Second post-Newtonian order radiative dynamics of inspiralling compact binaries in the effective field theory approach

Adam K. Leibovich,[1] Natália T. Maia,[1] Ira Z. Rothstein,[2] and Zixin Yang[1]

[1]*Pittsburgh Particle Physics Astrophysics and Cosmology Center, Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA*

[2]*Department of Physics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, USA*

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We use the effective field theory (EFT) framework to compute the mass quadrupole moment, the equation of motion, and the power loss of inspiralling compact binaries at the second order in the post-Newtonian (PN) approximation. We present expressions for the stress-energy pseudotensor components of the binary system in higher PN orders. The 2PN correction to the mass quadrupole moment as well as to the acceleration computed in the linearized harmonic gauge presented here are the ingredients needed for the calculation of the next-to-next-to leading order radiation reaction force, which will be presented elsewhere. While this paper reproduces known results, it supplies the building blocks necessary for future higher order calculations in the EFT methodology.

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I. INTRODUCTION

The successful detections of gravitational waves by LIGO and Virgo [1–8] and the consequent advent of multimessenger astronomy [9–11] have expedited the need for precise theoretical descriptions of the dynamics of binary inspirals. While numerical techniques are required for the late stages of inspirals, the early stage admits a perturbative treatment via the post-Newtonian (PN) approximation, which is an expansion in v^2/c^2, and can be matched onto numerical results for later stages of the inspiral. Generating higher order PN corrections will allow for more accurate parameter estimations.

In this paper, we will utilize the effective field theory (EFT) approach called nonrelativistic general relativity (NRGR), proposed in [12] (for reviews see [13–17]), as our calculation framework. To date, most of the results in the nonspinning sector of the EFT formalism have been geared towards the potential sector culminating in the present state of the art 4PN results [18,19], which agree with results previously derived using other methods [20–23]. In the radiation sector, the EFT results have only[1] been calculated to 1PN [26] as compared to the 3PN results calculated using more traditional GR methods [27]. Therefore, this paper is the next step in the calculation of higher order radiative effects in NRGR. In particular, in a separate paper we will use the results herein to calculate the next-to-next-to leading order radiation reaction force via the generation of an effective action.

[1]For spinning constituents the relevant multipole moments at 3PN for the flux [24] and 2.5 for the amplitude [25].

The radiation sector of NRGR, the topic of this paper, was first studied in [12]. The effective action that describes radiative effects is determined by the underlying symmetries—reparameterization and diffeomorphism invariances—and is applicable to arbitrary gravitational wave sources in the long wavelength approximation. The Wilson coefficients of the action, the multipole moments, cannot be determined by the symmetries and need to be fixed by a matching procedure. The expression for the effective action to all orders in the multipole expansion and the exact expressions of the multipole moments in terms of the components of the stress-energy tensor were presented in [28]. The NRGR framework provides a systematic way to compute the multipole moments of a binary system by integrating out the modes of the gravitational field that live in the near zone. The stress-energy tensor, whose moments are our targeted goal, is determined by calculating the radiation graviton one point function in the presence of the background potentials using Feynman diagrams. The number of Feynman diagrams grows rapidly with PN order.

The goal of this paper is to determine the 2PN correction to the mass quadrupole moment, which comes from various moments of the stress-energy pseudotensor. Each such contribution starts at different order in the PN expansion and only a few of these contributions can be derived from known quantities. We also derive the equation of motion of the binary system at 2PN order in Appendix B. Note that this acceleration was calculated previously in the EFT approach in [29], where the authors worked with Kaluza-Klein variables [30] in conjunction with harmonic coordinates. The 2PN acceleration derived here, on the other hand, is written in the linearized (background) harmonic gauge, which leaves a gauge invariant effective action for the radiation field after the potential modes are integrated out, and can be used in combination with previous results obtained in the EFT approach where the linearized harmonic gauge was used. Our results constitute the final missing part necessary for the computation of the next-to-next-to-leading order radiation reaction force as well as for the construction of spinning templates at 2.5PN order for the phase and 3PN order for the amplitude. These computations are ongoing and will be reported in a subsequent publication.

This paper is organized as follows. In Sec. II, we provide a summary of NRGR for binary systems of compact bodies with emphasis in the radiation sector, where we explicitly show how the mass quadrupole moment depends on the components of the pseudotensor in different PN orders. The contributions to the quadrupole that come from higher PN order components of the pseudotensor are computed in Sec. III, while the contributions coming from the lower PN order components are obtained in Sec. IV. We use the results obtained in these sections to write down, in Sec. V, the components of the pseudotensor that can be used to compute the multipole moments, which are shown to agree with the literature. The assembly of all contributions constitutes the 2PN correction to the mass quadrupole moment, presented in Sec. VI, in terms of the worldlines of the compact bodies and also in the center-of-mass (c.m.) frame. In Sec. VII we present our final remarks on the results presented in this paper. Appendix A is intended for readers interested in computing radiation effects in NRGR to higher orders. The necessary ingredients for the computation of the higher PN order components of the pseudotensor are presented therein. In Appendix B we show the result for the acceleration at 2PN order computed in the linearized harmonic gauge, which is necessary to compare the quadrupole moment obtained in this paper with the result in [27], as well as to compute the power loss at the second PN order.

We use the following definitions throughout this paper: $m = m_1 + m_2$, $\nu \equiv m_1 m_2/m^2$, and $\mu = m\nu$. The relative position is defined as $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2$, while $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{a} \equiv \mathbf{a}_1 - \mathbf{a}_2$ are the relative velocity and acceleration, respectively. If those relative quantities appear inside a sum over the particles indices $a, b = 1, 2$, they become dependent on the indices a, b instead, e.g., $\mathbf{r} \equiv \mathbf{x}_a - \mathbf{x}_b$. We adopt the mostly minus signature convention for $\eta^{\alpha\beta}$ and Latin indices are contracted with the Euclidean metric. We use $c = 1$ units and the Planck mass is defined as $m_{\rm Pl} \equiv 1/\sqrt{32\pi G}$.

II. EFT SETUP

During the inspiral stage, the physics of a binary system of compact bodies is naturally separated into three length scales: the typical size of the bodies of order of the Schwarzchild radius r_s, the orbital distance between the two bodies given by r, and the wavelength λ_{GW} of the gravitational radiation. As the relative velocity v of the bodies is small, those three length scales together constitute a hierarchical structure

$$r_s \ll r \ll \lambda_{GW}. \tag{2.1}$$

The first step is to "integrate out" the scale associated with the bodies' size.[2] Hence, the binary system can be initially described by the action

$$S = S_{EH} + S_{GF} + S_{pp}, \tag{2.2}$$

such that gravity is described by the Einstein-Hilbert (EH) action $S_{EH} = -2m_{\rm Pl}^2 \int d^4x \sqrt{-g} g_{\mu\nu} R^{\mu\nu}$ with a gauge fixing term S_{GF}, while the massive bodies are described by the point particle action $S_{pp} = -\sum_a m_a \int d\tau_a$. The index $a = 1, 2$ distinguishes the two bodies.

Next, the two different modes of the gravitational field are separated in a diffeomorphism invariant way[3] via

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) = \eta_{\mu\nu} + \bar{h}_{\mu\nu}(x) + H_{\mu\nu}(x). \tag{2.3}$$

The off-shell potential mode H obeys $\partial_0 H_{\mu\nu} \sim (\frac{v}{r}) H_{\mu\nu}$ and $\partial_i H_{\mu\nu} \sim (\frac{1}{r}) H_{\mu\nu}$ whereas the on-shell radiation mode obeys $\partial_\alpha \bar{h}_{\mu\nu} \sim (\frac{v}{r}) \bar{h}_{\mu\nu}$. Moreover, the radiation field $\bar{h}_{\mu\nu}(x)$ has to be Taylor expanded around a point inside the source (for instance c.m. of the binary system) at the level of the action in order to achieve a uniform power counting in the parameter $v^2 \sim \frac{r_s}{r}$ [32]. With these considerations, the action in (2.2) is then given as an expansion in the fields $\bar{h}_{\mu\nu}(x)$ and $H_{\mu\nu}(x)$, each of which scale homogeneously in v^2.

To describe the dynamics associated with gravitational waves, the potential mode of the gravitational field is integrated leaving an effective action that will depend only on the radiation field and the worldlines. This action will be diffeomorphism invariant if one chooses the linearized harmonic gauge when integrating out the potential field, via the gauge fixing action

$$S_{GF} = \int d^4x \sqrt{-\bar{g}} \bar{\Gamma}_\mu \bar{\Gamma}^\mu, \tag{2.4}$$

where $\bar{\Gamma}_\mu = \bar{\nabla}_\alpha H^\alpha_\mu - \frac{1}{2} \bar{\nabla}_\mu H^\alpha_\alpha$, with $\bar{\nabla}_\mu$ representing the covariant derivative associated with the background metric $\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{\bar{h}_{\mu\nu}(x)}{m_{\rm Pl}}$.

[2]Finite size effects are accounted for by inserting higher-dimensional operators in the effective action, respecting the symmetries of the system.

[3]Double counting subtleties arise at 4PN but can be systematically disentangled [31].

Moreover, as a result of the "elimination" of the degrees of freedom that live in the orbital scale, the binary system is then regarded as a single point particle coupled to its gravitational field and whose internal dynamics is described by a set of multipole moments. We present a brief review of the EFT radiation sector in the next section.

A. Radiation sector

The radiation action, which describes arbitrary gravitational wave sources in the long wavelength approximation, can be written in a diffeomorpshim invariant way in terms of multipoles. Specifically, it is a derivative expansion where higher order terms are suppressed by powers of the ratio between the size of the binary system over the wavelength of the radiation emitted. In the c.m. frame, the action of the radiation sector is

$$S_{\rm rad}[\bar{h}, x_a] = -\int dt\sqrt{\bar{g}_{00}}\left[m + \frac{1}{2}L_{ij}\omega_0^{ij} + \sum_{l=2}^{\infty}\left(\frac{1}{l!}I^L\nabla_{L-2}E_{i_{l-1}i_l} - \frac{2l}{(2l+1)!}J^L\nabla_{L-2}B_{i_{l-1}i_l}\right)\right], \tag{2.5}$$

where a multi-index representation $L = i_1 \ldots i_l$ is used. The first two terms generate the Kerr background in which the gravitational waves propagate. The multipole moments, which constitute the source of radiation, are coupled to the electric and the magnetic components of the Weyl tensor. To check the expressions of the terms present in the equation above, see Ref. [26].

To determine the moments, one performs a matching between the effective action (2.5) in the long wavelength limit and the action valid below the orbital scale (2.2), which depends on both radiation and potential modes of the gravitational field. The latter action is used in order to compute the one-graviton emission amplitude. As a result, by definition the resulting action takes the form

$$\Gamma[\bar{h}] = -\frac{1}{2m_{\rm Pl}}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}, \tag{2.6}$$

where $T^{\mu\nu}$ is the stress-energy pseudotensor of the system. Relations from the Ward identity $\partial_\mu T^{\mu\nu} = 0$ as well as the on-shell equations of motion can be used to bring both actions (2.5) and (2.6) in a comparable form. After that, a general form for the mass quadrupole moment is obtained in terms of the components of the stress-energy pseudotensor and its derivatives,

$$\begin{aligned} I^{ij} = \sum_{p=0}^{\infty}\frac{5!!}{(2p)!!(5+2p)!!}\Bigg\{&\left(1+\frac{2p(3+p)}{3}\right)\left[\int d^3\mathbf{x}\partial_0^{2p}T^{00}\mathbf{x}^{2p}\mathbf{x}^i\mathbf{x}^j\right]_{TF} + \left(1+\frac{p}{3}\right)\left[\int d^3\mathbf{x}\partial_0^{2p}T^{ll}\mathbf{x}^{2p}\mathbf{x}^i\mathbf{x}^j\right]_{TF} \\ &-\frac{4}{3}\left(1+\frac{p}{2}\right)\left[\int d^3\mathbf{x}\partial_0^{2p+1}T^{0l}\mathbf{x}^{2p}\mathbf{x}^l\mathbf{x}^i\mathbf{x}^j\right]_{TF} + \frac{1}{6}\left[\int d^3\mathbf{x}\partial_0^{2p+2}T^{kl}\mathbf{x}^{2p}\mathbf{x}^k\mathbf{x}^l\mathbf{x}^i\mathbf{x}^j\right]_{TF}\Bigg\}, \end{aligned} \tag{2.7}$$

where TF stands for trace free.[4] For the exact expressions for the multipole moments in all orders in the PN expansion, see [28]. The leading-order contribution to the mass quadrupole moment comes from just one term

$$I^{ij}_{\rm 0PN} = \left[\int d^3\mathbf{x}T^{00}_{\rm 0PN}\mathbf{x}^i\mathbf{x}^j\right]_{TF} = \sum_a m_a[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}, \tag{2.8}$$

while its 1PN correction [26] is given by four different contributions of the components of the stress-energy pseudotensor:

$$\begin{aligned} I^{ij}_{\rm 1PN} &= \left[\int d^3\mathbf{x}T^{00}_{\rm 1PN}\mathbf{x}^i\mathbf{x}^j\right]_{TF} + \left[\int d^3\mathbf{x}T^{ll}_{\rm 0PN}\mathbf{x}^i\mathbf{x}^j\right]_{TF} - \frac{4}{3}\left[\int d^3\mathbf{x}\partial_0T^{0l}_{\rm 0PN}\mathbf{x}^l\mathbf{x}^i\mathbf{x}^j\right]_{TF} + \frac{11}{42}\left[\int d^3\mathbf{x}\partial_0^2T^{00}_{\rm 0PN}\mathbf{x}^2\mathbf{x}^i\mathbf{x}^j\right]_{TF} \\ &= \sum_a m_a\left[\left(\frac{3}{2}\mathbf{v}_a^2 - \sum\nolimits_{b\neq a}\frac{Gm_b}{r}\right)\mathbf{x}_a^i\mathbf{x}_a^j + \frac{11}{42}\frac{d^2}{dt^2}(\mathbf{x}_a^2\mathbf{x}_a^i\mathbf{x}_a^j) - \frac{4}{3}\frac{d}{dt}(\mathbf{x}_a\cdot\mathbf{v}_a\mathbf{x}_a^i\mathbf{x}_a^j)\right]_{TF}. \end{aligned} \tag{2.9}$$

[4]More precisely, the multipole moments are symmetric trace-free (STF) quantities, but we are suppressing the "S" in the label to avoid redundancy since the general expression for the quadrupole moment is explicitly written as a symmetric tensor already.

The 2PN correction to the leading order mass quadrupole moment is given by

$$I^{ij}_{2\mathrm{PN}} = \left[\int d^3\mathbf{x} T^{00}_{2\mathrm{PN}} \mathbf{x}^i \mathbf{x}^j\right]_{TF} + \left[\int d^3\mathbf{x} T^{ll}_{1\mathrm{PN}} \mathbf{x}^i \mathbf{x}^j\right]_{TF} - \frac{4}{3}\left[\int d^3\mathbf{x} \partial_0 T^{0l}_{1\mathrm{PN}} \mathbf{x}^l \mathbf{x}^i \mathbf{x}^j\right]_{TF} + \frac{1}{6}\left[\int d^3\mathbf{x} \partial_0^2 T^{kl}_{0\mathrm{PN}} \mathbf{x}^k \mathbf{x}^l \mathbf{x}^i \mathbf{x}^j\right]_{TF} + \frac{11}{42}\left[\int d^3\mathbf{x} \partial_0^2 T^{00}_{1\mathrm{PN}} \mathbf{x}^2 \mathbf{x}^i \mathbf{x}^j\right]_{TF} + \frac{2}{21}\left[\int d^3\mathbf{x} \partial_0^2 T^{ll}_{0\mathrm{PN}} \mathbf{x}^2 \mathbf{x}^i \mathbf{x}^j\right]_{TF} - \frac{1}{7}\left[\int d^3\mathbf{x} \partial_0^3 T^{0l}_{0\mathrm{PN}} \mathbf{x}^2 \mathbf{x}^l \mathbf{x}^i \mathbf{x}^j\right]_{TF} + \frac{23}{1512}\left[\int d^3\mathbf{x} \partial_0^4 T^{00}_{0\mathrm{PN}} \mathbf{x}^4 \mathbf{x}^i \mathbf{x}^j\right]_{TF} + I^{ij}_{1\mathrm{PN}}(\mathbf{a}_{1\mathrm{PN}}). \tag{2.10}$$

Notice that the last term in the expression above arises from the last two terms of (2.9) after using the equations of motion. While $T^{00}_{0\mathrm{PN}}$ and $T^{0l}_{0\mathrm{PN}}$ are known, the higher PN order components $T^{00}_{2\mathrm{PN}}$, $T^{0i}_{1\mathrm{PN}}$, $T^{ll}_{1\mathrm{PN}}$ have yet to be obtained in the EFT formalism.

III. HIGHER ORDER STRESS-ENERGY TENSORS

Introducing the partial Fourier transform of the stress-energy pseudotensor $T^{\mu\nu}(t,\mathbf{k}) = \int d^3x T^{\mu\nu}(t,\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}$, we consider the long wavelength limit $\mathbf{k}\to 0$ to write

$$T^{\mu\nu}(t,\mathbf{k}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\left(\int d^3\mathbf{x} T^{\mu\nu}(t,\mathbf{x})\mathbf{x}^{i_1}\ldots\mathbf{x}^{i_n}\right)\mathbf{k}_{i_1}\ldots\mathbf{k}_{i_n}, \tag{3.1}$$

where each term in this expansion corresponds to a sum of Feynman diagrams that scale as a definite power of the parameter v. This partial Fourier transform is convenient since Feynman graphs are more easily handled in momentum space and, with the pseudotensor written in this way, we can read off the contributions to the mass quadrupole moment (2.10), the ultimate goal of this paper.

A. 2PN correction to T^{00}

The leading order and the next-to-leading order temporal components of the pseudotensor, obtained in [26] using the EFT techniques summarized in the previous section, are given by

$$T^{00}_{0\mathrm{PN}}(t,\mathbf{k}) = \sum_a m_a e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.2}$$

$$T^{00}_{1\mathrm{PN}}(t,\mathbf{k}) = \left[\sum_a \frac{1}{2} m_a \mathbf{v}_a^2 - \sum_{a\neq b}\frac{G m_a m_b}{2r} + O(\mathbf{k}) + \cdots\right] \times e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.3}$$

If we take into account the zeroth order term of the exponential expanded in the radiation momentum $\mathbf{k}$, we see that the leading order pseudotensor provides the total mass whereas the next-to-leading order represents the Newtonian energy of a dynamical two-body system. These quantities scale as mv^0 and mv^2, respectively. Hence, to obtain the 2PN correction to the leading order T^{00}, we have to calculate all Feynman diagrams that contribute to the one-graviton $\bar{h}_{00}$ emission and enter at order v^4.

The simplest contribution to the second PN correction for the temporal component of the stress-energy pseudotensor is illustrated in Fig. 1 and comes from the source action term (A7). Comparing this diagram against (2.6), we extract the following contribution to the pseudotensor:

$$T^{00}_{\mathrm{Fig1}}(t,\mathbf{k}) = \sum_a \frac{3}{8} m_a \mathbf{v}_a^4 e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.4}$$

By expanding the exponential up to the second order in the radiation momentum $\mathbf{k}$, we read off the contribution for the mass quadrupole moment:

$$\int d^3\mathbf{x} T^{00}_{\mathrm{Fig1}}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_a \frac{3}{8} m_a \mathbf{v}_a^4 [\mathbf{x}_a^i \mathbf{x}_a^j]_{TF}. \tag{3.5}$$

The diagrams that contain the exchange of one potential graviton are shown in Fig. 2 and are composed by the

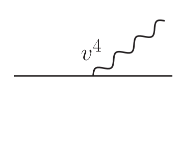


FIG. 1. No graviton exchange between the two particles, one external $\bar{h}^{00}$ momentum.

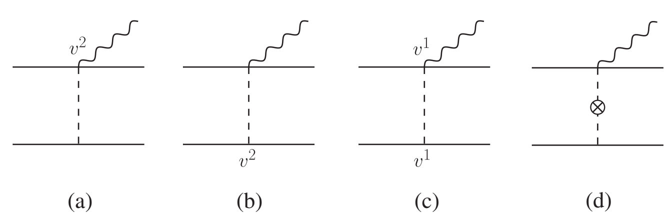


FIG. 2. One-graviton exchange with external $\bar{h}^{00}$ momentum.

couplings between the source action terms (A1)–(A6) and also the propagator (A15) and its correction (A16). Notice that we need not separate out all of the various terms that arise in the Feynman rules into different orders in the PN expansion as is done in Appendix A. We also calculated covariant vertices, as is done when calculating in the post-Minkowskian (PM) expansion (see e.g., [33]), and then expand in v, as a calculational check. However, for pedagogical purposes we have separated Feynman rules into given orders in the PN expansion. The results from Fig. 2 are given by

$$T^{00}_{\mathrm{Fig2}a}(t,\mathbf{k}) = \sum_{a\neq b} \frac{5}{2}\frac{Gm_a m_b}{r}\mathbf{v}_a^2 e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.6}$$

$$T^{00}_{\mathrm{Fig2}b}(t,\mathbf{k}) = \sum_{a\neq b} \frac{3}{2}\frac{Gm_a m_b}{r}\mathbf{v}_b^2 e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.7}$$

$$T^{00}_{\mathrm{Fig2}c}(t,\mathbf{k}) = -\sum_{a\neq b} 4\frac{Gm_a m_b}{r}\mathbf{v}_a\cdot\mathbf{v}_b e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.8}$$

$$T^{00}_{\mathrm{Fig2}d}(t,\mathbf{k}) = \sum_{a\neq b} \frac{Gm_a m_b}{2r}(-\mathbf{a}_b^i\mathbf{r}^i + \mathbf{v}_b^2 - (\mathbf{v}_b\cdot\mathbf{n})^2) e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.9}$$

Leaving

$$\int d^3\mathbf{x} T^{00}_{\mathrm{Fig}2a-2d}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_{a\neq b}\frac{Gm_a m_b}{2r}[(5\mathbf{v}_a^2 + 4\mathbf{v}_b^2 - 8\mathbf{v}_a\cdot\mathbf{v}_b - \mathbf{a}_b\cdot\mathbf{r} - (\mathbf{v}_b\cdot\mathbf{n})^2)\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}. \tag{3.10}$$

Note the implicit dependence on the indices a, b in the quantities $\mathbf{r} = \mathbf{x}_a - \mathbf{x}_b$, $r = |\mathbf{r}|$ and $\mathbf{n} = \frac{\mathbf{r}}{r}$ inside the sum. Additionally, notice the presence of an acceleration term in (3.9), which indicates that we are not using the equations of motion to reduce the accelerations. In fact, we will use the equations of motion to write the final expression for the mass quadrupole moment at 2PN order later on in this paper, after all contributions have been computed.

The graphs in Fig. 3 are composed by the source terms (A1)–(A3) together with the vertices (A17)–(A21) and (A16). Note that we multipole expand the denominators in $\mathbf{k}/\mathbf{q}\sim v$

$$\frac{1}{\mathbf{q}^2(\mathbf{q}+\mathbf{k})^2} = \frac{1}{\mathbf{q}^4} - \frac{2(\mathbf{q}\cdot\mathbf{k})}{\mathbf{q}^6} + \frac{4(\mathbf{q}\cdot\mathbf{k})^2}{\mathbf{q}^8} + \cdots. \tag{3.11}$$

In calculating the contributions to the mass quadrupole sourced by the temporal components of the pseudotensor at 2PN, we are allowed to drop terms depending on $\mathbf{k}^2$ in the expansion of the denominator, since those terms contribute to the trace part of the mass quadrupole, which is removed in the definition of the STF moment. The results are organized in orders of the radiation momentum, as it is shown below:

$$\begin{aligned}
T^{00}_{\mathrm{Fig3}a}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{4r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Big\{ & 2(\mathbf{v}^2 + \mathbf{a}\cdot\mathbf{r} - \dot{r}^2) + 5\mathbf{v}_a\cdot\mathbf{v}_b - 5\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} \\
& + i\mathbf{k}^i\Big[\Big(\mathbf{v}^2 + \mathbf{a}\cdot\mathbf{r} - \dot{r}^2 + \frac{5}{2}\mathbf{v}_a\cdot\mathbf{v}_b - \frac{5}{2}\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}\Big)\mathbf{r}^i \\
& + \Big(\frac{1}{2}r\dot{r} + \frac{5}{2}\mathbf{v}_b\cdot\mathbf{r}\Big)\mathbf{v}_b^i - \Big(2r\dot{r} + \frac{5}{2}\mathbf{v}_b\cdot\mathbf{r}\Big)\mathbf{v}_a^i - r^2(\mathbf{a}_a^i + \mathbf{a}_b^i)\Big] \\
& + \frac{1}{6}\mathbf{k}^i\mathbf{k}^j[-(2\mathbf{v}^2 + 5\mathbf{v}_a\cdot\mathbf{v}_b - 2\dot{r}^2 - 5\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + 2\mathbf{a}\cdot\mathbf{r})\mathbf{r}^i\mathbf{r}^j \\
& + (4\mathbf{v}_a\cdot\mathbf{r} + \mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{r}^j - (2\mathbf{v}_a\cdot\mathbf{r} + 8\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_b^i\mathbf{r}^j \\
& + r^2(-4\mathbf{v}^i\mathbf{v}^j - 7\mathbf{v}_a^i\mathbf{v}_b^j + 2\mathbf{a}_a^i\mathbf{r}^j + 4\mathbf{a}_b^i\mathbf{r}^j)]\Big\} + O(\mathbf{k}^3) + \cdots,
\end{aligned} \tag{3.12}$$

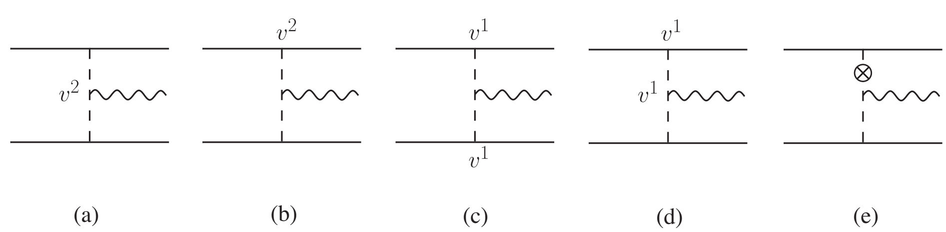


FIG. 3. Diagrams with two potential gravitons coupled to $\bar{h}_{00}$.

$$T^{00}_{\mathrm{Fig}3b}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{Gm_a m_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left[\frac{7}{4}\mathbf{v}_b^2+\frac{3}{4}\mathbf{v}_a^2-\frac{i}{2}\mathbf{k}^i\mathbf{v}_b^2\mathbf{r}^i+\frac{1}{2}\mathbf{k}^i\mathbf{k}^j\left(\frac{1}{2}\mathbf{v}_b^2\mathbf{r}^i\mathbf{r}^j+2\mathbf{r}^2\mathbf{v}_b^i\mathbf{v}_b^j\right)\right]+O(\mathbf{k}^3)+\cdots, \tag{3.13}$$

$$T^{00}_{\mathrm{Fig}3c}(t,\mathbf{k}) = \sum_{a\neq b}\frac{2Gm_a m_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}(2\mathbf{v}_a\cdot\mathbf{v}_b+\mathbf{k}^i\mathbf{k}^j\mathbf{r}^2\mathbf{v}_a^i\mathbf{v}_b^j)+O(\mathbf{k}^3)+\cdots, \tag{3.14}$$

$$\begin{aligned}T^{00}_{\mathrm{Fig}3d}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{2Gm_a m_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\bigg\{&-\frac{i}{2}\mathbf{k}^i[2\mathbf{r}^2\mathbf{a}_a^i+2\mathbf{v}_a^i(\mathbf{v}_b\cdot\mathbf{r}+\mathbf{v}_a\cdot\mathbf{r})]\\ &-\frac{1}{2}\mathbf{k}^i\mathbf{k}^j[\mathbf{r}^2(\mathbf{v}_a^i\mathbf{v}_a^j-\mathbf{v}_a^i\mathbf{v}_b^i-\mathbf{r}^j\mathbf{a}_a^i)-\mathbf{v}_a^i\mathbf{r}^j(\mathbf{v}_a\cdot\mathbf{r}+\mathbf{v}_b\cdot\mathbf{r})]\bigg\}+O(\mathbf{k}^3)+\cdots,\end{aligned} \tag{3.15}$$

$$\begin{aligned}T^{00}_{\mathrm{Fig}3e}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{Gm_a m_b}{4r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\bigg\{&6(-\mathbf{a}_b\cdot\mathbf{r}+\mathbf{v}_b^2-(\mathbf{v}_b\cdot\mathbf{n})^2)-\frac{3i}{2}\mathbf{k}^i[(\mathbf{a}_b\cdot\mathbf{r}-\mathbf{v}_b^2+(\mathbf{v}_b\cdot\mathbf{n})^2)\mathbf{r}^i-2\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_b^i+\mathbf{r}^2\mathbf{a}_b^i]\\ &-\frac{1}{2}\mathbf{k}^i\mathbf{k}^j[(-\mathbf{a}_b\cdot\mathbf{r}+\mathbf{v}_b^2-(\mathbf{v}_b\cdot\mathbf{n})^2)\mathbf{r}^i\mathbf{r}^j+4\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_b^i\mathbf{r}^j-2\mathbf{r}^2\mathbf{a}_b^i\mathbf{r}^j+2\mathbf{r}^2\mathbf{v}_b^i\mathbf{v}_b^j]\bigg\}+O(\mathbf{k}^3)+\cdots.\end{aligned} \tag{3.16}$$

Together, these quantities provide us with the following contribution:

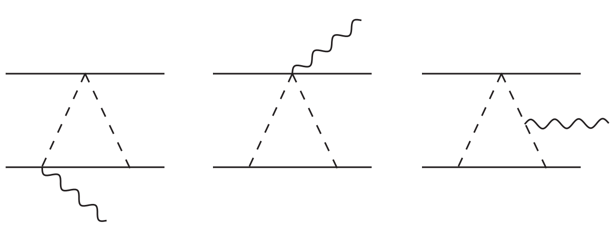
$$\begin{aligned}\int d^3\mathbf{x}T^{00}_{\mathrm{Fig}3a-3e}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_{a\neq b}\frac{Gm_a m_b}{12r}[&(-2\mathbf{v}_a^2-35\mathbf{v}_b^2+26\mathbf{v}_a\cdot\mathbf{v}_b-10\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}\\ &+3(\mathbf{v}_a\cdot\mathbf{n})^2+12(\mathbf{v}_b\cdot\mathbf{n})^2-4\dot{r}^2+\mathbf{a}_a\cdot\mathbf{r}+8\mathbf{a}_b\cdot\mathbf{r})\mathbf{x}_a^i\mathbf{x}_a^j\\ &+(\mathbf{v}_a^2+\mathbf{v}_a\cdot\mathbf{v}_b-5\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}+3(\mathbf{v}_a\cdot\mathbf{n})^2-2\dot{r}^2+\mathbf{a}_a\cdot\mathbf{r})\mathbf{x}_a^i\mathbf{x}_b^j\\ &+(\mathbf{v}_a\cdot\mathbf{r}+\mathbf{v}_b\cdot\mathbf{r})(-20\mathbf{v}_a^i\mathbf{x}_a^j+26\mathbf{v}_a^i\mathbf{x}_b^j)\\ &+\mathbf{r}^2(2\mathbf{v}_a^i\mathbf{v}_a^j-\mathbf{v}_a^i\mathbf{v}_b^j-22\mathbf{a}_a^i\mathbf{x}_a^j-23\mathbf{a}_a^i\mathbf{x}_b^j)]_{STF}.\end{aligned} \tag{3.17}$$

Contributions from Fig. 4 are composed of the source terms (A1), (A4), (A8) and (A9) and yield

$$T^{00}_{\mathrm{Fig}4a}(t,\mathbf{k}) = \sum_{a\neq b}\frac{G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.18}$$

$$T^{00}_{\mathrm{Fig}4b}(t,\mathbf{k}) = \sum_{a\neq b}\frac{3G^2m_am_b^2}{2r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.19}$$

$$T^{00}_{\mathrm{Fig}4c}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{3G^2m_am_bm}{2r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.20}$$



(a) (b) (c)

FIG. 4. Two-potential-graviton exchange with external $\bar{h}^{00}$ momentum.

which gives us

$$\int d^3\mathbf{x}T^{00}_{\mathrm{Fig}4a-4c}[\mathbf{x}^i\mathbf{x}^j]_{TF} = -\sum_{a\neq b}\frac{G^2m_a^2m_b}{2r^2}[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}. \tag{3.21}$$

The diagrams illustrated in Fig. 5 are composed of the three-potential-graviton vertices (A28)–(A30) as well as the three-potential-one-radiation-graviton vertex (A32)–(A33) in composition with (A1) and (A4) contribute to T^{00}_{2PN}. These diagrams give

$$T^{00}_{\mathrm{Fig}5a}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{G^2m_am_b^2}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.22}$$

$$T^{00}_{\mathrm{Fig}5b}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{2G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.23}$$

$$T^{00}_{\mathrm{Fig}5c}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left(\frac{1}{2}-\frac{7}{2}i\mathbf{k}^i\mathbf{r}^i+\frac{5}{3}\mathbf{k}^i\mathbf{k}^j\mathbf{r}^i\mathbf{r}^j\right)+O(\mathbf{k}^3)+\cdots, \tag{3.24}$$

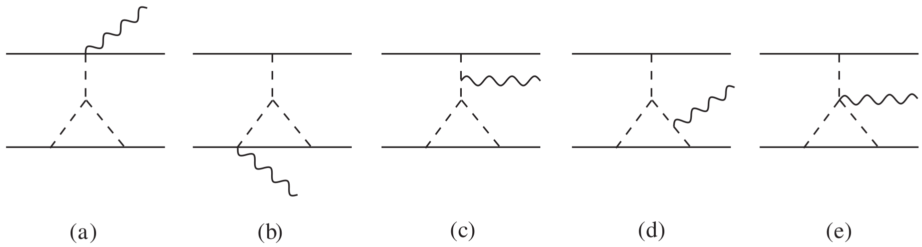


FIG. 5. Three-potential-graviton exchange with external $\bar{h}^{00}$ momentum.

$$T^{00}_{\mathrm{Fig}5d}(t,\mathbf{k})=\sum_{a\neq b}\frac{G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left(5-2i\mathbf{k}^i\mathbf{r}^i+\frac{2}{3}\mathbf{k}^i\mathbf{k}^j\mathbf{r}^i\mathbf{r}^j\right)+O(\mathbf{k}^3)+\cdots, \tag{3.25}$$

$$T^{00}_{\mathrm{Fig}5e}(t,\mathbf{k})=-\sum_{a\neq b}\frac{G^2m_am_b^2}{2r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.26}$$

Keeping terms to second order in the radiation momentum we have

$$\int d^3\mathbf{x}T^{00}_{\mathrm{Fig}5a-5e}[\mathbf{x}^i\mathbf{x}^j]_{TF}=\sum_{a\neq b}\frac{G^2m_am_b}{r^2}\left[\frac{3}{2}(m_a-m_b)\mathbf{x}_a^i\mathbf{x}_a^j-m_a\mathbf{x}_a^i\mathbf{x}_b^j+2m_a\mathbf{x}_b^i\mathbf{x}_b^j\right]_{TF}. \tag{3.27}$$

Summing the contributions (3.5), (3.10), (3.17), (3.21) and (3.27), the total contribution of $T^{00}_{2\mathrm{PN}}$ to the mass quadrupole is

$$\begin{aligned}\int d^3\mathbf{x}T^{00}_{2\mathrm{PN}}[\mathbf{x}^i\mathbf{x}^j]_{TF}=&\sum_a\frac{3}{8}m_a\mathbf{v}_a^4[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}+\sum_{a\neq b}\frac{Gm_am_b}{12r}\Bigg[\Bigg(28\mathbf{v}_a^2-11\mathbf{v}_b^2-22\mathbf{v}_a\cdot\mathbf{v}_b-10\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}\\&+3(\mathbf{v}_a\cdot\mathbf{n})^2+6(\mathbf{v}_b\cdot\mathbf{n})^2-4\dot{r}^2+\mathbf{a}_a\cdot\mathbf{r}+2\mathbf{a}_b\cdot\mathbf{r}+12\frac{Gm_a}{r}+6\frac{Gm_b}{r}\Bigg)\mathbf{x}_a^i\mathbf{x}_a^j\\&+\left(\mathbf{v}_a^2+\mathbf{v}_a\cdot\mathbf{v}_b-5\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}+3(\mathbf{v}_a\cdot\mathbf{n})^2-2\dot{r}^2+\mathbf{a}_a\cdot\mathbf{r}-12\frac{Gm_a}{r}\right)\mathbf{x}_a^i\mathbf{x}_b^j\\&+(\mathbf{v}_a\cdot\mathbf{r}+\mathbf{v}_b\cdot\mathbf{r})(-20\mathbf{v}_a^i\mathbf{x}_a^j+26\mathbf{v}_a^i\mathbf{x}_b^j)+\mathbf{r}^2(2\mathbf{v}_a^i\mathbf{v}_a^j-\mathbf{v}_a^i\mathbf{v}_b^j-22\mathbf{a}_a^i\mathbf{x}_a^j-23\mathbf{a}_a^i\mathbf{x}_b^j)\Bigg]_{STF}.\end{aligned} \tag{3.28}$$

B. 1PN correction to T^{0i}

The leading order T^{0i} component obtained in [26] is

$$T^{0i}_{0\mathrm{PN}}(t,\mathbf{k})=\sum_a m_a\mathbf{v}_a^ie^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.29}$$

The 1PN corrections enter at v^3 and are shown in Fig. 6.

To extract the T^{0i} contributions to the mass quadrupole moment, which is the third term in (2.10), the expansion of the denominator of vertices in Fig. 6(c) and 6(d) has to be carried out to third order. In addition, $\mathbf{k}^2$ terms cannot be dropped, since they contribute terms that cannot be included in the trace part of the quadrupole.

Comparing the diagrams illustrated in Fig. 6, which are composed of (A1), (A2), (A10), (A11) together with (A15), (A22) and (A23) we find

$$T^{0l}_{\mathrm{Fig}6a}(t,\mathbf{k})=\sum_a\frac{m_a}{2}\mathbf{v}_a^l\mathbf{v}_a^2e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.30}$$

$$T^{0l}_{\mathrm{Fig}6b}(t,\mathbf{k})=\sum_{a\neq b}\frac{Gm_am_b}{r}\mathbf{v}_a^le^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.31}$$

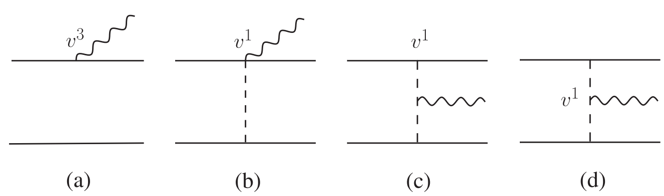


FIG. 6. All diagrams that contribute to $T^{0i}_{1\mathrm{PN}}$.

$$T^{0l}_{\mathrm{Fig6}c}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Big[-2\mathbf{v}_a^l + 2i\mathbf{k}^i(\mathbf{v}_a^i\mathbf{r}^l - \mathbf{r}^i\mathbf{v}_a^l) + \mathbf{k}^i\mathbf{k}^j(\mathbf{r}^i\mathbf{r}^j\mathbf{v}_a^l - \mathbf{v}_a^i\mathbf{r}^j\mathbf{r}^l) + \frac{i}{6}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^k(\mathbf{r}^2\delta^{ij}\mathbf{v}_a^k\mathbf{r}^l - \mathbf{r}^2\delta^{il}\mathbf{v}_a^j\mathbf{r}^k - 2\mathbf{v}_a^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l + 2\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{v}_a^l)\Big] + O(\mathbf{k}^4) + \cdots, \tag{3.32}$$

$$\begin{aligned} T^{0l}_{\mathrm{Fig6}d}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{4r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Big\{&\mathbf{v}_a^l + \mathbf{v}_b^l - \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}^l - \frac{i}{2}\mathbf{k}^i\left(3r\dot{r}\delta^{il} - \mathbf{r}^i(\mathbf{v}_a^l+\mathbf{v}_b^l) + \mathbf{v}^i\mathbf{r}^l + \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^l\right)\\ &+\frac{1}{6}\mathbf{k}^i\mathbf{k}^j\Big[-5\mathbf{r}^2(\mathbf{v}_a^i+\mathbf{v}_b^i)\delta^{jl} + (4\mathbf{v}_a\cdot\mathbf{r} - 5\mathbf{v}_b\cdot\mathbf{r})\mathbf{r}^i\delta^{jl} + (\mathbf{v}_a^i - 2\mathbf{v}_b^i)\mathbf{r}^j\mathbf{r}^l\\ &+(\mathbf{v}_a^l+\mathbf{v}_b^l)\left(\frac{1}{2}\delta^{ij}\mathbf{r}^2 - \mathbf{r}^i\mathbf{r}^j\right) + (\mathbf{v}_a\cdot\mathbf{r}+\mathbf{v}_b\cdot\mathbf{r})\left(\frac{1}{2}\delta^{ij}\mathbf{r}^l + \frac{1}{r^2}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^l\right)\Big]\\ &-\frac{i}{24}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^k\Big[\delta^{kl}(6\mathbf{r}^2\mathbf{v}_a^i\mathbf{r}^j + 14\mathbf{r}^2\mathbf{v}_b^i\mathbf{r}^j - 5\mathbf{v}_a\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^j + 7\mathbf{v}_b\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^j)\\ &+\delta^{ij}\mathbf{r}^2(3\delta^{kl}\mathbf{v}_a\cdot\mathbf{r} - 3\delta^{kl}\mathbf{v}_b\cdot\mathbf{r} - \mathbf{r}^k\mathbf{v}_b^l - \mathbf{r}^k\mathbf{v}_a^l + \mathbf{v}_a^k\mathbf{r}^l - \mathbf{v}_b^k\mathbf{r}^l) - \delta^{ij}(\mathbf{v}_a\cdot\mathbf{r}+\mathbf{v}_b\cdot\mathbf{r})\mathbf{r}^k\mathbf{r}^l\\ &+\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k(\mathbf{v}_a^l+\mathbf{v}_b^l) + (3\mathbf{v}_b^i - \mathbf{v}_a^i)\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l - \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\Big]\Big\} + O(\mathbf{k}^4) + \cdots. \end{aligned} \tag{3.33}$$

Expanding the exponentials up to the third order in the radiation momentum, we get

$$\begin{aligned} \int d^3\mathbf{x}\,\partial_0 T^{0l}_{1\mathrm{PN}}\mathbf{x}^l[\mathbf{x}^i\mathbf{x}^j]_{TF} = &\sum_a\frac{d}{dt}\left[\frac{1}{2}m_a\mathbf{v}_a^2\mathbf{v}_a\cdot\mathbf{x}_a\mathbf{x}_a^i\mathbf{x}_a^j\right]_{TF}\\ &+\sum_{a\neq b}\frac{d}{dt}\Big\{\frac{Gm_a m_b}{12r}\Big[(8\mathbf{r}^2 - 20\mathbf{r}\cdot\mathbf{x}_b)\mathbf{v}_a^i\mathbf{x}_a^j + (20\mathbf{r}^2 - 22\mathbf{r}\cdot\mathbf{x}_b)\mathbf{v}_a^i\mathbf{x}_b^j\\ &+\left(22\mathbf{v}_a\cdot\mathbf{x}_a - 30\mathbf{v}_b\cdot\mathbf{x}_a - 8\mathbf{v}_a\cdot\mathbf{x}_b + 8\mathbf{v}_b\cdot\mathbf{x}_b - \frac{2}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_a^j\\ &+\left(9\mathbf{v}_a\cdot\mathbf{x}_a - 7\mathbf{v}_a\cdot\mathbf{x}_b - \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_b^j\Big]\Big\}_{STF}. \end{aligned} \tag{3.34}$$

C. 1PN correction to T^{ii}

The leading order T^{ii} component obtained in [26] has the form

$$T^{ii}_{0\mathrm{PN}}(t,\mathbf{k}) = \left(\sum_a m_a\mathbf{v}_a^2 - \sum_{a\neq b}\frac{Gm_a m_b}{2r} + O(\mathbf{k}) + \cdots\right)\times e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.35}$$

Notice that, while $T^{0i}_{0\mathrm{PN}}$ in (3.29) is down by v^1 relative to $T^{00}_{0\mathrm{PN}}$ in (3.2), the leading order spatial component (3.35) is down by v^2 compared to $T^{00}_{0\mathrm{PN}}$; this fixes the PN hierarchy among the components T^{00}, T^{i0}, and T^{ij} of the pseudotensor.

To obtain $T^{ii}_{1\mathrm{PN}}$ as well as its contributions to $I^{ij}_{2\mathrm{PN}}$ we have to compute all diagrams that enter at v^4 with one $\bar{h}^{ii}$ external momentum. To compute the spatial component of the pseudotensor and to extract its contribution to the mass quadrupole moment we have to carry out the expansions up to the second order in the radiation momentum. As in Sec. III A, $\mathbf{k}^2$ may be dropped.

The diagrams illustrated in Fig. 7 involve (A1), (A8), (A12), (A13), (A15), and (A24) which give

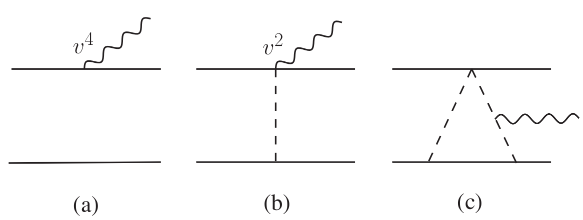


FIG. 7. Diagrams with $\bar{h}^{ii}$ external momentum.

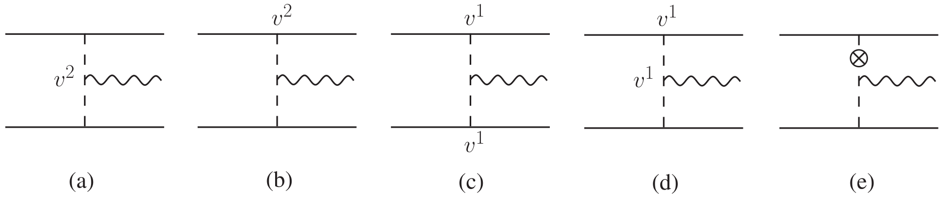


FIG. 8. One potential graviton exchange with $\bar{h}^{ii}$ external momentum.

$$T^{ll}_{\mathrm{Fig}7a}(t,\mathbf{k}) = \sum_a \frac{m_a}{2}\mathbf{v}_a^4 e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.36}$$

$$T^{ll}_{\mathrm{Fig}7b}(t,\mathbf{k}) = \sum_{a\neq b} \frac{Gm_a m_b}{r}\mathbf{v}_a^2 e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.37}$$

$$T^{ll}_{\mathrm{Fig}7c}(t,\mathbf{k}) = -\sum_{a\neq b} \frac{G^2 m_a m_b m}{2r^2} e^{-i\mathbf{k}\cdot\mathbf{x}_a}. \tag{3.38}$$

It is straightforward to extract the contribution for the mass quadrupole moment by expanding the exponentials up to the second order in the radiation momentum,

$$\int d^3\mathbf{x} T^{ll}_{\mathrm{Fig}7a-7c}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_a \frac{m_a}{2}\mathbf{v}_a^4[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF} + \sum_{a\neq b}\frac{Gm_a m_b}{r}\left(\mathbf{v}_a^2 - \frac{Gm}{2r}\right)[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}. \tag{3.39}$$

The computation of $T^{ii}_{1\mathrm{PN}}$ follows from the diagrams shown in Fig. 8 which involve (A1)–(A3) and (A15), (A16), (A24)–(A27),

$$\begin{aligned} T^{ll}_{\mathrm{Fig}8a}(t,\mathbf{k}) = \sum_{a\neq b}\frac{3Gm_a m_b}{4r} e^{-i\mathbf{k}\cdot\mathbf{x}_a}\bigg\{ & 2\mathbf{v}^2 + \mathbf{v}_a\cdot\mathbf{v}_b - 2\dot{r}^2 - \frac{1}{r^2}\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r} + 2\mathbf{a}\cdot\mathbf{r} \\ & + \frac{i}{2}\mathbf{k}^i\bigg[\left(2\mathbf{v}^2 + \mathbf{v}_a\cdot\mathbf{v}_b - 2\dot{r}^2 - \frac{1}{r^2}\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r} + 2\mathbf{a}\cdot\mathbf{r}\right)\mathbf{r}^i \\ & + \mathbf{v}_b^i(4\mathbf{v}_b\cdot\mathbf{r} - 3\mathbf{v}_a\cdot\mathbf{r}) + \mathbf{v}_a^i(3\mathbf{v}_b\cdot\mathbf{r} - 4\mathbf{v}_a\cdot\mathbf{r}) - 2r^2(\mathbf{a}_a^i + \mathbf{a}_b^i)\bigg] \\ & + \frac{1}{6}\mathbf{k}^i\mathbf{k}^j\bigg[\left(-2\mathbf{v}^2 - \mathbf{v}_a\cdot\mathbf{v}_b - 2\mathbf{a}\cdot\mathbf{r} + 2\dot{r}^2 + \frac{1}{r^2}\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r}\right)\mathbf{r}^i\mathbf{r}^j \\ & + (6\mathbf{v}_a\cdot\mathbf{r} - 8\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_b^i\mathbf{r}^j + (4\mathbf{v}_a\cdot\mathbf{r} - 3\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{r}^j \\ & + r^2(-4\mathbf{v}_a^i\mathbf{v}_a^j - 3\mathbf{v}_a^i\mathbf{v}_b^j - 4\mathbf{v}_b^i\mathbf{v}_b^j - 4\mathbf{a}^i\mathbf{r}^j + 6\mathbf{a}_a^i\mathbf{r}^j)\bigg]\bigg\} + O(\mathbf{k}^3) + \cdots, \end{aligned} \tag{3.40}$$

$$T^{ll}_{\mathrm{Fig}8b}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{r} e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left\{\frac{1}{4}(\mathbf{v}_a^2 + \mathbf{v}_b^2) - i\mathbf{v}_a^2\mathbf{k}^i\mathbf{r}^i - \mathbf{k}^i\mathbf{k}^j\left(\mathbf{r}^2\mathbf{v}_a^i\mathbf{v}_a^j - \frac{1}{2}\mathbf{v}_a^2\mathbf{r}^i\mathbf{r}^j\right)\right\} + O(\mathbf{k}^3) + \cdots, \tag{3.41}$$

$$\begin{aligned} T^{ll}_{\mathrm{Fig}8c}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{2r} e^{-i\mathbf{k}\cdot\mathbf{x}_a}[& -4\mathbf{v}_a\cdot\mathbf{v}_b - i\mathbf{k}^i(2\mathbf{v}_a\cdot\mathbf{v}_b\mathbf{r}^i + 4\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b^i - 4\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_a^i) \\ & + \mathbf{k}^i\mathbf{k}^j(2\mathbf{r}^2\mathbf{v}_a^i\mathbf{v}_b^j + \mathbf{v}_a\cdot\mathbf{v}_b\mathbf{r}^i\mathbf{r}^j - 2\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_a^i\mathbf{r}^j + 2\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b^i\mathbf{r}^j)] + O(\mathbf{k}^3) + \cdots, \end{aligned} \tag{3.42}$$

$$T^{ll}_{\mathrm{Fig8}d}(t,\mathbf{k}) = \sum_{a\neq b}\frac{Gm_a m_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x_a}}\left\{-4\left(\mathbf{v}\cdot\mathbf{v}_a+\mathbf{a}_a\cdot\mathbf{r}-\frac{1}{r^2}\mathbf{v}\cdot\mathbf{r}\mathbf{v}_a\cdot\mathbf{r}\right)\right.$$
$$-2i\mathbf{k}^i\left[\mathbf{r}^i\left(\mathbf{v}\cdot\mathbf{v}_a+\mathbf{a}_a\cdot\mathbf{r}-\frac{\dot{r}}{r}\mathbf{v}_a\cdot\mathbf{r}\right)+\mathbf{v}_a\cdot\mathbf{r}(\mathbf{v}^i-2\mathbf{v}_a^i)\right]$$
$$+\frac{1}{3}\mathbf{k}^i\mathbf{k}^j[\mathbf{r}^2(\mathbf{v}^i\mathbf{v}_a^j+\mathbf{a}_a\mathbf{r}^i)+2\mathbf{r}^i\mathbf{r}^j(\mathbf{v}\cdot\mathbf{v}_a+\mathbf{a}_a\cdot\mathbf{r})$$
$$\left.+r\dot{r}\mathbf{v}_a^i\mathbf{r}^j+2\mathbf{v}_a\cdot\mathbf{r}(2\mathbf{v}^i\mathbf{r}^j-3\mathbf{v}_a^i\mathbf{r}^j)-2\frac{\dot{r}}{r}\mathbf{v}_a\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^j]\right\}+O(\mathbf{k}^3)+\cdots, \tag{3.43}$$

$$T^{ll}_{\mathrm{Fig8}e}(t,\mathbf{k}) = -\sum_{a\neq b}\frac{Gm_a m_b}{4r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left\{2(-\mathbf{a}_b\cdot\mathbf{r}+\mathbf{v}_b^2-(\mathbf{v}_b\cdot\mathbf{n})^2)-\frac{i}{2}\mathbf{k}^i[(\mathbf{a}_b\cdot\mathbf{r}-\mathbf{v}_b^2+(\mathbf{v}_b\cdot\mathbf{n})^2)\mathbf{r}^i-2\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_b^i+\mathbf{r}^2\mathbf{a}_b^i]\right.$$
$$\left.+\frac{i^2}{6}\mathbf{k}^i\mathbf{k}^j[(-\mathbf{a}_b\cdot\mathbf{r}+\mathbf{v}_b^2-(\mathbf{v}_b\cdot\mathbf{n})^2)\mathbf{r}^i\mathbf{r}^j+4\mathbf{v}_b\cdot\mathbf{r}\mathbf{v}_b^i\mathbf{r}^j-2\mathbf{r}^2\mathbf{a}_b^i\mathbf{r}^j+2\mathbf{r}^2\mathbf{v}_b^i\mathbf{v}_b^j]\right\}+O(\mathbf{k}^3)+\cdots, \tag{3.44}$$

which provide us with

$$\int d^3\mathbf{x}T^{ll}_{\mathrm{Fig8}a-8e}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_{a\neq b}\frac{Gm_a m_b}{12r}\{(10\mathbf{v}_a^2-17\mathbf{v}_b^2-10\mathbf{v}_a\cdot\mathbf{v}_b$$
$$+5(\mathbf{v}_a\cdot\mathbf{n})^2+2\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}-8(\mathbf{v}_b\cdot\mathbf{n})^2-5\mathbf{a}_a\cdot\mathbf{r}+8\mathbf{a}_b\cdot\mathbf{r})\mathbf{x}_a^i\mathbf{x}_a^j$$
$$+(-5\mathbf{v}_a^2+7\mathbf{v}_a\cdot\mathbf{v}_b+5(\mathbf{v}_a\cdot\mathbf{n})^2-7\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n}-5\mathbf{a}_a\cdot\mathbf{r})\mathbf{x}_a^i\mathbf{x}_b^j$$
$$+(4\mathbf{v}_a\cdot\mathbf{r}-44\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{x}_a^j+(14\mathbf{v}_a\cdot\mathbf{r}-58\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{x}_b^j$$
$$+r^2(38\mathbf{v}_a^i\mathbf{v}_a^j-7\mathbf{v}_a^i\mathbf{v}_b^j+14\mathbf{a}_a^i\mathbf{x}_a^j+19\mathbf{a}_a^i\mathbf{x}_b^j)\}_{STF}. \tag{3.45}$$

Finally, the diagrams containing a three-potential-graviton exchange shown in Fig. 9 which involve (A1), (A31), and (A34) give

$$T^{ll}_{\mathrm{Fig9}a}(t,\mathbf{k}) = \sum_{a\neq b}\frac{G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left(-\frac{5}{2}+\frac{7}{2}i\mathbf{k}^i\mathbf{r}^i-\frac{4}{3}\mathbf{k}^i\mathbf{k}^j\mathbf{r}^i\mathbf{r}^j\right)+O(\mathbf{k}^3)+\cdots, \tag{3.46}$$

$$T^{ll}_{\mathrm{Fig9}b}(t,\mathbf{k}) = \sum_{a\neq b}\frac{G^2m_a^2m_b}{r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left(1-6i\mathbf{k}^i\mathbf{r}^i+\frac{7}{3}\mathbf{k}^i\mathbf{k}^j\mathbf{r}^i\mathbf{r}^j\right)+O(\mathbf{k}^3)+\cdots, \tag{3.47}$$

$$T^{ll}_{\mathrm{Fig9}c}(t,\mathbf{k}) = \sum_{a\neq b}\frac{7G^2m_am_b^2}{2r^2}e^{-i\mathbf{k}\cdot\mathbf{x}_a}, \tag{3.48}$$

which lead to

$$\int d^3\mathbf{x}T^{ll}_{\mathrm{Fig9}a-9c}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_{a\neq b}\frac{G^2m_am_b}{r^2}\left[\frac{3}{2}m\mathbf{x}_a^i\mathbf{x}_a^j-m_a\mathbf{x}_a^i\mathbf{x}_b^j\right]_{STF}. \tag{3.49}$$

With this, we now write the total contribution of $T^{ll}_{1\mathrm{PN}}$ to the mass quadrupole,

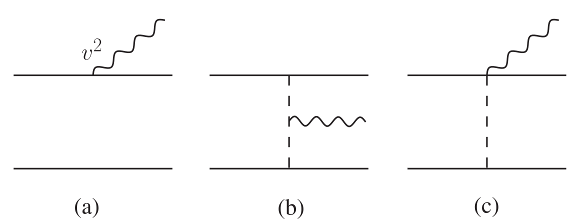


FIG. 9. Three-potential-graviton exchange with $\bar{h}^{ii}$ external momentum.

$$\int d^3\mathbf{x}T^{ll}_{1\mathrm{PN}}[\mathbf{x}^i\mathbf{x}^j]_{TF} = \sum_a \frac{m_a}{2}\mathbf{v}_a^4[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF} + \sum_{a\neq b}\frac{Gm_am_b}{12r}\Bigg\{\bigg(22\mathbf{v}_a^2 - 17\mathbf{v}_b^2 - 10\mathbf{v}_a\cdot\mathbf{v}_b$$
$$+5(\mathbf{v}_a\cdot\mathbf{n})^2 + 2\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} - 8(\mathbf{v}_b\cdot\mathbf{n})^2 - 5\mathbf{a}_a\cdot\mathbf{r} + 8\mathbf{a}_b\cdot\mathbf{r} + 12\frac{Gm}{r}\bigg)\mathbf{x}_a^i\mathbf{x}_a^j$$
$$+\bigg(-5\mathbf{v}_a^2 + 7\mathbf{v}_a\cdot\mathbf{v}_b + 5(\mathbf{v}_a\cdot\mathbf{n})^2 - 7\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} - 5\mathbf{a}_a\cdot\mathbf{r} - 12\frac{Gm_a}{r}\bigg)\mathbf{x}_a^i\mathbf{x}_b^j$$
$$+(4\mathbf{v}_a\cdot\mathbf{r} - 44\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{x}_a^j + (14\mathbf{v}_a\cdot\mathbf{r} - 58\mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a^i\mathbf{x}_b^j$$
$$+r^2(38\mathbf{v}_a^i\mathbf{v}_a^j - 7\mathbf{v}_a^i\mathbf{v}_b^j + 14\mathbf{a}_a^i\mathbf{x}_a^j + 19\mathbf{a}_a^i\mathbf{x}_b^j)\Bigg\}_{STF}. \tag{3.50}$$

IV. LOWER ORDER STRESS-ENERGY TENSORS

Although $T^{ij}_{0\mathrm{PN}}$, $T^{ii}_{0\mathrm{PN}}$, and $T^{00}_{1\mathrm{PN}}$ have been computed before in [26], to write an expression for the mass quadrupole moment at 2PN order, we need to expand them in the radiation momentum to higher order and terms depending on $\mathbf{k}^2$ must be kept.

To obtain the sixth term of (2.10) we need diagrams in Figs. 10(a) and 10(b). This gives us an expression for the leading order T^{ij}, as shown below:

$$T^{kl}_{0\mathrm{PN}}(t,\mathbf{k}) = \sum_a m_a\mathbf{v}_a^k\mathbf{v}_a^l e^{-i\mathbf{k}\cdot\mathbf{x}_a} + \sum_{a\neq b}\frac{Gm_am_b}{2r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Bigg\{-\frac{1}{r^2}\mathbf{r}^k\mathbf{r}^l - \frac{i}{2}\mathbf{k}^i\frac{1}{r^2}\mathbf{r}^i\mathbf{r}^k\mathbf{r}^l$$
$$+\frac{1}{12}\mathbf{k}^i\mathbf{k}^j\left[10\mathbf{r}^2(\delta^{kl}\delta^{ij} - \delta^{ik}\delta^{jl}) + \delta^{kl}\mathbf{r}^i\mathbf{r}^j + \delta^{ij}\mathbf{r}^k\mathbf{r}^l - 2\delta^{ik}\mathbf{r}^j\mathbf{r}^l + \frac{1}{r^2}2\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\right]$$
$$-\frac{i}{24}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^m\left[\mathbf{r}^2\mathbf{r}^m(10\delta^{ik}\delta^{jl} - 10\delta^{kl}\delta^{ij}) - \delta^{ij}\mathbf{r}^m\mathbf{r}^k\mathbf{r}^l + \mathbf{r}^i\mathbf{r}^j\left(2\delta^{mk}\mathbf{r}^l - \delta^{kl}\mathbf{r}^m - \frac{1}{r^2}\mathbf{r}^m\mathbf{r}^k\mathbf{r}^l\right)\right]$$
$$+\frac{1}{240}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^m\mathbf{k}^n\Bigg[\frac{16}{3}\mathbf{r}^4\delta^{mn}(\delta^{kl}\delta^{ij} - \delta^{ik}\delta^{jl}) + \mathbf{r}^2\delta^{ij}\delta^{mn}\mathbf{r}^k\mathbf{r}^l - 2\mathbf{r}^2\delta^{mn}\delta^{ik}\mathbf{r}^j\mathbf{r}^l$$
$$+\mathbf{r}^m\mathbf{r}^n\left(34\mathbf{r}^2\delta^{ik}\delta^{jl} - 33\mathbf{r}^2\delta^{kl}\delta^{ij} - 3\delta^{kl}\mathbf{r}^i\mathbf{r}^j + 6\delta^{ik}\mathbf{r}^j\mathbf{r}^l - 3\delta^{ij}\mathbf{r}^k\mathbf{r}^l - \frac{2}{r^2}\mathbf{r}^i\mathbf{r}^j\mathbf{r}^k\mathbf{r}^l\right)\Bigg]\Bigg\} + O(\mathbf{k}^5) + \cdots. \tag{4.1}$$

The first term in the expression above is related to Fig. 10(a), which comes from the simple source action term $-\sum\frac{m_a}{2m_{pl}}\int dt_a\mathbf{v}_a^i\mathbf{v}_a^j\bar{h}^{ij}(x_a)$. The other terms come from Fig. 10(b), which is composed of (A1) and (A17) by considering

$$F^{\langle H^{00}H^{00}\rangle}[q,k,\bar{h}^{ij}] = \bar{h}^{ij}\left[-\frac{1}{2}\mathbf{q}^i\mathbf{q}^j - \frac{1}{2}\mathbf{q}^i\mathbf{k}^j - \frac{1}{2}\mathbf{k}^i\mathbf{k}^j + \delta^{ij}\left(\frac{1}{4}\mathbf{q}^2 + \frac{1}{4}\mathbf{k}\cdot\mathbf{q} + \frac{1}{2}\mathbf{k}^2\right)\right], \tag{4.2}$$

where $F^{\langle H^{00}H^{00}\rangle}$ is defined in (A22).

Now, expanding the exponentials up to the fourth order in the radiation momentum, we extract the contribution

$$\int d^3\mathbf{x}\partial_0^2T^{kl}_{0\mathrm{PN}}\mathbf{x}^k\mathbf{x}^l[\mathbf{x}^i\mathbf{x}^j]_{TF} = \frac{d^2}{dt^2}\left[\sum\nolimits_a m_a(\mathbf{v}_a\cdot\mathbf{x}_a)^2\mathbf{x}_a^i\mathbf{x}_a^j\right]_{TF}$$
$$+\frac{d^2}{dt^2}\Bigg\{\sum_{a\neq b}\frac{Gm_am_b}{6r}\Bigg[\left(27\mathbf{r}^2 + \mathbf{x}_a^2 - 2\mathbf{x}_a\cdot\mathbf{x}_b - \frac{2}{r^2}\mathbf{r}\cdot\mathbf{x}_a\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_a^j$$
$$+\left(\frac{27}{2}\mathbf{r}^2 + \mathbf{x}_a^2 - \frac{1}{r^2}(\mathbf{r}\cdot\mathbf{x}_a)^2\right)\mathbf{x}_a^i\mathbf{x}_b^j\Bigg]\Bigg\}_{STF}. \tag{4.3}$$

Taking the trace of (4.1), we get

$$T^{ll}_{0\mathrm{PN}}(t,\mathbf{k}) = \sum_a m_a \mathbf{v}_a^2 e^{-i\mathbf{k}\cdot\mathbf{x}_a} + \sum_{a\neq b}\frac{Gm_am_b}{2r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left[-1-\frac{i}{2}\mathbf{k}^i\mathbf{r}^i+\frac{1}{4}\mathbf{k}^i\mathbf{k}^j(7\mathbf{r}^2\delta^{ij}+\mathbf{r}^i\mathbf{r}^j)\right.$$
$$+\frac{i}{24}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^m(21\mathbf{r}^2\delta^{ij}\mathbf{r}^m+2\mathbf{r}^i\mathbf{r}^j\mathbf{r}^m)$$
$$\left.+\frac{1}{144}\mathbf{k}^i\mathbf{k}^j\mathbf{k}^m\mathbf{k}^n(7\mathbf{r}^4\delta^{ij}\delta^{mn}-42\mathbf{r}^2\delta^{ij}\mathbf{r}^m\mathbf{r}^n-3\mathbf{r}^i\mathbf{r}^j\mathbf{r}^m\mathbf{r}^n)\right]+O(\mathbf{k}^5)+\cdots, \tag{4.4}$$

which contributes to the quadrupole in the form below:

$$\int d^3\mathbf{x}\partial_0^2 T^{ll}_{0\mathrm{PN}}\mathbf{x}^2[\mathbf{x}^i\mathbf{x}^j]_{TF} = \frac{d^2}{dt^2}\left[\sum\nolimits_a m_a\mathbf{v}_a^2\mathbf{x}_a^2\mathbf{x}_a^i\mathbf{x}_a^j\right]_{TF}$$
$$+\frac{d^2}{dt^2}\left\{\sum\nolimits_{a\neq b}\frac{Gm_am_b}{12r}[(-104\mathbf{x}_a^2+196\mathbf{x}_a\cdot\mathbf{x}_b-98\mathbf{x}_b^2)\mathbf{x}_a^i\mathbf{x}_a^j-49\mathbf{r}^2\mathbf{x}_a^i\mathbf{x}_b^j]\right\}_{STF}. \tag{4.5}$$

To be able to compute the fifth contribution in (2.10), we need an expression for $T^{00}_{1\mathrm{PN}}$ up to the fourth order in the radiation momentum. We regard the source action term $-\sum\frac{m_a}{4m_{pl}}\int dt_a\mathbf{v}_a^2\bar{h}^{00}(x_a)$ and also (A15), (A1), (A4), and (A18) to solve the diagrams in Figs. 10(a)–10(c). With this, we get an expression for $T^{00}_{1\mathrm{PN}}$ and its contribution to the mass quadrupole at 2PN, respectively:

$$T^{00}_{1\mathrm{PN}}(t,\mathbf{k}) = \sum_a\frac{1}{2}m_a\mathbf{v}_a^2e^{-i\mathbf{k}\cdot\mathbf{x}_a}+\sum_{a\neq b}\frac{Gm_am_b}{r}e^{-i\mathbf{k}\cdot\mathbf{x}_a}\left[-\frac{1}{2}\right.$$
$$\left.-\frac{3}{8}\mathbf{k}^2r^2\left(1+\frac{i}{2}\mathbf{k}^i\mathbf{r}^i-\frac{1}{6}\mathbf{k}^i\mathbf{k}^j\mathbf{r}^i\mathbf{r}^j+\frac{r^2}{36}\mathbf{k}^i\mathbf{k}^j\delta^{ij}\right)\right]$$
$$+O(\mathbf{k}^5)+\cdots, \tag{4.6}$$

$$\int d^3\mathbf{x}\partial_0^2T^{00}_{1\mathrm{PN}}\mathbf{x}^2[\mathbf{x}^i\mathbf{x}^j]_{TF}$$
$$=\frac{d^2}{dt^2}\left\{\sum_a\frac{1}{2}m_a\mathbf{v}_a^2\mathbf{x}_a^2[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}\right\}$$
$$+\frac{d^2}{dt^2}\left\{\sum_{a\neq b}\frac{Gm_am_b}{r}\left[\frac{7}{4}\mathbf{r}^2(2\mathbf{x}_a^i\mathbf{x}_a^j+\mathbf{x}_a^i\mathbf{x}_b^j)-\frac{1}{2}\mathbf{x}_a^2\mathbf{x}_a^i\mathbf{x}_a^j\right]_{STF}\right\}. \tag{4.7}$$

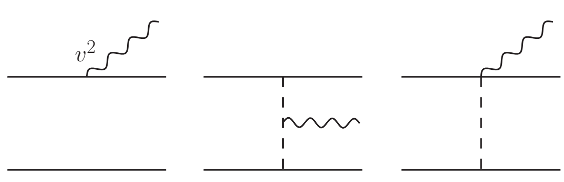


FIG. 10. Diagrams (a) and (b) contribute to $T^{ij}_{0\mathrm{PN}}$ when the external leg is $\bar{h}^{ij}(x)$, while diagrams (a), (b), and (c) contribute to $T^{00}_{1\mathrm{PN}}$ when we consider $\bar{h}^{00}(x)$ as the external leg.

Moreover, considering the expansion up to the fifth and sixth orders in the radiation momentum at (3.29) and (3.2), respectively, in addition to taking time derivatives, we get

$$\int d^3\mathbf{x}\partial_0^3T^{0l}_{0\mathrm{PN}}\mathbf{x}^2\mathbf{x}^l[\mathbf{x}^i\mathbf{x}^j]_{TF}=\frac{d^3}{dt^3}\left[\sum\nolimits_a m_a\mathbf{v}_a\cdot\mathbf{x}_a\mathbf{x}_a^2\mathbf{x}_a^i\mathbf{x}_a^j\right]_{TF}, \tag{4.8}$$

$$\int d^3\mathbf{x}\partial_0^4T^{00}_{0\mathrm{PN}}\mathbf{x}^4[\mathbf{x}^i\mathbf{x}^j]_{TF}=\frac{d^4}{dt^4}\left[\sum\nolimits_a m_a\mathbf{x}_a^4\mathbf{x}_a^i\mathbf{x}_a^j\right]_{TF}. \tag{4.9}$$

Before writing the final expression for the 2PN correction to the mass quadrupole moment, we still need to write the contribution of $I^{ij}_{1\mathrm{PN}}(\mathbf{a}_{1\mathrm{PN}})$, which is given by the two terms

$$\left[\int d^3\mathbf{x}\partial_0T^{0l}_{0\mathrm{PN}}\mathbf{x}^l\mathbf{x}^i\mathbf{x}^j\right]_{TF}\overset{2\mathrm{PN}}{\to}\sum_a m_a\mathbf{a}_{1\mathrm{PN}a}\cdot\mathbf{x}_a[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}, \tag{4.10}$$

$$\left[\int d^3\mathbf{x}\partial_0^2T^{00}_{0\mathrm{PN}}\mathbf{x}^2\mathbf{x}^i\mathbf{x}^j\right]_{TF}$$
$$\overset{2\mathrm{PN}}{\to}\sum_a 2m_a[\mathbf{x}_a^2\mathbf{a}^i_{1\mathrm{PN}a}\mathbf{x}_a^j+\mathbf{a}_{1\mathrm{PN}a}\cdot\mathbf{x}_a\mathbf{x}_a^i\mathbf{x}_a^j]_{STF}, \tag{4.11}$$

where the 1PN correction to the acceleration, for instance obtained in [34] using the EFT framework, is given by

$$
\begin{aligned}
\mathbf{a}^i_{1\mathrm{PN}(1)} = \frac{Gm_2}{2r^2}\Bigg\{ & \mathbf{n}^i\left[\frac{2Gm}{r} - 3(\mathbf{v}_1^2+\mathbf{v}_2^2) + 7(\mathbf{v}_1\cdot\mathbf{v}_2) + 3(\mathbf{v}_1\cdot\mathbf{n})(\mathbf{v}_2\cdot\mathbf{n})\right] \\
& - \mathbf{v}_1^i(\mathbf{v}_2\cdot\mathbf{n}) - (\mathbf{v}_1\cdot\mathbf{n})\mathbf{v}_2^i + \dot{r}(6\mathbf{v}_1^i - 7\mathbf{v}_2^i - \mathbf{n}^i(\mathbf{v}_2\cdot\mathbf{n})) \\
& - 6r\mathbf{a}_1^i + 7r\mathbf{a}_2^i + (\mathbf{v}^i - \mathbf{n}^i\dot{r})(\mathbf{v}_2\cdot\mathbf{n}) + r\mathbf{n}^i(\mathbf{a}_2\cdot\mathbf{n}) + \mathbf{n}^i(\mathbf{v}_2\cdot(\mathbf{v}-\mathbf{n}\dot{r}))\Bigg\} - \frac{1}{2}\mathbf{a}_1^i\mathbf{v}_1^2 - \mathbf{v}_1^i(\mathbf{v}_1\cdot\mathbf{a}_1). \qquad (4.12)
\end{aligned}
$$

V. CONSISTENCY TESTS

Now we check the expressions for the components $T^{00}_{2\mathrm{PN}}$, $T^{0i}_{1\mathrm{PN}}$, and $T^{ll}_{1\mathrm{PN}}$, which were obtained here for the first time in the EFT approach, with previous results derived using different methods.

The results presented in Sec. III A allow us to write down an expression for the temporal component of the pseudotensor up to 2PN order,[5]

$$
\begin{aligned}
T^{00}(t,\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Bigg\{ & \sum_a m_a\left(1+\frac{1}{2}\mathbf{v}_a^2+\frac{3}{8}\mathbf{v}_a^4\right) + \sum_{a\neq b}\frac{Gm_am_b}{2r}\Bigg\{-1+\mathbf{v}^2+\frac{7}{2}\mathbf{v}_a^2-\frac{5}{2}\mathbf{v}_b^2+\frac{5}{2}\mathbf{v}_a\cdot\mathbf{v}_b \\
& -\frac{5}{2}\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + 2(\mathbf{v}_b\cdot\mathbf{n})^2 - \dot{r}^2 + \mathbf{a}\cdot\mathbf{r} + 2\mathbf{a}_b\cdot\mathbf{r} + \frac{G}{r}(4m_a-3m_b) \\
& +\frac{1}{2}i\mathbf{k}^i\Bigg[\left(\mathbf{v}^2+\frac{1}{2}\mathbf{v}_b^2+\frac{5}{2}\mathbf{v}_a\cdot\mathbf{v}_b - \dot{r}^2 - \frac{5}{2}\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + \frac{3}{2}(\mathbf{v}_b\cdot\mathbf{n})^2 + \mathbf{a}\cdot\mathbf{r} + \frac{3}{2}\mathbf{a}_b\cdot\mathbf{r}\right)\mathbf{r}^i \\
& +\left(8\mathbf{v}_a\cdot\mathbf{r} + \frac{11}{2}\mathbf{v}_b\cdot\mathbf{r} - 2r\dot{r}\right)\mathbf{v}_a^i + \left(\frac{9}{2}\mathbf{v}_b\cdot\mathbf{r} + \frac{1}{2}r\dot{r}\right)\mathbf{v}_b^i + r^2(7\mathbf{a}_a^i - 2\mathbf{a}_b^i) + \frac{6Gm}{r}\Bigg]\Bigg\} + O(\mathbf{k}^2) + \cdots\Bigg\}. \qquad (5.1)
\end{aligned}
$$

We can use (3.1) to read off different contributions of T^{00} to the dynamics of the binary system. For instance, at zeroth order in the radiation momentum, we can read off the mechanical energy of the system. It is straightforward to see in (5.1) that the leading order terms in the PN approximation reproduce the total mass of the two-body system, while the next-to-leading order terms provide us with the Newtonian energy. The terms that account for the next-to-next-to-leading order (2PN) correction to this pseudotensor, which were calculated in Sec. III A, give us the following contribution to the conserved energy:

$$
E_{1\mathrm{PN}} = \int d^3\mathbf{x}T^{00}_{2\mathrm{PN}}(x) = \frac{3}{8}\sum_a m_a\mathbf{v}_a^4 + \sum_{a\neq b}\frac{Gm_am_b}{4r}\left[6\mathbf{v}_a^2 - 7(\mathbf{v}_a\cdot\mathbf{v}_b) - (\mathbf{v}_a\cdot\mathbf{n})(\mathbf{v}_b\cdot\mathbf{n}) + \frac{Gm}{r}\right]. \qquad (5.2)
$$

This result is equal to the first correction to the Newtonian energy presented in Eq. (205) of [35] and can also be calculated computing the Hamiltonian function using the Lagrangian obtained by Einstein, Infeld, and Hoffman in [36].

Regarding the 2PN terms in Eq. (5.1), we can read off the correction to the c.m. position

$$
\begin{aligned}
\mathbf{G}_{2\mathrm{PN}} &= \int d^3\mathbf{x}T^{00}_{2\mathrm{PN}}(x)\mathbf{x} \\
&= \frac{3}{8}\sum_a m_a\mathbf{v}_a^4\mathbf{x}_a + \sum_{a\neq b}\frac{Gm_am_b}{4r}\Bigg\{\Bigg[\frac{19}{2}\mathbf{v}_a^2 - 7\mathbf{v}_a\cdot\mathbf{v}_b - \frac{7}{2}\mathbf{v}_b^2 - \mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} \\
&\quad -\frac{1}{2}(\mathbf{v}_a\cdot\mathbf{n})^2 + \frac{1}{2}(\mathbf{v}_b\cdot\mathbf{n})^2 - 5\frac{Gm_a}{r} + 7\frac{Gm_b}{r}\Bigg]\mathbf{x}_a - 7(\mathbf{v}_a\cdot\mathbf{r} + \mathbf{v}_b\cdot\mathbf{r})\mathbf{v}_a\Bigg\}, \qquad (5.3)
\end{aligned}
$$

which agrees with the result presented in Eq. (B2c) of [37], where $\frac{d\mathbf{G}}{dt} = \mathbf{P}$, the total conserved linear momentum, such that the c.m. frame is defined by $\mathbf{G} = 0$. By solving this equation iteratively, using the equations of motion to reduce the second time derivatives of the position, we get the 2PN correction to the c.m. frame,

[5]To have an expression for T^{00} containing terms of second order in the radiation momentum, we would have to include $\mathbf{k}^2$ terms, but we discarded those terms since they are not needed to extract the contribution of $T^{00}_{2\mathrm{PN}}$ to the mass quadrupole moment. Nevertheless, it is enough to consider terms up the first order in the radiation momentum to perform the consistency tests on $T^{00}_{2\mathrm{PN}}$ in this section.

$$\delta\mathbf{r}_{2\mathrm{PN}} = \frac{\nu\delta m}{m}\left\{\mathbf{r}\left[\left(\frac{3}{8}-\frac{3\nu}{2}\right)\mathbf{v}^4 + \frac{Gm}{r}\left(\left(\frac{19}{8}+\frac{3\nu}{2}\right)\mathbf{v}^2\right)+\left(-\frac{1}{8}+\frac{3\nu}{4}\right)\dot{r}^2 + \left(\frac{7}{4}-\frac{\nu}{2}\right)\frac{Gm}{r}\right] - \mathbf{v}\left[\frac{7}{4}Gm\dot{r}\right]\right\}, \tag{5.4}$$

with $\delta m = m_1 - m_2$, which agrees with (B4a), (B4b), and (B5b) of [37].

Let us now consider the results obtained in Sec. III B to write down an expression for T^{0l} up to 1PN order,

$$\begin{aligned} T^{0l}(t,\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Bigg\{&\sum_a m_a\mathbf{v}_a^l\left(1+\frac{1}{2}\mathbf{v}_a^2\right) + \sum_{a\neq b}\frac{Gm_am_b}{4r}\Bigg[-3\mathbf{v}_a^l + \mathbf{v}_b^l - \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}^l \\ &-\frac{i}{2}\mathbf{k}^i\left(\mathbf{v}^i\mathbf{r}^l - 16\mathbf{v}_a^i\mathbf{r}^l + 15\mathbf{r}^i\mathbf{v}_a^l - \mathbf{r}^i\mathbf{v}_b^l + 3r\dot{r}\delta^{il} + \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}^i\mathbf{r}^l\right)\Bigg] + O(\mathbf{k}^2)+\cdots\Bigg\}. \end{aligned} \tag{5.5}$$

Taking into account only terms of order zero in the radiation momentum, we obtain the 1PN correction to the linear momentum of the binary system,

$$\mathbf{P}_{1\mathrm{PN}} = \int d^3\mathbf{x}T^{0l}_{1\mathrm{PN}}(x) = -\left[\frac{Gm_1m_2}{2r^3}(\mathbf{v}_1+\mathbf{v}_2)\cdot\mathbf{r}\right]\mathbf{x}_1^l + \left[\frac{m_1}{2}\mathbf{v}_1^2 - \frac{Gm_1m_2}{2r}\right]\mathbf{v}_1^l + 1\leftrightarrow 2. \tag{5.6}$$

The result above agrees with Eqs. (B1) and (B2b) of Ref. [37]. Considering all linear terms in the radiation momentum in (5.5), we are able to obtain the 1PN correction to the angular momentum of the binary system,

$$\mathbf{L}^i_{1\mathrm{PN}} = -\frac{1}{2}\epsilon^{ilk}\int d^3\mathbf{x}(T^{0l}_{1\mathrm{PN}}\mathbf{x}^k - T^{0k}_{1\mathrm{PN}}\mathbf{x}^l) = \frac{1}{2}\nu m(\mathbf{r}\times\mathbf{v})^i\left[(1-3\nu)\mathbf{v}^2 + \frac{Gm}{r}(6+2\nu)\right], \tag{5.7}$$

which agrees with Eq. (2.9b) of Ref. [38].

Furthermore, considering the result obtained in Sec. III C, we provide an expression for $T^{ll}(t,\mathbf{k})$ up to 1PN order:

$$\begin{aligned} T^{ll}(t,\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{x}_a}\Bigg\{&\sum_a m_a\mathbf{v}_a^2\left(1+\frac{1}{2}\mathbf{v}_a^2\right) + \sum_{a\neq b}\frac{Gm_am_b}{4r}\Bigg\{-2-5\mathbf{v}_a^2+5\mathbf{v}_b^2-\mathbf{v}_a\cdot\mathbf{v}_b - 6\dot{r}^2 \\ &-3\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + 2(\mathbf{v}_b\cdot\mathbf{n})^2 + 16\dot{r}\mathbf{v}_a\cdot\mathbf{n} - 10\mathbf{a}_a\cdot\mathbf{r} - 4\mathbf{a}_b\cdot\mathbf{r} + \frac{G}{r}(-8m_a+12m_b) \\ &+ i\mathbf{k}^i\Bigg[\mathbf{r}^i\left(-9\mathbf{v}_a^2 + \frac{5}{2}\mathbf{v}_b^2 - \frac{1}{2}\mathbf{v}_a\cdot\mathbf{v}_b - \frac{3}{2}\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + \frac{1}{2}(\mathbf{v}_b\cdot\mathbf{n})^2\right. \\ &\left.-3\dot{r}^2 + 8\dot{r}\mathbf{v}_a\cdot\mathbf{n} - 5\mathbf{a}_a\cdot\mathbf{r} - \frac{5}{2}\mathbf{a}_b\cdot\mathbf{r} - \frac{10Gm_a}{r}\right) + \left(2\mathbf{v}_a\cdot\mathbf{r} + \frac{25}{2}\mathbf{v}_b\cdot\mathbf{r}\right)\mathbf{v}_a^i \\ &+\left(-\frac{9}{2}\mathbf{v}_a\cdot\mathbf{r} + 5\mathbf{v}_b\cdot\mathbf{r}\right)\mathbf{v}_b^i - \mathbf{r}^2\left(3\mathbf{a}_a^i + \frac{5}{2}\mathbf{a}_b^i\right)\Bigg]\Bigg\} + O(\mathbf{k}^2)+\cdots\Bigg\}. \end{aligned} \tag{5.8}$$

We can use the moment relation

$$\int d^3\mathbf{x}T^{ll} = \frac{1}{2}\frac{d^2}{dt^2}\int d^3\mathbf{x}T^{00}\mathbf{x}^2 \tag{5.9}$$

to prove the self-consistency of our results. At leading order in the PN expansion, it is trivial to prove that this relation holds using (5.1) and (5.8), while at next-to-leading order more computation is required. From (5.8) we can read off up to 1PN,

$$\int d^3\mathbf{x}T^{ll} = \sum_a m_a\mathbf{v}_a^2\left(1+\frac{1}{2}\mathbf{v}_a^2\right) + \sum_{a\neq b}\frac{Gm_am_b}{r}\left[-\frac{1}{2}-\frac{1}{4}\mathbf{v}_a\cdot\mathbf{v}_b + \frac{3}{2}(\mathbf{v}_a\cdot\mathbf{n})^2 - \frac{7}{4}\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + \frac{5}{2}\frac{Gm_a}{r}\right]. \tag{5.10}$$

To check if the result above satisfies (5.9), we need a complete expression for $T^{00}(t,\mathbf{k})$ up to 1PN order and which contains all terms up to the quadratic order in the radiation momentum. In other words, we cannot discard terms proportional to $\mathbf{k}^2$ as we did in Sec. III A, where we dropped these terms that would not contribute to the trace-free quadrupole moment.

Therefore, the expression that we need for $T^{00}(t,\mathbf{k})$ is the sum of (3.2) with (4.6), which provides us with the following result up to 1PN order:

$$\frac{1}{2}\frac{d^2}{dt^2}\int d^3\mathbf{x}T^{00}\mathbf{x}^2 = \frac{1}{2}\frac{d^2}{dt^2}\left[\sum_a m_a\left(1+\frac{1}{2}\mathbf{v}_a^2\right)\mathbf{x}_a^2 + \sum_{a\neq b}\frac{Gm_am_b}{r}\left(-\frac{1}{2}\mathbf{x}_a^2+\frac{9}{4}r^2\right)\right]. \tag{5.11}$$

At this point, it is straightforward to show, after taking the second order time derivative and imposing the leading and next-to-leading order equations of motion, that (5.9) holds, as we expected.

VI. MASS QUADRUPOLE MOMENT AT 2PN ORDER

We are now ready to sum the contributions (3.28), (3.34), (3.50), (4.3), (4.5), (4.7), (4.8), (4.9) and to write down the expression for the 2PN correction to the mass quadrupole moment in a general orbit,

$$\begin{aligned} I^{ij}_{2\mathrm{PN}} &= \sum_a m_a f^{ij}_{1(a)} + \sum_{a\neq b}\frac{Gm_am_b}{r}f^{ij}_{2(a,b)} + \frac{d}{dt}\left[\sum_a m_a f^{ij}_{3(a)} + \sum_{a\neq b}\frac{Gm_am_b}{r}f^{ij}_{4(a,b)}\right] \\ &\quad + \frac{d^2}{dt^2}\left[\sum_a m_a f^{ij}_{5(a)} + \sum_{a\neq b}\frac{Gm_am_b}{r}f^{ij}_{6(a,b)}\right] + \frac{d^3}{dt^3}\left[\sum_a m_a f^{ij}_{7(a)}\right] + \frac{d^4}{dt^4}\left[\sum_a m_a f^{ij}_{8(a)}\right], \end{aligned} \tag{6.1}$$

where we have defined the following quantities for convenience:

$$f^{ij}_{1(a)} \equiv \left[\frac{7}{8}\mathbf{v}_a^4\mathbf{x}_a^i\mathbf{x}_a^j + \frac{11}{21}\mathbf{x}_a^2\mathbf{a}^i_{1\mathrm{PN}a}\mathbf{x}_a^j - \frac{17}{21}\mathbf{a}_{1\mathrm{PN}a}\cdot\mathbf{x}_a\mathbf{x}_a^i\mathbf{x}_a^j\right]_{STF}, \tag{6.2}$$

$$\begin{aligned} f^{ij}_{2(a,b)} \equiv \frac{1}{12}\Bigg[&\Bigg(50\mathbf{v}_a^2 - 28\mathbf{v}_b^2 - 32\mathbf{v}_a\cdot\mathbf{v}_b - 4\dot{r}^2 - 24\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} \\ &+8(\mathbf{v}_a\cdot\mathbf{n})^2 + 14(\mathbf{v}_b\cdot\mathbf{n})^2 - 4\mathbf{a}_a\cdot\mathbf{r} + 10\mathbf{a}_b\cdot\mathbf{r} + 24\frac{Gm_a}{r} + 18\frac{Gm_b}{r}\Bigg)\mathbf{x}_a^i\mathbf{x}_a^j \\ &+\left(-4\mathbf{v}_a^2 + 8\mathbf{v}_a\cdot\mathbf{v}_b - 12\mathbf{v}_a\cdot\mathbf{n}\mathbf{v}_b\cdot\mathbf{n} + 8(\mathbf{v}_a\cdot\mathbf{n})^2 - 2\dot{r}^2 - 4\mathbf{a}_a\cdot\mathbf{r} - 24\frac{Gm_1}{r}\right)\mathbf{x}_a^i\mathbf{x}_b^j \\ &+\mathbf{v}_a^i\mathbf{x}_a^j(-16\mathbf{v}_a\cdot\mathbf{r} - 64\mathbf{v}_b\cdot\mathbf{r}) + \mathbf{v}_a^i\mathbf{x}_b^j(40\mathbf{v}_a\cdot\mathbf{r} - 32\mathbf{v}_b\cdot\mathbf{r}) \\ &+\mathbf{r}^2(40\mathbf{v}_a^i\mathbf{v}_a^j - 8\mathbf{v}_a^i\mathbf{v}_b^j - 8\mathbf{a}_a^i\mathbf{x}_a^j - 4\mathbf{a}_a^i\mathbf{x}_b^j)\Bigg]_{STF}, \end{aligned} \tag{6.3}$$

$$f^{ij}_{3(a)} \equiv -\frac{2}{3}\mathbf{v}_a^2\mathbf{v}_a\cdot\mathbf{x}_a[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}, \tag{6.4}$$

$$\begin{aligned} f^{ij}_{4(a,b)} \equiv -\frac{1}{9}\Bigg[&(8\mathbf{r}^2 - 20\mathbf{r}\cdot\mathbf{x}_b)\mathbf{v}_a^i\mathbf{x}_a^j + (20\mathbf{r}^2 - 22\mathbf{r}\cdot\mathbf{x}_b)\mathbf{v}_a^i\mathbf{x}_b^j \\ &+\left(22\mathbf{v}_a\cdot\mathbf{x}_a - 30\mathbf{v}_b\cdot\mathbf{x}_a - 8\mathbf{v}_a\cdot\mathbf{x}_b + 8\mathbf{v}_b\cdot\mathbf{x}_b - \frac{2}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_a^j \\ &+\left(9\mathbf{v}_a\cdot\mathbf{x}_a - 7\mathbf{v}_a\cdot\mathbf{x}_b - \frac{1}{r^2}(\mathbf{v}_a+\mathbf{v}_b)\cdot\mathbf{r}\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_b^j\Bigg]_{STF}, \end{aligned} \tag{6.5}$$

$$f^{ij}_{5(a)} \equiv \left(\frac{1}{6}(\mathbf{v}_a\cdot\mathbf{x}_a)^2 + \frac{19}{84}\mathbf{v}_a^2\mathbf{x}_a^2\right)[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}, \tag{6.6}$$

$$f^{ij}_{6(a,b)} \equiv \left[\left(\frac{31}{42}\mathbf{x}_a^2 - \frac{11}{6}\mathbf{x}_a\cdot\mathbf{x}_b + \frac{8}{9}\mathbf{x}_b^2 - \frac{1}{18}\frac{1}{r^2}\mathbf{r}\cdot\mathbf{x}_a\mathbf{r}\cdot\mathbf{x}_b\right)\mathbf{x}_a^i\mathbf{x}_a^j + \left(\frac{4}{9}\mathbf{r}^2 + \frac{1}{36}\mathbf{x}_a^2 - \frac{1}{36}\frac{1}{r^2}(\mathbf{r}\cdot\mathbf{x}_a)^2\right)\mathbf{x}_a^i\mathbf{x}_b^j\right]_{STF}, \tag{6.7}$$

$$f^{ij}_{7(a)} \equiv -\frac{1}{7}\mathbf{v}_a\cdot\mathbf{x}_a\mathbf{x}_a^2[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}, \tag{6.8}$$

$$f^{ij}_{8(a)} \equiv \frac{23}{1512}\mathbf{x}_a^4[\mathbf{x}_a^i\mathbf{x}_a^j]_{TF}. \tag{6.9}$$

With the exception of the accelerations in (6.2) which are of 1PN order, all other accelerations in $I^{ij}_{2\mathrm{PN}}$ should be taken as the Newtonian acceleration.

In order to write the 2PN correction of the mass quadrupole moment in the c.m. frame, we must have in mind that the positions of the compact bodies in this frame are given by

$$\mathbf{x}_1 = \frac{m_2}{m}\mathbf{r} + \delta\mathbf{r}_{1\mathrm{PN}} + \cdots, \tag{6.10}$$

$$\mathbf{x}_2 = -\frac{m_1}{m}\mathbf{r} + \delta\mathbf{r}_{1\mathrm{PN}} + \cdots, \tag{6.11}$$

where $\delta\mathbf{r}_{1\mathrm{PN}}$ accounts for the 1PN correction to the c.m. frame, which can be obtained following the procedure presented through (5.3) and (5.4) but this time using (4.6). Thus, the corrections to the c.m. frame necessary to write the 2PN mass quadrupole are

$$\delta\mathbf{r}_{1\mathrm{PN}} = \frac{\nu\delta m}{2m}\mathbf{r}\left(\mathbf{v}^2 - \frac{Gm}{r}\right), \tag{6.12}$$

$$\delta\mathbf{r}_{2\mathrm{PN}} = \frac{\nu\delta m}{2m}\left\{\mathbf{r}\left[\left(\frac{3}{4} - 3\nu\right)\mathbf{v}^4 + \frac{Gm}{r}\left(\left(\frac{19}{4} + 3\nu\right)\mathbf{v}^2\right) + \left(-\frac{1}{4} + \frac{3\nu}{2}\right)\dot{r}^2 + \left(\frac{7}{2} - \nu\right)\frac{Gm}{r}\right] - \mathbf{v}\left[\frac{7}{2}Gm\dot{r}\right]\right\}. \tag{6.13}$$

Applying (6.10) and (6.11) to (2.8) and (2.9), we obtain the following contributions at 2PN order:

$$I^{ij}_{0\mathrm{PN}+2\mathrm{PN}} = \frac{\nu^2\delta m^2}{4m}\left(\mathbf{v}^4 - 2\mathbf{v}^2\frac{Gm}{r} + \frac{G^2m^2}{r^2}\right)[\mathbf{r}^i\mathbf{r}^j]_{TF}, \tag{6.14}$$

$$I^{ij}_{1\mathrm{PN}+1\mathrm{PN}} = \frac{\nu^2\delta m^2}{21m}\left\{\left[-29\mathbf{v}^4 + \frac{Gm}{r}\left(41\mathbf{v}^2 + \frac{17}{2}\dot{r}^2 - 12\frac{Gm}{r}\right)\right]\mathbf{r}^i\mathbf{r}^j + \left(24\mathbf{v}^2 - 19\frac{Gm}{r}\right)r\dot{r}\mathbf{v}^i\mathbf{r}^j + \left(-22\mathbf{v}^2 + 22\frac{Gm}{r}\right)r^2\mathbf{v}^i\mathbf{v}^j\right\}_{STF}. \tag{6.15}$$

Adding these two contributions to (6.1) after applying (6.10) and (6.11), we finally obtain the expression for the 2PN correction to the mass quadrupole moment in the c.m. frame,

$$\begin{aligned} I^{ij}_{2\mathrm{PN}} = m\nu\Bigg\{&\mathbf{r}^i\mathbf{r}^j\left[\frac{1}{252}(653 - 1906\nu + 337\nu^2)\frac{G^2m^2}{r^2} + \frac{1}{756}(2021 - 5947\nu - 4883\nu^2)\frac{Gm}{r}v^2\right.\\ &\left.- \frac{1}{756}(131 - 907\nu + 1273\nu^2)\frac{Gm}{r}\dot{r}^2 + \frac{1}{504}(253 - 1835\nu + 3545\nu^2)v^4\right]\\ &- r\dot{r}\mathbf{r}^i\mathbf{v}^j\left[\frac{1}{378}(1085 - 4057\nu - 1463\nu^2)\frac{Gm}{r} + \frac{1}{63}(26 - 202\nu + 418\nu^2)v^2\right]\\ &+\mathbf{v}^i\mathbf{v}^j\left[\frac{1}{189}(742 - 335\nu - 985\nu^2)\frac{Gm}{r} + \frac{1}{126}(41 - 337\nu + 733\nu^2)v^2 + \frac{5}{63}(1 - 5\nu + 5\nu^2)\dot{r}^2\right]\Bigg\}_{STF}. \end{aligned} \tag{6.16}$$

We can use the result above to compute, for instance, the 2PN correction to the power loss, whose expression in terms of the multipole moments is given by [28]

$$P = -\frac{G}{5}\left\{I^{(3)}_{ij}I^{(3)}_{ij} - \frac{5}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{5}{9072}I^{(5)}_{ijkl}I^{(5)}_{ijkl} + \frac{16}{9}J^{(3)}_{ij}J^{(3)}_{ij} - \frac{5}{84}J^{(4)}_{ijk}J^{(4)}_{ijk} + \cdots\right\}. \tag{6.17}$$

The expressions for these multipole moments below 2PN order are known and can be found for instance in [16]. Considering all terms which contribute to the power loss at 2PN order in the expression above, making use of (6.16) and the 2PN acceleration (B10) obtained in Appendix B, we get

$$P_{EFT}^{2PN} = -\frac{8}{15}\frac{G^3m^4\nu^2}{r^4}\left\{\frac{2}{3}(-253+1026\nu-56\nu^2)\frac{G^3m^3}{r^3}+\left[\frac{1}{756}(245185+81828\nu+4368\nu^2)v^2 - \frac{1}{252}(97247+9798\nu+5376\nu^2)\dot{r}^2\right]\frac{G^2m^2}{r^2}+\left[\frac{1}{21}(-4446+5237\nu-1393\nu^2)v^4 + \frac{1}{7}(4987-8513\nu+2165\nu^2)v^2\dot{r}^2-\frac{1}{63}(33510-60971\nu+14290\nu^2)\right]\frac{Gm}{r} + \frac{1}{42}(1692-5497\nu+4430\nu^2)v^6-\frac{1}{14}(1719-10278\nu+6292\nu^2)v^4\dot{r}^2 + \frac{1}{14}(2018-15207\nu+7572\nu^2)v^2\dot{r}^4-\frac{1}{42}(2501-20234\nu+8404\nu^2)\dot{r}^6\right\}. \tag{6.18}$$

At this point we can see that (6.16) and (6.18) seem to be in disagreement with the results presented at [39] and [40] where the Epstein-Wagoner formalism and multipolar post-Minkowskian approach of Blanchet, Damour, and Iyer (BDI) were used, respectively. For instance, the mass quadrupole moment presented in this paper and the ones in the mentioned references differ by a factor of $-\frac{4G^2m^2}{r^2}[m\nu\mathbf{r}^i\mathbf{r}^j]_{TF}$. The power loss shown above and the energy fluxes at (6.13d) in [39] and at (3.5d) in [40] differ by a global minus sign, as well as by the numerical factors on terms depending on $\frac{G^5m^6\nu^2v^2}{r^6}$ and $\frac{G^5m^6\nu^2\dot{r}^2}{r^6}$. The difference in the global sign comes from the relation $P=-\frac{dE}{dt}$, which is actually a matter of convention on how the energy flux is defined. For this reason, we consider instead $|P|=|\frac{dE}{dt}|$ and compare the result for the power loss obtained here against the ones in the literature, and we find the following difference:

$$P_{EFT}-\bar{P}=\frac{32}{5}\frac{G^5m^6\nu^2}{r^6}(4v^2-3\dot{r}^2), \tag{6.19}$$

where $\bar{P}$ is the modulus of the energy flux computed via the Epstein-Wagoner and BDI approaches.[6]

Furthermore, the 2PN acceleration obtained in Appendix B is also different from the one presented in [40], which was computed via the BDI formalism. It turns out that these differences should not be a surprise since the gauge choice adopted here and in other formalisms are not the same: in the BDI and in the Epstein-Wagoner approaches the harmonic gauge is used, while in the EFT approach we use the linearized harmonic gauge (2.4), which depends on the background field metric. The different gauge choices for fixing the gravity action imply different coordinate systems. In fact, the difference between the mass quadrupole moments suggests a coordinate transformation of the form

$$\mathbf{r}_{EFT}\to\bar{\mathbf{r}}-\frac{2G^2m^2}{r^2}\bar{\mathbf{r}}, \tag{6.20}$$

where $\bar{\mathbf{r}}$ is the coordinate used in the BDI and Epstein-Wagoner approaches. When this transformation is applied to the power loss (6.18), we can verify that

$$P_{EFT}(\bar{\mathbf{r}})=\bar{P}. \tag{6.21}$$

An analogous comparison holds for the mass quadrupole moment and the 2PN acceleration, showing the agreement between our results and the literature. It should also be noticed that this coordinate transformation was already brought to attention in [12] when the authors used NRGR to calculate the spacetime metric generated by a point mass at rest.

VII. FINAL REMARKS

In this paper, we provided an independent computation of the 2PN correction to the mass quadrupole moment of a binary system of compact bodies moving in general orbits, using the EFT approach in the linearized harmonic gauge. We calculated high order corrections to the components of the pseudo-stress-energy tensor, which were used to obtain the mass quadrupole moment correction as well as the 1PN correction to the conserved energy and to the linear and angular momenta of the system and the 2PN correction to the c.m. frame. We used these quantities to perform tests that confirmed the consistency of our results within the EFT formalism itself and with results presented in the literature computed using different formalisms. Therefore, we not only extracted the contributions of the stress-energy pseudotensor to the 2PN correction to the mass quadrupole, but we provided the expressions for the components of the pseudotensor with higher order corrections that will be useful for future calculations on the dynamics of a compact binary system.

We also calculated the 2PN correction to the equation of motion in the linearized harmonic gauge that was used, together with the mass quadrupole moment obtained in this paper, to write down the power loss due to the emission of gravitational waves. We thus compared our results against the literature and we showed that the 2PN correction to the mass quadrupole moment, to the relative acceleration of the

[6] If the power is expressed in terms of the gauge invariant frequency of a circular orbit $P=\bar{P}$.

two-body system, and to the power loss obtained in this paper are in agreement with the results computed via the BDI and in the Epstein-Wagoner formalisms once a coordinate transformation is performed.

Although the 2PN correction to the mass quadrupole and to the equation of motion of compact binary systems obtained here were known in the literature, this derivation establishes the ground work for higher order calculations in the EFT formalism. Finally, these are the final missing ingredients necessary for the analysis of the radiation reaction of the binary system at the next-to-next-to-leading order in the EFT approach, which will be presented in a future paper.

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APPENDIX A: COMPONENTS IN THE CALCULATION OF THE PSEUDOTENSOR

In this appendix we show the ingredients used to compute the components of the pseudotensor. We used the package xAct [41] from *Mathematica* for the extraction of the vertices from the action.

1. Source terms

The source action terms needed to compute the contributions to $T^{00}_{2\mathrm{PN}}$ are given below:

$$S^{\mathbf{v}^0} = -\sum_a \frac{m_a}{2m_{\mathrm{Pl}}} \int dt_a H^{00}(x_a), \tag{A1}$$

$$S^{\mathbf{v}^1} = \sum_a \frac{m_a}{m_{\mathrm{Pl}}} \int dt_a \mathbf{v}_a^i H^{0i}(x_a), \tag{A2}$$

$$S^{\mathbf{v}^2} = -\sum_a \frac{m_a}{2m_{\mathrm{Pl}}} \int dt_a \left(\frac{\mathbf{v}_a^2}{2} H^{00}(x_a) + \mathbf{v}_a^i \mathbf{v}_a^j H^{ij}(x_a) \right), \tag{A3}$$

$$S^{\mathbf{v}^0}_{\bar{h}^{00}} = \sum_a \frac{m_a}{4m_{\mathrm{Pl}}^2} \int dt_a H^{00}(x_a) \bar{h}^{00}(x_a), \tag{A4}$$

$$S^{\mathbf{v}^1}_{\bar{h}^{00}} = -\sum_a \frac{m_a}{2m_{\mathrm{Pl}}^2} \int dt_a \mathbf{v}_a^i H^{0i}(x_a) \bar{h}^{00}(x_a), \tag{A5}$$

$$S^{\mathbf{v}^2}_{\bar{h}^{00}} = \sum_a \frac{m_a}{8m_{\mathrm{Pl}}^2} \int dt_a (3\mathbf{v}_a^2 H^{00}(x_a) + 2\mathbf{v}_a^i \mathbf{v}_a^j H^{ij}(x_a)) \bar{h}^{00}(x_a), \tag{A6}$$

$$S^{\mathbf{v}^4}_{\bar{h}^{00}} = -\sum_a \frac{3m_a}{16m_{\mathrm{Pl}}} \int dt_a \mathbf{v}_a^4 \bar{h}^{00}(x_a), \tag{A7}$$

$$S^{\mathbf{v}^0}_{H^2} = \sum_a \frac{m_a}{8m_{\mathrm{Pl}}^2} \int dt_a H^{00}(x_a) H^{00}(x_a), \tag{A8}$$

$$S^{\mathbf{v}^0}_{H^2 \bar{h}^{00}} = -\sum_a \frac{3m_a}{16m_{\mathrm{Pl}}^3} \int dt_a H^{00}(x_a) H^{00}(x_a) \bar{h}^{00}(x_a). \tag{A9}$$

In addition, to write down the contributions for $T^{0i}_{1\mathrm{PN}}$ we must to consider

$$S^{\mathbf{v}^1}_{\bar{h}^{0i}} = -\sum_a \frac{m_a}{2m_{\mathrm{Pl}}^2} \int dt_a \mathbf{v}_a^i H^{00}(x_a) \bar{h}^{0i}(x_a), \tag{A10}$$

$$S^{\mathbf{v}^3}_{\bar{h}^{0i}} = \sum_a \frac{m_a}{2m_{\mathrm{Pl}}} \int dt \mathbf{v}_a^2 \mathbf{v}_a^i \bar{h}^{0i}(x_a), \tag{A11}$$

whereas for $T^{ll}_{1\mathrm{PN}}$ the following terms are also necessary:

$$S^{\mathbf{v}^2}_{\bar{h}^{ij}} = \sum_a \frac{m_a}{4m_{pl}^2} \int dt_a \mathbf{v}_a^i \mathbf{v}_a^j H^{00}(x_a) \bar{h}^{ij}(x_a), \tag{A12}$$

$$S^{\mathbf{v}^4}_{\bar{h}^{ij}} = -\sum_a \frac{m_a}{4m_{pl}} \int dt_a \mathbf{v}_a^2 \mathbf{v}_a^i \mathbf{v}_a^j \bar{h}^{ij}(x_a). \tag{A13}$$

Although all the sources terms above are conveniently expressed in position space, effectively we perform the partial Fourier transform[7]

$$H^{\mu\nu}(t, \mathbf{q}) = \int d^3x H^{\mu\nu}(t, \mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}, \tag{A14}$$

to carry out the Feynman diagrams in momentum space.

2. Vertices

From the EH action expanded in the radiation and potential fields and fixed with the background gauge, we obtain the propagator

$$\langle H_{\mu\nu}(t, \mathbf{q}) H_{\alpha\beta}(t', \mathbf{q}') \rangle = -i(2\pi)^3 P_{\mu\nu\alpha\beta} \delta(t - t') \delta^3(\mathbf{q} + \mathbf{q}') \frac{1}{\mathbf{q}^2}, \tag{A15}$$

as well as its correction

$$\langle H_{\mu\nu}(t, \mathbf{q}) H_{\alpha\beta}(t', \mathbf{q}') \rangle_{v^2} = -i(2\pi)^3 P_{\mu\nu\alpha\beta} \frac{d^2}{dt dt'} \delta(t - t') \delta^3(\mathbf{q} + \mathbf{q}') \frac{1}{\mathbf{q}^4}. \tag{A16}$$

The two-potential-one-radiation vertex regarded inside the momentum integrals of the internal potential momenta coupled to the particles has the form

[7]We consider the partial Fourier transform for the radiation field as well.

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3}\int \frac{d^3\mathbf{q}'}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{x}_1} e^{-i\mathbf{q}'\cdot\mathbf{x}_2}\langle iS_{\bar{h}H^2}\rangle = -\frac{i}{m_{\rm Pl}}\delta(t-t')\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{F[q,k,\bar{h}]}{\mathbf{q}^2(\mathbf{q}+\mathbf{k})^2}, \quad (A17)$$

for which the different contractions necessary to write down the contributions to $T^{00}_{2\rm PN}$ are

$$F^{\langle H^{00}H^{00}\rangle}[q,k,\bar{h}^{00}] = \bar{h}^{00}\left[\frac{3}{4}(\mathbf{q}^2+\mathbf{k}\cdot\mathbf{q}) - \frac{5}{4}q_0^2 - \frac{5}{4}k_0q_0 - \frac{1}{2}k_0^2\right], \quad (A18)$$

$$F^{\langle \mathbf{v}_1^k H^{0k}H^{00}\rangle}[q,k,\bar{h}^{00}] = \bar{h}^{00}\mathbf{v}_1^k\left[-\mathbf{q}^k\left(q_0+\frac{1}{2}k_0\right)\right], \quad (A19)$$

$$F^{\langle \mathbf{v}_1^k\mathbf{v}_1^l H^{kl}H^{00}\rangle}[q,k,\bar{h}^{00}] = \bar{h}^{00}\mathbf{v}_1^k\mathbf{v}_1^l\left[\frac{1}{4}\delta^{kl}(\mathbf{q}^2+3\mathbf{k}\cdot\mathbf{q}) - \frac{1}{2}\mathbf{k}^k\mathbf{k}^l\right], \quad (A20)$$

$$F^{\langle \mathbf{v}_1^k H^{0k}H^{0l}\mathbf{v}_2^l\rangle}[q,k,\bar{h}^{00}] = \bar{h}^{00}\mathbf{v}_1^k\mathbf{v}_2^l\left[-\frac{1}{4}\delta^{kl}(\mathbf{q}^2+\mathbf{k}\cdot\mathbf{q}) + \frac{1}{4}\mathbf{k}^k\mathbf{k}^l\right]. \quad (A21)$$

On the other hand, to compute the contributions to $T^{0i}_{1\rm PN}$, the contractions required are

$$F^{\langle H^{00}H^{00}\rangle}[q,k,\bar{h}^{0i}] = \bar{h}^{0i}\left[q_0\left(\mathbf{q}^i+\frac{1}{2}\mathbf{k}^i\right)+k_0\left(\frac{1}{2}\mathbf{q}^i+\mathbf{k}^i\right)\right], \quad (A22)$$

$$F^{\langle \mathbf{v}_1^k H^{0k}H^{00}\rangle}[q,k,\bar{h}^{0i}] = \bar{h}^{0i}\mathbf{v}_1^k\left[-\delta^{ik}\left(\frac{1}{2}\mathbf{q}^2+\mathbf{k}\cdot\mathbf{q}\right)+\mathbf{q}^i\mathbf{k}^k+\frac{1}{2}\mathbf{k}^i\mathbf{k}^k\right], \quad (A23)$$

whereas for $T^{ll}_{1\rm PN}$ we need

$$F^{\langle H^{00}H^{00}\rangle}[q,k,\bar{h}^{ll}] = \bar{h}^{ll}\left[\frac{1}{4}\mathbf{q}^2+\frac{1}{4}\mathbf{k}\cdot\mathbf{q}-\frac{3}{4}(q_0^2+k_0q_0+2k_0^2)\right], \quad (A24)$$

$$F^{\langle \mathbf{v}_1^k H^{0k}H^{00}\rangle}[q,k,\bar{h}^{ll}] = \bar{h}^{ll}\mathbf{v}_1^k\left[-k_0\left(\mathbf{q}^k+\frac{1}{2}\mathbf{k}^k\right)\right], \quad (A25)$$

$$F^{\langle \mathbf{v}_1^k\mathbf{v}_1^l H^{kl}H^{00}\rangle}[q,k,\bar{h}^{ll}] = \bar{h}^{ll}\mathbf{v}_1^k\mathbf{v}_1^l\left[\delta^{kl}\left(-\frac{1}{4}\mathbf{q}^2+\frac{1}{4}\mathbf{k}\cdot\mathbf{q}\right)-\frac{1}{2}\mathbf{k}^k\mathbf{k}^l\right], \quad (A26)$$

$$F^{\langle \mathbf{v}_1^k H^{0k}H^{0l}\mathbf{v}_2^l\rangle}[q,k,\bar{h}^{ll}] = \bar{h}^{ll}\mathbf{v}_1^k\mathbf{v}_2^l\left[\frac{1}{4}\delta^{kl}(\mathbf{q}^2+\mathbf{k}\cdot\mathbf{q})-\frac{1}{2}\mathbf{q}^l\mathbf{k}^k+\frac{1}{2}\mathbf{q}^k\mathbf{k}^l+\frac{1}{4}\mathbf{k}^k\mathbf{k}^l\right]. \quad (A27)$$

The three-graviton vertex, in turn, comes naturally in a simple form even not integrated on the internal momenta:

$$\langle H^{00}_{\mathbf{q}_1}H^{00}_{\mathbf{q}_2}H^{00}_{\mathbf{q}_3}\rangle = -\frac{(2\pi)^3}{4m_{\rm Pl}}\delta(t_2-t_1)\delta(t_3-t_1)\delta^3(\boldsymbol{q}_1+\boldsymbol{q}_2+\boldsymbol{q}_3)\times\frac{(\boldsymbol{q}_1^2+\boldsymbol{q}_2^2+\boldsymbol{q}_3^2)}{\boldsymbol{q}_1^2\boldsymbol{q}_2^2\boldsymbol{q}_3^2}. \quad (A28)$$

In the composition of the three-potential-graviton vertex with two-potential-one-radiation-graviton vertex, after integrating in the third momentum, the integrand takes the form

$$\frac{F[\mathbf{q}_1,\mathbf{q}_2,\mathbf{k},\bar{h}]}{\mathbf{q}_1^2\mathbf{q}_2^2(\mathbf{q}_1+\mathbf{k})^2(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k})^2}, \quad (A29)$$

in which the numerators for the contractions needed to compute the contributions for $T^{00}_{2\rm PN}$ and $T^{ll}_{1\rm PN}$ are, respectively,

$$F^{\langle H^{00}H^{00}H^{00}\rangle}[\mathbf{q}_1,\mathbf{q}_2,\mathbf{k},\bar{h}^{00}] = \frac{1}{4}\bar{h}_{00}\left[\mathbf{q}_1^4+\frac{5}{2}\mathbf{q}_1^2(\mathbf{q}_1\cdot\mathbf{q}_2)+\frac{5}{2}\mathbf{q}_1^2\mathbf{q}_2^2+(\mathbf{q}_1+\mathbf{q}_2)^2(\mathbf{q}_1\cdot\mathbf{k}) +\frac{5}{2}\mathbf{q}_1^2(\mathbf{q}_2\cdot\mathbf{k})+3(\mathbf{q}_1\cdot\mathbf{k})(\mathbf{q}_2\cdot\mathbf{k})-(\mathbf{q}_1\cdot\mathbf{k})^2+(\mathbf{q}_2\cdot\mathbf{k})^2\right], \quad (A30)$$

$$F^{\langle H^{00}H^{00}H^{00}\rangle}[\mathbf{q}_1,\mathbf{q}_2,\mathbf{k},\bar{h}^{ll}] = -\frac{\bar{h}^{ll}}{8}[2\mathbf{q}_1^4-\mathbf{q}_1^2\mathbf{q}_1\cdot\mathbf{q}_2+10\mathbf{q}_1^2\mathbf{q}_1\cdot\mathbf{k}+10(\mathbf{q}_1\cdot\mathbf{k})^2-\mathbf{q}_1^2\mathbf{q}_2^2 -\mathbf{q}_1^2\mathbf{q}_2\cdot\mathbf{k}-2(\mathbf{q}_1\cdot\mathbf{k})(\mathbf{q}_2\cdot\mathbf{k})-2(\mathbf{q}_2\cdot\mathbf{k})^2]. \quad (A31)$$

The three-potential-one-radiation-graviton vertex integrated in the internal momenta can be expressed in this way:

$$\prod_{i=1}^3\int\frac{d^3\mathbf{q}_i}{(2\pi)^9}e^{i\mathbf{q}_i\cdot\mathbf{x}_i}\langle iS_{\bar{h}H^3}\rangle = -\frac{1}{m_{\rm Pl}^2}\delta(t_2-t_1)\delta(t_3-t_1)\int\frac{d^3\mathbf{q}_2}{(2\pi)^3}\times\int\frac{d^3\mathbf{q}_3}{(2\pi)^3}\frac{e^{i(\mathbf{q}_2+\mathbf{q}_3)\cdot\mathbf{x}}F[\mathbf{q}_2,\mathbf{q}_3,\mathbf{k},\bar{h}]}{\mathbf{q}_2^2\mathbf{q}_3^2(\mathbf{q}_2+\mathbf{q}_3+\mathbf{k})^2}, \quad (A32)$$

where we have chosen to integrate on $\mathbf{q}_1$, for instance coupled to particle 1, and leaving the momenta $\mathbf{q}_2$ and $\mathbf{q}_3$, both coupled to particle 2, to be integrated in the process of solving the diagrams. For this case, the contractions required to write down the contribution for T^{00}_{2PN} and T^{ll}_{1PN}, respectively, are given by

$$F^{\langle H^{00}H^{00}H^{00}\rangle}[\mathbf{q}_2,\mathbf{q}_3,\mathbf{k},\bar{h}^{00}] = -\frac{1}{8}\bar{h}^{00}(\mathbf{q}_2^2+\mathbf{q}_3^2+\mathbf{q}_2\cdot\mathbf{q}_3+\mathbf{q}_2\cdot\mathbf{k}+\mathbf{q}_3\cdot\mathbf{k}), \quad \text{(A33)}$$

$$F^{\langle H^{00}H^{00}H^{00}\rangle}[\mathbf{q}_2,\mathbf{q}_3,\mathbf{k},\bar{h}^{ll}] = -\frac{7}{8}\bar{h}^{ll}(\mathbf{q}_2^2+\mathbf{q}_3^2+\mathbf{q}_2\cdot\mathbf{q}_3+\mathbf{q}_2\cdot\mathbf{k}+\mathbf{q}_3\cdot\mathbf{k}). \quad \text{(A34)}$$

3. Integrals

To solve integrals in the momentum space, it is helpful to use some general relations that can be obtained by using Feynman parameters [42]. If we consider a spacetime of d dimensions, then for $D=d-1$ we have

$$\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{(\mathbf{k}^2)^a} = \frac{1}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}-a)}{\Gamma(a)}\left(\frac{r^2}{4}\right)^{a-\frac{D}{2}}, \quad \text{(A35)}$$

$$\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{1}{[\mathbf{k}^2]^a[(\mathbf{k}-\mathbf{p})^2]^b} = \frac{(\mathbf{p}^2)^{\frac{D}{2}-a-b}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(a+b-\frac{D}{2})}{\Gamma(a)\Gamma(b)}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)}{\Gamma(D-a-b)}, \quad \text{(A36)}$$

$$\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{\mathbf{k}^i}{[\mathbf{k}^2]^a[(\mathbf{k}-\mathbf{p})^2]^b} = \frac{\mathbf{p}^i(\mathbf{p}^2)^{\frac{D}{2}-a-b}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(a+b-\frac{D}{2})}{\Gamma(a)\Gamma(b)}\frac{\Gamma(\frac{D}{2}-a+1)\Gamma(\frac{D}{2}-b)}{\Gamma(D-a-b+1)}, \quad \text{(A37)}$$

$$\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{\mathbf{k}^i\mathbf{k}^j}{[\mathbf{k}^2]^a[(\mathbf{k}-\mathbf{p})^2]^b} = \frac{1}{(4\pi)^{\frac{D}{2}}}\frac{(\mathbf{p}^2)^{\frac{D}{2}-a-b}}{\Gamma(a)\Gamma(b)\Gamma(D-a-b+2)} \times\left\{\frac{g^{ij}\mathbf{p}^2}{2}\Gamma\left(a+b-1-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-a+1\right)\Gamma\left(\frac{D}{2}-b+1\right) +\mathbf{p}^i\mathbf{p}^j\Gamma\left(a+b-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-b\right)\Gamma\left(\frac{D}{2}-a+2\right)\right\}. \quad \text{(A38)}$$

These integrals are especially important to solve diagrams that have a composition of the three-potential-graviton vertex with the two-potential-one-radiation vertex, where an analysis of the integrals in an arbitrary dimension D is required to handle divergences.

APPENDIX B: 2PN ACCELERATION

In this appendix we present the result for the 2PN acceleration computed via the EFT approach in the linearized harmonic gauge.

To write down the equation of motion of the binary system at 2PN order, we need to obtain the Lagrangian by integrating out the potential modes of the gravitational fields in the action (2.2). Below the diagrams which contribute to the dynamics at 2PN order are presented.

The simplest contribution to the 2PN Lagrangian comes from the diagram show in Fig. 11, which gives the following contribution:

$$L_{\text{Fig11}} = \sum_a \frac{1}{16}m_a\mathbf{v}_a^6. \quad \text{(B1)}$$

Next, we have the diagrams with one-graviton exchange illustrated in Fig. 12. Summing those diagrams together yields

$$L_{\text{Fig12}} = \sum_{a\neq b}\frac{Gm_am_b}{16r^3}\Big\{15r^4\mathbf{a}_a\cdot\mathbf{a}_b + r^2[14\mathbf{v}_a^2 - 20\mathbf{v}_a^2\mathbf{v}_a\cdot\mathbf{v}_b + 2(\mathbf{v}_a\cdot\mathbf{v}_b)^2 + 3\mathbf{v}_a^2\mathbf{v}_b^2 + 2\mathbf{v}_b^2\mathbf{a}_a\cdot\mathbf{r} - \mathbf{a}_a\cdot\mathbf{r}\mathbf{a}_b\cdot\mathbf{r} + 28\mathbf{a}_b\cdot\mathbf{v}_a\mathbf{v}_a\cdot\mathbf{r} + 24\mathbf{a}_a\cdot\mathbf{v}_a\mathbf{v}_b\cdot\mathbf{r}] + 2(\mathbf{a}_b\cdot\mathbf{r} - \mathbf{v}_b^2)(\mathbf{v}_a\cdot\mathbf{r})^2 + 12(\mathbf{v}_a\cdot\mathbf{v}_b - \mathbf{v}_a^2)\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r} + \frac{3}{r^2}(\mathbf{v}_a\cdot\mathbf{r})^2(\mathbf{v}_b\cdot\mathbf{r})^2\Big\}. \quad \text{(B2)}$$

In Fig. 13 we show all diagrams with two-graviton exchange that enter at the second PN order. The sum of those diagrams is

$$L_{\text{Fig13}} = \sum_{a\neq b}\frac{G^2m_a^2m_b}{4r^4}(6r^2\mathbf{v}_a^2 + 7r^2\mathbf{v}_b^2 - 14r^2\mathbf{v}_a\cdot\mathbf{v}_b + 2r\dot{r}\mathbf{v}_a\cdot\mathbf{r} - 2\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r}). \quad \text{(B3)}$$

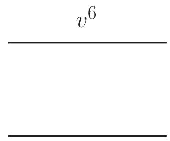


FIG. 11. Diagram with no graviton exchange.

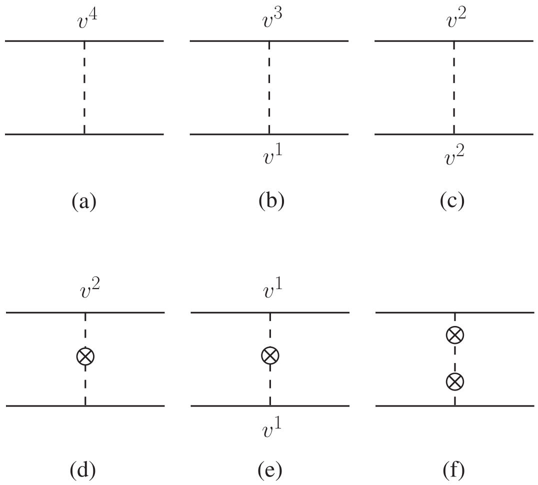


FIG. 12. Diagrams with one-graviton exchange.

There is also the diagram with a three-graviton source term as well as two other diagrams with combinations of the two-graviton source, as shown in Fig. 14. Their contribution to the Lagrangian is

$$L_{\mathrm{Fig14}} = -\sum_{a\neq b} \frac{G^3 m_a^2 m_b}{2r^3}(m_a + 3m_b). \tag{B4}$$

The diagrams which contain three-graviton vertices are illustrated in Fig. 15 and give

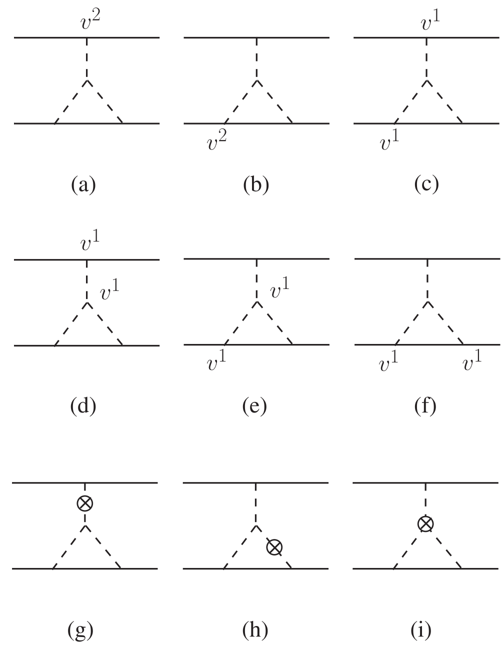


FIG. 15. Diagrams with three-graviton exchange.

$$L_{\mathrm{Fig15}} = \sum_{a\neq b} \frac{G^2 m_a^2 m_b}{2r^4}[r^2(5\mathbf{v}_a^2 - 6\mathbf{v}_a\cdot\mathbf{v}_b + 2\mathbf{v}_b^2 + 2\mathbf{a}_b\cdot\mathbf{r}) - 9(\mathbf{v}_a\cdot\mathbf{r})^2 + 14\mathbf{v}_a\cdot\mathbf{r}\mathbf{v}_b\cdot\mathbf{r} - 3(\mathbf{v}_b\cdot\mathbf{r})^2]. \tag{B5}$$

In Fig. 16, we show diagrams with a four-graviton vertex that enter at the 2PN order and, together, yield the result

$$L_{\mathrm{Fig16}} = \sum_{a\neq b} \frac{G^3 m_a^3 m_b}{r^3}. \tag{B6}$$

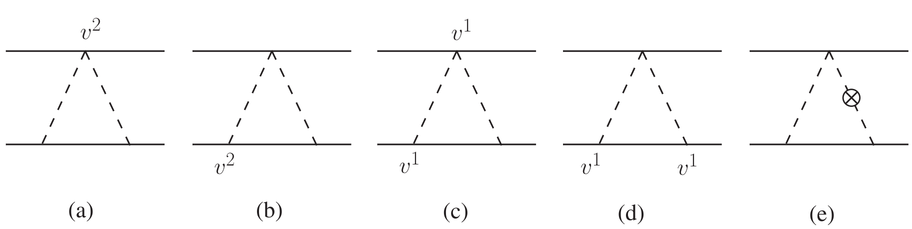


FIG. 13. Diagrams with two-graviton exchange.

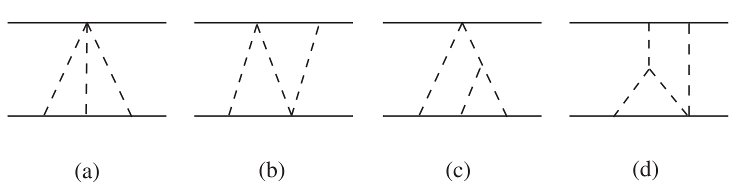
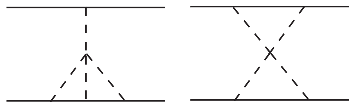


FIG. 14. (a) Three-graviton emission from one of the bodies; (b) symmetric three-graviton exchange; (c) composition of a three-graviton vertex with a two-graviton vertex in the source term.

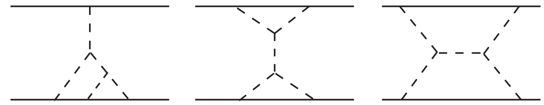


(a) (b)

FIG. 16. Diagrams with four-graviton vertex.

Lastly, the diagrams with five propagators are shown in Fig. 17 and provide us with the following result:

$$L_{\mathrm{Fig17}} = \sum_{a\neq b} \frac{G^3 m_a^2 m_b}{r^3}(m_b - 2m_a). \tag{B7}$$



(a) (b) (c)

FIG. 17. Diagrams with five propagators.

Summing up all contributions from Fig. 11 to Fig. 17, we write down the Lagrangian at 2PN order in the linearized harmonic gauge:

$$\begin{aligned}
L_{\mathrm{2PN}} = {} & \frac{1}{16} m_1 \mathbf{v}_1^6 - \frac{G^3 m_1 m_2}{2r^3}(3m_1^2 + m_1 m_2) + \frac{G^2 m_1 m_2}{4r^2}\Big[(16m_1 + 11m_2)\mathbf{v}_1^2 \\
& -13m\mathbf{v}_1\cdot\mathbf{v}_2 - 4m_2\mathbf{a}_1\cdot\mathbf{r} - \frac{2}{r^2}(8m_1 + 3m_2)(\mathbf{v}_1\cdot\mathbf{r})^2 + \frac{12}{r^2} m\mathbf{v}_1\cdot\mathbf{r}\mathbf{v}_2\cdot\mathbf{r}\Big] \\
& + \frac{G m_1 m_2}{8r}\Big[\frac{15}{2} r^2 \mathbf{a}_1\cdot\mathbf{a}_2 + 7\mathbf{v}_1^4 - 10\mathbf{v}_1^2\mathbf{v}_1\cdot\mathbf{v}_2 + (\mathbf{v}_1\cdot\mathbf{v}_2)^2 + \frac{3}{2}\mathbf{v}_1^2\mathbf{v}_2^2 \\
& + \mathbf{a}_1\cdot\mathbf{r}\mathbf{v}_2^2 - 14\mathbf{a}_1\cdot\mathbf{v}_2\mathbf{v}_2\cdot\mathbf{r} + 12\mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_2\cdot\mathbf{r} - \frac{1}{2}\mathbf{a}_1\cdot\mathbf{r}\mathbf{a}_2\cdot\mathbf{r} - \frac{1}{r^2}\mathbf{a}_1\cdot\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})^2 \\
& + \frac{1}{r^2}\Big(6\mathbf{v}_1\cdot\mathbf{r}\mathbf{v}_2\cdot\mathbf{r}\mathbf{v}_1\cdot\mathbf{v}_2 - (\mathbf{v}_1\cdot\mathbf{r})^2\mathbf{v}_2^2 - 6\mathbf{v}_1\cdot\mathbf{r}\mathbf{v}_2\cdot\mathbf{r}\mathbf{v}_2^2 + \frac{3}{2r^2}(\mathbf{v}_1\cdot\mathbf{r})^2(\mathbf{v}_2\cdot\mathbf{r})^2\Big)\Big] + 1 \leftrightarrow 2.
\end{aligned} \tag{B8}$$

We use the Lagrangian above to determine the equations of motion of the two-body system at the second PN order. Below we show the acceleration for one of the objects in the binary:

$$\begin{aligned}
\mathbf{a}_1^{\mathrm{2PN}} = {} & \frac{1}{8}\frac{Gm_2}{r^3}\mathbf{r}\Big\{\frac{G^2}{r^2}(-2m_1^2 - 20m_1m_2 + 16m_2^2) + \frac{G}{r}\Big[(18m_1 + 56m_2)\mathbf{v}_1^2 - (84m_1 + 128m_2)\mathbf{v}_1\cdot\mathbf{v}_2 + (58m_1 + 64m_2)\mathbf{v}_2^2 \\
& + 30m_1\mathbf{a}_1\cdot\mathbf{r} - 12m\mathbf{a}_2\cdot\mathbf{r} + \frac{28}{r^2}(m_1 - 4m_2)\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_1\cdot\mathbf{r} - 2\mathbf{v}_2\cdot\mathbf{r}) - \frac{1}{r^2}(56m_1 + 176m_2)(\mathbf{v}_2\cdot\mathbf{r})^2\Big] \\
& + 2\mathbf{v}_1^4 - 16(\mathbf{v}_1\cdot\mathbf{v}_2)^2 - 16\mathbf{v}_2^4 + 32\mathbf{v}_1\cdot\mathbf{v}_2\mathbf{v}_2^2 - 2\mathbf{v}_1^2\mathbf{a}_2\cdot\mathbf{r} - 2\mathbf{v}_2^2\mathbf{a}_2\cdot\mathbf{r} \\
& -4\mathbf{a}_2\cdot\mathbf{v}_2\mathbf{v}_2\cdot\mathbf{r} + \frac{(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2}(12\mathbf{v}_1^2 - 48\mathbf{v}_1\cdot\mathbf{v}_2 + 36\mathbf{v}_2^2) - 15\frac{(\mathbf{v}_2\cdot\mathbf{r})^4}{r^4}\Big\} \\
& + \frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_1\Big\{\frac{G}{r}[(48m_2 - 15m_1)\mathbf{v}_1\cdot\mathbf{r} + (23m_1 - 40m_2)\mathbf{v}_2\cdot\mathbf{r}] + \mathbf{v}_2\cdot\mathbf{r}(4\mathbf{v}_1^2 + 16\mathbf{v}_1\cdot\mathbf{v}_2 - 20\mathbf{v}_2^2) \\
& - 24\frac{\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2} + 18\frac{(\mathbf{v}_2\cdot\mathbf{r})^3}{r^2} + \mathbf{v}_1\cdot\mathbf{r}(8\mathbf{v}_1^2 - 16\mathbf{v}_1\cdot\mathbf{v}_2 + 16\mathbf{v}_2^2 - 2\mathbf{a}_2\cdot\mathbf{r}) + 2r^2(12\mathbf{a}_1 - 7\mathbf{a}_2)\cdot\mathbf{v}_1\Big\} \\
& + 2\mathbf{a}_1\cdot\mathbf{v}_1\mathbf{v}_1^2\mathbf{v}_1 + \frac{1}{4}\mathbf{a}_1\Big(49\frac{G^2m_1m_2}{r^2} + 36\frac{G^2m_2^2}{r^2} + 12\frac{Gm_2}{r}\mathbf{v}_1^2 + \mathbf{v}_1^4\Big) \\
& + \frac{1}{4}\frac{Gm_2}{r^3}\mathbf{v}_2\Big\{\frac{G}{r}[(31m_1 - 24m_2)\mathbf{v}_1\cdot\mathbf{r} + (40m_2 - 9m_1)\mathbf{v}_2\cdot\mathbf{r}] + \mathbf{v}_2\cdot\mathbf{r}(-4\mathbf{v}_1^2 - 16\mathbf{v}_1\cdot\mathbf{v}_2 + 20\mathbf{v}_2^2) \\
& + 24\frac{\mathbf{v}_1\cdot\mathbf{r}(\mathbf{v}_2\cdot\mathbf{r})^2}{r^2} - 18\frac{(\mathbf{v}_2\cdot\mathbf{r})^3}{r^2} + \mathbf{v}_1\cdot\mathbf{r}(16\mathbf{v}_1\cdot\mathbf{v}_2 - 16\mathbf{v}_2^2) - 14r^2\mathbf{a}_2\cdot\mathbf{v}_2\Big\} - \frac{7}{4}\frac{Gm_2}{r}\mathbf{a}_2\Big(6\frac{Gm}{r} + \mathbf{v}_1^2 + \mathbf{v}_2^2\Big).
\end{aligned} \tag{B9}$$

All accelerations in the right-hand side of the equality above should be regarded as Newtonian accelerations if we want the entire expression to be of definite 2PN order. To write the acceleration in the c.m. frame, we have to consider, in addition to (B9), the reduced contribution from applying the equation of motion inside (4.12) as well as the PN corrections to the c.m. frame (6.10) and (6.11). Adding these contributions together, we finally obtain the expression for the relative acceleration of the two-body system in the c.m. frame, at the second PN order, in the linearized harmonic gauge:

$$\mathbf{a}_{2\mathrm{PN}} = -\frac{Gm}{8r^3}\Big\{\mathbf{r}\Big[(56+174\nu)\frac{G^2m^2}{r^2} - (32+52\nu-16\nu^2)\frac{Gm}{r}v^2 + (112-200\nu-16\nu^2)\frac{Gm}{r}\dot{r}^2 + (24\nu-32\nu^2)v^4 - (36\nu-48\nu^2)v^2\dot{r}^2 + (15\nu-45\nu^2)\dot{r}^4\Big] + 4r\dot{r}\mathbf{v}\Big[(-12+41\nu+8\nu^2)\frac{Gm}{r} - (15\nu+4\nu^2)v^2 + (9\nu+6\nu^2)\dot{r}^2\Big]\Big\}. \tag{B10}$$

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