-negative $p + w \geq 0$ (which also follows from the WEC for the perfect fluid [3]), and by virtue of (17) and (18) one can conclude that both expressions in (19) are non-negative for arbitrary coefficients of decomposition α and β, which depend on the a^k choice, for arbitrary coefficients f which depend on the fluid dynamics and also for an arbitrary power index m. Therefore the statement of non-negativity of the quadratic form, constructed with the powers of the stress-energy tensor deviator, holds not only for the vacuum nonlinear electrodynamics, but also for the perfect fluid.

IV. VERIFICATION FOR A SCALAR FIELD

Now let us check the statement for the real scalar field ϕ, under the assumption of an arbitrary coupling between the Lagrangian and the kinetic term. Such type of matter is widely discussed in general relativity as a k-essence model of the dark energy [22]. In this case the action functional can be represented in the form

$$S = \int \sqrt{-g}\mathcal{L}(X, \phi) d^4x, \tag{20}$$

where $X = \nabla_k\phi\nabla^k\phi$ is the kinetic term. For the action (20) it is easy to derive the symmetric stress-energy tensor

$$T_{ik} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{ik}} = 2\frac{\partial\mathcal{L}}{\partial X}\nabla_i\phi\nabla_k\phi - \mathcal{L}g_{ik}, \tag{21}$$

and its deviator, which is close by the form to the similar expression for the perfect fluid:

$$\mathcal{D}_{ik} = 2\frac{\partial\mathcal{L}}{\partial X}\left(\nabla_i\phi\nabla_k\phi - \frac{X}{4}g_{ik}\right). \tag{22}$$

However, we should note, that now there is no restriction on the value of X and it can be zero or negative, in contradiction to $u_k u^k > 0$ in the perfect fluid model. As earlier in Secs. II and III, we obtain the general expressions for the deviator powers, which we use to construct a quadratic form with the causal vector a^k. Omitting auxiliary calculations, we get

$$\mathcal{D}_{ik}^{(2m)} a^i a^k = \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m} \frac{X^{2m-1}}{2^{2m}} \times \{(3^{2m}-1)(\nabla_k\phi a^k)^2 + X a_k a^k\}, \tag{23}$$

$$\mathcal{D}_{ik}^{2m+1} a^i a^k = \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m+1} \frac{X^{2m}}{2^{2m+1}} \times \{(3^{2m+1}+1)(\nabla_k\phi a^k)^2 - X a_k a^k\}. \tag{24}$$

In order to determine the sign of the expression for each quadratic form, we perform decomposition of the covariant derivative on the basis of the isotropic tetrade (15):

$$\nabla_k\phi = f_1 l_k + f_2 n_k + f_3(m_k + \bar{m}_k) + i f_4(\bar{m}_k - m_k), \tag{25}$$

where the coefficients f depends on the scalar field configuration, and as earlier in Sec. III we use the following decomposition for the causal vector: $a_k = \alpha l_k + \beta n_k$, with $\alpha\beta \geq 0$. Representation (25) allows us to obtain useful auxiliary expressions:

$$X = \nabla_k\phi\nabla^k\phi = 2(f_1 f_2 - f_3^2 - f_4^2),$$
$$(\nabla_k\phi a^k)^2 = (\alpha f_2 + \beta f_1)^2, \tag{26}$$

substitution of which into Eq. (23) leads to

$$\mathcal{D}_{ik}^{(2m)} a^i a^k = \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m} \frac{X^{2m-1}}{2^{2m}} \times \{(3^{2m}-1)(\alpha f_2 + \beta f_1)^2 + 4\alpha\beta(f_1 f_2 - f_3^2 - f_4^2)\}, \tag{27}$$

$$\mathcal{D}_{ik}^{2m+1}a^i a^k = \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m+1}\frac{X^{2m}}{2^{2m+1}}\{(3^{2m+1}-1)\times(\alpha f_2+\beta f_1)^2+4\alpha\beta(f_3^2+f_4^2)+2(\alpha^2 f_2^2+\beta_2^2 f_1^2)\}. \tag{28}$$

It should be taken into account that due to the requirement of the ghosts lack (unitarity condition), the derivative of the Lagrangian over the kinetic term $\partial\mathcal{L}/\partial X \geq 0$ is not negative [23] and this governs the sign of the first multiplier in (27) and (28) also to be not negative. It is easy to see that the second multiplier and the expression in the curly brackets in (28) are non-negative for all m starting from zero and for all α, β, and f. Based on this, we can conclude that the quadratic form constructed with the odd power of the stress-energy tensor deviator is non-negative on the set of causal vectors $\mathcal{D}_{ik}^{(2m+1)}a^i a^k \geq 0$. This fact supports our statement about the deviator powers; however, consideration of the expression for an even powers (27) gives an opposite result. Let us discuss this in detail and consider two cases of the kinetic term sign.

When $X \geq 0$, from the expression (26) follows that $f_1 f_2 \geq f_3^2+f_4^2$, so all the terms in the curly brackets in (27) are non-negative, and therefore this quadratic form is also non-negative.

In the opposite case, when $X < 0$, the inequalities in Eq. (26) leads to the estimation:

$$(\alpha f_2+\beta f_1)^2 < (\alpha f_2)^2+(\beta f_1)^2+2\alpha\beta(f_3^2+f_4^2). \tag{29}$$

Moreover, the quadratic form, based on the even power of the deviatior (27) can be rewritten as

$$\mathcal{D}_{ik}^{(2m)}a^i a^k = \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m}\frac{X^{2m-1}}{2^{2m}}\{(3^{2m}+1)(\alpha f_2+\beta f_1)^2-2(\alpha f_2)^2-2(\beta f_1)^2-4\alpha\beta(f_3^2+f_4^2)\}, \tag{30}$$

where the first term in the curly brackets can be bounded by using (29), which leads to the estimation from above:

$$\mathcal{D}_{ik}^{(2m)}a^i a^k < \left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m}\frac{X^{2m-1}}{2^{2m}}\{(3^{2m}+1)[(\alpha f_2)^2+(\beta f_1)^2+2\alpha\beta(f_3^2+f_4^2)]-2(\alpha f_2)^2-2(\beta f_1)^2-4\alpha\beta(f_3^2+f_4^2)\}$$
$$= (3^{2m}-1)\left(\frac{\partial\mathcal{L}}{\partial X}\right)^{2m}\frac{X^{2m-1}}{2^{2m}}\{(\alpha f_2)^2+(\beta f_1)^2+2\alpha\beta(f_3^2+f_4^2)\} < 0. \tag{31}$$

Due to $X < 0$ the right-hand side of the last expression is always negative for any coefficients α and β, which are correspondent to the causal vector $\alpha\beta \geq 0$ and this result indicates that the statement about positive definiteness of the deviator powers can be violated.

V. CONCLUSION

In the paper we have considered the properties of the stress-energy tensor deviator. We have investigated the expressions that can be interpreted as nonlinear generalization of the energy conditions, which are commonly used in general relativity. It was revealed that, for an arbitrary model of vacuum nonlinear electrodynamics the quadratic form constructed with an arbitrary power of the deviator can never be negative on the set of causal vectors. To find out whether this feature is characteristic only for nonlinear electrodynamics or it has a more common character, we checked this property for an isotropic perfect fluid model and for an arbitrary real scalar field. In the case of the perfect fluid, the noted feature of the quadratic form is also valid for any fluid dynamics. However, for the scalar field the statement cannot always be confirmed. It was shown that for the scalar field configuration with negative kinetic term (for instance, which corresponds to the case of static inhomogeneous field), the quadratic form for the even powers of the deviator will be negative. The possibility of violating the positive definiteness of the quadratic form indicates that the models of vacuum nonlinear electrodynamics are specific, since this violation does not take place for them.

It should be noted that, despite the fact that in accordance with Eq. (31), the quadratic form for a scalar field is not sign-specified, the same expression leads to another remarkable property. The sign of the quadratic form is completely determined by the field configuration and does not depend on the choice of the causal vector a^k, which is usually interpreted as the four-velocity of an observer which tests the features of the field. Hence, for each considered field configuration any observer will find that the sign of a quadratic form which is constructed with an arbitrary power of the stress-energy tensor deviator will not depend on the choice of the four-velocity of the observer. This statement is valid for vacuum nonlinear electrodynamics, the perfect fluid model, and the scalar field with an arbitrary coupling to the kinetic term in the Lagrangian; however, it should be verified for the other types of matter in each particular case.

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