# General geometric operators in all dimensional loop quantum gravity

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(Received 9 March 2020; accepted 26 March 2020; published 15 April 2020)

Two strategies for constructing general geometric operators in all dimensional loop quantum gravity are proposed. The different constructions mainly come from the two different regularization methods for the basic building blocks of the spatial geometry. The first regularization method is a generalization of the regularization of the length operator in standard (1 + 3)-dimensional loop quantum gravity, while the second method is a natural extension of those for standard (D − 1)-area and usual D-volume operators. Two versions of general geometric operators to measure arbitrary $m$-areas are constructed, and their properties are discussed and compared. They serve as valuable candidates to study the quantum geometry in arbitrary dimensions.

DOI: 10.1103/PhysRevD.101.084032

## I. INTRODUCTION

As a nonperturbative and background-independent approach to unify general relativity (GR) and quantum physics, loop quantum gravity (LQG) has made remarkable progress [1–4]. An important prediction of this theory is the quantum discreteness of spatial geometry at Planck scale, since the spectrums of the geometric operators, such as volume and area, are discrete [5–7]. A key step in the procedure of constructing these geometric operators is to regularize the classical geometric quantities in terms of holonomy and flux which have direct quantum analogs. Different choices of regularization strategies may lead to different versions of a geometric operator, e.g., the two versions of volume operator [5,7,8]. Some consistency checks [9–11] on different regularization methods have been done in order to choose suitable construction and fix the regularization ambiguity. It turns out that many geometric quantities, including length [12–14], area [5,6], volume [5,7], angle [15], metric components [16], and spatial Riemann curvature scalar [17], have been quantized as well-defined operators in the kinematic Hilbert space of LQG [6,18–20]. The starting point of LQG is the Ashtekar-Barbero connection dynamics of (1 + 3)-dimensional GR, and this theory will be called (1 + 3)-dimensional standard LQG in the following part of this paper. However, this Hamiltonian connection formulation depends on the dimensions of space such that the internal gauge group is $SU(2)$, since its definition representation and adjoint representation have same dimension. This structure cannot be directly extended to the higher dimensional case. An alternative connection dynamics of GR in arbitrary (1 + D)-dimensions was proposed by Bodendorfer, Thiemann and Thurn [21,22]. In their scheme, the connection formulation is achieved by extending the Arnowitt-Deser-Misner (ADM) phase space (1 + D)-dimensional GR as a Yang-Mills phase space with gauge group $SO(D+1)$, and extra Gaussian constraint and simplicity constraint is introduced to eliminate the gauge degrees of freedom. By such a scheme, the connection formulation with compact gauge group $SO(D+1)$ is valid for both Euclidean and Lorentzian signatures. The difference of signatures is reflected by the plus and minus signs respectively in front of the Hamiltonian constraints in the two cases. In the present paper, the construction of geometric operators in arbitrary dimensional LQG based on this alternative formalism will be studied.

The phase space of the classical connection theory is coordinated by a canonical pair $(A_a^{IJ}, \pi^b_{KL})$ with nontrivial Poisson bracket,

$$\{A_a^{IJ}(x), \pi^b_{KL}(y)\} = 2\kappa\beta\delta_a^b\delta_K^{[I}\delta_L^{J]}\delta^{(D)}(x-y), \qquad (1)$$

where $\kappa = 16\pi G^{(D+1)}$ is the gravitational constant in (1 + D)-dimensional space-time, $\beta$ is the Barbero-Immirzi parameter in this theory, the spatial indices read $a, b, c, \ldots \in \{1, \ldots, D\}$, internal indices read $I, J, K, \ldots \in \{1, \ldots, D+1\}$ and $x, y, \ldots$ are coordinates on a D-dimensional spatial manifold $\sigma$. This phase space is subject to the Gaussian constraint

$$\mathcal{G}^{IJ} := \partial_a\pi^{aIJ} + 2A_{aK}^{[I}\pi^{|aK|J]} = 0, \qquad (2)$$

and simplicity constraint

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$$S^{ab}_{IJKL} := \pi^a_{[IJ}\pi^b_{KL]} = 0, \tag{3}$$

which induce gauge transformations, as well as spatial diffeomorphism constraint and Hamiltonian constraint which give the spacetime diffeomorphism transformations. The solution of simplicity constraint links the momentum $\pi^a_{IJ}$ and spatial geometry by

$$\pi^a_{IJ} \hat{=} \sqrt{q}\, n_{[I} e^a_{J]}, \tag{4}$$

where $\hat{=}$ represents "equal on simplicity constraint surface," $e^a_J$ is the "D-frame" related to the inverse spatial metric $q^{ab}$ by $q^{ab} = e^a_I e^{bI}$, $q$ is the determinant of the spatial metric $q_{ab}$, and $n^I$ is an internal normalized vector such that $e^a_J n^J = 0$. Also, on the constraint surface of the simplicity constraint, one can define the spin connection field $\Gamma^{IJ}_a$ by [21]

$$\Gamma_{aIJ} = n_{[I}\partial_a n_{J]} + e_{b[I}\partial_a e^b_{J]} + \Gamma^b_{ac} e_{b[I} e^c_{J]}, \tag{5}$$

where $e^I_a = q_{ab} e^{bI}$, and $\Gamma^c_{ab}$ is the Levi-Civita connection. Then on the constraint surface of the Gaussian and simplicity constraints, the extrinsic curvature $K_{ab}$ of the spatial hypersurface $\sigma$ is related to the $SO(D+1)$ connection by

$$K^b_a = \frac{1}{\beta\sqrt{q}}(A_{aIJ} - \Gamma_{aIJ})\pi^{bIJ}. \tag{6}$$

Based on these relations, the symplectic reduction of the $SO(D+1)$ Yang-Mills phase space with respect to Gaussian and simplicity constraints can be identified with the familiar ADM phase space of GR in $(1+\mathrm{D})$-dimensional space-time. Especially, the transformations induced by the simplicity constraint only change some components of $A^{IJ}_a$, while the Gaussian constraint induces the local $SO(D+1)$ rotations. This classical connection theory can be quantized following the standard loop quantization methods, and the resulting all dimensional LQG is equipped with a kinematic Hilbert space $\mathcal{H} = L^2(\bar{A}, d\mu_0)$ and the corresponding quantum constraints. The kinematic Hilbert space is the completion of the space of cylindrical functions and spanned by a basis of states each of which is given by a network of holonomies, with a specific $SO(D+1)$ representation assigned to each edge of the network, and a specific coupling between the neighboring $SO(D+1)$ representations assigned to each vertex of the network. The basic operators, including holonomy operator and flux operator, act on a cylindrical function $f_\gamma(A)$ in $\mathcal{H}$ as

$$\hat{h}_e(A)\cdot f_\gamma(A) := h_e(A) f_\gamma(A), \tag{7}$$

$$\hat{\pi}^{IJ}(S)\cdot f_{\gamma_S}(A) := -\mathbf{i}\hbar\kappa\beta \sum_{e\in E(\gamma_S)} \epsilon(e,S) R^{IJ}_e f_{\gamma_S}(A), \tag{8}$$

where $h_e(A)$ is the holonomy of $A_{aIJ}$ field along edge $e$, $\hat{\pi}^{IJ}(S)$ is the standard flux operator corresponding to the classical flux $\pi^{IJ}(S) := \int_S dS \pi^{aIJ} n_a(S)$, with $dS$ and $n_a(S)$ being the measure and normal covector field on $(\mathrm{D}-1)$-surface $S$ respectively, $\gamma_S$ denotes a graph adapted to $S$ and equivalent to $\gamma$, $R^{IJ}_e := \mathrm{tr}((\tau^{IJ} h_e(0,1))^{\mathrm{T}} \frac{\partial}{\partial h_e(0,1)})$ is the right invariant vector fields on $SO(D+1)$ associated to the edge $e$ of $\gamma_S$ with $\tau^{IJ}$ being an element of $so(D+1)$, and T representing the transposition of the matrix, and $\epsilon(e,S)$ is defined by

$$\epsilon(e,S) = \begin{cases} +1 & \text{if } e \text{ lies above the surface } S \text{ and } b(e)\in S; \\ -1 & \text{if } e \text{ lies below the surface } S \text{ and } b(e)\in S; \quad e\in E(\gamma_S), \\ 0 & \text{if } e\cap S = \emptyset \text{ or } e \text{ lies in } S. \end{cases}$$

Here $\gamma_S$ is such a graph that there are only outgoing edges at each true vertex, and $b(e)$ denotes the beginning point of the edge $e$. In comparison with the standard LQG in $(1+3)$-dimensions, a subtle issue in the arbitrary dimensional LQG is how to solve the simplicity constraint [23,24]. Up to now, the edge simplicity constraint operator has been solved by restricting the representation labeled to each edge being simple representation, while the anomalous vertex simplicity constraint has several alternative solutions which are called simple intertwiners. Besides, the construction of geometric operators in $\mathcal{H}$ is also a complicated issue. Although the candidates of the $(\mathrm{D}-1)$-area operator and the D-volume operator in $(1+\mathrm{D})$-dimensional LQG were proposed following the same procedure in the construction of 2-area operator and 3-volume operator in standard $(1+3)$-dimensional LQG, a systematic method to construct more general geometric operators is lacking. The general construction method is crucial, since there are more and more geometric quantities as the increasing of spatial dimensions. Notice that for a given geometric quantity, several different classical expressions with the basic conjugate variables could exist. Hence there would be different ways to construct the corresponding geometric operators. Notice that the spatial metric $q_{ab}$ is determined by the momentum variable by $qq^{ab} \hat{=} \frac{1}{2}\pi^{aIJ}\pi^{b}_{IJ}$, $q$ denotes the determinant of $q_{ab}$. Then we can define the de-densitized dual momentum $\sqrt{q}\pi^{IJ}_{b}$ as a function of $\pi^{a}_{IJ}$ on simplicity constraint surface, with $\pi^{IJ}_{a}$ as the inverse of $\pi^{b}_{KL}$ satisfying $\frac{1}{2}\pi^{a}_{IJ}\pi^{IJ}_{b} \hat{=} \delta^{a}_{b}$. Now we can give the relation between the spatial metric $q_{ab}$ and de-densitized dual momentum $\sqrt{q}\pi^{IJ}_{a}$ by

$$q_{ab} \hat{=} \frac{1}{2}\sqrt{q}\pi_{aIJ}\sqrt{q}\pi^{IJ}_{b}, \tag{9}$$

where $\sqrt{q}\pi_{aIJ}$ plays the role of coframe. Therefore, if one could construct an operator corresponding to the de-densitized dual momentum $\sqrt{q}\pi^{IJ}_{a}$, a building block for all geometric operators in arbitrary dimensions would be ready. In this paper, two strategies to construct such a building block operator will be proposed. In the first strategy, we employ the expression

$$\sqrt{q}\pi^{IJ}_{a}(x) \hat{=} \frac{(D-1)}{\beta\kappa}\{A^{IJ}_{a}(x), V(x,\square)\}, \tag{10}$$

where $V(x,\square) \coloneqq \int_{\square} d^{D}y\sqrt{q}(y)$ and $\square \ni x$ is a proper small open D-dimensional region. In the second strategy, $\sqrt{q}\pi^{IJ}_{a}$ is purely expressed by conjugate momentum $\pi^{a}_{IJ}$.

This paper is organized as follows. In Section II, we will construct general geometric operators by the first strategy mentioned above. First, Thiemann's construction of length operator in standard $(1+3)$-dimensional LQG will be extended to construct a length operator in the all dimensional case. Then, we will construct an alternative two-dimensional area operator by using cotriad as building blocks in the standard $(1+3)$-dimensional LQG. The method can naturally be extended to construct a 2-area operator in $(1+\mathrm{D})$-dimensional LQG. Finally, following the construction procedure of the 2-area operator, by using the de-densitized dual momentum as building blocks, general $m$-area operators for $m$-dimensional surfaces in D-dimensional space will be proposed. In Sec. III, certain special cases of the general "$m$-area" operators and the problems related to their construction will be discussed. The consistency of the alternative flux operator, which is used to construct the general $m$-area operators, with the standard flux operator will also be checked. The second strategy to construct general geometric operators will be discussed in Sec. IV. The de-densitized dual momentum is totally given by the conjugate momentum in this strategy. By suitable regularization, its components can be expressed in terms of flux and volume properly. Then it becomes an operator by replacing the flux and volume by their quantum analogs. By using this well-defined dual momentum operator as building blocks, we will get the general geometric operators corresponding to the $m$-areas which are totally composed with flux operator and volume operator. Certain special cases of the general geometric operators and their virtues and problems will be also discussed. Our results will be summarized and discussed in the final section.

As two frameworks of connection dynamics and several geometric operators are involved, it is necessary to clarify some expressions and indices appeared in this paper. We denote by $A^{i}_{a}$, $E^{b}_{j}$, and $e^{i}_{a}$ as the Ashtekar-Barbero connection, densitized triad (which is also the conjugate momentum in this theory), and cotriad respectively in $(1+3)$-dimensional standard LQG, where $q_{ab} = e^{i}_{a}e_{bi}$ is the spatial metric in this formulation. We denote by $A^{IJ}_{a}$, $\pi^{b}_{KL}$, and $\sqrt{q}\pi^{IJ}_{a}$ as the connection, conjugate momentum, and de-densitized dual momentum respectively in all dimensional LQG. Besides, in the following part of this paper, the 2-area and 3-volume operator constructed in [6,7] will be called standard 2-area and usual 3-volume operator (or just area and volume operator) in $(1+3)$-dimensional standard LQG respectively, which are totally constructed with the standard flux operators in this theory and the usual volume operator takes a special internal regularization. Similarly, the $(\mathrm{D}-1)$-area and D-volume operator constructed in [22] will be called standard $(\mathrm{D}-1)$-area and usual D-volume operator (or just area and volume operator) in all dimensional LQG respectively, which are also totally constructed with the standard flux operators.

## II. GEOMETRIC OPERATOR IN ALL DIMENSIONAL LQG: FIRST STRATEGY

In the standard $(1+3)$-dimensional LQG the volume operator, area operator and angle operator were directly constructed by the standard flux operator $\hat{E}^{i}(S)$, while the length operators were constructed in several different ways [12–14]. The construction of these length operators involves two steps, the classical length of a curve is expressed by cotriad $e^{i}_{a}$ in the first step. In the second step, different expressions for $e^{i}_{a}$ are used in different ways. In Thiemann's construction of length operator, $e^{i}_{a}$ is expressed as [12]

$$e^{i}_{a}(x) = \frac{2}{\kappa\gamma_{\mathrm{BI}}}\{A^{i}_{a}(x), V(x,\square)\}, \tag{11}$$

wherein $\gamma_{\mathrm{BI}}$ is the Barbero-Immrizi parameter in standard $(1+3)$-dimensional LQG. In the other construction $e^{i}_{a}$ is expressed as [13,14]

$$e^{i}_{a}(x) = \frac{\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}E^{b}_{j}E^{c}_{k}}{\mathrm{sgn}(\det(E))\sqrt{|\det(E)|}}. \tag{12}$$

In contrast to the standard flux operator $\hat{E}^{i}(S)$ corresponds to the classical flux $E^{i}(S) \coloneqq \int_{S} E^{ai}n_{a}(S)$ with $n_{a}(S)$ being the normal covector field of 2-surface $S$, the expressions (11) of cotriad imply an alternative flux operator $\hat{E}^{i}_{\mathrm{alt}}(S)$ whose expression involves the commutator of holonomy and volume operator [9]. We will introduce its detail in

Sec. II B. The consistency checking of $\hat{E}^i_{\rm alt}(S)$ with $\hat{E}^i(S)$ indicated a suitable volume operator $\hat{V}$ and fixed its regularization ambiguity [11]. We will refer to this $\hat{V}$ as the usual volume operator. Thus it is also reasonable to consider alternative ways in the construction of the general geometric operators in all dimensional LQG, which are based on two different expressions of the de-densitized dual momentum $\sqrt{q}\pi^{IJ}_a$. In this section, we will discuss how to construct general geometric operators based on the expression (10), which is similar to that of Thiemann's length operator in standard $(1+3)$-dimensional LQG.

Let $e_\epsilon$ be a small segment of a curve $e$ with coordinate length $\epsilon \to 0$. The dual de-densitized momentum can be smeared over $e_\epsilon$ as

$$\underline{\pi}(e_\epsilon) \coloneqq \int_{e_\epsilon} \sqrt{q}\pi^{IJ}_a \tau_{IJ} \dot{e}^a_\epsilon ds \doteq -\frac{(D-1)}{\beta\kappa} h_{e_\epsilon}\{h^{-1}_{e_\epsilon}, V(v,\Box)\}, \tag{13}$$

where $\tau_{IJ}$ is the basis of Lie algebra $so(D+1)$, $h_{e_\epsilon}$ denotes the holonomy of the connection $A^{IJ}_a$ along $e_\epsilon$, $v$ is the starting point of $e_\epsilon$, and $s$ is the parameter of $e_\epsilon$. This smeared quantity can be quantized directly as

$$\hat{\underline{\pi}}(e_\epsilon) \coloneqq -\frac{(D-1)}{\mathbf{i}\beta\kappa\hbar} h_{e_\epsilon}[h^{-1}_{e_\epsilon}, \hat{V}(v,\Box)], \tag{14}$$

which is called smeared de-densitized dual momentum operator, and where $\hat{V}(v,\Box)$ is the usual D-volume operator which is totally constructed by flux operators [22]. It will be used as building blocks to construct general geometric operators in all dimensional LQG.

### A. The first length operator in all dimensional LQG

Classically, the length of a curve $e$ reads

$$L_e = \int_e ds \sqrt{q_{ab}\dot{e}^a \dot{e}^b(s)}. \tag{15}$$

Partitioning of the curve $e$ as a composition of $T$ segments $\{e^\epsilon_t, t\in\mathbb{N}, 0\le t\le T\}$, i.e.,

$$e = e^\epsilon_1 \circ e^\epsilon_2 \circ \cdots \circ e^\epsilon_t \circ \cdots \circ e^\epsilon_T, \tag{16}$$

wherein $\circ$ is a composition of composable curves which can be carried out with

$$e^\epsilon_t : [(t-1)\epsilon, t\epsilon] \to \sigma; \qquad s_t \mapsto e^\epsilon_t(s_t), \tag{17}$$

and $\epsilon = \frac{1}{T}$. Then, we have

$$L_e = \lim_{\epsilon\to 0} \sum_{t=1}^{T} L_{e^\epsilon_t}, \tag{18}$$

where one has up to $\mathcal{O}(\epsilon^2)$, $L_{e^\epsilon_t} \doteq \sqrt{\frac{1}{2}(\int_{e^\epsilon_t} ds\sqrt{q}\dot{e}^a \pi_{aIJ})(\int_{e^\epsilon_t} ds\sqrt{q}\dot{e}^b \pi^{IJ}_b)}$. Then our task turns to be constructing the length operator $\hat{L}_{e^\epsilon_t}$ of a small curve $e^\epsilon_t$. By Eq. (10) one has

$$q_{ab}(x) \doteq \frac{-(D-1)^2}{2(\beta\kappa)^2} \mathrm{tr}(\tau_{IJ}\tau_{KL})\{A^{IJ}_a(x), V(x,\Box)\}\{A^{KL}_b(x), V(x,\Box)\}. \tag{19}$$

It is easy to see that in the limit $\epsilon \to 0$, we have

$$L_{e_\epsilon} \doteq \sqrt{-\frac{1}{2}\mathrm{tr}(\underline{\pi}(e_\epsilon)\underline{\pi}(e_\epsilon))}. \tag{20}$$

Hence, by Eq. (14), $L(e_\epsilon)$ can be quantized as

$$\begin{aligned}\hat{L}_{e_\epsilon} &= \sqrt{-\frac{1}{2}\mathrm{tr}(\hat{\underline{\pi}}(e_\epsilon)\hat{\underline{\pi}}(e_\epsilon))} = \frac{(D-1)}{\sqrt{2}\beta\kappa\hbar}\sqrt{\mathrm{tr}(h_{e_\epsilon}[h^{-1}_{e_\epsilon}, \hat{V}(v,\Box)]h_{e_\epsilon}[h^{-1}_{e_\epsilon}, \hat{V}(v,\Box)])} \\ &= \frac{(D-1)}{\sqrt{2}\beta\kappa\hbar}\sqrt{(D+1)\hat{V}^2(v,\Box) - \mathrm{tr}(h_{e_\epsilon}\hat{V}(v,\Box)h^{-1}_{e_\epsilon}\hat{V}(v,\Box)) - \mathrm{tr}(\hat{V}(v,\Box)h_{e_\epsilon}\hat{V}(v,\Box)h^{-1}_{e_\epsilon}) + \mathrm{tr}(h_{e_\epsilon}\hat{V}^2(v,\Box)h^{-1}_{e_\epsilon})}.\end{aligned} \tag{21}$$

Denoting $\hat{\ell}_{e_\epsilon} \coloneqq \mathbb{I}\hat{V}(v,\Box) - h_{e_\epsilon}\hat{V}(v,\Box)h^{-1}_{e_\epsilon}$, we have

$$\hat{L}(e_\epsilon) = \frac{(D-1)}{\sqrt{2}\beta\kappa\hbar}\sqrt{\mathrm{tr}(\hat{\ell}_{e_\epsilon}\hat{\ell}_{e_\epsilon})}. \tag{22}$$

Note that $\hat{\ell}_{e_\epsilon} = \hat{\ell}^\dagger_{e_\epsilon}$ because of $h^\dagger_{e_\epsilon} = h^{-1}_{e_\epsilon}$ and $\hat{V}(v, \square) = \hat{V}^\dagger(v, \square)$. Therefore $\hat{L}_{e_\epsilon}$ is a positive and symmetric operator. Then the length operator for the curve $e$ can be defined as

$$\hat{L}_e = \lim_{\epsilon\to 0} \sum_{t=1}^{T} \hat{L}_{e^\epsilon_t}. \tag{23}$$

Note that, although the expression of $\hat{L}_e$ contains the summation of infinite terms at the limit $\epsilon \to 0$, only a finite number of terms are nonvanishing when it acts on a cylindrical state $f_\gamma$ since the volume operator only acts on nontrivial vertices $v$ of $\gamma$. Thus the regulator $\epsilon$ can be removed in the graph-dependent manner. In the rest of the paper, all the limits of this kind of infinite summation of operators can be understood in this way. The domain of $\hat{L}_e$ is the Dth order differentiable cylindrical functions satisfying the simplicity constraint in $\mathcal{H}_{\rm kin}$, and the demonstration of its cylindrical consistency is similar to that in the standard (1 + 3)-dimensional LQG [12].

### B. 2-area operator in all dimensional LQG

Although the area operator of 2-surface can be defined by the flux operator naturally in the standard (1 + 3)-dimensional LQG, the construction cannot be directly extended to higher dimensional cases. In order to construct a 2-area operator in the all dimensional case, let us come back to the (1 + 3)-dimensional theory to construct an alternative 2-area operator.

We first consider the alternative flux $E^i_{\rm (alt)}({}^2S_{\diamond_{12}}) \coloneqq \int_{^2\mathrm{S}} d({}^2\mathrm{S}) E^{ai}_{\rm (alt)} n_a({}^2S_{\diamond_{12}})$ [9,11], where $E^{ai}_{\rm (alt)} \coloneqq \frac{1}{2}\epsilon^i{}_{jk}\epsilon^{abc} e^j_b \mathcal{S} e^k_c$, $n_a({}^2S_{\diamond_{12}}) = \frac{1}{2}\epsilon_{abc}\dot{e}^b_\imath \dot{e}^c_\jmath \epsilon^{\imath\jmath}$ with $\imath, \jmath, \imath', \jmath', = 1,\, 2$, and $\mathcal{S} \coloneqq \mathrm{sgn}(\det(\mathrm{e}))$. Here $e^\epsilon_1, e^\epsilon_2$ are two linearly independent segments beginning at $v$ with coordinate length $\epsilon$, $\dot{e}^b_1$, $\dot{e}^c_2$ are their tangent vectors respectively, and ${}^2S_{\diamond_{12}}$ is a proper open 2-surface with coordinate area $\epsilon^2$ and containing $e^\epsilon_1, e^\epsilon_2$ and $v$. Then at the order of $\mathcal{O}(\epsilon^2)$, we have

$$\begin{aligned} E^{\rm (alt)}_i({}^2S_{\diamond_{12}}) &= \frac{\epsilon^2}{-2\times 2}\mathcal{S}\mathrm{tr}(\tau_i\tau_j\tau_k) e^j_b e^k_c \epsilon^{\imath\jmath}\dot{e}^b_\imath \dot{e}^c_\jmath \\ &= -\frac{1}{(\kappa\gamma_{\rm BI})^2}\mathcal{S}\mathrm{tr}(\tau_i h_{e^\epsilon_\imath}\{h^{-1}_{e^\epsilon_\imath}, V(x,\square)\} h_{e^\epsilon_\jmath}\{h^{-1}_{e^\epsilon_\jmath}, V(x,\square)\})\epsilon^{\imath\jmath} \\ &= -\frac{1}{(\kappa\gamma_{\rm BI})^2}\mathrm{tr}(\tau_i h_{e^\epsilon_\jmath}\{h^{-1}_{e^\epsilon_\imath}, V(x,\square)\}\mathcal{S}\{h^{-1}_{e^\epsilon_\jmath}, V(x,\square)\} h_{e^\epsilon_\imath})\epsilon^{\imath\jmath}. \end{aligned} \tag{24}$$

Notice that in the third step of Eq. (24), the ordering of holonomies $h_{e^\epsilon_\imath}$ and $h_{e^\epsilon_\jmath}$ was changed, while the contraction of their indices was kept unchanged. Classically, it is easy to see that $E^{ai}_{\rm (alt)} = E^{ai}$, or $E^{\rm (alt)}_i({}^2S_{\diamond_{12}}) = E_i({}^2S_{\diamond_{12}})$, wherein $E^{ai}$ is the density triad in standard (1 + 3)-dimensional LQG. Thus we can define the alternative regulated flux operator by

$$\hat{E}^{\rm (alt)}_i({}^2S_{\diamond_{12}}) = \frac{1}{(\kappa\gamma_{\rm BI}\hbar)^2}\mathrm{tr}(\tau_i h_{e^\epsilon_\jmath}[h^{-1}_{e^\epsilon_\imath}, \hat{V}(x,\square)]\hat{\mathcal{S}}[h^{-1}_{e^\epsilon_\jmath}, \hat{V}(x,\square)] h_{e^\epsilon_\imath})\epsilon^{\imath\jmath}. \tag{25}$$

Then a corresponding symmetric operator can be defined as

$$\hat{E}^{\rm alt}_i({}^2S_{\diamond_{12}}) = \frac{1}{2}(\hat{E}^{\rm (alt)}_i({}^2S_{\diamond_{12}}) + \hat{E}^{\rm (alt)\dagger}_i({}^2S_{\diamond_{12}})), \tag{26}$$

where we ordered all the variables following the scheme in [9,11]. This ordering ensures that $\hat{E}^{\rm alt}_i({}^2S_{\diamond_{12}})$ is consistent with the standard flux operator $\hat{E}^i({}^2S_{\diamond_{12}})$ at least in certain cases. Now, the classical identity $\mathrm{Ar}({}^2S_{\diamond_{12}}) \approx \sqrt{E^i({}^2S_{\diamond_{12}})E^j({}^2S_{\diamond_{12}})\delta_{ij}}$ indicates that we can define an alternative area operator by

$$\widehat{\mathrm{Ar}}_{\rm alt}({}^2S_{\diamond_{12}}) = \sqrt{\hat{E}^{\rm alt}_i({}^2S_{\diamond_{12}})\hat{E}^{\rm alt}_j({}^2S_{\diamond_{12}})\delta^{ij}}. \tag{27}$$

The alternative area operator can also be understood in another perspective of geometry. The classical corresponding expression of $\widehat{\mathrm{Ar}}_{\rm alt}({}^2S_{\diamond_{12}})$ can be written as

$$\mathrm{Ar}({}^2S_{\diamond_{12}}) \approx \sqrt{\delta_{ij}E^j_{\rm alt}({}^2S_{\diamond_{\imath\jmath}})E^i_{\rm alt}({}^2S_{\diamond_{12}})} \approx \sqrt{\frac{\epsilon^4}{2}q_{bd}q_{ce}\epsilon^{\imath\jmath}\dot{e}^b_\imath\dot{e}^c_\jmath\epsilon^{\imath'\jmath'}\dot{e}^d_{\imath'}\dot{e}^e_{\jmath'}}. \tag{28}$$

Hence we have

$$\mathrm{Ar}({}^2S_{\diamond_{\imath\jmath}}) \approx \sqrt{L^2(e_1^\epsilon)L^2(e_2^\epsilon) - (\epsilon\dot{e}_1^a(x)q_{ab}\epsilon\dot{e}_2^b(x))^2}, \quad (29)$$

where $L(e_\imath^\epsilon) \coloneqq \int_{e_\imath^\epsilon} ds(e_\imath^\epsilon)\sqrt{q_{ab}\dot{e}_\imath^a\dot{e}_\imath^b} \approx \epsilon\sqrt{q_{ab}\dot{e}_\imath^a\dot{e}_\imath^b(v)}$. Note that we also have

$$\cos\theta_{12} = \lim_{\epsilon\to 0}\frac{\epsilon^2\dot{e}_1^a(v)q_{ab}\dot{e}_2^b(v)}{L(e_1^\epsilon)L(e_2^\epsilon)}, \quad (30)$$

where $\theta_{12}$ is the angle between $\dot{e}_1^a(v)$ and $\dot{e}_2^b(v)$. Therefore we obtain

$$\mathrm{Ar}({}^2S_{\diamond_{12}}) \approx L(e_1^\epsilon)L(e_2^\epsilon)\sin\theta_{12}, \quad (31)$$

which is the standard expression of the area of ${}^2S_{\diamond_{\imath\jmath}}$ in Euclidean space.

The advantage of the alternative 2-area operator (27) is that its construction can be extended to the all dimensional case naturally. Let us define an alternative "flux" operator suitable to (1 + D)-dimensional cases as

$$\begin{aligned}\hat{\tilde{E}}^{\bar{M}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}}) &\equiv \hat{\tilde{E}}^{I_1\ldots I_{D-1}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}}) \\ &= \frac{\mathrm{C}}{(\kappa\beta\hbar)^2}\epsilon^{I_1\ldots I_{D-1}KL}\mathrm{tr}^{e_1,e_2}\sum_{\imath,\jmath}(\tau^{e_\imath}_{KI}\tau^{e_\jmath I}{}_L \\ &\quad\times h_{e_\jmath^\epsilon}[h^{-1}_{e_\imath^\epsilon},\hat{V}(v,\square)]\hat{\mathcal{S}}[h^{-1}_{e_\jmath^\epsilon},\hat{V}(v,\square)]h_{e_\imath^\epsilon})\epsilon^{\imath\jmath},\end{aligned} \quad (32)$$

where $\mathrm{C} = \frac{(D-1)^2}{2\sqrt{(D-1)!}}$, $\bar{M}$ is $(D-1)$-tuple totally asymmetric indices, $\epsilon^{I_1\ldots I_{D+1}}$ is the Levi-Civita symbols in the internal space, $\mathrm{tr}^{e_1,e_2}$ represents tracing the indices of $\tau^{e_1}_{I_{D-2}I}, h_{e_1^\epsilon}, h^{-1}_{e_1^\epsilon}$ and $\tau^{e_2}_{I_{D-2}I}, h_{e_2^\epsilon}, h^{-1}_{e_2^\epsilon}$ separately. Note that $\mathcal{S}$ takes the value of 1 if $(D+1)$ is odd while takes the value of 1 with a sign of $\det(\pi) \coloneqq \frac{1}{2D!}\epsilon_{aa_1b_1\ldots a_nb_n}\epsilon_{IJI_1J_1\ldots I_nJ_n}\pi^{aIJ}\pi^{a_1I_1K_1}\pi^{b_1J_1}{}_{K_1}\ldots\pi^{a_nI_nK_n}\pi^{b_nJ_n}{}_{K_n}$ if $(D+1)$ is even. Here $\epsilon_{a_1\ldots a_D}$ is the Levi-Civita symbol in the external space. The quantization $\hat{S}$ of $S$ will be introduced in the Appendix. Then the area $\mathrm{Ar}({}^2S_{\diamond_{12}})$ of the 2-surface ${}^2S_{\diamond_{12}}$ can be promoted as an operator in the all dimensional case by

$$\widehat{\mathrm{Ar}({}^2S_{\diamond_{12}})} = \sqrt{\hat{\tilde{E}}^{\mathrm{gen}}_{\bar{M}}({}^2S_{\diamond_{12}})\hat{\tilde{E}}^{\bar{M}}_{\mathrm{gen}}({}^2S_{\diamond_{12}})}, \quad (33)$$

where

$$\hat{\tilde{E}}^{\bar{M}}_{\mathrm{gen}}({}^2S_{\diamond_{12}}) = \frac{1}{2}(\hat{\tilde{E}}^{\bar{M}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}}) + \hat{\tilde{E}}^{\bar{M}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}})^\dagger). \quad (34)$$

Now we need to show that the classical analog of $\widehat{\mathrm{Ar}({}^2S_{\diamond_{12}})}$ is exactly the 2-area. Note that at the order of $\mathcal{O}(\epsilon^2)$ the classical analog of $\hat{\tilde{E}}^{\bar{M}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}})$ reads

$$\begin{aligned}\tilde{E}^{\bar{M}}_{(\mathrm{gen})}({}^2S_{\diamond_{12}}) &= -\frac{\mathrm{C}}{(\kappa\beta)^2}\epsilon^{I_1\ldots I_{D-1}KL}\mathrm{tr}^{e_1,e_2}\sum_{\imath,\jmath}(\tau^{e_\imath}_{KI}\tau^{e_\jmath I}{}_L h_{e_\jmath^\epsilon}\{h^{-1}_{e_\imath^\epsilon},V(v,\square)\}\mathcal{S}\{h^{-1}_{e_\jmath^\epsilon},V(v,\square)\}h_{e_\imath^\epsilon})\epsilon^{\imath\jmath} \\ &= \frac{\mathrm{C}}{(\kappa\beta)^2}\epsilon^2\epsilon^{I_1\ldots I_{D-1}KL}\mathrm{tr}(\tau_{KI}\tau^{J_1J_1'})\mathrm{tr}(\tau_L{}^I\tau^{J_2J_2'})\{A_{aJ_1J_1'}\dot{e}^a_\imath,V(v,\square)\}\mathcal{S}\{A_{bJ_2J_2'}\dot{e}^b_\jmath,V(v,\square)\})\epsilon^{\imath\jmath} \\ &= -\frac{\mathrm{C}}{(D-1)^2}\epsilon^2\epsilon^{I_1\ldots I_{D-1}KL}\dot{e}^a_\imath\sqrt{q}\pi_{aK}{}^I\mathcal{S}\dot{e}^b_\jmath\sqrt{q}\pi_{bIL}\epsilon^{\imath\jmath},\end{aligned} \quad (35)$$

which is the analog of the alternative flux (24) of (1 + 3)-dimensional standard LQG. Hence, at the order of $\mathcal{O}(\epsilon^4)$, we have

$$\begin{aligned}\delta_{\bar{M}\bar{M}'}\tilde{E}^{\bar{M}}_{(\mathrm{gen})}\tilde{E}^{\bar{M}'}_{(\mathrm{gen})}({}^2S_{\diamond_{12}}) &= \frac{\mathrm{C}^2}{(D-1)^4}\epsilon^4(D-1)!2!\delta^{[K}_{K'}\delta^{L]}_{L'}\dot{e}^a_\imath\sqrt{q}\pi_{aK}{}^I\dot{e}^b_\jmath\sqrt{q}\pi_{bIL}\epsilon^{\imath\jmath}\dot{e}^{a'}_{\imath'}\sqrt{q}\pi^{K'}_{a'I'}\dot{e}^{b'}_{\jmath'}\sqrt{q}\pi_{b'}{}^{I'L'}\epsilon^{\imath'\jmath'} \\ &= \frac{1}{2}\epsilon^4 q_{aa'}q_{bb'}\dot{e}^a_\imath\dot{e}^b_\jmath\epsilon^{\imath\jmath}\dot{e}^{a'}_{\imath'}\dot{e}^{b'}_{\jmath'}\epsilon^{\imath'\jmath'} \\ &= (L(e_1^\epsilon)L(e_2^\epsilon)\sin\theta_{12})^2,\end{aligned} \quad (36)$$

which is extended from the calculation of $\delta_{ij}E^j_{\mathrm{alt}}({}^2S_{\diamond_{\imath\jmath}})E^i_{\mathrm{alt}}({}^2S_{\diamond_{12}})$ that comes from Eq. (28). Therefore, the classical analog of $\widehat{\mathrm{Ar}({}^2S_{\diamond_{12}})}$ does correspond to the classical 2-area expression $L(e_1^\epsilon)L(e_2^\epsilon)\sin\theta_{12}$.

In the above construction of $\mathrm{Ar}(\widehat{{}^2S_{\diamond_{12}}})$, we generalized the alternative 2-area operator in the standard $(1+3)$-dimensional LQG to the higher dimensional case, and kept the authentic ordering of its constituents, which has been shown to be consistent with the standard flux operator in certain cases.

### C. General *m*-area operators in all dimensional LQG

The above construction of the 2-area operator inspires us to consider the more general case. We will use a similar way to construct $m$-dimensional $(1\leq m\leq D)$ area operators in $(1+\mathrm{D})$-dimensional LQG. Consider a partition ${}^mS=\bigcup_{t=1}^{T}{}^m\bar{S}^t_{\diamond_{1m}}$ of an open $m$-surface ${}^mS$, with ${}^m\bar{S}^t_{\diamond_{1m}}$ being closed $m$-surface with open interior ${}^mS^t_{\diamond_{1m}}$, $t$ being the labeling number of these component $m$-surfaces in this partition, and $T$ being the total number of them. The arbitrary small $m$-surfaces ${}^mS_{\diamond_{1m}}$ has coordinate area $\epsilon^m$ and contains $e^\epsilon_1, e^\epsilon_2, \ldots, e^\epsilon_m$ and $v$, where the $m$ small segments $e^\epsilon_1, e^\epsilon_2, \ldots, e^\epsilon_m$ have common beginning point $v$ and coordinate length $\epsilon$. Their tangent vectors $\dot{e}^a_1(v), \dot{e}^a_2(v), \ldots, \dot{e}^a_m(v)$ at point $v$ span an $m$-dimensional vector space. A local right-handed coordinate system $\{s_1,\ldots,s_m\}$ can be defined such that $v=(0,\ldots,0)$, $t(e^\epsilon_\iota)=(0,\ldots,s_\iota,\ldots,0)|_{s_\iota=\epsilon}$, $\dot{e}^a_\iota=(\frac{\partial}{\partial s_\iota})^a|_{e^\epsilon_\iota}$, and $e^\epsilon_1, e^\epsilon_2, \ldots, e^\epsilon_m$ are its positive oriented coordinate axis, where $\iota=1,\ldots,m$, and $t(e^\epsilon_\iota)$ is the target point of $e^\epsilon_\iota$. The classical expression of the $m$-area of ${}^mS$ reads

$$\mathrm{Ar}({}^mS)=\sum\mathrm{Ar}({}^mS_{\diamond_{1m}})=\sum\int_{{}^mS_{\diamond_{1m}}}\sqrt{\det({}^mq)}ds_1\ldots ds_m, \tag{37}$$

where

$$\det({}^mq)=\frac{1}{m!}{}^mq_{a_1a'_1}\ldots{}^mq_{a_ma'_m}\dot{e}^{a_1}_{\iota_1}\ldots\dot{e}^{a_m}_{\iota_m}\epsilon^{\iota_1\ldots\iota_m}\dot{e}^{a'_1}_{\iota'_1}\ldots\dot{e}^{a'_m}_{\iota'_m}\epsilon^{\iota'_1\ldots\iota'_m} \tag{38}$$

denotes the determinant of the metric ${}^mq_{ab}$ on ${}^mS_{\diamond_{1m}}$ induced by $q_{ab}$. To construct the $m$-area operator, we need to consider the following two cases separately.

#### 1. Case I: m is even

Taking account of the identity (9), we define the $m$-form component

$$\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}\coloneqq\frac{1}{\sqrt{m!}}\mathcal{S}\sqrt{q}\pi^{I_1J_1}_{a_1}\delta_{J_1J_2}\sqrt{q}\pi^{I_2J_2}_{a_2}\ldots\times\sqrt{q}\pi^{I_{m-1}J_{m-1}}_{a_{m-1}}\delta_{J_{m-1}J_m}\sqrt{q}\pi^{I_mJ_m}_{a_m}\dot{e}^{a_1}_{\iota_1}\ldots\dot{e}^{a_m}_{\iota_m}\epsilon^{\iota_1\ldots\iota_m} \tag{39}$$

such that

$$\det({}^mq)\hat{=}\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}\tilde{E}_{(\mathrm{gen})I_1\ldots I_m}. \tag{40}$$

The general flux of $\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}$ is defined as

$$\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}({}^mS_{\diamond_{1m}})\coloneqq\int_{e^\epsilon_1}\ldots\int_{e^\epsilon_m}\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}ds_1\ldots ds_m. \tag{41}$$

Then, up to $\mathcal{O}(\epsilon^{m+1})$ we have

$$\begin{aligned}\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}({}^mS_{\diamond_{1m}})=&\frac{(-1)^m}{\sqrt{m!}}\int_{e^\epsilon_1}\ldots\int_{e^\epsilon_m}\mathrm{tr}(\tau^{I_1J_1}\tau_{I''_1J''_1})\mathcal{S}\sqrt{q}\pi^{I''_1J''_1}_{a_1}\delta_{J_1J_2}\mathrm{tr}(\tau^{I_2J_2}\tau_{I''_2J''_2})\sqrt{q}\pi^{I''_2J''_2}_{a_2}\ldots\\&\times\delta_{J_{m-1}J_m}\mathrm{tr}(\tau^{I_mJ_m}\tau_{I''_mJ''_m})\sqrt{q}\pi^{I''_mJ''_m}_{a_m}\dot{e}^{a_1}_{\iota_1}\ldots\dot{e}^{a_m}_{\iota_m}\epsilon^{\iota_1\ldots\iota_m}ds_1\ldots ds_m\\\hat{=}&\left(\frac{(D-1)^m}{(\beta\kappa)^m\sqrt{m!}}\right)\mathrm{tr}^{e^\epsilon_1\ldots e^\epsilon_m}(\tau^{I_1J_1}_{e^\epsilon_{\iota_1}}\delta_{J_1J_2}\tau^{I_2J_2}_{e^\epsilon_{\iota_2}}\ldots\delta_{J_{m-1}J_m}\tau^{I_mJ_m}_{e^\epsilon_{\iota_m}}\mathcal{S}h_{e^\epsilon_{\iota_1}}\{h^{-1}_{e^\epsilon_{\iota_1}},V(v,\square)\}\ldots h_{e^\epsilon_{\iota_m}}\{h^{-1}_{e^\epsilon_{\iota_m}},V(v,\square)\})\epsilon^{\iota_1\ldots\iota_m}.\end{aligned} \tag{42}$$

Inspired by the ordering of the alternative 2-area operator, the general flux can be quantized as

$$\widehat{\tilde{E}^{I_1\ldots I_m}_{(\mathrm{gen})}}({}^mS_{\diamond_{1m}})=\left(\frac{(D-1)^m}{(\mathbf{i}\beta\kappa\hbar)^m\sqrt{m!}}\right)\mathrm{tr}^{e^\epsilon_1\ldots e^\epsilon_m}(\tau^{I_1J_1}_{e^\epsilon_{\iota_1}}\delta_{J_1J_2}\tau^{I_2J_2}_{e^\epsilon_{\iota_2}}\ldots\delta_{J_{m-1}J_m}\tau^{I_mJ_m}_{e^\epsilon_{\iota_m}}\widehat{\mathrm{Per.}})\epsilon^{\iota_1\ldots\iota_m},\qquad\epsilon\mapsto0, \tag{43}$$

where

$$\widehat{\mathrm{Per.}}\equiv h_{e^\epsilon_1}h_{e^\epsilon_2}\ldots h_{e^\epsilon_{m-1}}h_{e^\epsilon_m}\hat{V}_{\mathrm{tot}}(v,\square)h^{-1}_{e^\epsilon_1}h^{-1}_{e^\epsilon_2}\ldots h^{-1}_{e^\epsilon_{m-1}}h^{-1}_{e^\epsilon_m}. \tag{44}$$

Here we denote

$$\hat{V}_{\rm tot}(v,\square)=\sum_{\iota_1,\ldots,\iota_m}(-1)^{\vartheta(\iota_1,\ldots,\iota_m)}\hat{V}_{\iota_1 1}(v,\square)\hat{V}_{\iota_2 2}(v,\square)\ldots\hat{V}_{\iota_{[m/2]},[m/2]}(v,\square)\hat{\mathcal{S}}\hat{V}_{\iota_{[m/2]+1},[m/2]+1}(v,\square)\ldots\hat{V}_{\iota_m m}(v,\square), \tag{45}$$

where $\iota_1,\ldots,\iota_m$ takes values from $+$ and $-$, $\vartheta(\iota_1,\ldots,\iota_m)$ is the total numbers of $-$ in $\{\iota_1,\ldots,\iota_m\}$, and thus $\hat{V}_{\iota\imath}(v,\square)$ denotes one of the elements of the matrix

$$\begin{pmatrix}\hat{V}_{+1}(v,\square) & \hat{V}_{+2}(v,\square) & \cdots & \hat{V}_{+(m-1)}(v,\square) & \hat{V}_{+m}(v,\square)\\ \hat{V}_{-1}(v,\square) & \hat{V}_{-2}(v,\square) & \cdots & \hat{V}_{-(m-1)}(v,\square) & \hat{V}_{-m}(v,\square)\end{pmatrix}. \tag{46}$$

Notice that $\hat{V}_{\iota\imath}(v,\square)$ is the D-volume operator and the index $\iota$, $\imath$ are used to label how it acts on the holonomies in Eq. (44). Our requirement is that $\hat{V}_{+\imath}(v,\square)$ acts on the holonomies except $h^{-1}_{e^\epsilon_\imath}$ on its right, while $\hat{V}_{-\imath}(v,\square)$ acts on all the holonomies on its right. An explicit expression for them can be given by $\hat{V}_{+\imath}(v,\square)=h^{-1}_{e^\epsilon_\imath}\hat{V}(v,\square)h_{e^\epsilon_\imath}$ and $\hat{V}_{-\imath}(v,\square)=\hat{V}(v,\square)$. The symmetric version of the general flux operator reads

$$\widehat{\tilde{E}^{I_1\ldots I_m}_{\rm gen}}({}^mS_{\diamond_{1m}})=\frac{1}{2}\left(\widehat{\tilde{E}^{I_1\ldots I_m}_{(\rm gen)}}({}^mS_{\diamond_{1m}})+\widehat{\tilde{E}^{I_1\ldots I_m}_{(\rm gen)}}{}^{\dagger}({}^mS_{\diamond_{1m}})\right). \tag{47}$$

Since classically one has

$$\mathrm{Ar}({}^mS)=\lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\mathrm{Ar}({}^mS_{\diamond_{1m}})=\lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\tilde{E}^{I_1\ldots I_m}_{(\rm gen)}({}^mS_{\diamond_{1m}})\tilde{E}_{(\rm gen)I_1\ldots I_m}({}^mS_{\diamond_{1m}})}, \tag{48}$$

we propose the $m$-area operator as

$$\widehat{\mathrm{Ar}({}^mS)}=\lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\widehat{\mathrm{Ar}({}^mS_{\diamond_{1m}})}=\lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\widehat{\tilde{E}^{I_1\ldots I_m}_{\rm gen}}({}^mS_{\diamond_{1m}})\widehat{\tilde{E}^{\rm gen}_{I_1\ldots I_m}}({}^mS_{\diamond_{1m}})}\,. \tag{49}$$

Similar to the construction of the length operator $\hat{L}_e$, the regulator $\epsilon$ can be removed in the graph-dependent way.

### 2. Case II: m is odd

Similar to case I, classically we define the $m$-form component:

$$\tilde{E}^{II_1\ldots I_m}_{(\rm gen)}\coloneqq\frac{1}{\sqrt{2m!}}\mathcal{S}\sqrt{q}\pi^{II_1}_{a_1}\sqrt{q}\pi^{I_2J_2}_{a_2}\delta_{J_2J_3}\sqrt{q}\pi^{I_3J_3}_{a_3}\ldots\sqrt{q}\pi^{I_{m-1}J_{m-1}}_{a_{m-1}}\delta_{J_{m-1}J_m}\sqrt{q}\pi^{I_mJ_m}_{a_m}\dot{e}^{a_1}_{\iota_1}\ldots\dot{e}^{a_m}_{\iota_m}\epsilon^{\iota_1\ldots\iota_m}. \tag{50}$$

Note that there are $m+1$ internal indices in (50), while (39) contains only $m$ internal indices. Then it is easy to see that

$$\det({}^mq)\hat{=}\tilde{E}^{II_1\ldots I_m}_{(\rm gen)}\tilde{E}_{(\rm gen)II_1\ldots I_m}. \tag{51}$$

The generalized flux of the $m$-form can be expressed up to $\mathcal{O}(\epsilon^{m+1})$ as

$$\begin{aligned}\tilde{E}^{II_1\ldots I_m}_{(\rm gen)}({}^mS_{\diamond_{1m}})&\coloneqq\int_{e^\epsilon_1}\ldots\int_{e^\epsilon_m}\tilde{E}^{II_1\ldots I_m}_{(\rm gen)}ds_1\ldots ds_m\\&=\frac{(-1)^m}{\sqrt{2m!}}\int_{e^\epsilon_1}\ldots\int_{e^\epsilon_m}\mathrm{tr}(\tau^{II_1}\tau_{I''I''_1})\mathcal{S}\sqrt{q}\pi^{I''I''_1}_{a_1}\mathrm{tr}(\tau^{I_2J_2}\tau_{I''_2J''_2})\sqrt{q}\pi^{I''_2J''_2}_{a_2}\delta_{J_2J_3}\ldots\\&\quad\times\delta_{J_{m-1}J_m}\mathrm{tr}(\tau^{I_mJ_m}\tau_{I''_mJ''_m})\sqrt{q}\pi^{I''_mJ''_m}_{a_m}\dot{e}^{a_1}_{\iota_1}\ldots\dot{e}^{a_m}_{\iota_m}\epsilon^{\iota_1\ldots\iota_m}ds_1\ldots ds_m\\&\hat{=}\left(\frac{(D-1)^m}{(\beta\kappa)^m\sqrt{2m!}}\right)\mathrm{tr}^{e^\epsilon_1\ldots e^\epsilon_m}(\tau^{II_1}_{e^\epsilon_{\iota_1}}\tau^{I_2J_2}_{e^\epsilon_{\iota_2}}\delta_{J_2J_3}\ldots\delta_{J_{m-1}J_m}\tau^{I_mJ_m}_{e^\epsilon_{\iota_m}}\mathcal{S}h_{e^\epsilon_{\iota_1}}\{h^{-1}_{e^\epsilon_{\iota_1}},V(v,\square)\}\ldots h_{e^\epsilon_{\iota_m}}\{h^{-1}_{e^\epsilon_{\iota_m}},V(v,\square)\})\epsilon^{\iota_1\ldots\iota_m}.\end{aligned} \tag{52}$$

Following the same quantization procedures as case I, we have

$$\widehat{\tilde{E}^{II_1\ldots I_m}_{(\mathrm{gen})}}({}^mS_{\diamond_{1m}}) \coloneqq \left(\frac{(D-1)^m}{(\mathbf{i}\beta\kappa\hbar)^m\sqrt{2m!}}\right)\mathrm{tr}^{e^\epsilon_1\ldots e^\epsilon_m}(\tau^{II_1}_{e^\epsilon_{\imath_1}}\tau^{I_2J_2}_{e^\epsilon_{\imath_2}}\delta_{J_2J_3}\ldots\delta_{J_{m-1}J_m}\tau^{I_mJ_m}_{e^\epsilon_{\imath_m}}\widehat{\mathrm{Per.}})\epsilon^{\imath_1\ldots\imath_m}, \tag{53}$$

where $\widehat{\mathrm{Per.}}$ was defined by (44). The symmetric generalized flux operator can be defined by

$$\widehat{\tilde{E}^{II_1\ldots I_m}_{\mathrm{gen}}}({}^mS_{\diamond_{1m}}) \coloneqq \frac{1}{2}\left(\widehat{\tilde{E}^{II_1\ldots I_m}_{(\mathrm{gen})}}({}^mS_{\diamond_{1m}}) + \widehat{\tilde{E}^{II_1\ldots I_m}_{(\mathrm{gen})}}{}^{\dagger}({}^mS_{\diamond_{1m}})\right). \tag{54}$$

Again, classically one has the area expression

$$\mathrm{Ar}({}^mS) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\mathrm{Ar}({}^mS_{\diamond_{1m}}) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\tilde{E}^{II_1\ldots I_m}_{(\mathrm{gen})}({}^mS_{\diamond_{1m}})\tilde{E}_{(\mathrm{gen})II_1\ldots I_m}({}^mS_{\diamond_{1m}})}. \tag{55}$$

Hence the $m$-area operator for this case is proposed as

$$\widehat{\mathrm{Ar}({}^mS)} = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\widehat{\mathrm{Ar}({}^mS_{\diamond_{1m}})} = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\widehat{\tilde{E}^{II_1\ldots I_m}_{\mathrm{gen}}}({}^mS_{\diamond_{1m}})\widehat{\tilde{E}^{\mathrm{gen}}_{II_1\ldots I_m}}({}^mS_{\diamond_{1m}})}. \tag{56}$$

## III. ISSUES OF THE GENERAL $m$-AREA OPERATORS

### A. The ambiguity in the construction of geometric operators

Let us consider two special cases of the general $m$-area operator where an ambiguity in the construction of geometric operators will appear. In the special case of $m = 1$, the general $m$-area operator of ${}^mS_{\diamond_{1m}}$ becomes a length operator of $e^\epsilon$ as

$$\widehat{L_{\mathrm{alt}}(e^\epsilon)} = \sqrt{\widehat{\tilde{E}^{II_1}_{\mathrm{gen}}}(e^\epsilon)\widehat{\tilde{E}^{\mathrm{gen}}_{II_1}}(e^\epsilon)}, \tag{57}$$

where

$$\widehat{\tilde{E}^{II_1}_{\mathrm{gen}}}(e^\epsilon) = \frac{-(D-1)}{\mathbf{i}\beta\kappa\hbar\sqrt{2}}\mathrm{tr}(\tau^{II_1}_{e^\epsilon}h_{e^\epsilon}\hat{V}(v,\Box)h^{-1}_{e^\epsilon}). \tag{58}$$

From Eq. (57) we get

$$\widehat{L_{\mathrm{alt.}}(e^\epsilon)} = \frac{(D-1)}{\sqrt{2}\beta\kappa\hbar}\sqrt{(h_{e^\epsilon})^I{}_J\hat{V}(v,\Box)\hat{V}(v,\Box)(h^{-1}_{e^\epsilon})^J{}_I - (h_{e^\epsilon})_I{}^J\hat{V}(v,\Box)(h^{-1}_{e^\epsilon})_J{}^K(h_{e^\epsilon})^I{}_{J'}\hat{V}(v,\Box)(h^{-1}_{e^\epsilon})^{J'}{}_K}. \tag{59}$$

Recall that the generalization of Thiemann's length operator was given by Eq. (21), which is different from Eq. (59) formally. In fact, this difference comes from the different choices of the ordering of the holonomies and volumes in the expressions of $q_{ab}$.

In the case of $m = D$, the general $m$-area operator of ${}^mS_{\diamond_{1m}}$ becomes alternative $D$-volume operators as

$$\widehat{\mathrm{Vol}_{\mathrm{alt}}}({}^DS_{\diamond_{1D}}) = \sqrt{\widehat{\tilde{E}^{II_1\ldots I_m}_{\mathrm{gen}}}({}^DS_{\diamond_{1D}})\widehat{\tilde{E}^{\mathrm{gen}}_{II_1\ldots I_m}}({}^DS_{\diamond_{1D}})} \tag{60}$$

for odd $D$, and

$$\widehat{\mathrm{Vol}_{\mathrm{alt}}}({}^DS_{\diamond_{1D}}) = \sqrt{\widehat{\tilde{E}^{I_1\ldots I_m}_{\mathrm{gen}}}({}^DS_{\diamond_{1D}})\widehat{\tilde{E}^{\mathrm{gen}}_{I_1\ldots I_m}}({}^DS_{\diamond_{1D}})} \tag{61}$$

for even $D$. The alternative $D$-volume operators (60) and (61) are totally constructed by the dual momentum operator which involves the usual $D$-volume operator [22]. There is an analogous alternative volume operator $\widehat{\mathrm{Vol}^{\mathrm{sta}}_{\mathrm{alt}}}$ in the standard $(1+3)$-dimensional LQG [25]. Consider the case of $D = 3$ and denote by $\widehat{\mathrm{Vol}^{\mathrm{all}}_{\mathrm{alt}}}$ the operators (60) and (61) in this case. It is interesting to compare $\widehat{\mathrm{Vol}^{\mathrm{all}}_{\mathrm{alt}}}$ with $\widehat{\mathrm{Vol}^{\mathrm{sta}}_{\mathrm{alt}}}$. There are the following two main differences between them. First, the dual momentum used to construct $\widehat{\mathrm{Vol}^{\mathrm{all}}_{\mathrm{alt}}}$ is $so(4)$-valued, while the cotriad used to construct $\widehat{\mathrm{Vol}^{\mathrm{sta}}_{\mathrm{alt}}}$ is Lie algebra $su(2)$ valued. Second, the construction schemes and ingredients of the two operators are different. To construct $\widehat{\mathrm{Vol}^{\mathrm{sta}}_{\mathrm{alt}}}$, one employed the classical identity

$\mathrm{Vol}^{\mathrm{sta}}_{\mathrm{alt}}(R) := \int_R d^3x|\det(e)|$ with Eq. (11), where $V(v,\square)$ is quantized as the usual volume operator [7] in standard (1 + 3)-dimensional LQG. To construct $\widehat{\mathrm{Vol}^{\mathrm{all}}_{\mathrm{alt}}}$, we employed the classical identity $\mathrm{Vol}^{\mathrm{all}}_{\mathrm{alt}}(R) := \int_R d^Dx\sqrt{\det(q)}$ with Eqs. (9) and (10), where $V(v,\square)$ is quantized as the usual volume operator [22] in all dimensional LQG. It should be noted that, since all the spatially geometric quantities can be classically expressed by the frame or flux, while there are alternative regularizations for them, the ambiguity which appeared in the construction of the above $m$-area operators is rather general in the construction of other geometric operators.

### B. The issue of simplicity constraint

In the construction of geometric operators in all dimensional LQG, there is the issue of how to carry out the simplicity constraint. Classically, on the simplicity constraint surface of the phase space, one has $\pi^a_{IJ} = 2\sqrt{q}n_{[I}e^a_{J]}$ and $\pi^{IJ}_a = 2\sqrt{q}^{-1}n^{[I}e^{J]}_a$. Hence the identity (10) holds. By quantization, one expects that this "simple" property be transformed as the requirement to the right invariant vector fields such that $\hat{N}^{[I}R^{JK]}_{e_\imath} = 0,\ \forall\, b(e_\imath) = v$, where $\hat{N}^I$ is the operator of an auxiliary internal vector field which plays the role of $n^I$, and $b(e_i)$ denotes the beginning point of $e_i$. In the construction of a general geometric operator, usually there would appear the following term acting on a state $f_\gamma$ as

$$h^{-1}_{e^{\imath_1}_\epsilon}\dots h^{-1}_{e^{\imath_2}_\epsilon}\dots\hat{V}^{m_1}_{\square_\epsilon}h_{e^{\imath_1}_\epsilon}\dots\hat{V}^{m_2}_{\square_\epsilon}h_{e^{\imath_2}_\epsilon}\dots f_\gamma, \tag{62}$$

where $f_\gamma$ is supposed to satisfy the simplicity constraint by labeling its edges by the simple representation of $SO(D+1)$ and its vertices by the simple intertwiners. Note that the volume operator can keep its geometric meaning only on the state satisfying the simplicity constraint. However, the holonomy operator may change the simple intertwiner into a nonsimple one. Suppose that $f_\gamma$ satisfy the quantum simplicity constraint. Then we have

$$\hat{N}^{[I}R^{JK]}_e h_e(A)\cdot f_\gamma(A) = f_\gamma(A)\hat{N}^{[I}R^{JK]}_e h_e(A). \tag{63}$$

Equation (63) does not vanish unless $\hat{N}^{[I}R^{JK]}_e h_e(A) = 0$. This condition could not be satisfied for the general holonomy $h_e(A)$ in the construction of the general geometric operator. Hence, the operator (62) already lost its geometric meaning. One possible solution to this problem is to introduce a projection operator $\hat{\mathbb{P}}_S$, which projects the space of the kinematic states into the solution space of simplicity constraint, and insert it into the two sides of each volume operator in (62) to define

$$h^{-1}_{e^{\imath_1}_\epsilon}\dots h^{-1}_{e^{\imath_2}_\epsilon}\dots\hat{\mathbb{P}}_S\hat{V}^{m_1}_{\square_\epsilon}\hat{\mathbb{P}}_S h_{e^{\imath_1}_\epsilon}\dots\hat{\mathbb{P}}_S\hat{V}^{m_2}_{\square_\epsilon}\hat{\mathbb{P}}_S h_{e^{\imath_2}_\epsilon}\dots f_\gamma. \tag{64}$$

Generally, the degrees of freedom that should be eliminated by the simplicity constraints in the construction of a geometric operator are still unclear. This issue needs further investigation. Moreover, there is the issue of anomaly for the quantum simplicity constraint [22,23]. It is argued that only the weak solutions of the quantum simplicity constraints have the reasonable physical degrees of freedom [24]. In the next section, we will introduce another scheme for constructing general geometric operators, which leads to a better behavior of the operators concerning the issue of simplicity constraints.

### C. Consistency of the alternative flux and the standard flux operators

In the special case of $m = D-1$ the $m$-area operator introduced in the last section is alternative to the $(D-1)$-area operator defined by the standard flux operator. It is worth checking whether the two versions of area operators are consistent with each other. Now we consider the case that $(D-1)$ is even. Since the alternative $(D-1)$-area operator consists of the alternative flux operator,

$$\hat{\pi}^{IJ}_{\mathrm{alt}}({}^{(D-1)}S_{\diamond_{(D-1)}}) := \frac{1}{\sqrt{2(D-1)!}}\epsilon^{IJ}{}_{I_1\dots I_{D-1}}\widehat{\tilde{E}^{I_1\dots I_{D-1}}_{\mathrm{gen}}}({}^{(D-1)}S_{\diamond_{(D-1)}}), \tag{65}$$

the necessary condition for the consistency of the two versions of area operator is the consistency of $\hat{\pi}^{IJ}_{\mathrm{alt}}({}^{(D-1)}S_{\diamond_{(D-1)}})$ with the standard flux operator $\hat{\pi}^{IJ}({}^{(D-1)}S_{\diamond_{(D-1)}})$. Now we check this issue. Note that the action of volume operator in the expression of the standard flux on a cylindrical function $f_\gamma$ is given by

$$\hat{V}(v,\square)\cdot f_\gamma = (\hbar\kappa\beta)^{\frac{D}{D-1}}\left|c_{\mathrm{reg.}}\frac{\mathbf{i}^D}{D!}\sum_{e_1,\dots,e_D\in E(\gamma),e_1\cap\dots\cap e_D=v}q_{e_1,\dots,e_D}\right|^{\frac{1}{D-1}}\cdot f_\gamma, \tag{66}$$

where

$$q_{e_1,\dots,e_D} = \frac{1}{2}\mathrm{sgn}(\det(\dot{e}_1(v),\dots,(\dot{e}_D(v)))\epsilon_{IJI_1J_1I_2J_2\dots I_nJ_n}\times R^{IJ}_e R^{I_1K_1}_{e_1}R^{J_1}_{e'_1K_1}\dots R^{I_nK_n}_{e_n}R^{J_n}_{e'_nK_n}, \tag{67}$$

with $R^{IJ}_e := \mathrm{tr}((\tau^{IJ}h_e(A))^T\frac{\partial}{\partial h_e(A)})$. Here the set $(e_1,\dots,e_D)$ is relabeled as $(e,e_1,e'_1,\dots,e_n,e'_n)$ in the right-hand side of Eq. (67). Let $T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}$ be a spin network state which intersects the surface ${}^{(D-1)}S_{\diamond_{1,(D-1)}}$ by an inner point $v$ of its edge $e_0$. By the identity

$$[\hat{V}(v,\square)]^{\frac{D-1}{2}}\hat{\mathcal{S}}[\hat{V}(v,\square)]^{\frac{D-1}{2}} = (\mathbf{i}\hbar\kappa\beta)^D\frac{c_{\text{reg.}}}{D!}\sum_{e_1,\ldots,e_D\in E(\gamma),e_1\cap\ldots\cap e_D=v} q_{e_1,\ldots,e_D}, \tag{68}$$

the action of $\tilde{E}^{\widehat{I_1\ldots I_{D-1}}}_{\text{gen}}({}^{(D-1)}S_{\diamond_{(D-1)}})$ on $T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}$ reads

$$\begin{aligned}\tilde{E}^{\widehat{I_1\ldots I_{D-1}}}_{\text{gen}}({}^{(D-1)}S_{\diamond_{(D-1)}})\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}} &= (\mathbf{i}\hbar\kappa\beta)^D\frac{c_{\text{reg.}}}{D!}\left(\frac{(D-1)^{(D-1)}}{(\mathbf{i}\hbar\kappa\beta)^{(D-1)}\sqrt{(D-1)!}}\right)\text{tr}^{e_1^\epsilon\ldots e^\epsilon_{(D-1)}}(\tau^{I_1J_1}_{e^\epsilon_{\iota_1}}\delta_{J_1J_2}\tau^{I_2J_2}_{e^\epsilon_{\iota_2}}\ldots\delta_{J_{(D-2)}J_{(D-1)}}\tau^{I_{(D-1)}J_{(D-1)}}_{e^\epsilon_{\iota_{(D-1)}}}\\ &\quad\times h_{e_1^\epsilon}h_{e_2^\epsilon}\ldots h_{e^\epsilon_{(D-2)}}h_{e^\epsilon_{(D-1)}}\hat{\mathbb{P}}_S\epsilon_{I'J'I'_1J'_1\ldots I'_nJ'_n}R^{I'J'}_{e_0}R^{I'_1K'_1}_{e_1}R^{J'_1}_{e'_1K'_1}\cdots\\ &\quad\times R^{I'_nK'_n}_{e_n}R^{J'_n}_{e'_nK'_n}\hat{\mathbb{P}}_S h^{-1}_{e_1^\epsilon}h^{-1}_{e_2^\epsilon}\ldots h^{-1}_{e^\epsilon_{(D-2)}}h^{-1}_{e^\epsilon_{(D-1)}})\epsilon^{\iota_1\ldots\iota_{(D-1)}}\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}\\ &\sim(\mathbf{i}\hbar\kappa\beta)^D\frac{1}{D!}\frac{c_{\text{reg.}}(D-1)^{(D-1)}}{(\mathbf{i}\hbar\kappa\beta)^{(D-1)}\sqrt{(D-1)!}}\left(\frac{1}{4}\right)^{\frac{D-1}{2}}(D-1)!R^{I'J'}_{e_0}\epsilon_{I'J'}{}^{I_1I_2\ldots I_{D-1}}\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}},\end{aligned} \tag{69}$$

as $\epsilon\to 0$, where $e_1 = e_1^\epsilon$, $e'_1 = e_2^\epsilon$, $e_2 = e_3^\epsilon$, $e'_2 = e_4^\epsilon,\ldots,e_n = e^\epsilon_{D-2}$, $e'_n = e^\epsilon_{D-1}$, $n=\frac{D-1}{2}$, equation $\lim_{\epsilon\to0}\text{tr}(\tau_{IJ}h_{e_\iota^\epsilon}R^{KL}_{e_\iota^\epsilon}h^{-1}_{e_\iota^\epsilon}) = \delta^K_{[I}\delta^L_{J]}$ was used, and the symbol $\sim$ represents “be proportional to.” Hence we obtain

$$\hat{\pi}^{IJ}_{\text{alt}}({}^{(D-1)}S_{\diamond_{(D-1)}})\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}\sim c_{\text{reg.}}\sqrt{2}\frac{\mathbf{i}\hbar\kappa\beta}{D}\left(\frac{D-1}{2}\right)^{(D-1)}R^{IJ}_{e_0}\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}. \tag{70}$$

Notice that there is the ambiguity of choosing a suitable projection operator $\hat{\mathbb{P}}_S$ depending on how to solve the quantum simplicity constraint. This leads to the undetermined factor $c_{\text{reg.}}$ in Eq. (70), which is still an open issue for the alternative flux operator. Recall that the action of the standard flux operator reads

$$\hat{\pi}^{IJ}({}^{(D-1)}S_{\diamond_{(D-1)}})\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}} = 2\mathbf{i}\hbar\kappa\beta R^{IJ}_{e_0}\cdot T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}. \tag{71}$$

Therefore, the actions of $\hat{\pi}^{IJ}_{\text{alt}}({}^{(D-1)}S_{\diamond_{(D-1)}})$ and $\hat{\pi}^{IJ}({}^{(D-1)}S_{\diamond_{(D-1)}})$ on $T_{\gamma,{}^{(D-1)}S_{\diamond_{1,(D-1)}}}$ are equivalent up to an undetermined factor in the above case.

## IV. GENERAL GEOMETRIC OPERATOR: SECOND STRATEGY

Another way to construct general geometric operators in all dimensional LQG is to express the de-densitized dual momentum by the momentum variable $\pi^a_{IJ}$ as

$$\sqrt{q}\pi^{IJ}_a \doteq \frac{\frac{1}{(D-1)!}\epsilon_{aa_1b_1\ldots a_nb_n}\epsilon^{IJI_1J_1\ldots I_nJ_n}\pi^{a_1}_{I_1K_1}\pi^{b_1K_1}_{J_1}\ldots\pi^{a_n}_{I_nK_n}\pi^{b_nK_n}_{J_n}}{\text{sgn}(\det(\pi))|\det(\pi)|^{\frac{D-2}{D-1}}} \tag{72}$$

for $D=2n+1$ is odd, where

$$\det(\pi) \coloneqq \frac{1}{2D!}\epsilon_{aa_1b_1\ldots a_nb_n}\epsilon^{IJI_1J_1\ldots I_nJ_n}\pi^a_{IJ}\pi^{a_1}_{I_1K_1}\pi^{b_1K_1}_{J_1}\ldots\pi^{a_n}_{I_nK_n}\pi^{b_nK_n}_{J_n}; \tag{73}$$

and

$$\sqrt{q}\pi_{a_1I_1K_1} \doteq \frac{\frac{2}{(D-1)!}\epsilon_{a_1b_1\ldots a_nb_n}V^I\epsilon_{I[I_1|J_1\ldots I_nJ_n|}\pi^{b_1J_1}{}_{K_1]}\pi^{a_2I_2K_2}\pi^{b_2J_2}{}_{K_2}\ldots\pi^{a_nI_nK_n}\pi^{b_nJ_n}{}_{K_n}}{\text{ddet}(\pi)^{\frac{2D-3}{2D-2}}} \tag{74}$$

for $D=2n$ is even, where

$$V^I := \frac{1}{D!}\epsilon_{a_1b_1\ldots a_nb_n}\epsilon^{II_1J_1\ldots I_nJ_n}\pi^{a_1}_{I_1K_1}\pi^{b_1K_1}_{J_1}\pi^{a_2}_{I_2K_2}\pi^{b_2K_2}_{J_2}\ldots \times \pi^{a_n}_{I_nK_n}\pi^{b_nK_n}_{J_n}, \quad (75)$$

and $\mathrm{ddet}(\pi) := V^IV_I \hat{=} (\sqrt{q})^{2D-2}$. Then, we can regularize and quantize them through the flux operators, volume operator and so on, by taking account of Eqs. (9), (77) and (80). This strategy is similar to that used to construct the other two versions of length operator [13,14] in the standard $(1+3)$-dimensional LQG. In this section, we will first extend the construction of the length operator in [14] to all dimensional theory, and then follow a similar strategy to construct general geometric operators.

## A. The second length operator in all dimensional LQG

Let us recall the classical expression (18) of the length $L_e$ of a curve $e$. The length segment $L_{e^\epsilon}$ related to an arbitrary segment $e^\epsilon$ can be rexpressed by fluxes following a partition of the neighborhood of $e^\epsilon$ in $\sigma$ as follows. Choose a set of $(D-1)$-faces $({}^{D-1}S_1, \ldots, {}^{D-1}S_i, \ldots, {}^{D-1}S_{D-1})$, i.e., $(D-1)$-hypercubes, with coordinate volume $\epsilon^{(D-1)}$ intersecting at $e_\epsilon$. The normal covectors $(n^1_a, \ldots, n^i_a, \ldots, n^{D-1}_a)$ of these $(D-1)$-faces are chosen to be linearly independent so that

$$\dot{e}^a_\epsilon \epsilon_{aa_1a_2\ldots a_{D-1}} = \epsilon_{i_1\ldots i_{D-1}} n^{i_1}_{a_1}\ldots n^{i_{D-1}}_{a_{D-1}}, \quad (76)$$

where $\epsilon_{i_1\ldots i_{D-1}}$ is the $(D-1)$-dimensional Levi-Civita symbol. Taking account of the expressions (72) and (74) for $\sqrt{q}\pi_{aIJ}$, we can define the smeared quantity

$$l_{e_\epsilon,IJ} := \frac{\epsilon_{i_1\ldots i_{D-1}}\epsilon_{IJI_1J_1\ldots I_nJ_n}\pi^{I_1K_1}({}^{D-1}S^{i_1})\pi^{J_1}{}_{K_1}({}^{D-1}S^{i_2})\ldots\pi^{I_nK_n}({}^{D-1}S^{i_{D-2}})\pi^{J_n}{}_{K_n}({}^{D-1}S^{i_{D-1}})}{(D-1)!V^{D-2}_{\square_\epsilon}} \quad (77)$$

for $D=2n+1$ is odd, where $V_{\square_\epsilon} = \int_{\square_\epsilon} d^Dx|\det\pi|^{\frac{1}{D-1}}$, and $\square_\epsilon$ is the D-hypercube which contains point $v$ and has coordinate volume $\epsilon^D$. Here $\det(\pi)$ was smeared as $\det(\pi)(p) = \pi(p, \triangle_1, \ldots, \triangle_D)$ with

$$\pi(p,\triangle_1,\ldots,\triangle_D) := \frac{1}{\mathrm{vol}(\triangle_1)\ldots\mathrm{vol}(\triangle_D)}\int_\sigma d^Dx_1\ldots\int_\sigma d^Dx_D\chi_{\triangle_1}(p,x_1)\chi_{\triangle_2}(2p,x_1+x_2)\ldots\chi_{\triangle_D}(Dp,x_1+\cdots+x_D) \times \frac{1}{2D!}\epsilon_{aa_1b_1\ldots a_nb_n}\epsilon^{IJI_1J_1\ldots I_nJ_n}\pi^a_{IJ}\pi^{a_1}_{I_1K_1}\pi^{b_1K_1}_{J_1}\ldots\pi^{a_n}_{I_nK_n}\pi^{b_nK_n}_{J_n}, \quad (78)$$

where $\chi_\triangle(p,x)$ denotes the characteristic function in the coordinate $x$ of a hypercube with center $p$, which is spanned by the D right-handed vectors $\vec{\triangle}^{\tilde{i}} := \triangle^{\tilde{i}}\vec{v}^{\tilde{i}}, \tilde{i}=1,\ldots D$, with $\vec{v}^{\tilde{i}}$ being a normal vector in the frame under consideration, and has coordinate volume $\mathrm{vol} = \triangle^1\ldots\triangle^D\det(\vec{v}^1,\ldots,\vec{v}^D) = \epsilon^D$. Thus one has

$$\chi_\triangle(p,x) = \prod_{\tilde{i}=1}^D \Theta\left(\frac{\triangle^{\tilde{i}}}{2} - |\langle\vec{v}^{\tilde{i}}, x-p\rangle|\right), \quad (79)$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product and $\Theta(y)=1$ for $y>0$ and zero otherwise. Also, in Eq. (78) we used the lower indices $\triangle_I = (\triangle^1_I, \ldots, \triangle^D_I)$ to label different hypercubes, see [22]. Similarly, we have the smeared quantity

$$l_{e^\epsilon,I_1K_1} := \frac{2}{(D-1)!}\epsilon_{i_1\ldots i_{D-1}}V^I(\square^{D-1}_\epsilon)\epsilon_{I[I_1|J_1\ldots I_nJ_n|}\pi^{J_1}{}_{K_1]}({}^{D-1}S^{i_1})\pi^{I_2K_2}({}^{D-1}S^{i_2})\pi^{J_2}{}_{K_2}({}^{D-1}S^{i_3})\ldots \times \pi^{I_nK_n}({}^{D-1}S^{i_{D-2}})\pi^{J_n}{}_{K_n}({}^{D-1}S^{i_{D-1}})V_{\square_\epsilon}{}^{3-2D} \quad (80)$$

for $D=2n$ is even, where $V^I(p)$ is also smeared as $V^I(\square^{D-1}_\epsilon) := [\mathrm{vol}(\square_\epsilon)]^{D-2}\int_{\square_\epsilon}V^I(p)dp^D$, with $V^I(p) = V^I(p,\triangle_1,\ldots,\triangle_D)$ and

$$V^I(p,\triangle_1,\ldots,\triangle_D) := \frac{1}{\mathrm{vol}(\triangle_1)\ldots\mathrm{vol}(\triangle_D)}\int_\sigma d^Dx_1\ldots\int_\sigma d^Dx_D\chi_{\triangle_1}(p,x_1)\chi_{\triangle_2}(2p,x_1+x_2)\ldots\chi_{\triangle_D}(Dp,x_1+\cdots+x_D) \times \frac{1}{D!}\epsilon_{a_1b_1\ldots a_nb_n}\epsilon^{II_1J_1\ldots I_nJ_n}\pi^{a_1}_{I_1K_1}\pi^{b_1K_1}_{J_1}\pi^{a_2}_{I_2K_2}\pi^{b_2K_2}_{J_2}\ldots\pi^{a_n}_{I_nK_n}\pi^{b_nK_n}_{J_n} \quad (81)$$

similar to the definition of $\pi(p,\triangle_1,\ldots,\triangle_D)$. With these smeared quantities, we can reexpress the length of a curve as

$$L_e = \lim_{\epsilon\to 0}\sum_{e^\epsilon} L_{e_\epsilon} = \lim_{\epsilon\to 0}\sum_{e^\epsilon}\sqrt{\frac{1}{2} l^{IJ}_{e^\epsilon} l^{e^\epsilon}_{IJ}}. \quad (82)$$

Correspondingly, the length operator based on this expression is given by

$$\hat{L}_e = \lim_{\epsilon\to 0}\sum_{e^\epsilon}\sqrt{\frac{1}{2}\hat{l}^{IJ}_{e^\epsilon}(\hat{l}_{e^\epsilon,IJ})^\dagger}. \quad (83)$$

An alternative formulation reads

$$\hat{L}_e = \lim_{\epsilon\to 0}\sum_{e^\epsilon}\frac{1}{2}\sqrt{\frac{1}{2}(\hat{l}^{IJ}_{e^\epsilon} + (\hat{l}_{e^\epsilon,IJ})^\dagger)(\hat{l}^{IJ}_{e^\epsilon} + (\hat{l}_{e^\epsilon,IJ})^\dagger)}. \quad (84)$$

Now we need to define the operator $\hat{l}^{IJ}_{e^\epsilon}$. In all dimensional LQG [22], the fluxes and volume can be promoted as operators immediately. The action of a flux operator on a cylindrical function $f_\gamma$ reads

$$\hat{\pi}^{IJ}({}^{D-1}S^i)\cdot f_\gamma = \mathbf{i}\hbar\kappa\beta \sum_{e_\imath\in E(\gamma[{}^{D-1}S^i])} \epsilon(e_\imath, {}^{D-1}S^i) R^{IJ}_{e_\imath}\cdot f_\gamma, \quad (85)$$

where $E(\gamma[{}^{D-1}S^i])$ denotes the collection of the edges intersecting the face ${}^{D-1}S^i$, and $R^{IJ}_e$ is the right invariant vector field on $SO(D+1)\ni h_e(A)$. When $D$ is even, the action of the volume operator is given by

$$\hat{V}_{\square_\epsilon}\cdot f_\gamma = (\hbar\kappa\beta)^{\frac{D}{D-1}} \sum_{v\in V(\gamma)\cap\square_\epsilon} \hat{V}_{v,\gamma}\cdot f_\gamma, \quad (86)$$

where

$$\hat{V}_{v,\gamma} = (\hat{V}^I_{v,\gamma}\hat{V}_{Iv,\gamma})^{\frac{1}{2D-2}}, \quad (87)$$

with

$$\hat{V}^I_{v,\gamma} = \frac{\mathbf{i}^D}{D!} \sum_{e_1,\ldots,e_D\in E(\gamma), e_1\cap\ldots\cap e_D = v} s(e_1,\ldots,e_D)\hat{q}^I_{e_1,\ldots,e_D},$$

$$\hat{q}^I_{e_1,\ldots,e_D} = \epsilon_{II_1J_1I_2J_2\ldots I_nJ_n} R^{I_1K_1}_{e_1} R^{J_1}_{e'_1K_1}\ldots R^{I_nK_n}_{e_n} R^{J_n}_{e'_nK_n}, \quad (88)$$

wherein the set $(e_1,\ldots,e_D)$ is relabeled as $(e_1, e'_1, \ldots, e_n, e'_n)$. When $D$ is odd, the action of the volume operator is also given by

$$\hat{V}_{\square_\epsilon}\cdot f_\gamma = (\hbar\kappa\beta)^{\frac{D}{D-1}} \sum_{v\in V(\gamma)\cap\square_\epsilon} \hat{V}_{v,\gamma}\cdot f_\gamma, \quad (89)$$

with

$$\hat{V}_{v,\gamma} = \left|\frac{\mathbf{i}^D}{D!} \sum_{e_1,\ldots,e_D\in E(\gamma), e_1\cap\ldots\cap e_D = v} s(e_1,\ldots,e_D)\hat{q}_{e_1,\ldots,e_D}\right|^{\frac{1}{D-1}}, \quad (90)$$

with

$$\hat{q}_{e_1,\ldots,e_D} = \frac{1}{2}\epsilon_{IJI_1J_1I_2J_2\ldots I_nJ_n} R^{IJ}_e R^{I_1K_1}_{e_1} R^{J_1}_{e'_1K_1}\ldots R^{I_nK_n}_{e_n} R^{J_n}_{e'_nK_n}. \quad (91)$$

In the above equations we used the inventions that $V(\gamma)$ is the collection of vertices of the graph $\gamma$, $s(e1,\ldots,e_D) \coloneqq \mathrm{sgn}(\det(\dot{e}_1(v),\ldots,\dot{e}_D(v)))$, and $v$ is the intersection point of the D-tuple of edges $(e_1, e_2, \ldots, e_D)$. It is understood that we only sum over the D-tuples of edges which are incident at a common vertex. Similarly, we can quantize $V^I(\square^{D-1}_\epsilon)$ in (80) as

$$\begin{aligned}\hat{V}^I(\square^{D-1}_\epsilon)\cdot f_\gamma &= (\mathrm{vol}(\square_\epsilon))^{D-2} \sum_{v\in V(\gamma)\cap\square_\epsilon}\int_{\square_\epsilon} dp \sum_{e_1,\ldots,e_D}\frac{(\mathbf{i}\hbar\kappa\beta)^D s(e_1,\ldots,e_D)}{D!\mathrm{vol}(\triangle_1)\ldots\mathrm{vol}(\triangle_{D-1})}\chi_{\triangle_1}(p,v)\ldots\chi_{\triangle_{D-1}}(p,v)\hat{q}^I_{e_1,\ldots,e_D}\cdot f_\gamma \\ &= (\kappa\beta\hbar)^D \sum_{v\in V(\gamma)\cap\square_\epsilon}\hat{V}^I_{v,\gamma}\cdot f_\gamma. \end{aligned} \quad (92)$$

Hence the operator $\hat{l}^{IJ}_{e^\epsilon}$ is well defined by replacing the components in its classical expression with the corresponding quantum operators. Several remarks are listed below on the replacement. First, the expression involves the inverse of the local volume operator $\hat{V}_{\square_\epsilon}$ which is noninvertible as it has a huge kernel. To overcome this problem, we can introduce an operator $\widehat{V^{-1}}_{\square_\epsilon}$ similar to the "inverse" volume operator in $(1+3)$-dimensional standard LQG, which is defined as the limit

$$\widehat{V^{-1}_{\square_\epsilon}} = \lim_{\varepsilon\to 0}(\hat{V}^2_{\square_\epsilon} + \varepsilon^2(l_p^{(D+1)})^{2D})^{-1}\hat{V}_{\square_\epsilon}, \quad (93)$$

where $l_p^{(D+1)}$ is the Plank length in $(1+\mathrm{D})$-dimensional space-time. The existence of the "inverses" volume operator $\widehat{V^{-1}_{\square_\epsilon}}$ indicates that the length operator will be nonvanishing only on the vertex which does not vanish the volume operator. Second, although the prequantized smeared quantities are well defined in some limit, they are not yet background-independent because of the appearing of $k_{\mathrm{pre}}(e_{\imath_1},\ldots,e_{\imath_{D-1}},\theta) \coloneqq \epsilon_{i_1\ldots i_{D-1}}\epsilon(e_{\imath_1}, {}^{D-1}S^{i_1})\ldots\epsilon(e_{\imath_{D-1}}, {}^{D-1}S^{i_{D-1}})$ after we replace fluxes by flux operators in Eqs. (77) and (80), where $\theta = ({}^{D-1}S^1,\ldots,{}^{D-1}S^{D-1})$ represents the choice of the set of (D − 1)-surfaces ${}^{D-1}S$ in the partition of the neighborhood of $e^\epsilon$. The background structure can be removed by suitably "averaging" the regularized operator over it following a strategy similar to the treatment of the volume operator and the length operator in $(1+3)$-dimensional standard LQG [7,14]. The averaging is taken over the $(D-1)^2$-dimensional space $\mathcal{B}$ of all the choices of $({}^{D-1}S^1, \ldots, {}^{D-1}S^{D-1})$ in the partition of the neighborhood of $e^\epsilon$, and it results in

$$k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}}) = \int_{\mathcal{B}} d\theta\mu(\theta) k_{\mathrm{pre}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}}, \theta), \tag{94}$$

where $\theta \in \mathcal{B}$, and $d\theta\mu(\theta)$ is a suitable normalized measure on $\mathcal{B}$. We take into account the fact that for any finite set $e_\epsilon, e_1, \ldots, e_{n_v}$ which intersected at vertex $v$, such that each two of them are not tangent at $v$, the functions $k_{\mathrm{pre}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}}, \theta)$ with $\iota_1 < \iota_2 \ldots < \iota_{D-1}$ and the constant function constitute a set of linearly independent functions on $\mathcal{B}$ [7]. The averaging result $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ has the following properties: (1) $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ depends only on the segments of the edges $(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ that are located in the neighborhood of $e^\epsilon$; (2) $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ is totally antisymmetric in $\iota_1, \ldots, \iota_{D-1}$, i.e., $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}}) = k_{\mathrm{av}}(e_{[\iota_1}, \ldots, e_{\iota_{D-1}]})$; (3) in the limit $\epsilon \to 0$, $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ is nonvanishing only if $e_{\iota_1}, \ldots, e_{\iota_{D-1}}$ and $e^\epsilon$ intersect at a vertex $v$ and their tangential directions are linearly independent there; (4) the choice of the measure $d\theta\mu(\theta)$ ensures that $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}}) = k_{\mathrm{av}}(e'_{\iota_1}, \ldots, e'_{\iota_{D-1}})$ if $(e, e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ and $(e, e'_{\iota_1}, \ldots, e'_{\iota_{D-1}})$ are related by an orientation preserving diffeomorphism of $\sigma$. These properties ensure that $k_{\mathrm{av}}(e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ can be given by $k_{\mathrm{av}} \cdot \varsigma(e_\epsilon, e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ uniquely, wherein $k_{\mathrm{av}}$ is a constant, and $\varsigma(e_\epsilon, e_{\iota_1}, \ldots, e_{\iota_{D-1}})$ is the orientation function which equals $+1$ (or $-1$) if the tangential directions of $e_\epsilon, e_{\iota_1}, \ldots, e_{\iota_{D-1}}$ are linearly independent at the vertex $v$ dual to $\Box_\epsilon$ and oriented positively (or negatively), or equals zero otherwise. Third, the following noncommutative relations generally hold:

$$[\hat{l}^{IJ}_{e^\epsilon}, \hat{V}_{\Box_\epsilon}] \neq 0, \tag{95}$$

where $v \in e^\epsilon$ is the vertex dual to $\Box_\epsilon$, and

$$[\hat{L}_{e^\epsilon_i}, \hat{L}_{e^\epsilon_j}] \neq 0, \tag{96}$$

where $e^\epsilon_i$ and $e^\epsilon_j$ intersect at a true vertex which is dual to a nonvanishing volume. This result indicates that we should choose a "nice" extended curve to define its length operator [13].

Based on the above treatment the operator $\hat{l}_{e_\epsilon,IJ}$ can be given by

$$\hat{l}_{e_\epsilon,IJ} \cdot f_\gamma := \frac{(\mathbf{i}\kappa\beta\hbar)^{D-1}}{(D-1)!} \sum_{e_{\iota_1}} \ldots \sum_{e_{\iota_{D-1}}} k_{\mathrm{av}} \cdot \varsigma(e_\epsilon, e_{\iota_1}, \ldots, e_{\iota_{D-1}}) \epsilon_{IJI_1J_1\ldots I_nJ_n} R^{I_1K_1}_{e_{\iota_1}} R^{J_1}_{e_{\iota_2}\,K_1} \ldots R^{I_nK_n}_{e_{\iota_{D-2}}} R^{J_n}_{e_{\iota_{D-1}}\,K_n} (\widehat{V^{-1}_{\Box_\epsilon}})^{D-2} \cdot f_\gamma, \tag{97}$$

for $D = 2n+1$ is odd, and

$$\begin{aligned} \hat{l}_{e_\epsilon,I_1K_1} \cdot f_\gamma := {} & \frac{2(\mathbf{i}\kappa\beta\hbar)^{D-1}}{(D-1)!} \sum_{e_{\iota_1}} \ldots \sum_{e_{\iota_{D-1}}} k_{\mathrm{av}} \cdot \varsigma(e_\epsilon, e_{\iota_1}, \ldots, e_{\iota_{D-1}}) \\ & \times \hat{V}^I(\Box^{D-1}_\epsilon) \epsilon_{I[I_1|J_1\ldots I_nJ_n|} R^{J_1}_{e_{\iota_1}\,K_1]} R^{I_2K_2}_{e_{\iota_2}} R^{J_2}_{e_{\iota_3}\,K_2} \ldots R^{I_nK_n}_{e_{\iota_{D-2}}} R^{J_n}_{e_{\iota_{D-1}}\,K_n} \widehat{V^{-1}_{\Box_\epsilon}}^{2D-3} \cdot f_\gamma, \end{aligned} \tag{98}$$

for $D = 2n$ is even. The final formulation of the second length operator is given by Eq. (83) or Eq. (84).

### B. The second version of general $m$-area operators

The above procedure of constructing the length operator can be extended to construct the general geometric operators measuring the $m$-area of a $m$-dimensional surface ${}^mS$. By the partition ${}^mS = \sum_{t\in\mathbb{N}, 0\leq t\leq T} {}^m\bar{S}^t_{\diamond_{1m}}$ of an open $m$-surface ${}^mS$, the $m$-area $\mathrm{Ar}({}^mS_{\diamond_{1m}})$ can be reexpressed by fluxes following a partition of the neighborhood of ${}^mS_{\diamond_{1m}}$ in $\sigma$ as follows. Suppose that the $(D-m)$-tuple of the (D − 1)-surface ${}^{D-1}S_i$ $(i = 1, \ldots, D-m)$ with coordinate (D − 1)-area $\epsilon^{D-1}$ intersects at the $m$-dimensional region ${}^mS_{\diamond_{1m}}$. The normal covectors $(n^1_a, \ldots, n^i_a, \ldots, n^{D-m}_a)$ of ${}^mS_{\diamond_{1m}}$ span a $(D-m)$-dimensional vector space and satisfy

$$\epsilon^{\iota_1\ldots\iota_m} \dot{e}^{a_1}_{\iota_1} \ldots \dot{e}^{a_m}_{\iota_m} = \frac{1}{(D-m)!} \epsilon^{a_1\ldots a_m a_{m+1}\ldots a_D} n^{i_1}_{a_{m+1}} \ldots n^{i_{D-m}}_{a_D} \epsilon_{i_1\ldots i_{D-m}}. \tag{99}$$

We consider the following two cases.

*Case I: $\bar{m} := D-m$ is even*

Define

$$\begin{aligned} \bar{E}_{K_1\ldots K_{\bar{m}}} := {} & \frac{1}{\sqrt{\bar{m}!}} \pi^{b_1}_{K_1L_1} \delta^{L_1L_2} \pi^{b_2}_{K_2L_2} \ldots \pi^{b_{\bar{m}-1}}_{K_{\bar{m}-1}L_{\bar{m}-1}} \delta^{L_{\bar{m}-1}L_{\bar{m}}} \pi^{b_{\bar{m}}}_{K_{\bar{m}}L_{\bar{m}}} \\ & \times n^{i_1}_{b_1} \ldots n^{i_{\bar{m}}}_{b_{\bar{m}}} \epsilon_{i_1\ldots i_{\bar{m}}} |\det(\pi)|^{\frac{1-\bar{m}}{D-1}}, \end{aligned} \tag{100}$$

for $D$ is odd, and

$$\bar{E}_{K_1\ldots K_{\bar{m}}} := \frac{1}{\sqrt{\bar{m}!}}\pi^{b_1}_{K_1L_1}\delta^{L_1L_2}\pi^{b_2}_{K_2L_2}\ldots\pi^{b_{\bar{m}-1}}_{K_{\bar{m}-1}L_{\bar{m}-1}}\delta^{L_{\bar{m}-1}L_{\bar{m}}}\pi^{b_{\bar{m}}}_{K_{\bar{m}}L_{\bar{m}}}n^{i_1}_{b_1}\ldots n^{i_{\bar{m}}}_{b_{\bar{m}}}\epsilon_{i_1\ldots i_{\bar{m}}}|\mathrm{ddet}(\pi)|^{\frac{1-\bar{m}}{2D-2}}, \tag{101}$$

for $D$ is even. Both of them satisfy

$$\det({}^mq) = \bar{E}^{K_1\ldots K_{\bar{m}}}\bar{E}_{K_1\ldots K_{\bar{m}}}, \tag{102}$$

which gives $\det(q) = \det({}^mq)\det({}^{\bar{m}}q)$, and

$$\det({}^{\bar{m}}q)^{-1} := \frac{1}{\bar{m}!}n^{i'_1}_{a_1}\ldots n^{i'_{\bar{m}}}_{a_{\bar{m}}}\epsilon_{i_1\ldots i_{\bar{m}}}q^{a_1b_1}\ldots q^{a_{\bar{m}}b_{\bar{m}}}n^{i_1}_{b_1}\ldots n^{i_{\bar{m}}}_{b_{\bar{m}}}\epsilon_{i_1\ldots i_{\bar{m}}}. \tag{103}$$

*Case II*: $\bar{m} := D - m$ *is odd*

Similar to the last case, we can define

$$\bar{E}_{IJK_1\ldots K_{\bar{m}-1}} := \frac{1}{\sqrt{2\bar{m}!}}\pi^{b}_{IJ}\pi^{b_1}_{K_1L_1}\delta^{L_1L_2}\pi^{b_2}_{K_2L_2}\ldots\pi^{b_{\bar{m}-2}}_{K_{\bar{m}-2}L_{\bar{m}-2}}\delta^{L_{\bar{m}-2}L_{\bar{m}-1}}\pi^{b_{\bar{m}-1}}_{K_{\bar{m}-1}L_{\bar{m}-1}}n^{i}_{b}n^{i_1}_{b_1}\ldots n^{i_{\bar{m}-1}}_{b_{\bar{m}-1}}\epsilon_{ii_1\ldots i_{\bar{m}}}|\det(\pi)|^{\frac{1-\bar{m}}{D-1}}, \tag{104}$$

for $D$ is odd, and

$$\bar{E}_{IJK_1\ldots K_{\bar{m}-1}} := \frac{1}{\sqrt{2\bar{m}!}}\pi^{b}_{IJ}\pi^{b_1}_{K_1L_1}\delta^{L_1L_2}\pi^{b_2}_{K_2L_2}\ldots\pi^{b_{\bar{m}-2}}_{K_{\bar{m}-2}L_{\bar{m}-2}}\delta^{L_{\bar{m}-2}L_{\bar{m}-1}}\pi^{b_{\bar{m}-1}}_{K_{\bar{m}-1}L_{\bar{m}-1}}n^{i}_{b}n^{i_1}_{b_1}\ldots n^{i_{\bar{m}-1}}_{b_{\bar{m}-1}}\epsilon_{ii_1\ldots i_{\bar{m}}}|\mathrm{ddet}(\pi)|^{\frac{1-\bar{m}}{2D-2}}, \tag{105}$$

for $D$ is even. They also satisfy

$$\det({}^mq) = \bar{E}^{IJK_1\ldots K_{\bar{m}-1}}\bar{E}_{IJK_1\ldots K_{\bar{m}-1}}. \tag{106}$$

Similar to the construction of the length operator, we define

$$\bar{\mathcal{E}}^{K_1\ldots K_{\bar{m}}} := \frac{1}{\epsilon^m\sqrt{\bar{m}!}}\pi^{K_1L_1}({}^{D-1}S^{i_1})\delta_{L_1L_2}\pi^{K_2L_2}({}^{D-1}S^{i_2})\ldots\pi^{K_{\bar{m}-1}L_{\bar{m}-1}}({}^{D-1}S^{i_{\bar{m}-1}})\delta_{L_{\bar{m}-1}L_{\bar{m}}}\pi^{K_{\bar{m}}L_{\bar{m}}}({}^{D-1}S^{i_{\bar{m}}})\epsilon_{i_1\ldots i_{\bar{m}}}V^{(1-\bar{m})}_{\square_\epsilon}, \tag{107}$$

for $\bar{m}$ is even, and

$$\begin{aligned}\bar{\mathcal{E}}^{IJK_1\ldots K_{\bar{m}-1}} := \frac{1}{\sqrt{2\bar{m}!}}&\pi^{IJ}({}^{D-1}S^{i})\pi^{K_1L_1}({}^{D-1}S^{i_1})\delta_{L_1L_2}\pi^{K_2L_2}({}^{D-1}S^{i_2})\ldots\\ &\times\pi^{K_{\bar{m}-2}L_{\bar{m}-2}}({}^{D-1}S^{i_{\bar{m}-2}})\delta_{L_{\bar{m}-2}L_{\bar{m}-1}}\pi^{K_{\bar{m}-1}L_{\bar{m}-1}}({}^{D-1}S^{i_{\bar{m}-1}})\epsilon_{ii_1\ldots i_{\bar{m}-1}}V^{(1-\bar{m})}_{\square_\epsilon},\end{aligned} \tag{108}$$

for $\bar{m}$ is odd, where $\square_\epsilon$ is a D-dimensional box with coordinate volume $\epsilon^D$ containing the tuple of ${}^{D-1}S^i$. Then, the $m$-area $\mathrm{Ar}({}^mS)$ can be reexpressed as

$$\mathrm{Ar}({}^mS) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\mathrm{Ar}({}^mS_{\diamond_{1m}}) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\bar{\mathcal{E}}^{K_1\ldots K_{\bar{m}}}\bar{\mathcal{E}}_{K_1\ldots K_{\bar{m}}}} \tag{109}$$

for $\bar{m}$ is even, and

$$\mathrm{Ar}({}^mS) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\mathrm{Ar}({}^mS_{\diamond_{1m}}) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\bar{\mathcal{E}}^{IJK_1\ldots K_{\bar{m}-1}}\bar{\mathcal{E}}_{IJK_1\ldots K_{\bar{m}-1}}} \tag{110}$$

for $\bar{m}$ is odd. Since all the components in Eqs. (107) and (108) have clear quantum analogs, we can obtain the general geometric operators as

$$\widehat{\mathrm{Ar}}({}^mS) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\widehat{\mathrm{Ar}}({}^mS_{\diamond_{1m}}) = \lim_{\epsilon\to 0}\sum_{{}^mS_{\diamond_{1m}}}\sqrt{\hat{\bar{\mathcal{E}}}^{K_1\ldots K_{\bar{m}}}\hat{\bar{\mathcal{E}}}^{\dagger}_{K_1\ldots K_{\bar{m}}}} \tag{111}$$

for $\bar{m}$ is even, and

$$\widehat{\mathrm{Ar}}({}^{m}S)=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\widehat{\mathrm{Ar}}({}^{m}S_{\diamond_{1m}})=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\sqrt{\hat{\hat{\mathcal{E}}}^{IJK_1\ldots K_{\bar{m}-1}}\hat{\hat{\mathcal{E}}}^{\dagger}_{IJK_1\ldots K_{\bar{m}-1}}} \tag{112}$$

for $\bar{m}$ is odd. Also, an alternative formulation can be given as

$$\widehat{\mathrm{Ar}}({}^{m}S)=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\widehat{\mathrm{Ar}}({}^{m}S_{\diamond_{1m}})=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\frac{1}{2}\sqrt{(\hat{\hat{\mathcal{E}}}_{\epsilon}^{K_1\ldots K_{\bar{m}}}+\hat{\hat{\mathcal{E}}}^{\dagger}_{K_1\ldots K_{\bar{m}}})(\hat{\hat{\mathcal{E}}}_{\epsilon}^{K_1\ldots K_{\bar{m}}}+\hat{\hat{\mathcal{E}}}^{\dagger}_{K_1\ldots K_{\bar{m}}})} \tag{113}$$

for $\bar{m}$ is even, and

$$\widehat{\mathrm{Ar}}({}^{m}S)=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\widehat{\mathrm{Ar}}({}^{m}S_{\diamond_{1m}})=\lim_{\epsilon\to 0}\sum_{{}^{m}S_{\diamond_{1m}}}\frac{1}{2}\sqrt{(\hat{\hat{\mathcal{E}}}^{IJK_1\ldots K_{\bar{m}-1}}+\hat{\hat{\mathcal{E}}}^{\dagger}_{IJK_1\ldots K_{\bar{m}-1}})(\hat{\hat{\mathcal{E}}}^{IJK_1\ldots K_{\bar{m}-1}}+\hat{\hat{\mathcal{E}}}^{\dagger}_{IJK_1\ldots K_{\bar{m}-1}})} \tag{114}$$

for $\bar{m}$ is odd. Note that we defined

$$\begin{aligned}\hat{\hat{\mathcal{E}}}^{K_1\ldots K_{\bar{m}}}&:=\frac{1}{\sqrt{\bar{m}!}}\hat{\pi}^{K_1L_1}({}^{D-1}S^{i_1})\delta_{L_1L_2}\hat{\pi}^{K_2L_2}({}^{D-1}S^{i_2})\ldots\hat{\pi}^{K_{\bar{m}-1}L_{\bar{m}-1}}({}^{D-1}S^{i_{\bar{m}-1}})\delta_{L_{\bar{m}-1}L_{\bar{m}}}\hat{\pi}^{K_{\bar{m}}L_{\bar{m}}}({}^{D-1}S^{i_{\bar{m}}})\epsilon_{i_1\ldots i_{\bar{m}}}\widehat{V^{-1}_{\square_\epsilon}}^{(\bar{m}-1)}\\&=\frac{(\mathbf{i}\hbar\kappa\beta)^{\bar{m}}}{\sqrt{\bar{m}!}}\sum_{e_{\imath_1},\ldots,e_{\imath_{\bar{m}}}}\epsilon(e_{\imath_1},{}^{D-1}S^{i_1})\ldots\epsilon(e_{\imath_{\bar{m}}},{}^{D-1}S^{i_{\bar{m}}})\ldots R^{K_1L_1}_{e_{\imath_1}}\delta_{L_1L_2}R^{K_2L_2}_{e_{\imath_2}}\ldots R^{K_{\bar{m}-1}L_{\bar{m}-1}}_{e_{\imath_{\bar{m}-1}}}\delta_{L_{\bar{m}-1}L_{\bar{m}}}R^{K_{\bar{m}}L_{\bar{m}}}_{e_{\imath_{\bar{m}}}}\epsilon_{i_1\ldots i_{\bar{m}}}\widehat{V^{-1}_{\square_\epsilon}}^{(\bar{m}-1)},\end{aligned} \tag{115}$$

for $\bar{m}$ is even, and

$$\begin{aligned}\hat{\hat{\mathcal{E}}}^{IJK_1\ldots K_{\bar{m}-1}}&:=\frac{1}{\sqrt{2\bar{m}!}}\hat{\pi}^{IJ}({}^{D-1}S^{i})\hat{\pi}^{K_1L_1}({}^{D-1}S^{i_1})\delta_{L_1L_2}\pi^{K_2L_2}({}^{D-1}S^{i_2})\ldots\\&\quad\times\hat{\pi}^{K_{\bar{m}-2}L_{\bar{m}-2}}({}^{D-1}S^{i_{\bar{m}-2}})\delta_{L_{\bar{m}-2}L_{\bar{m}-1}}\hat{\pi}^{K_{\bar{m}-1}L_{\bar{m}-1}}({}^{D-1}S^{i_{\bar{m}-1}})\epsilon_{ii_1\ldots i_{\bar{m}-1}}\widehat{V^{-1}_{\square_\epsilon}}^{(\bar{m}-1)}\\&=\frac{(\mathbf{i}\hbar\kappa\beta)^{\bar{m}}}{\sqrt{2\bar{m}!}}\sum_{e_\imath,e_{\imath_1},\ldots,e_{\imath_{\bar{m}-1}}}\epsilon(e_\imath,{}^{D-1}S^{i})\epsilon(e_{\imath_1},{}^{D-1}S^{i_1})\ldots\epsilon(e_{\imath_{\bar{m}-1}},{}^{D-1}S^{i_{\bar{m}-1}})\\&\quad\times R^{IJ}_{e_\imath}R^{K_1L_1}_{e_{\imath_1}}\delta_{L_1L_2}R^{K_2L_2}_{e_{\imath_2}}\ldots R^{K_{\bar{m}-2}L_{\bar{m}-2}}_{e_{\imath_{\bar{m}-2}}}\delta_{L_{\bar{m}-2}L_{\bar{m}-1}}R^{K_{\bar{m}-1}L_{\bar{m}-1}}_{e_{\imath_{\bar{m}-1}}}\epsilon_{ii_1\ldots i_{\bar{m}-1}}\widehat{V^{-1}_{\square_\epsilon}}^{(\bar{m}-1)},\end{aligned} \tag{116}$$

for $\bar{m}$ is odd. Here we can also remove the background structure by suitably averaging the regularized operators. The average of $\epsilon_{i_1\ldots i_{\bar{m}}}\epsilon(e_{\imath_1},{}^{D-1}S^{i_1})\ldots\epsilon(e_{\imath_{\bar{m}}},{}^{D-1}S^{i_{\bar{m}}})$ gives ${}^{\bar{m}}k_{\mathrm{av}}\cdot\varsigma(e^{\epsilon}_1,\ldots,e^{\epsilon}_m,e_{\imath_1},\ldots,e_{\imath_{\bar{m}}})$ for $\bar{m}$ is even, and that of $\epsilon_{ii_1\ldots i_{\bar{m}-1}}\epsilon(e_\imath,{}^{D-1}S^{i})\epsilon(e_{\imath_1},{}^{D-1}S^{i_1})\ldots\epsilon(e_{\imath_{\bar{m}-1}},{}^{D-1}S^{i_{\bar{m}-1}})$ gives ${}^{\bar{m}}k_{\mathrm{av}}\cdot\varsigma(e^{\epsilon}_1,\ldots,e^{\epsilon}_m,e_\imath,e_{\imath_1},\ldots,e_{\imath_{\bar{m}-1}})$ for $\bar{m}$ is odd, wherein ${}^{\bar{m}}k_{\mathrm{av}}$ is a constant, $(e^{\epsilon}_1,\ldots,e^{\epsilon}_m)$ is the set of edges to give ${}^{m}S_{\diamond_{1m}}$, and $\varsigma(e^{\epsilon}_1,\ldots,e^{\epsilon}_m,e_{\imath_1},\ldots,e_{\imath_{\bar{m}}})$ or $\varsigma(e^{\epsilon}_1,\ldots,e^{\epsilon}_m,e_\imath,e_{\imath_1},\ldots,e_{\imath_{\bar{m}-1}})$ is the orientation function.

We have constructed the background-independent "elementary" general geometric operators in all dimensional LQG. The operators (111), (112), (113) and (114) are symmetric. The overall undetermined factor ${}^{\bar{m}}k_{\mathrm{av}}$ is expected to be fixed by semiclassical consistency. It should be noted that in the special case of $\bar{m}=D-1$ the general geometric operators become some length operators. However, they are not exactly the same as (83) and (84). Nevertheless, the two versions of length operators can be identified by certain operator reordering. Also, the (D − 1)-area operator which is constructed with flux operators directly can be given as the special case of $\bar{m}=1$ from the general geometric operators, and the usual D-volume operator can be given as the special case of $\bar{m}=0$. Thus the construction strategy of general geometric operators is the extension of those for the standard (D − 1)-area operator and usual D-volume operator.

It is easy to see that the elementary geometric operator $\widehat{\mathrm{Ar}}({}^{m}S_{\diamond_{1m}})$ does not commute with the D-volume operator $\hat{V}_{\square_\epsilon}$ if they both contain a same vertex $v$. This implies that these elementary geometric operators are generally noncommutative,

$$[\widehat{\mathrm{Ar}}({}^mS_{\diamond_{1m}}), \widehat{\mathrm{Ar}}({}^mS'_{\diamond_{1m}})] \neq 0, \tag{117}$$

for ${}^mS_{\diamond_{1m}}$ and ${}^mS'_{\diamond_{1m}}$ contain the same vertex $v$ which is dual to $\square_\epsilon$. Hence we can only define the $m$-area operator of nice extended $m$-surfaces based on the elementary geometric operators as suggested in Ref. [13]. Also, we leave the operator ordering issue of our general geometric operators for further study [24].

## V. CONCLUDING REMARKS

In the previous sections, we constructed two kinds of length operators for all dimensional LQG by extending the constructions in standard (1 + 3)-dimensional LQG. Based on the two different strategies, we also constructed two kinds of general geometric operators to measure arbitrary $m$-areas in all dimensional LQG. In the first strategy, by Eq. (10) the de-densitized dual momentum $\sqrt{q}\pi_a^{IJ}$ is regularized as Eq. (13). Then the general geometric quantities with $\underline{\pi}(e_\epsilon)$ as building blocks can be quantized by this regularization and suitable choices of operator ordering. In the second strategy, as the de-densitized dual momentum can be expressed by the momentum $\pi^a_{IJ}$ and the volume element by Eqs. (72) and (74), it can also be regularized as Eqs. (77) and (80). For the general geometric quantities, the $m$-area element can be regularized by the flux of $\pi^a_{IJ}$ through Eqs. (109) and (110). Then they can be quantized by the regularization and introducing the inverse volume operator. To get well-defined and background-independent general geometric operators, the averaging of the regularizations has to be also introduced.

Several remarks on the two kinds of general geometric operators are listed in order. First, the first kind of general geometric operators was constructed in Sec. II with the so-called (de-densitized) dual momentum, whose smeared version was expressed by the holonomy of connection. This construction would lead to some problem if the simplicity constraint was taken into account, since the action of a holonomy could change a state satisfying the constraint into a nonsatisfying one. To solve the problem, some projection operators should be introduced in the construction. Different from the first one, the second kind of general geometric operators constructed in Sec. IV would have a good behavior even if the simplicity constraint was considered, since these kinds of operators and the simplicity constraint are both totally composed of the flux operators. In this sense, the second kind of general geometric operators is expected to be a better choice than the first one in the consideration of obtaining the semiclassical spatial geometry from all dimensional LQG. Second, the second kind of general geometric operators contains the standard (D − 1)-area operator and usual D-volume operator as some special cases. Hence, its construction could be regarded as a natural extension of those of standard (D − 1)-area operator and usual D-volume operator. Different from the second one, the construction of the first kind of general geometric operators is completely different from those of standard (D − 1)-area and usual D-volume operators. Thus it deserves checking the consistency between them in future work. Note that a similar consistency check was performed in (1 + 3)-dimensional standard LQG [25]. Third, in the construction of the first kind of general geometric operators, the choice of the operator ordering is inspired by that of the alternative flux operator in (1 + 3)-dimensional standard LQG [9,10]. The consistency between the alternative flux operator $\hat{\pi}^{IJ}_{\mathrm{alt.}}({}^{(D-1)}S_{\diamond_{(D-1)}})$ and the standard flux operator $\hat{\pi}^{IJ}({}^{(D-1)}S_{\diamond_{(D-1)}})$ in (1 + D)-dimensional LQG was checked in Sec. III.

Moreover, the properties of these general geometric operators are worth further studying. Though it is hard to obtain the spectra of the general geometric operators, one may consider the semiclassical behavior of these operators. For instance, one can study the actions of the general geometric operators on the semiclassical states that are equipped with the simple coherent intertwiners [26]. The undetermined regularization constants in these general geometric operators are also expected to be fixed in such a kind of semiclassical consistency check.

## ACKNOWLEDGMENTS

We benefited greatly from our numerous discussions with Norbert Bodendorfer, Shupeng Song and Cong Zhang. This work is supported by the National Natural Science Foundation of China (NSFC) with Grants No. 11875006 and No. 11961131013.

## APPENDIX: QUANTIZATION OF $\mathcal{S}$

Recall that in all dimensional LQG, $\mathcal{S}$ takes a value of 1 if $(D+1)$ is odd while it takes a value of 1 with a sign of $\det(\pi) \coloneqq \frac{1}{2D!}\epsilon_{aa_1b_1\ldots a_nb_n}\epsilon_{IJI_1J_1\ldots I_nJ_n}\pi^{aIJ}\pi^{a_1I_1K_1}\pi^{b_1J_1}{}_{K_1}\ldots\pi^{a_nI_nK_n}\pi^{b_nJ_n}{}_{K_n}$ if $(D+1)$ is even. Let us focus on the case that $(D+1)$ is even now. Notice that $\det(\pi)$ is smeared as $\pi(p,\triangle_1,\ldots,\triangle_D)$ which is defined in Eq. (78) and then we have

$$\mathcal{S}(p) = \mathrm{sgn}(\det(\pi)(p)) = \mathrm{sgn}(\pi(p,\triangle_1,\ldots,\triangle_D)), \quad \epsilon\to 0. \tag{A1}$$

Also we have the volume of the box $\square_\epsilon$ which is given by $V_{\square_\epsilon} = \int_{\square_\epsilon} d^Dx|\det\pi|^{\frac{1}{D-1}}$ and it can be transformed as

$$\begin{aligned} V_{\square_\epsilon} &= \int_{\square_\epsilon} d^Dp|\pi(p,\triangle_1,\ldots,\triangle_D)|^{\frac{1}{D-1}} \\ &= |\pi(p,\triangle_1,\ldots,\triangle_D)|^{\frac{1}{D-1}}\cdot\epsilon^D, \qquad \epsilon\to 0. \end{aligned} \tag{A2}$$

It should be noticed that

$$\mathcal{S}(p)\cdot\pi(p,\triangle_1,\ldots,\triangle_D)=|\pi(p,\triangle_1,\ldots,\triangle_D)|. \tag{A3}$$

Then, similar to the discussion of the signum operator in [9], the $\mathcal{S}$ must be identified with the signum that appears inside the absolute value under the $(D-1)$-degree roots in the definition of the volume $V_{\square_\epsilon}$ in the classical theory. This meaning of $\mathcal{S}$ can be extended to the quantum case naturally. Recall the expression, Eqs. (89) and (90), of the volume operator for $D$ is odd and consider the case $\epsilon\to 0$: we can immediately conclude that Eq. (68) holds, where the right-hand side of Eq. (68) is basically the expression inside the absolute value in the definition of the volume operator and $\hat{\mathcal{S}}$ represents the signum of the expression inside the absolute values in the volume operator.


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