Range of novel black hole phase transitions via massive gravity: Triple points and *N*-fold reentrant phase transitions

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Massive gravities in anti–de Sitter spacetime can be viewed as effective dual field theories of different phases of condensed matter systems with broken translational symmetry such as solids, (perfect) fluids, and liquid crystals. Motivated by this fact, we explore the black hole chemistry (BHC) of these theories and find a new range of novel phase transitions close to realistic ones in ordinary physical systems. We find that the equation of state of topological black holes (TBHs) at their inflection point(s) in d-dimensional spacetime reduces to a polynomial equation of degree $(d-4)$, which yields up to $n = (d-4)$ critical points. As a result, for (neutral) TBHs, we observe triple-point phenomena with the associated first-order phase transitions (in $d \geq 7$), and a new phenomenon we call an N-fold reentrant phase transition, in which several (N) regions of thermodynamic phase space exhibit distinct reentrant phase transitions, with associated virtual triple points and zeroth-order phase transitions (in $d \geq 8$), as well as Van der Waals transitions (in $d \geq 5$) and reentrant (in $d \geq 6$) behavior. We conclude that BHC in higher-dimensional massive gravity is very likely to offer further new surprises.

I. INTRODUCTION

From a classical field theory perspective, de Rham–Gabadadze–Tolley (dRGT) massive gravity [1,2] is a consistent extension of general relativity with an explicit mass term. By giving the graviton a mass, massive gravity could provide a possible explanation for the accelerated expansion of the Universe without the requirement of dark energy [3–5]. Moreover it has recently been shown that massive spin-2 particles can explain the current observations related to dark matter [6]. Furthermore, the new observations from LIGO [7] imply that the graviton mass is bounded to be $m_g \leq 7.7 \times 10^{-23}$ eV/c^2, and so such an assumption remains empirically viable, since models of massive gravity typically yield the bound $m_g \leq 10^{-30}$–10^{-33} eV/c^2, whose observable effects are out of reach of LIGO [8].

Perhaps the most challenging problem in building a massive gravitational field theory is the appearance of ghost instabilities [so-called Boulware-Deser (BD) ghosts] [9]. However dRGT massive gravity has been shown to be ghost-free in the decoupling limit to all orders in the nonlinearities; away from the decoupling limit, the

possibility of ghost fields is excluded at least up to and including quadratic order in the nonlinearities [2]. Any pathological BD ghost is eliminated at the full nonlinear level due to the Hamiltonian constraint and the existence of a nontrivial secondary constraint [10]. This ghost analysis generalizes both to arbitrary massive couplings [11] and higher dimensions [12,13].

Beyond these achievements, dRGT massive gravity in anti–de Sitter (AdS) spacetime with a singular, degenerate reference metric is proving useful in building holographic models for normal conductors that are close to realistic ones, with finite DC conductivity [14–16]. Recent developments have shown that they can be regarded as dual to effective field theories of different phases of matter, particularly homogeneous and isotropic condensed matter systems with broken translational invariance [17–19].

For these reasons we study here the chemistry of AdS black holes (BHC) in this context, where the thermodynamic pressure $P = -\Lambda/8\pi$, and $\Lambda < 0$ is the cosmological constant [20]. We concentrate on higher-dimensional massive gravity as an alternative candidate to Einstein's general relativity, investigate the extended phase space thermodynamics of the AdS BH solutions in detail, and bring some new perspectives on BH thermodynamics with massive gravitons. The phase structure of this class of theories has yet to be considered in higher dimensions at all orders. Our purpose is to discover which properties of black holes are universal and which show a dependence on the

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spacetime dimension. In so doing we find a new range of critical phenomena that may also be present in nature. Specifically, we find that topological black holes (TBHs) with massive gravitons can mimic the critical behavior of everyday substances in nature without the inclusion of any extra or unusual matter fields in the gravitational action. Furthermore, besides van der Waals (vdW) [21,22] and reentrant [23] phase behavior, in massive gravity a gravitational triple point can emerge for AdS BHs with various horizon topologies in spacetime dimensions with $d \geq 7$.

Even more remarkably, in $d \geq 8$ we discover a novel and more complex phenomenon: N-fold reentrant phase transitions (RPTs), for all types of TBHs. By "N-fold RPTs" we mean that a number of RPTs are present in distinct regions of thermodynamic phase space, i.e., for regions $P \in (P_{T_{r_i}}, P_{Z_i})$ and $T \in (T_{T_{r_i}}, T_{Z_i})$ the associated RPT_i occurs. This is in contrast to the situation in Lovelock gravity (first seen in Ref. [24]), in which hyperbolic black holes exhibit double (or multiple) RPTs (with the associated sequence of LBH → SBH → LBH → SBH... phase transitions, where SBH denotes small black hole, LBH denotes large black hole and IBH denotes intermediate black hole) as the temperature decreases along a single line in phase space; this critical behavior is similar to observed double reentrant transitions for smectic and nematic phases of liquid crystals [25–28], which schematically have the form $A \rightarrow B \rightarrow A \rightarrow B$. By contrast, N-fold RPTs refer to the phenomenon of RPTs (multiple or not) occurring along more than one line of temperature in phase space.

This situation, in some aspects, resembles that of a (experimentally confirmed) scenario for liquid crystals [29–31], but in a slightly different way. In liquid crystals, reentrance is encountered as the temperature is lowered monotonically, with other thermodynamic quantities kept fixed. For massive gravity TBHs we find that several RPTs occur at various locations in phase space. From the experimental point of view, since it is not easy to obtain RPTs from any theory [30], we believe that these kinds of exact and simple relations in modified gravity could possibly shed some light on the microscopic structure of this strange phenomenon and establish a new link between BH physics and the realm of statistical mechanics of many-body systems.

N-fold RPTs are thus a generic feature of higher-dimensional TBHs in massive gravity. We explicitly show that the analogue of triple points in SBH/IBH/LBH phase transitions and virtual triple points in N-fold RPTs can be obtained by adding the fifth-, sixth-, and higher-order graviton self-interactions besides the first four terms that usually appear in the literature. Although we do not find evidence for any quadruple critical points [32], their existence is not ruled out; whether or not such phenomena exist for BHs (in massive gravity or elsewhere) remains an open question.

II. ACTION, FIELD EQUATIONS AND AdS BLACK HOLES

The bulk action for massive gravity on the d-dimensional background manifold $\mathcal{M}$ in the presence of a negative cosmological constant can be written as [1,2]

$$\mathcal{I}_b = -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^d x \sqrt{-g} \left[R - 2\Lambda + m_g^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i(g,f) \right]$$

(1)

where the overall minus ensures that the semiclassical partition function yields consistent results for the thermodynamic quantities. Here m_g is the graviton mass, c_i's are arbitrary massive couplings, and

$$\mathcal{U}_i = \sum_{y=1}^{i} (-1)^{y+1} \frac{(i-1)!}{(i-y)!} \mathcal{U}_{i-y}[\mathcal{K}^y],$$

(2)

where $\mathcal{U}_{i-y} = 1$ if $i = y$. The massive graviton self-interactions $\mathcal{U}_i$ are constructed from a $d \times d$ matrix $\mathcal{K}^{\nu}{}_{\mu}$, which is posited to have the following explicit form:

$$\mathcal{K}^{\mu}{}_{\nu} = (\sqrt{\mathcal{K}})^{\mu}{}_{\lambda} (\sqrt{\mathcal{K}})^{\lambda}{}_{\nu} = \sqrt{g^{\mu\lambda} f_{\lambda\nu}}.$$

(3)

$g_{\mu\nu}$ is the dynamical (physical) metric, and $f_{\lambda\nu}$ is the auxiliary reference metric, needed to define the mass term for gravitons.

We seek here a holographic system (AdS BH) that mimics the physics of solids, liquids, (perfect) fluids, and especially liquid crystals, i.e., a system with broken (spatial) translational symmetry as a key ingredient [33]. To this end, following Refs. [17–19], we consider a subclass of dRGT massive gravity [14,15,34], having a dynamical (physical) metric $g_{\lambda\nu}$ and a degenerate (nonphysical) reference metric $f_{\lambda\nu} = \partial_\lambda \phi^a \partial_\nu \phi^b f_{ab}(\phi)$, in the configuration space of scalar Stückelberg fields ϕ^a ($a = 1, 2, ..., d$), where spatial inhomogeneities are substituted with graviton mass terms [14,15,34]. This is equivalent to working with $(d-2)$ Stückelberg fields for restoring translational symmetry in spatial directions: there is a gravitational sector with massless gravitons, encoding $d(d-3)/2$ physical modes, and a scalar sector with $(d-2)$ Stückelberg fields minimally coupled to gravity [34] that encode $(d-2)$ physical degrees of freedom; in all there are $(d(d-1)-4)/2$ degrees of freedom.

By gauge fixing (e.g., working in the unitary gauge $\phi^a = \delta^a_\mu x^\mu$), general covariance is preserved in the (t, r) coordinates and breaks in the other spatial coordinates $(x_1, x_2, ..., x_{d-2})$. Consequently, the dual gauge theory on the AdS boundary will have a conserved energy without conserved momentum currents [14,15].

We are looking for a holographic system (AdS BH) that mimics the physics of solids, liquids, (perfect) fluids, and especially liquid crystals, i.e., a system with broken (spatial) translational symmetry as a key ingredient [33]. To do so, we need a system of massless scalar fields interacting with pure (Einstein) gravity that can be separated into different phases of physical matter [17–19]. This theory can be reformulated in an equivalent way via dRGT massive gravity with a singular (degenerate) reference metric, as we shall assume.

We employ the static ansatz

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2 h_{ij}dx_i dx_j \quad (4)$$

for the dynamical metric $g_{\mu\nu}$. The degenerate (spatial) background [14,35] is chosen for the reference metric as $f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij})$ where c_0 is a positive constant, with $h_{ij}dx_i dx_j = dx_1^2 + \frac{\sin^2(\sqrt{k}x_1)}{k}\sum_{i=2}^{d-2}dx_i^2\prod_{j=2}^{i-1}\sin^2 x_j$ representing spherical ($k = 1$), planar ($k = 0$), and hyperbolic ($k = -1$) horizon geometries of constant curvature $d_1 d_2 k$ and volume $\omega_{d_2}^{(k)}$ (in what follows we will use the convention $d_n = d - n$). With appropriate identifications these become compact surfaces of higher genus, yielding TBHs [36]. The interaction terms $\mathcal{U}_i$ are $\mathcal{U}_i = (c_0/r)^i \prod_{j=2}^{i+1}d_j$; there exist $(d-2)$ potential terms in a d-dimensional spacetime.

Varying the bulk action (1), including the Gibbons-Hawking surface term ($\mathcal{I}_{GH}$), with respect to the dynamical metric ($g_{\mu\nu}$) yields

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} = 0, \quad (5)$$

where

$$\mathcal{X}_{\mu\nu} = -\sum_{i=1}^{d-2}\frac{c_i}{2}\left[\mathcal{U}_i g_{\mu\nu} + \sum_{y=1}^{i}(-1)^y\frac{i!}{(i-y)!}\mathcal{U}_{i-y}\mathcal{K}_{\mu\nu}^y\right]. \quad (6)$$

The above gravitational field equations can be analytically solved using the metric ansatz (4), and, the metric function $V(r)$ is obtained as

$$V(r) = k + \frac{r^2}{\ell^2} - \frac{m}{r^{d_3}} + m_g^2\sum_{i=1}^{d-2}\left(\frac{c_0^i c_i}{d_2 r^{i-2}}\prod_{j=2}^{i}d_j\right), \quad (7)$$

where the Arnowitt-Deser-Misner (ADM) mass of the black hole is [35]

$$M = \frac{d_2\omega_{d_2}^{(k)}}{16\pi}m \quad (8)$$

where for up to four interaction potentials ($i = 1, 2, 3, 4$)

$$m = kr_+^{d_3} + \frac{r_+^{d_1}}{\ell^2} + m_g^2 r_+^{d_3}\left(\frac{c_0 c_1}{d_2}r_+ + c_0^2 c_2\right.$$
$$\left. + \frac{d_3 c_0^3 c_3}{r_+} + \frac{d_3 d_4 c_0^4 c_4}{r_+^2}\right) \quad (9)$$

with $V(r_+) = 0$; the extension to higher-order potentials is straightforward.

Note that although the Λ-term of the metric function (7) is dominant for large r and the curvature tensor approaches that of pure AdS spacetime, the asymptotic symmetry group is not necessarily that of pure AdS. For example charged black hole solutions with a degenerate reference metric [35] have the same form as Eq. (7) but break the global symmetries of AdS. To our knowledge there has been no thorough analysis in the literature regarding the asymptotic symmetries of solutions in massive gravity with a negative cosmological constant, and we shall not consider a full analysis of the asymptotic behavior of our solutions here.

The Hawking temperature of the BH spacetime can be obtained by employing the Euclidean formalism. By the analytic continuation of the Lorentzian metric (4) to Euclidean signature and requiring the regularity condition near the horizon, we obtain

$$T = \beta^{-1} = \frac{V'(r)}{4\pi}\bigg|_{r=r_+}$$
$$= \frac{1}{4\pi r_+}\left[d_3 k + d_1\frac{r_+^2}{\ell^2} + m_g^2\sum_{i=1}^{d-2}\left(\frac{c_0^i c_i}{r_+^{i-2}}\prod_{j=3}^{i+1}d_j\right)\right] \quad (10)$$

for the Hawking temperature.

III. EUCLIDEAN ACTION AND FREE ENERGY

We assume the gravitational partition function of the massive AdS BH could be defined by a Euclidean path integral over a dynamical metric (tensor field $g_{\mu\nu}$) as

$$\mathcal{Z} = \int \mathcal{D}[g]e^{-\mathcal{I}_E[g]} \simeq e^{-\mathcal{I}_{\text{on-shell}}} \quad (11)$$

whose most dominant contribution originates from substituting the classical solutions of the action, i.e., the so-called on-shell action, by applying the saddle-point approximation. The on-shell action can be evaluated using the Hawking-Witten prescription (the so-called subtraction method) [37,38], and the divergence in the partition function will be canceled. That leads to $F \equiv \Delta F = \beta^{-1}(\mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}})$ for the free energy difference, in which the zero-point energy of the boundary gauge theory is eliminated due to our renormalization method. We choose the thermal AdS background in massive gravity as the ground state with the period β_0, which is different from the

period of a massive AdS BH, β. Note that this background solution is given by setting $m = 0$ in Eq. (7), which is not pure AdS; it will later become evident that this is the appropriate background. As usual in massive gravity, cosmological and black hole solutions are modified at long distances relative to their counterparts in Einstein gravity. For a sufficiently tiny graviton mass m_g, the black hole solution tends to the Schwarzschild-AdS case at short distances.

In deriving the on-shell action, we have made use of some necessary ingredients, which we briefly explain. The Ricci scalar in the bulk action (1) can be written in terms of the cosmological constant and massive potential terms. Contraction of the field equation (5) yields

$$R = \frac{2}{d_2}[\Lambda d + m_g^2 \mathcal{X}], \qquad (\mathcal{X} \equiv g^{\mu\nu}\mathcal{X}_{\mu\nu}). \qquad (12)$$

The second term of the above equation can be summed with the graviton's interaction Lagrangians, and then recast in a compact form as

$$\frac{2}{d_2}m_g^2 \mathcal{X} + m_g^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i = m_g^2 \sum_{i=1}^{d-2}(i-2)\frac{c_0^i c_i}{r^i}\prod_{j=3}^{i+1} d_j, \qquad (13)$$

in which we have made use of the identity

$$2\prod_{j=3}^{i+1} d_j + \sum_{y=1}^{i}(-1)^y \frac{i!}{(i-y)!}\prod_{j=2}^{i-y+1} d_j = (i-2)\prod_{j=3}^{i+1} d_j, \qquad (14)$$

where $\prod_x^y \ldots = 1$ if $x > y$. As a result, the on-shell action for the massive AdS BH is

$$\mathcal{I}_{\mathrm{BH}} = \frac{\beta\omega_{d_2}^{(k)}}{16\pi G_d}\left[\frac{2r^{d_1}}{\ell^2} - m_g^2 \sum_{i=1}^{d-2}\frac{(i-2)c_0^i c_i r^{d_{i+1}}}{d-i-1}\prod_{j=3}^{i+1} d_j\right]_{r_+}^{R} \qquad (15)$$

where R is an upper cutoff to regularize the on-shell actions. Repeating the same procedure for the thermal AdS background in massive gravity yields

$$\mathcal{I}_{\mathrm{AdS}} = \frac{\beta_0 \omega_{d_2}^{(k)}}{16\pi G_d}\left[\frac{2R^{d_1}}{\ell^2} - m_g^2 \sum_{i=1}^{d-2}\frac{(i-2)c_0^i c_i R^{d_{i+1}}}{d-i-1}\prod_{j=3}^{i+1} d_j\right]. \qquad (16)$$

Fixing the temperature of both the AdS and BH configurations at $r = R$, i.e., $\beta_0 V_0(R)^{1/2} = \beta V(R)^{1/2}$ so that both the AdS and BH spacetimes must have the same geometry at $r = R$ gives $\beta_0 = \beta(1 - \frac{m\ell^2}{2R^{d-1}} + O(R^{-2(d-1)}))$. Subtracting the on-shell action of the AdS background yields

$$\mathcal{I} \equiv \lim_{R\to\infty}(\mathcal{I}_{\mathrm{BH}} - \mathcal{I}_{\mathrm{AdS}})$$
$$= \frac{\beta\omega_{d_2}^{(k)} r_+^{d_3}}{16\pi G_d}\left[k - \frac{r_+^2}{\ell^2} + m_g^2 \sum_{i=1}^{d-2}\left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}}\prod_{j=3}^{i} d_j\right)\right] \qquad (17)$$

where the following identity has been used:

$$\frac{1}{d_2}\prod_{j=2}^{i} d_j + \frac{i-2}{d-i-1}\prod_{j=3}^{i+1} d_j = (i-1)\prod_{j=3}^{i} d_j. \qquad (18)$$

IV. THERMODYNAMICS

Thermodynamic quantities associated with TBH spacetimes can be directly extracted via the partition function, Eqs. (11) and (17). The mean energy of thermal radiation, $\langle E \rangle$, is given by

$$M \equiv \langle E \rangle = -\frac{\partial}{\partial\beta}\ln \mathcal{Z}$$
$$= \frac{d_2\omega_{d_2}^{(k)}}{16\pi}\left[k + \frac{r_+^2}{\ell^2} + m_g^2 \sum_{i=1}^{d-2}\left(\frac{c_0^i c_i}{d_2 r_+^{i-2}}\prod_{j=2}^{i} d_j\right)\right]r_+^{d_3}, \qquad (19)$$

which is in agreement with the ADM mass [39,40] of the BH spacetime (setting $G_d = 1$). Noting that the pressure is $P = -\Lambda/8\pi = d_1 d_2/16\pi\ell^2$ (implying that the BH mass is interpreted as the enthalpy, $M \equiv H$ [41]), we find

$$V = \left(\frac{\partial M}{\partial P}\right)_{X_i} = \frac{\omega_{d_2}^{(k)}}{d_1}r_+^{d_1} \qquad (20)$$

for the thermodynamic volume, where X_i denotes the extensive quantities. The Gibbs free energy, which depends on the quantities T and P, is given by

$$G \equiv \beta^{-1}\ln \mathcal{Z}(\beta, P)$$
$$= \frac{\omega_{d_2}^{(k)} r_+^{d_3}}{16\pi}\left[k - \frac{16\pi P r_+^2}{d_1 d_2} + m_g^2 \sum_{i=1}^{d-2}\left(\frac{(i-1)c_0^i c_i}{r_+^{i-2}}\prod_{j=3}^{i} d_j\right)\right]. \qquad (21)$$

Finally, the entropy of TBHs is calculated as

$$S = \beta M - \mathcal{I} = \frac{\omega_{d_2}^{(k)}}{4}r_+^{d_2}. \qquad (22)$$

These quantities obey the Smarr formula

$$(d-3)M = (d-2)TS - 2PV + \sum_{i=1}^{d-2}(i-2)\mathcal{C}_i c_i, \qquad (23)$$

with $\mathcal{C}_i = \left(\frac{\partial M}{\partial c_i}\right)_{S,P,c_{i\neq j}} = \frac{m_g^2\omega_{d_2}^{(k)}}{16\pi}c_0^i r_+^{d-i-1}\prod_{j=2}^{i} d_j$.

Note that Eq. (23) (which follows from Eulerian scaling [41]) proves that variations in the massive couplings

(c_i) are required for consistency of the first law with the Smarr formula, so massive couplings are not fixed *a priori*. This implies that the thermodynamic quantities (M, T, S, P, V, $\mathcal{C}_i$ and c_i) analytically satisfy the first law of thermodynamics in the enthalpy representation, i.e., $dM = T dS + V dP + \sum_{i=1}^{d-2} \mathcal{C}_i dc_i$.

V. EQUATION OF STATE AND PHASE STRUCTURE

The equation of state is simply obtained from Eq. (10) as

$$P = \frac{d_2 T}{4r_+} - \frac{d_2 d_3 k}{16\pi r_+^2} - \frac{m_g^2}{16\pi} \sum_{i=1}^{d-2} \left(\frac{c_0^i c_i}{r_+^i} \prod_{j=2}^{i+1} d_j \right). \quad (24)$$

The critical point occurs at the spike-like divergence of the specific heat at constant pressure (i.e., an inflection point in the $P - V$ diagrams) and can be found from the relations

$$\left(\frac{\partial P}{\partial v} \right)_T = \left(\frac{\partial^2 P}{\partial v^2} \right)_T = 0 \rightarrow \left(\frac{\partial P}{\partial r_+} \right)_T = \left(\frac{\partial^2 P}{\partial r_+^2} \right)_T = 0,$$
$$(25)$$

which yield

$$2k + m_g^2 \sum_{i=1}^{d-2} \left(\frac{i(i-1) c_0^i c_i}{r_+^{i-2}} \prod_{j=4}^{i+1} d_j \right) = 0, \quad (26)$$

where the specific volume $v = 4 r_+ \ell_P^{d_2}/d_2$ is proportional to r_+ [42,43].

Since the expression on the left-hand side of Eq. (26) is a real and inhomogeneous polynomial equation of degree $(d-4)$ in r_+, an arbitrary number of critical points could be produced by adjusting the spacetime dimension d. Consequently higher-dimensional TBHs within the framework of massive gravity can exhibit as many as $n = (d-4)$ critical points in d dimensions. Note that criticality does not take place in $d = 4$; the inclusion of $U(1)$ charge can yield critical behavior in this case. Interestingly, in $d \geq 7$ dimensions, with up to five graviton self-interaction potentials ($\mathcal{U}_1, \ldots, \mathcal{U}_5$), we have

$$(k + m_g^2 c_0^2 c_2) r_+^3 + 3 d_4 m_g^2 c_0^3 c_3 r_+^2 + 6 d_4 d_5 m_g^2 c_0^4 c_4 r_+$$
$$+ 10 d_4 d_5 d_6 m_g^2 c_0^5 c_5 = 0, \quad (27)$$

which can have three positive roots, indicating the existence of a triple point and an analogue of the solid/liquid/gas phase transition for uncharged-AdS BHs in massive gravity (without loss of generality, we set $c_0 = 1$ hereafter). This situation is similar to that seen in multispinning ($d \geq 6$) Kerr-AdS BHs [44]. In a d-dimensional spacetime, an inhomogeneous polynomial equation of degree $(d-4)$ could at most have $(d-4)$ positive roots. This indicates the possibility of finding more than three different BH phases in massive gravity.
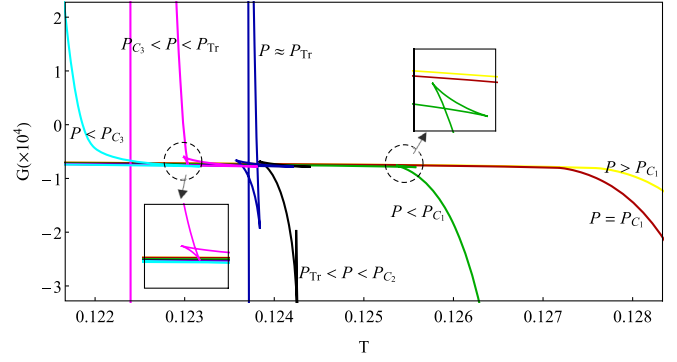

FIG. 1. Analogue of a triple point: $G - T$ diagram for various pressures; we have set $k = 0$, $d = 10$, $m_g = 1$, $c_0 = 1$, $c_1 = 1$, $c_2 = 1.6$, $c_3 = -2.1$, $c_4 = 0.35$, $c_5 = 2.6$, $c_6 = -2.9$, $c_7 = 0$, and $c_8 = 0$. Critical data: ($T_{C_1} = 0.127185$, $P_{C_1} = 0.002482$), ($T_{C_2} = 0.124382$, $P_{C_2} = 0.001624$), and ($T_{\mathrm{Tr}} = 0.123577$, $P_{\mathrm{Tr}} = 0.001532$).

VI. TRIPLE POINT AND SBH/IBH/LBH PHASE TRANSITION

We now consider a ten-dimensional BH spacetime with a flat ($k = 0$) geometry for its event horizon. This leads to a boundary dual gauge theory with a Minkowski metric. To observe the analogue of a triple point, we have tuned the massive couplings to produce three critical points, as shown in Fig. 1, where we depict the $G - T$ diagram corresponding to the SBH/IBH/LBH phase transition that resembles the solid/liquid/gas phase transition in usual substances. The isobar corresponding to $P_{c_2} < P < P_{c_1}$ displays the swallowtail (vdW) behavior which indicates a first-order phase transition. For pressures with $P_{\mathrm{Tr}} < P < P_{c_2}$ we observe two swallowtails, indicating the appearance of two first-order phase transitions, implying three-phase behavior. By decreasing the pressure, the two swallowtails eventually merge and a triple point ($T_{\mathrm{Tr}}, P_{\mathrm{Tr}}$) appears.

VII. N-FOLD RPTs

Next we present an analogue of four critical points in the context of BH thermodynamics. By further tuning the parameters, we show in Fig. 2 that four critical points can be created in the thermodynamic phase space of ten-dimensional planar BHs. As seen in Fig. 2, two distinct reentrant phase transitions take place in the phase space, indicating the appearance of two virtual triple points (referred to as P_{Tr_1} and P_{Tr_2}). In fact, for a fixed pressure in ranges of $P_{Tr_1} < P < P_{Z_1}$ and $P_{Tr_2} < P < P_{Z_2}$, each RPT takes place along a single horizontal transition line in the $P - T$ diagram (Fig. 3) and we may observe multiple phase transitions indicating N-fold reentrant phase transition behavior. In these two distinct regions of phase space, as temperature decreases monotonically, a first-order phase transition is observed, and then, a finite jump (discontinuity) appears in the global minimum of the Gibbs free
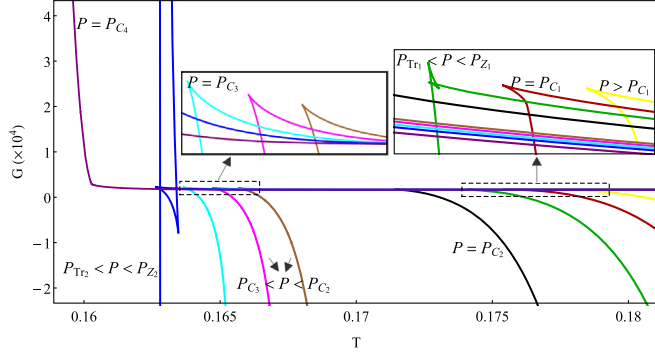
FIG. 2.   Twofold RPTs: $G - T$ diagram for various pressures; we have set $k = 0, d = 10, m_g = 1, c_0 = 1, c_1 = 1, c_2 = 3, c_3 = -4.3$, $c_4 = 2.6$, $c_5 = 1.2$, $c_6 = -3.5$, $c_7 = 2.6$, and $c_8 = 0$. Critical data: $(T_{C_1} = 0.175778$, $P_{C_1} = 0.007513)$, $(T_{C_2} = 0.172218$, $P_{C_2} = 0.005847)$, $(T_{C_3} = 0.1653102$, $P_{C_3} = 0.003276)$, $(T_{C_4} = 0.160285$, $P_{C_4} = 0.002343)$, $(T_{Tr_1} = 0.174034$, $P_{Tr_1} = 0.006794)$, $(T_{Z_1} = 0.174203$, $P_{Z_1} = 0.006882)$, $(T_{Tr_2} = 0.162519$, $P_{Tr_2} = 0.0029496)$, and $(T_{Z_2} = 0.162572$, $P_{Z_2} = 0.002965)$.

energy, which displays the zeroth-order phase transition. Then we observe a twofold reentrant phenomenon, which in some aspects is reminiscent of the sequence of reentrant phase transitions in liquid crystals [29–31]. But an important difference remains here. In the thermodynamic phase space of the obtained TBHs, two RPTs are observed in distinct regions of phase space, i.e., the first RPT occurs for a given pressure in $P_{Tr_1} < P < P_{Z_1}$ as temperature is lowered monotonically in the region $T_{Tr_1} < T < T_{Z_1}$, and the second RPT occurs in the same way with temperature in the region $T_{Tr_2} < T < T_{Z_2}$ and a given pressure in $P_{Tr_2} < P < P_{Z_2}$. But, in liquid crystals and multicomponent fluids, multiple RPTs (especially the double
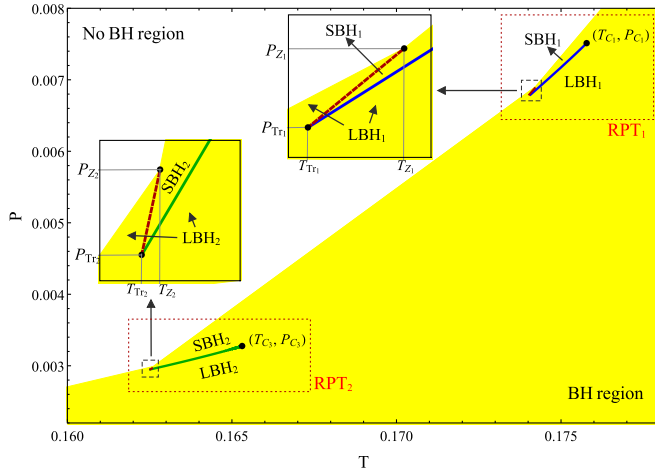

FIG. 3.   Phase diagram for twofold RPTs: The corresponding coexistence lines of twofold RPTs, presented in Fig. 2, in the $P - T$ diagram. Dashed and solid lines represent zeroth- and first-order phase transitions, respectively. Qualitatively, this behavior is generic for any twofold RPTs.

reentrance sequence $A \to B \to A \to B$ [25–28]) take place as the temperature is lowered with other thermodynamic quantities (such as pressure) held fixed. This situation is more similar to that seen in $(d = 7)$ Lovelock-AdS BHs with hyperbolic horizon topology [24], in which a double RPT (LBH → SBH → LBH → SBH) has been observed solely by lowering the temperature. These remarks indicate the importance of studying the critical behavior of TBHs in the other gravitational alternatives such as Lovelock gravity.

Summarizing, we have demonstrated the existence of N-fold RPTs within massive gravity. These depend on the number of inflection points. Our investigations show that the existence of N-fold RPTs with corresponding (N) virtual triple points is a generic feature of all types of TBHs in higher-dimensional massive gravity, which is a step forward in understanding BH phase transitions via modified gravity. Perhaps these new kinds of N-fold RPTs are present in many-body systems; they certainly merit further exploration.

VIII. CLOSING REMARKS

Since condensed matter systems usually break translational invariance in nature, we have considered holographic duals of such systems with homogeneity and isotropy properties using the language of dRGT massive gravity with a singular, degenerate reference metric. We obtained TBH solutions that are free from pathological behavior and dual to matter with broken translational symmetry property; so, in principle, they can be dual to liquid crystals [33,45,46]. These TBHs yield up to $n = (d - 4)$ critical points in d dimensions. Consequently, a number of new interesting phenomena emerge, in particular N-fold reentrant phase transitions, indicating multiple phases in $d \geq 7$ dimensions. Moreover, in the grand canonical ensemble of charged-AdS BHs, this behavior persists and holds in higher dimensions as well. Since (multiple) RPTs are typical of liquid crystals, elementally, holographic models for them using BHC could probably simulate realistic critical behavior and perhaps be implemented to predict new features of criticality, such as N-fold RPTs, in nature. No proposal has yet been provided to experimentally verify TBH phase transitions, but it is conceivable that analogue gravity simulations [47] may one day get to this point.

From the molecular point of view, (multiple) RPTs appear in compounds (especially in liquid crystals) and only approximate qualitative explanations exist for them. Regarding this, we believe that the analytic TBH equations of state (24) in $d \geq 8$ and their counterparts in other gravitational alternatives such as Lovelock gravity [24] could possibly shed some light on the microscopic structure of multiple RPTs (or perhaps N-fold RPTs if they exist) in liquid crystals and multicomponent fluids [29–31] which

merits further investigation in the future. Furthermore, we expect that the exact massive TBHs with a range of novel phase transitions constructed here have potential applications in the context of the AdS/CFT correspondence; in particular, a holographic interpretation of them remains an open question.

Finally, we note that our considerations are valid for all types of TBHs, which can be manifest by introducing the effective topological factor $k_{\text{eff}} = (k + m_g^2 c_0^2 c_2)$ appearing in Eqs. (7), (10), (21), (24), and (26). The only necessity is that the same value must be provided for the combination k_{eff} (by varying the massive constant c_2) while keeping other parameters fixed. As a result, the same critical points with the same critical behavior are found for the cases of the spherical, planar, and hyperbolic BHs.

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