Pseudoinvariance and the extra degree of freedom in $f(T)$ gravity

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Nonlinear generalizations of teleparallel gravity entail the modification of a Lagrangian that is pseudoinvariant under local Lorentz transformations of the tetrad field. This procedure consequently leads to the loss of the local pseudoinvariance and the appearance of additional degrees of freedom (d.o.f.). The constraint structure of $f(T)$ gravity suggests the existence of one extra d.o.f. when compared with general relativity, which should describe some aspect of the orientation of the tetrad. The purpose of this article is to better understand the nature of this extra d.o.f. by means of a toy model that mimics essential features of $f(T)$ gravity. We find that the nonlinear modification of a Lagrangian L possessing a local rotational pseudoinvariance produces two types of solutions. In one case the original gauge-invariant variables—the analogue of the metric in teleparallelism—evolve like when governed by the (nondeformed) Lagrangian L; these solutions are characterized by a (selectable) constant value of its Lagrangian, which is the manifestation of the extra d.o.f. In the other case, the solutions do contain new dynamics for the original gauge-invariant variables, but the extra d.o.f. does not materialize because the Lagrangian remains invariant on-shell. Coming back to $f(T)$ gravity, the first case includes solutions where the torsion scalar T is a constant, to be chosen at the initial conditions (extra d.o.f.), and no new dynamics for the metric is expected. The latter case covers those solutions displaying a genuine modified gravity; T is not a constant, but it is (on-shell) invariant under Lorentz transformations depending only on time. Both kinds of $f(T)$ solutions are exemplified in a flat Friedmann-Lemaître-Robertson-Walker universe. Finally, we present a toy model for a higher-order Lagrangian with rotational invariance [analogous to $f(R)$ gravity] and derive its constraint structure and number of d.o.f.

I. INTRODUCTION

The common notion that gravity can only be represented through the curvature of spacetime has being challenged by at least two different approaches, where either the torsion or the nonmetricity provide physically and mathematically equivalent versions of general relativity (GR). These two theories correspond to the teleparallel equivalent of general relativity (TEGR) [1] and the symmetric teleparallel equivalent of general relativity (STEGR) [2,3], and their dynamical variables are the torsion tensor and the nonmetricity tensor, respectively. The description of general relativity in terms of curvature, torsion and nonmetricity has incidentally being called the "geometrical trinity of gravity" [4,5], and it consists in an intriguing starting point to formulate extensions of Einstein's gravity. The TEGR as a starting point for building extensions to general relativity

has gained wide attention in the recent years, particularly for its versatility to predict novel consequences in the realm of cosmology, giving rise to the $f(T)$ gravity paradigm [6,7], where T is the torsion scalar. Equivalently, in STEGR the nonmetricity scalar Q is used, giving rise to the very recent $f(Q)$ theories of gravity [8].

Recent interest has emerged for understanding the issue of the number and nature of the degrees of freedom in modified gravity theories based on a teleparallel framework. Some early attempts to understand $f(T)$ gravity as TEGR plus a minimally coupled scalar field through conformal transformations were documented in Refs. [9,10], where it was shown that it is not possible to cleanly obtain a teleparallel Einstein frame, due to the appearance of Lorentz-breaking terms. However, later it was shown through a full Hamiltonian analysis, that $f(T)$ gravity has a unique extra degree of freedom (d.o.f.) [11], which consequently cannot be attributed to a conformal field redefinition of the theory. In this regard, recently disformal transformations were studied in order to obtain a clean

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isolation of such an extra d.o.f., but these efforts have been unsuccessful [12]. Other attempts to understand the issue of the d.o.f. worth considering in this discussion are the studies of the linearized approximation around Minkowski spacetime [13–15], which do not show the extra d.o.f. Also, propagating modes do not appear in linear cosmological perturbations around a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe [16–22]. In the light of the perturbative analysis, there are concerns in the community regarding the pathological behavior and strong coupling problem in $f(T)$ gravity [18,19,23–25]. The disappearance of degrees of freedom at the perturbative level is a behavior shared with other modified gravitational theories such as massive, bimetric and Hořava gravities. Nonetheless, an important distinction between these theories and $f(T)$ gravity is that the former use the metric as the dynamical field; in contrast $f(T)$ gravity is a tetrad-based physical theory. The extra d.o.f. can be roughly interpreted as a scalar field that has a role in selecting preferred reference frames that are solutions of the equations of motion [26], exhibiting in this way the loss of local Lorentz invariance (LLI). So, it is still unclear if it should dynamically manifest at the perturbative level, putting in doubt concerns about the strong coupling problem.

Another road to understanding the important matter of the lack of LLI in these theories comes from the analysis of pseudoinvariance in TEGR. It is widely known that TEGR is a pseudoinvariant (also called *quasi*-invariant [27]) theory under local Lorentz transformations (LLT) on the tetrad field. This means that the TEGR Lagrangian changes by a boundary term under LLT, or in other words, the difference between the Ricci scalar R from GR and the torsion scalar T in TEGR is a boundary term. Therefore, in the nonlinear modification of the TEGR Lagrangian, we cannot integrate out this boundary term, giving rise to the modification of a pseudoinvariant system. Boundary terms are very common in GR, such as topological invariants that are nontrivial in higher dimensions or the Gibbons-Hawking-York term,[1] but the nonlinear modifications of these terms are not commonly used for model building. In this regard, we have a very unique case of modified pseudoinvariance in modifications to gravity based on the teleparallel formalism. Our aim is to analyze the properties of pseudoinvariant systems and their nonlinear modifications through toy models, which will be very helpful to understand the disappearance of the extra d.o.f. in $f(T)$ gravity for some solutions and its general behavior.

This work is organized as follows. In Sec. II we introduce the basic concepts and definitions of teleparallel and modified teleparallel gravity. In Sec. III we present the

Hamiltonian analysis of a toy model with rotational pseudoinvariance, and the analysis of the nonlinear modification of it. We compare the outcome and generic features of the toy model with the $f(T)$ gravity case in Sec. IV, and classify a couple of qualitatively different cosmological backgrounds. In Sec. V we display a different toy model that shares some features with $f(R)$ gravity. Section VI is devoted to the conclusions.

II. TELEPARALLEL AND MODIFIED TELEPARALLEL GRAVITY

A. Teleparallel geometry

We begin by introducing the basic notation and main expressions for understanding the teleparallel formalism. Let us consider a manifold $\mathcal{M}$, a basis $\{\mathbf{e}_a\}$ in the tangent space $T_p(\mathcal{M})$, and the dual basis $\{\mathbf{E}^a\}$ in the cotangent space $T_p^*(\mathcal{M})$. This pair of basis/cobasis accomplishes $\mathbf{E}^a(\mathbf{e}_b) = \delta_b^a$. When expanded in a coordinate basis as $\mathbf{e}_a = e_a^\mu \partial_\mu$ and $\mathbf{E}^a = E_\mu^a dx^\mu$, the duality relationship looks like

$$E_\mu^a e_b^\mu = \delta_b^a, \qquad e_a^\mu E_\nu^a = \delta_\nu^\mu. \tag{1}$$

Our notation is such that greek letters $\mu, \nu, \ldots = 0, \ldots, n-1$ represent spacetime coordinate indices, and latin letters $a, b, \ldots, g, h = 0, \ldots, n-1$ are for Lorentzian tangent space indices. A *vielbein* (vierbein or tetrad in $n = 4$ dimensions) is a basis that encodes the metric structure of the spacetime through the expression

$$\mathbf{g} = \eta_{ab}\mathbf{E}^a \otimes \mathbf{E}^b \tag{2}$$

[$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski symbol]. This allows to write

$$\mathbf{E}^a \cdot \mathbf{E}^b = \mathbf{g}(\mathbf{E}^a, \mathbf{E}^b) = \eta_{ab}, \tag{3}$$

which indicates that the vielbein is an orthonormal basis. In component notation, the former expressions are written as

$$g_{\mu\nu} = \eta_{ab}E_\mu^a E_\nu^b, \qquad \eta_{ab} = g_{\mu\nu}e_a^\mu e_b^\nu, \tag{4}$$

from which the relation between the metric volume and the determinant of the matrix E_μ^a can be derived, giving

$$\sqrt{|g|} = \det[E_\mu^a] \doteq E. \tag{5}$$

TEGR comes from the formulation of a dynamical theory of spacetime geometry for the vielbein field, encoding the metric structure of spacetime. The Lagrangian density for TEGR is

$$L = ET, \tag{6}$$

where T is the torsion scalar or Weitzenböck invariant,

$$T \doteq T^\rho_{\ \mu\nu}S_\rho^{\ \mu\nu}, \tag{7}$$

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[1]It has been claimed that the surface term from TEGR has the same contribution, once varied, as the Gibbons-Hawking-York term, erasing in the same way the unwanted contributions to the Einstein equations of motion when spacetime boundaries are considered [28].

which is made up of the *torsion tensor*

$$T^\mu{}_{\nu\rho} \doteq e_a{}^\mu(\partial_\nu E^a_\rho - \partial_\rho E^a_\nu),\qquad(8)$$

and the so-called *superpotential*

$$S_\rho{}^{\mu\nu} \doteq \frac{1}{2}(K^{\mu\nu}{}_\rho + T^\mu\delta^\nu_\rho - T^\nu\delta^\mu_\rho),\qquad(9)$$

where $T^\mu \doteq T_\lambda{}^{\lambda\mu}$ is the *torsion vector*. In the latter, we define the contortion tensor as

$$K^{\mu\nu}{}_\rho \doteq \frac{1}{2}(T_\rho{}^{\mu\nu} - T^{\mu\nu}{}_\rho + T^{\nu\mu}{}_\rho),\qquad(10)$$

which is the difference between the Levi-Civita connection and a general connection. The field strength (8) is the torsion associated with the Weitzenböck connection $\Gamma^\mu_{\nu\rho} \doteq e_a{}^\mu\partial_\nu E^a_\rho$. The Weitzenböck connection is the simplest choice that cancels out the Riemann tensor, rendering a curvatureless spacetime where the parallel transport does not depend on the path: it is absolute. However, other choices for the connection are possible. A modern summary and criticism of these approaches can be found in Ref. [29]. The equations of motion for the Lagrangian (6) are obtained by varying L with respect to the tetrad field; they are

$$4e\partial_\mu(Ee_a^\lambda S_\lambda{}^{\mu\nu}) + 4e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu} - e_a^\nu T = -2\kappa e_a^\lambda \mathcal{T}_\lambda{}^\nu,\quad(11)$$

where $\mathcal{T}_\lambda{}^\nu$ is the energy-momentum tensor coming from a matter field. Equation (11) can be proved to be equivalent to the Einstein equations when written in terms of the metric tensor. TEGR is equivalent to GR not only in this sense, but also at the level of the Lagrangians. This is because the torsion scalar T and the Levi-Civita scalar curvature R are related by a boundary term

$$R = -T + 2e\partial_\mu(ET^\mu),\qquad(12)$$

which is integrated out once in the action, yielding the equivalence between the TEGR and GR Lagrangians.

B. Modified teleparallel gravity

If our starting point to describe the gravitational interactions is the TEGR Lagrangian, then the simplest way to a theory of modified gravity is to replace the TEGR Lagrangian by a nonlinear function of it, in the same way that $f(R)$ gravity is the simplest generalization of GR. If we try to deform gravity in this way, we can define the following action:

$$S = \frac{1}{2\kappa}\int d^4x\, E(f(T) + L_m[E^a_\mu]),\qquad(13)$$

where L_m is a Lagrangian for matter. The dynamical equations of motion of this action are found by varying in terms of the tetrad field. It is obtained that

$$4e\partial_\mu(f'(T)Ee_a^\lambda S_\lambda{}^{\mu\nu}) + 4f'(T)e_a^\lambda T^\sigma{}_{\mu\lambda}S_\sigma{}^{\mu\nu} - e_a^\nu f(T)$$
$$= -2\kappa e_a^\lambda \mathcal{T}_\lambda{}^\nu.\qquad(14)$$

The equations of motion (14) possess an unusual feature: while they are invariant under *global* Lorentz transformations of the tetrad field, they are sensitive to the *local* orientation of the tetrad. This means that they endow the spacetime with preferred parallelizations, which relate each other through a subset of LLT [30]. The breakdown of the LLI is irrelevant for the metric, since the components of the metric tensor are not affected by either global or local Lorentz transformations of the tetrad field. Then this loss of LLI is not a proper Lorentz violation in the sense of other explicitly Lorentz-breaking gravitational theories, but implies the existence of an extra degree of freedom [11] that could be only detected through interactions of matter with the tetrad field instead of the metric.

The growing interest in $f(T)$ gravity mainly lies in its success in the cosmological arena. In fact, a Born-Infeld-like $f(T)$ is able to smooth spacetime singularities, leading to a maximum attainable Hubble factor in the early Universe, and so driving an inflationary epoch without the need of an inflaton field [6]. At the far end, the theory can explain the accelerated expansion of the Universe by means of a power law in the torsion scalar. In this work we are interested in understanding how the extra degree of freedom of $f(T)$ gravity manifests itself in simple flat FLRW cosmological backgrounds; we present a couple of solutions of this kind in what comes next.

C. Branching of cosmological solutions

Recently it has been noticed that two different types of solutions can be obtained when using $f(T)$ gravity in the context of flat FLRW geometries, which present qualitatively different values for the torsion scalar. On the one hand, the simplest and best-known solution is [6,7]

$$\mathbf{E}^0 = \mathbf{dt},\quad \mathbf{E}^1 = a(t)\mathbf{dx},\quad \mathbf{E}^2 = a(t)\mathbf{dy},\quad \mathbf{E}^3 = a(t)\mathbf{dz},\qquad(15)$$

which easily proves to be a solution of the system of equations (14). The torsion scalar for this solution is

$$T = -6H^2 = -6\left(\frac{\dot{a}}{a}\right)^2,\qquad(16)$$

and the scale factor $a(t)$ satisfies the dynamical equations[2] coming from replacing Eq. (15) in Eq. (14), giving

$$-2T^{\frac{3}{2}}\frac{d}{dT}(T^{-\frac{1}{2}}f(T))|_{T=-6H^2} = 2\kappa\rho,$$
$$-8\dot{H}T^{\frac{1}{2}}\frac{d}{dT}\left(T^{\frac{1}{2}}\frac{df}{dT}\right)\Big|_{T=-6H^2} = 2\kappa(\rho+p).\qquad(17)$$

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[2]Incidentally, notice that these equations are invariant under the change $f(T) \to f(T) + A\sqrt{T}$.

The dynamics of the scale factor $a(t)$ is subject to the choice of the function f; therefore this is the way the metric behavior departs from general relativity.

On the other hand the flat FLRW geometry also allows for a family of solutions that reads

$$\mathbf{E}^0 = \cosh \lambda \mathbf{dt} + a(t) \sinh \lambda \mathbf{dr},$$

$$\mathbf{E}^1 = \sinh \lambda \mathbf{dt} + a(t) \cosh \lambda \mathbf{dr}, \qquad \mathbf{E}^2 = a(t) r \mathbf{d}\theta,$$

$$\mathbf{E}^3 = a(t) r \sin \theta \mathbf{d}\varphi, \tag{18}$$

where

$$\lambda(t, r) = \psi(ra(t)) + \frac{t}{2ra(t)} - \frac{ra(t)}{4} \int (T_o + 6H^2) dt, \tag{19}$$

where T_o is a constant, and ψ is an arbitrary function of the radial distance $ra(t)$. In this case, the torsion scalar is constant,

$$T = T_o, \tag{20}$$

and the scale factor $a(t)$ satisfies the dynamical equations

$$6H^2 - T_o + \frac{f(T_o)}{f'(T_o)} = \frac{2\kappa}{f'(T_o)}\rho, \quad -4\dot{H} = \frac{2\kappa}{f'(T_o)}(\rho + p), \tag{21}$$

which are nothing but the equations of general relativity for a cosmological constant $\Lambda = (T_o - f(T_o)/f'(T_o))/2$ and an effective Newton constant $\hat{G} = G/f'(T_o)$. Then, this other type of solution comes with an integration constant T_o—it appears in the radial boost governed by the function λ—that affects the effective values of the fundamental constants of the cosmology. This fact was first reported in Ref. [31] for a vanishing value of the torsion scalar, through the null tetrad approach developed in Ref. [32].

We will employ a mechanical toy model to explain why the solutions of the original (GR) theory actually coexist with the expected new solutions of the modified $f(T)$ theory. We are interested in knowing how many degrees of freedom are involved in each case, and which is the remnant gauge freedom kept by the tetrad.

III. MODIFYING A MECHANICAL SYSTEM WITH ROTATIONAL PSEUDOINVARIANCE

A. Counting degrees of freedom in constrained Hamiltonian systems

We will summarize Dirac's procedure for constrained Hamiltonian systems [33–35] for later use in a couple of toy models. We consider a Lagrangian $L = L(q^k, \dot{q}^k)$ such that the equations defining the canonical momenta

$p_k = \partial L/\partial \dot{q}^k$ cannot be unambiguously solved for all the velocities. If so, the momenta are not independent but there exist some relations among the p_k's and q^k's,

$$\phi_\rho(q^k, p_k) = 0, \qquad \rho = 1, ..., P, \tag{22}$$

which will be called *primary constraints*.[3] The constraints (22) define a subspace Γ_p of the phase space—the *constraint surface*—where the dynamics of the system will remain confined. The *primary Hamiltonian*

$$H_p \equiv H_c + u^\rho \phi_\rho, \tag{23}$$

is the sum of the canonical Hamiltonian $H_c = \dot{q}^k p_k - L(q^k, \dot{q}^k)$ and a linear combination of the primary constraints. The Lagrange multipliers $u^\rho(t)$ are free functions that can be varied independently to ensure the primary constraints. They leave H_p with a degree of ambiguity that comes from the fact that the velocities cannot be uniquely solved in terms of the canonical momenta.

The condition that the primary constraints be preserved over time leads to the following system of equations:

$$\dot{\phi}_\sigma = \{\phi_\sigma, H_p\} = \{\phi_\sigma, H_c\} + \{\phi_\sigma, \phi_\rho\}$$

$$u^\rho \equiv h_\sigma + C_{\sigma\rho} u^\rho \overset{!}{\approx} 0, \tag{24}$$

where ≈ 0 means *weakly* zero (i.e., "zero on the constraint surface"); h_σ and $C_{\sigma\rho}$ are implicitly defined. These *consistency* equations could be accomplished by solving them for the functions u^ρ. However if $\det C_{\sigma\rho} \approx 0$ and $h_\sigma \not\approx 0$, the consistency equations cannot be entirely solved for the functions u^ρ. In such a case, *secondary* constraints will be needed to ensure that the primary constraints remain weakly zero while the system evolves.[4] Thus, the procedure should be iterated for the consistency of the secondary constraints, which could lead to more secondary constraints. The algorithm finishes when the set of primary and secondary constraints,

$$\phi_\rho \approx 0, \qquad \rho = 1, ..., P,$$

$$\phi_{\bar{\rho}} \approx 0, \qquad \bar{\rho} = P + 1, ..., P + S, \tag{25}$$

can be forced to consistently evolve by merely fixing some of the Lagrange multipliers u^ρ. We can wonder how many Lagrange multipliers will be fixed, since some of the consistency equations could be automatically satisfied without imposing any condition on the Lagrange multipliers. For simplicity let us call $\phi_{\hat{\rho}}$, $\hat{\rho} = 1, ..., P + S$, the

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[3]We will assume that the $\phi_\rho(q, p)$'s are independent functions.
[4]Secondary constraints will appear each time that $w_a^\sigma h_\sigma \neq 0$, where w_a^σ is a null eigenvector of the $P \times P$ matrix $C_{\sigma\rho}$ ($w_a^\sigma C_{\sigma\rho} \approx 0$).

complete set of independent constraints defining the constraint surface Γ. The consistency equations are

$$\dot{\phi}_{\hat{\rho}} = h_{\hat{\rho}} + C_{\hat{\rho}\rho}u^{\rho} \approx 0. \qquad (26)$$

If the rank of the $S \times P$ matrix $C_{\hat{\rho}\rho}$ is $K < P$, then there will be $P - K$ *right* null eigenvectors V_a^{ρ},

$$C_{\hat{\rho}\rho}V_a^{\rho} \approx 0, \quad a = 1, \ldots, P - K. \qquad (27)$$

Therefore the replacement $u^{\rho} \to u^{\rho} + v^a V_a^{\rho}$, with arbitrary functions $v^a(t)$, will not alter the equation (26). As a consequence, whenever the rank of $C_{\hat{\rho}\rho}$ is less than P then it will remain an undetermined sector in the primary Hamiltonian (23) associated with the constraints

$$\phi_a \equiv V_a^{\rho}\phi_{\rho} \approx 0. \qquad (28)$$

Let us call *first class* any phase space function $F(q, p)$ having weakly vanishing Poisson brackets with all the constraints $\phi_{\hat{\rho}}$; otherwise it will be *second class*. Remarkably, the constraints ϕ_a are first class.[5] Also H_p is first class due to the consistency relations. The constraints $\phi_{\hat{\rho}}$ can be linearly combined to get a maximum number of independent first-class constraints "$\gamma_{\tilde{A}}$." A set of second-class constraints "χ_A" will complete the set of $P + S$ constraints characterizing the constraint surface Γ. Since both $C_{a\rho}$ and $C_{\tilde{A}\rho}$ are weakly zero, the consistency equations for all the first-class constraints imply nothing for the u^{ρ}'s. So, let us pay attention to the consistency equations for the second-class constraints. We notice that the square matrix $\Delta_{AB} = \{\chi_A, \chi_B\}$ must be invertible; otherwise, there would still be first-class constraints among the χ_A's. Since the determinant of the antisymmetric matrix Δ_{AB} is different from zero, we also conclude that the number of second-class constraints is even. Let us check the consistency of the second-class constraints and the consequences for the Lagrange multipliers; we start from

$$\dot{\chi}_A = \{\chi_A, H_p\} \approx h_A + u^{\rho}\{\chi_A, \chi_{\rho}\} = h_A + u^{\rho}\Delta_{A\rho} \approx 0. \qquad (29)$$

Then, by multiplying with Δ^{BA}

$$0 \approx \Delta^{BA}h_A + u^{\rho}\delta_{\rho}^{B}. \qquad (30)$$

Therefore, if the index B alludes to a secondary constraint it is

$$0 \approx \Delta^{BA}h_A, \qquad (31)$$

which should already be a secondary constraint, since we have assumed that the algorithm is finished (all the secondary constraints have been found). On the other hand, if the index B alludes to a primary constraint it is

$$u^{\rho} = -\Delta^{\rho A}h_A. \qquad (32)$$

These two results imply that the primary Hamiltonian can be written as[6]

$$H_p = H_c + v^a\phi_a + h_A\Delta^{AB}\chi_B. \qquad (33)$$

The ambiguity associated with the free functions $v(t)$ implies that only first-class phase-space functions will unambiguously evolve. For any other phase space the evolution will be determined modulo *gauge transformations* generated by the ϕ_a's. Dirac conjectured that not only the primary first-class constraints but all the $\gamma_{\tilde{A}}$'s generate gauge transformations. Because of this reason it is a common practice to use instead the *extended* Hamiltonian

$$H_E = H_c + v^{\tilde{A}}\gamma_{\tilde{A}} + h_A\Delta^{AB}\chi_B \qquad (34)$$

without damaging the evolution of the first-class phase-space functions.

The gauge freedom involved in H_E can be fully frozen by accompanying the $\gamma_{\tilde{A}}$'s with an equal number of independent *gauge-fixing conditions* $\xi_{\tilde{A}}(q, p) \approx 0$.[7] If the gauge-fixing conditions fulfill $\det\{\gamma_{\tilde{A}}, \xi_{\tilde{B}}\} \not\approx 0$, then the $v^{\tilde{A}}$'s will be completely fixed by the requirement that the gauge-fixing conditions must be consistent with the evolution of the system. Actually $\det\{\gamma_{\tilde{A}}, \xi_{\tilde{B}}\} \not\approx 0$ means that the $\gamma_{\tilde{A}}$'s and the $\xi_{\tilde{B}}$'s form a second-class set. In fact no first-class constraint remains since the gauge freedom has been completely frozen. Not only the gauge-invariant functions—the *observables*—but any phase-space function will so evolve without ambiguities. Thus the phase space is restricted by the set of conditions $\gamma_{\tilde{A}} \approx 0$, $\xi_{\tilde{A}} \approx 0$, $\chi_A \approx 0$. Each pair of conditions eliminates one degree of freedom. Therefore, the d.o.f. are counted by considering the number of pairs of canonical variables (q^n, p_n) and the number of first-class (f.c.) and second-class (s.c.) constraints through the following formula:

number of d.o.f. = number of (p, q)

$$- \text{number of } f.c. \text{ constraints}$$

$$- \frac{1}{2} \text{ number of } s.c. \text{ constraints.} \qquad (35)$$

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[5]$\{\phi_{\hat{\rho}}, \phi_a\} \approx \{\phi_{\hat{\rho}}, \phi_{\rho}\}V_a^{\rho} = C_{\hat{\rho}\rho}V_a^{\rho} \approx 0$. The ϕ_a's are a complete set of first-class primary constraints, since no linearly independent solutions to the former equation are left on Γ.

[6]This Hamiltonian is usually called the *total* Hamiltonian, since it recognizes the ambiguity associated with the functions ϕ_a.

[7]The conditions $\xi_{\tilde{A}}(q, p) \approx 0$ must be attainable by means of gauge transformations generated by the $\gamma_{\tilde{A}}$'s.

We will make extensive use of this algorithm in the following subsections.

B. Rotationally pseudoinvariant Lagrangian

We will propose a toy model that mimics some general features of TEGR theory, so later we can study its modification, which will possess several features also present in $f(T)$ gravity. Let us study the following mechanical Lagrangian[8]:

$$L = 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 - U(z\bar{z}) + \dot{z}\frac{\partial}{\partial z}g(z,\bar{z}) + \dot{\bar{z}}\frac{\partial}{\partial \bar{z}}g(z,\bar{z}),$$

(36)

where z, $\bar{z}$ are complex-conjugate canonical variables. As can be seen, L is a Lagrangian governing the evolution of a sole dynamical variable: $z\bar{z}$. In fact the last two terms are just the total derivative $dg(z,\bar{z})/dt$ and they do not influence the Lagrange equations. Besides, the first term is a kinetic energy for $\sqrt{z\bar{z}}$,

$$2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 = \frac{1}{2z\bar{z}}(\bar{z}\dot{z}+z\dot{\bar{z}})^2,$$

and U is a potential for $z\bar{z}$. This means that the Lagrange equations will govern the evolution of the modulus of the complex variable z, but the evolution of its phase $z/|z|$ will remain undetermined. We can notice this fact also at the level of the symmetries of the Lagrangian, which is *pseudoinvariant* under ("local") time-dependent rotations (it is invariant except for a total derivative):

$$z \to e^{i\alpha(t)}z, \qquad \bar{z} \to e^{-i\alpha(t)}\bar{z} \Rightarrow \delta L = \frac{d}{dt}\delta g(z,\bar{z}).$$

(37)

We can recognize some features that resemble the TEGR theory. In fact, the TEGR Lagrangian is pseudoinvariant under LLT of the tetrad, so it only governs the dynamics of the metric, but it is unable to determine the "orientation" of the tetrad. The analogy is not complete because the boundary term in this toy model just contains first-order derivatives of the canonical variables, differing from the case of TEGR in which the boundary term contains second-order derivatives of the tetrad.

Now let us pass to the Hamiltonian formalism, and look for the constraint algebra. The canonical momenta are defined as

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[8]This model and some of the conclusions drawn here were first presented in Ref. [36]. However, explicit calculations are given in the present article.

$$p_z \equiv \frac{\partial L}{\partial \dot{z}} = \frac{1}{z}\frac{d}{dt}(z\bar{z}) + \frac{\partial}{\partial z}g(z,\bar{z}),$$

$$p_{\bar{z}} \equiv \frac{\partial L}{\partial \dot{\bar{z}}} = \frac{1}{\bar{z}}\frac{d}{dt}(z\bar{z}) + \frac{\partial}{\partial \bar{z}}g(z,\bar{z}),$$

(38)

from which it is easily seen that the *primary* constraint is

$$G^{(1)} \equiv z\left(p_z - \frac{\partial g}{\partial z}\right) - \bar{z}\left(p_{\bar{z}} - \frac{\partial g}{\partial \bar{z}}\right) \approx 0,$$

(39)

which fulfills

$$\{G^{(1)}, z\bar{z}\} = 0.$$

(40)

In Eq. (39) one recognizes the form of the angular momentum, so $G^{(1)}$ is the generator of rotations. Equation (40) then says that $z\bar{z}$ is invariant under rotations. As it happens in any theory having invariance under rotations, the angular momentum is conserved; however the conservation here is not the result of the dynamical equations but it appears in the form of a constraint among the canonical variables. This means that the (conserved) value of the angular momentum cannot be freely chosen by manipulating the initial conditions; instead the initial conditions are restricted to satisfy the sole allowed value $G^{(1)} = 0$. The reason why the angular momentum behaves in such way is because not only is the Lagrangian (pseudo) invariant under rotations, like the Lagrangian of a particle in a central potential, but its very dynamical variable $z\bar{z}$ is already invariant under (even local) rotations. These are the features characterizing the so-called gauge systems, i.e., those systems whose Lagrangians do not give dynamics to each canonical variable, but only govern some combinations of variables, which can be recognized through their invariance under (local) *gauge* transformations. Noticeably, in the case under study, the angular momentum $G^{(1)}$ has contributions coming from those terms in L that are linear in $\dot{z}, \dot{\bar{z}}$ (the terms we added to L to make it pseudoinvariant). As we know, these extra terms do not affect the fulfillment of the Lorentz algebra in theories of gravity such as TEGR [37]; however they are essential to establish the number of degrees of freedom of the "deformed" theories, as we are going to see in the next subsection.

The canonical Hamiltonian is

$$H = \dot{z}p_z + \dot{\bar{z}}p_{\bar{z}} - L = 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 + U(z\bar{z})$$

$$= \frac{1}{8z\bar{z}}\left[z\left(p_z - \frac{\partial g}{\partial z}\right) + \bar{z}\left(p_{\bar{z}} - \frac{\partial g}{\partial \bar{z}}\right)\right]^2 + U(z\bar{z}),$$

(41)

while the *primary* Hamiltonian is

$$H_p = H + u(t)G^{(1)},$$

(42)

TABLE I. Comparison between the rotationally pseudoinvariant toy model and TEGR.

	Toy model	TEGR
Coordinates	$z, \bar{z}$	E^a_μ
Gauge parameter	$\alpha(t)$	$\Lambda^a_b(x)$
Gauge symmetry	Rotations	Lorentz transformations
Primary constraint(s)	$G^{(1)}$	$G^{(1)}_{ab}$
Degrees of freedom	1	2
Observable	$|z|$	Two polarizations of $g_{\mu\nu}$

where $u(t)$ is a Lagrange multiplier. The presence of the last term has a twofold meaning. On the one hand it means that the form of the Hamiltonian is ambiguous on the constraint surface, since one can rewrite some of the canonical variables by using the constraint. On the other hand it implies that only the quantities $\mathcal{O}(z, \bar{z}, p_z, p_{\bar{z}})$ having rotational invariance (i.e., $\{G^{(1)}, \mathcal{O}\} \approx 0$) will unambiguously evolve. The other (non-gauge-invariant) quantities are not *observables*; their evolution will remain ambiguous as long as the function $u(t)$ remains unknown.

For consistency reasons, $G^{(1)}$ should evolve without leaving the constraint surface; i.e., $\dot{G}^{(1)} = \{G^{(1)}, H_p\} \approx 0$. If this condition were not fulfilled, then one would impose new (*secondary*) constraints to have a consistent evolution. Since $\{G^{(1)}, z\bar{z}\}$ is zero, then

$$\{G^{(1)}, H_p\} \approx 0 \Leftrightarrow \left\{ G^{(1)}, zp_z + \bar{z}p_{\bar{z}} - z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}} \right\} \approx 0. \tag{43}$$

As can be easily verified $\dot{G}^{(1)} = \{G^{(1)}, H_p\} = 0$ (i.e., it is zero throughout the phase space, not only on the constraint surface). Therefore, no *secondary* constraints appear in this example. There is a unique (necessarily) *first-class* constraint, so one degree of freedom is removed, and the system is left with just one genuine degree of freedom [33–35]. As said, $z\bar{z}$ is the gauge invariant (observable) associated to the unique physical degree of freedom.

The analogies between the toy model and the TEGR theory are summarized in Table I. Notice that in the table we do not list all TEGR constraints; the discussion on the number and physical interpretation of them can be found in Sec. IV.

C. Modified pseudoinvariant rotational Lagrangian

Let us deform the mechanical toy model given by the Lagrangian (36) and replace the pseudoinvariant Lagrangian L with a function of itself:

$$\mathcal{L} = f(L). \tag{44}$$

The theory described by the Lagrangian $\mathcal{L} = f(L)$ is dynamically equivalent to the one governed by the Jordan-frame Lagrangian that includes an additional dynamical variable ϕ:

$$\mathcal{L} = \phi L - V(\phi). \tag{45}$$

In fact, the Lagrange equation for ϕ is

$$L = V'(\phi). \tag{46}$$

So, the dynamics says that $\mathcal{L}$ in Eq. (45) is the Legendre transform of $V(\phi)$; therefore, $\mathcal{L}$ is a function $f(L)$. Each choice of V equals a choice of f; the inverse Legendre transform then implies

$$\phi = f'(L). \tag{47}$$

On the other hand, the Lagrange equation (46) also says that the dynamics of ϕ is completely determined by the dynamics of $z(t)$ and $\bar{z}(t)$ through the function $L(z(t), \bar{z}(t), \dot{z}(t), \dot{\bar{z}}(t))$. We remark that, although L comes with a total derivative, $\mathcal{L}$ is not pseudoinvariant because the total derivative in Eq. (45) is multiplied by ϕ. So, the system described by the Lagrangian $\mathcal{L}$ has, in principle, two degrees of freedom, let us say z and $\bar{z}$. Nevertheless, there is a particular case where the number of degrees of freedom reduces to one: when the function g in Eq. (36) has the form $g(z, \bar{z}) = v(z\bar{z})$. In that case $\mathcal{L}$ in Eq. (45) depends only on $z\bar{z}$ and ϕ; but, as already said, the dynamics for ϕ is linked to the one for $z\bar{z}$. This alternative (one or two degrees of freedom) should be reflected by the Dirac-Bergmann algorithm for the Hamiltonian formalism of this system.

Let us compute the canonical momenta associated with ϕ, z and $\bar{z}$:

$$G^{(1)}_\pi \equiv \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \approx 0, \tag{48}$$

$$p_z = \frac{\phi}{z}\frac{d}{dt}(z\bar{z}) + \phi\frac{\partial}{\partial z}g(z, \bar{z}),$$

$$p_{\bar{z}} = \frac{\phi}{\bar{z}}\frac{d}{dt}(z\bar{z}) + \phi\frac{\partial}{\partial \bar{z}}g(z, \bar{z}). \tag{49}$$

We easily get the angular momentum constraint

$$G^{(1)} \equiv z\left(p_z - \phi\frac{\partial g}{\partial z}\right) - \bar{z}\left(p_{\bar{z}} - \phi\frac{\partial g}{\partial \bar{z}}\right) \approx 0. \tag{50}$$

Notice that the piece of $G^{(1)}$ which comes from the boundary term in L is now multiplied by ϕ. In consequence, the dynamical system defined by Eq. (45) has two primary constraints whose Poisson bracket is

$$\{G^{(1)}, G^{(1)}_\pi\} = -z\frac{\partial g}{\partial z} + \bar{z}\frac{\partial g}{\partial \bar{z}}. \tag{51}$$

The canonical Hamiltonian is

$$\mathcal{H} = \dot{z}p_z + \dot{\bar{z}}p_{\bar{z}} - \mathcal{L} = 2\phi\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 + \phi U(z\bar{z}) + V(\phi)$$

$$= \frac{1}{8\phi z\bar{z}}\left[z\left(p_z - \phi\frac{\partial g}{\partial z}\right) + \bar{z}\left(p_{\bar{z}} - \phi\frac{\partial g}{\partial \bar{z}}\right)\right]^2$$

$$+ \phi U(z\bar{z}) + V(\phi), \tag{52}$$

and the primary Hamiltonian is

$$\mathcal{H}_p = \mathcal{H} + u^\pi(t)G_\pi^{(1)} + u(t)G^{(1)}, \tag{53}$$

where u^π, u are Lagrange multipliers. We must evaluate the evolution of the primary constraints to look for secondary constraints:

$$\dot{G}_\pi^{(1)} = \{G_\pi^{(1)}, \mathcal{H}_p\} = \{\pi, \mathcal{H}_p\}$$

$$= \frac{(zp_z + \bar{z}p_{\bar{z}})^2}{8\phi^2 z\bar{z}} - \frac{1}{8z\bar{z}}\left(z\frac{\partial g}{\partial z} + \bar{z}\frac{\partial g}{\partial \bar{z}}\right)^2$$

$$- U(z\bar{z}) - V'(\phi) + u(t)\left(z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}}\right)$$

$$= L - V'(\phi) + \left(u(t) - \frac{d}{dt}\ln\sqrt{\frac{z}{\bar{z}}}\right)\left(z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}}\right), \tag{54}$$

$$\dot{G}^{(1)} = \{G^{(1)}, \mathcal{H}_p\} = u^\pi\{G^{(1)}, G_\pi^{(1)}\} = -u^\pi\left(z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}}\right). \tag{55}$$

Therefore there are three different ways to guarantee the consistency of the evolution, which we proceed to study in three separate cases.

1. Case (i)

If $g(z, \bar{z}) \neq v(z\bar{z})$, i.e., $z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}} \neq 0$, then we guarantee the consistency of the evolution by choosing the Lagrange multipliers in the following way:

$$u^\pi = 0, \qquad u(t) = -\frac{L(t) - V'(\phi(t))}{z(t)\frac{\partial g}{\partial z}(t) - \bar{z}(t)\frac{\partial g}{\partial \bar{z}}(t)} + \frac{d}{dt}\ln\sqrt{\frac{z(t)}{\bar{z}(t)}}. \tag{56}$$

The system has no secondary constraints; the only constraints $G^{(1)}$, $G_\pi^{(1)}$ are second class, since $\{G^{(1)}, G_\pi^{(1)}\}$ in Eq. (51) is different from zero. So, they remove only one degree of freedom [33–35]; there are two genuine degrees of freedom among the variables $(z, \bar{z}, \phi)$. Notice that, differing from $f(T)$ gravity, no gauge freedom is left in this system since both Lagrange multipliers have been fixed, so the primary Hamiltonian completely determines

the evolution of the variables. In particular, the evolution of ϕ is given by the equation

$$\dot{\phi} = \{\phi, \mathcal{H}_p\} = u^\pi = 0, \tag{57}$$

which means that ϕ is a free constant. The equation for $z(t)$ is

$$\dot{z} = \{z, \mathcal{H}_p\} = \frac{1}{4\phi\bar{z}}\left(zp_z + \bar{z}p_{\bar{z}} - \phi z\frac{\partial g}{\partial z} - \phi\bar{z}\frac{\partial g}{\partial \bar{z}}\right)$$

$$+ u(t)z = \dot{z} - z\frac{L - V'(\phi)}{z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}}}, \tag{58}$$

so the Lagrange equation $L - V'(\phi) = 0$ [see Eq. (46)] is obtained. The Lagrange multiplier u then becomes

$$u(t) = \frac{d}{dt}\ln\sqrt{z(t)/\bar{z}(t)}. \tag{59}$$

By combining Eqs. (46) and (57) one gets

$$L = \text{const} \quad \text{and} \quad \mathcal{L} = \text{const}. \tag{60}$$

Instead of $\dot{p}_z$, let us compute the evolution of the rotationally invariant quantity $zp_z - \phi z\partial g/\partial z$:

$$\frac{d}{dt}\left(zp_z - \phi z\frac{\partial g}{\partial z}\right)$$

$$= \left\{zp_z - \phi z\frac{\partial g}{\partial z}, \mathcal{H}_p\right\} = \left\{zp_z - \phi z\frac{\partial g}{\partial z}, \mathcal{H}\right\}$$

$$= \frac{1}{8\phi z\bar{z}}\left(zp_z + \bar{z}p_{\bar{z}} - \phi z\frac{\partial g}{\partial z} - \phi\bar{z}\frac{\partial g}{\partial \bar{z}}\right)^2 - \phi z\frac{\partial U(z\bar{z})}{\partial z}$$

$$= \mathcal{H} - \phi U(z\bar{z}) - V(\phi) - \phi z\bar{z}U'(z\bar{z}). \tag{61}$$

By replacing this with Eqs. (49), (52) and (57), one obtains

$$\frac{d^2}{dt^2}(z\bar{z}) = 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 - z\bar{z}U'(z\bar{z}), \tag{62}$$

or

$$4\frac{d^2}{dt^2}\sqrt{z\bar{z}} = -\frac{dU}{d\sqrt{z\bar{z}}}, \tag{63}$$

which amounts to the conservation of $h = 2(d(\sqrt{z\bar{z}})/dt)^2 + U$. Equation (62) coincides with the Lagrange equation for the system described by the Lagrangian L.[9] However the Lagrangian $\mathcal{L}$ still makes a difference with L. This is because the dynamics governed by $\mathcal{L}$ imposes a new

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[9]Analogously, in $f(T)$ gravity the solutions with constant T satisfy the Einstein equations (although the cosmological and gravitational constants are shifted).

constant of motion besides h: L has to be a constant as well. While Eq. (62) only fixes the evolution of $|z(t)|$, the condition $L = \text{const}$ is an additional requirement that involves the phase of $z(t)$ in the total derivative term of L.[10] Therefore, differing from L, the Lagrangian $\mathcal{L}$ governs the evolutions of both the modulus and the phase of z (notice, however, that the initial phase is irrelevant due to the *global* rotational symmetry).

In sum, the system described by the Lagrangian (45) has two degrees of freedom: one of them is $|z|^2 = z\bar{z}$ whose dynamics does not differ from the one described by the Lagrangian (36); in both cases we arrive at the conserved quantity $h = 2(d(\sqrt{z\bar{z}})/dt)^2 + U$. Once the evolution of $|z|$ is determined by the choices of the initial value $|z(t_o)|$ and the constant of motion h, the evolution of the phase of z, which is the remaining degree of freedom, is determined by the condition $L(t) = \text{const}$. The value of this constant connects with the value of ϕ through Eq. (46). There is no other physics associated with ϕ, over and above the one related to the phase of z. In the analogy with $f(T)$ gravity, ϕ could then be regarded as a variable carrying information about the "orientation" of the tetrad, which would be partially determined by the dynamical equations. We will discuss this issue more later.

2. Case (ii)

If $g(z, \bar{z}) = v(z\bar{z})$, i.e., $z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial \bar{z}} = 0$, then $\mathcal{L}$ depends only on $|z|^2 = z\bar{z}$ and ϕ. The constraints $G^{(1)}$, $G_\pi^{(1)}$ commute [see Eq. (51)], and the Lagrange multipliers u, u^π are not determined by the Eqs. (54)–(55). While $\dot{G}^{(1)}$ is zero, the consistency of $G_\pi^{(1)}$ leads to the *secondary* constraint

$$G^{(2)} = L - V'(\phi) \approx 0, \qquad (64)$$

which recovers Eq. (46), and tells that the values of ϕ are now linked to those of $z\bar{z}$.

If $g(z, \bar{z}) = v(z\bar{z})$, then L is invariant under rotations. Therefore

$$\{G^{(1)}, G^{(2)}\} = 0. \qquad (65)$$

Besides

$$\{G_\pi^{(1)}, G^{(2)}\} = V''(\phi). \qquad (66)$$

Let us examine the consistency of $G^{(2)}$ under the time evolution of the system:

$$\dot{G}^{(2)} = \{G^{(2)}, \mathcal{H}_p\} = \{G^{(2)}, \mathcal{H}\} + u\{G^{(2)}, G^{(1)}\}$$
$$+ u^\pi \{G^{(2)}, G_\pi^{(1)}\} = \{L, \mathcal{H}\} - u^\pi V''(\phi). \qquad (67)$$

If $V''(\phi) \neq 0$, then the consistency can be guaranteed by choosing u^π,[11]

$$u^\pi(t) = V''(\phi)^{-1}\{L, \mathcal{H}\} = V''(\phi)^{-1}\frac{dL}{dt}. \qquad (68)$$

In such a case we are left with a first-class constraint $G^{(1)}$ and two second-class constraints $G_\pi^{(1)}$, $G^{(2)}$. So, two degrees of freedom are suppressed by the constraint structure. Since we started with three dynamical variables, z, $\bar{z}$ and ϕ, the system has one genuine degree of freedom. The observable (gauge-invariant) variable is $z\bar{z}$. The phase of z remains as a gauge freedom; it is not determined by the evolution since the Lagrange multiplier $u(t)$ has not been fixed.

The dynamical equation for ϕ,

$$\dot{\phi} = \{\phi, \mathcal{H}_p\} = u^\pi(t) = V''(\phi)^{-1}\frac{dL}{dt} = f''(L)\frac{dL}{dt}$$
$$= \frac{d}{dt}f'(L), \qquad (69)$$

does not contain new information since it can also be obtained by differentiating Eq. (64); in particular, it does not constrain L to be a constant. The evolution of ϕ is then entirely determined by the evolution of $z\bar{z}$ through Eq. (64). On the other hand, the evolution of $z\bar{z}$ will be different than in case (i); this is because ϕ is no longer a constant [as it is in case (i)]. This does not mean that ϕ does not have a role; the reader must remember that ϕ exists because the Lagrangian L has been replaced with $\mathcal{L} = f(L)$. In fact, the lhs of Eq. (61), which is the equation we used to obtain the evolution of $z\bar{z}$, will now generate an additional term associated with $\dot{\phi}$. Since $zp_z - \phi z\partial g/\partial z = \phi d(z\bar{z})/dt$ [see Eq. (49)], the new term will be $\dot{\phi}d(z\bar{z})/dt$. Thus, the dynamical equation (62) for $z\bar{z}$ will now read

$$\phi^{-1}\dot{\phi}\frac{d}{dt}(z\bar{z}) + \frac{d^2}{dt^2}(z\bar{z}) = 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 - z\bar{z}U'(z\bar{z}), \quad (70)$$

or

$$4\frac{d}{dt}\ln f'(L)\frac{d}{dt}(\sqrt{z\bar{z}}) + 4\frac{d^2}{dt^2}\sqrt{z\bar{z}} = -\frac{dU}{d\sqrt{z\bar{z}}}. \qquad (71)$$

This is the result we were expecting, because it is the Lagrange equation for a Lagrangian $\mathcal{L} = f(L)$ that depends exclusively on $z\bar{z}$.

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[10]In $f(T)$ gravity, it involves the orientation of the tetrad which affects the boundary term of the TEGR Lagrangian.

[11]In the Legendre transform it is $V''(\phi)^{-1} = f''(L)$.

3. Case (iii)

As can be seen in Eqs. (54) and (55), the consistency of the evolutions of both primary constraints $G_\pi^{(1)}$ and $G^{(1)}$ are affected by the quantity $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z}$. This quantity vanishes if $g(z,\bar{z}) = v(z\bar{z})$, as considered in case (ii), which means that $\mathcal{L} = f(L)$ becomes invariant under local rotations, and the system is left with only one degree of freedom. However, we could still consider another possibility: the condition

$$z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial\bar{z}} = 0 \qquad (72)$$

is satisfied only in some region of the constraint surface. For instance, let us consider a function $g(z,\bar{z}) = g(z+\bar{z})$; then

$$g(z,\bar{z}) = g(z+\bar{z}) \Rightarrow z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial\bar{z}} = (z-\bar{z})g'. \qquad (73)$$

We see that the relevant quantity for our analysis vanishes if z is real. Therefore the real solutions, if they exist, would work as in case (ii). Since the phase of z has been frozen to be zero, no extra d.o.f. would be left in these solutions.

The condition (72) defines a hypersurface in the phase space. The intersection of this hypersurface with the constraint surface, if it exists, would constitute a subspace where the degree of freedom associated with the phase of z does not manifest itself, since the Lagrangian $\mathcal{L} = f(L)$ would turn out to be invariant under *infinitesimal* local rotations $\delta z = i\alpha(t)z$:

$$\delta\mathcal{L} = \delta f(L) = f'(L)\delta L = f'(L)\frac{d}{dt}\delta g$$

$$= f'(L)\frac{d}{dt}\left[\delta z\frac{\partial g}{\partial z} + \delta\bar{z}\frac{\partial g}{\partial\bar{z}}\right]$$

$$= if'(L)\frac{d}{dt}\left[\alpha(t)\left(z\frac{\partial g}{\partial z} - \bar{z}\frac{\partial g}{\partial\bar{z}}\right)\right] = 0. \qquad (74)$$

Thus, we should wonder about the existence of solutions to the equations of motion lying on the subspace defined by Eq. (72) and the constraints. These solutions should not contain a d.o.f. associated with the phase of z; they would remain as solutions to the equations of motion under infinitesimal local rotations. These solutions would evidence just one degree of freedom: the one related to the modulus of z. Therefore, the Lagrangian $\mathcal{L} = f(L)$ could lead to solutions displaying one or two degrees of freedom, depending on which region of the constraint surface they occupy [i.e., depending on whether they satisfy the condition (72) or not].

In sum, in the Jordan frame we rewrite the Lagrangian $\mathcal{L} = f(L)$ as $\mathcal{L} = \phi L - V(\phi)$. If the boundary term $\dot{g}(z,\bar{z})$ present in L is such that $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z} \neq 0$, then an extra degree of freedom associated with the phase of z will

manifest itself. In the Jordan frame, the extra degree of freedom comes from the free choice of the *constant* ϕ which, on its side, determines the phase of z through the condition $L = V'(\phi) = \text{const}$. Instead, if $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z} = 0$, then $\mathcal{L} = f(L)$ will not be sensitive to the phase of z, so ϕ cannot be associated with an extra degree of freedom but will be entirely determined by $z\bar{z}$ through the equation $G^{(2)} = L - V'(\phi) \approx 0$ without imposing any condition on the value of L. However the fact that ϕ is not constrained to be a constant will imply an additional term in the dynamical equation for the modulus of z, as can be straightforwardly verified in the Lagrange equations for the Lagrangian $\mathcal{L} = f(L)$. Besides, if there were solutions such that $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z}$ cancels out, then these solutions will remain as solutions of the equations of motion under infinitesimal local perturbations of the phase of z; therefore they would just exhibit the degree of freedom associated with $z\bar{z}$.

IV. $f(T)$ GRAVITY: A MODIFIED LORENTZIAN PSEUDOINVARIANT LAGRANGIAN

A. Summary of d.o.f. counting in $f(T)$ gravity

The modified rotationally pseudoinvariant system of Sec. III is useful to understand several features of $f(T)$ gravity, since the latter consists in the modification of the Lorentzian pseudoinvariant TEGR Lagrangian. Due to the inherent complications of the dynamical equations of $f(T)$ gravity, the Jordan-frame formalism has been used for the analysis of the constraint algebra and the counting of d.o.f. [11,38]. Reference [38] used the first-order Hamiltonian formalism developed in Refs. [39,40] as a base for computing the constraint structure of $f(T)$ gravity. Instead, Ref. [11] used the canonical Hamiltonian formalism for TEGR described in Ref. [37].[12] While in Ref. [38] the authors claimed that $f(T)$ gravity has $n-1$ extra d.o.f. in dimension n, the outcome of the counting of d.o.f. in Ref. [11] gave only one extra d.o.f. in arbitrary dimension. More evidence that speaks in favor of only one d.o.f. can be found in Ref. [26], where the extra d.o.f. was identified with a scalar field which partially determines the orientation of the tetrad field. Other classes of modified teleparallel gravities might have a different number of d.o.f. [41,42].

In what follows we will summarize some key findings that are essential for the understanding of the counting of degrees of freedom in $f(T)$ gravity. The notation in what comes next will be borrowed from Ref. [11]; the reader can find all the definitions and details there. The constraints of $f(T)$ gravity can be counted and classified as follows:

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[12]In Ref. [37] the TEGR Lagrangian was expressed in the form $L_{\text{TEGR}} = ET = (1/2)E\partial_\mu E_\nu^g\partial_\rho E_\lambda^h e_c^\mu e_e^\nu e_d^\rho e_f^\lambda M_{gh}{}^{cedf}$, where $M_{gh}{}^{cedf} \doteq 2\eta_{gh}\eta^{c[d}\eta^{f]e} - 4\delta_g^{[d}\eta^{f][c}\delta_h^{e]} + 8\delta_g^{[c}\eta^{e][d}\delta_h^{f]}$ is the so-called supermetric.

(1) One primary constraint $G_\pi^{(1)}$ coming from the vanishing of the momentum conjugate to the auxiliary scalar field ϕ.

(2) n primary constraints $G_a^{(1)}$ coming from the absence of $\partial_0 E_0^c$ in the Lagrangian (analogous to electromagnetism).

(3) $n(n-1)/2$ primary constraints $G_{ab}^{(1)}$ associated with Lorentz invariance (also appearing in TEGR).

(4) n secondary constraints $G_\mu^{(2)}$ due to the diffeomorphism invariance (same constraints as in GR).

From the whole set of primary and secondary constraints of the theory, there are only two nonvanishing Poisson brackets. These correspond to

$$\{G_{ab}^{(1)}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx F_{ab}\delta(\mathbf{x} - \mathbf{y}), \qquad (75)$$

and

$$\{G_0^{(2)}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx F_\phi\delta(\mathbf{x} - \mathbf{y}), \qquad (76)$$

where F_{ab}, F_ϕ are

$$F_{ab} = 4E\partial_i E_j^c e_{[b}^0 e_a^i e_{c]}^j, \qquad F_\phi = E(T - V'(\phi)). \qquad (77)$$

The functions F_{ab}, F_ϕ are key in determining the number of physical d.o.f. of the theory. They enter the matrix of Poisson brackets $C_{\hat\rho\rho}$, so they determine the rank of $C_{\hat\rho\rho}$ and the separation of the constraints into first and second class. These functions can be arranged to compose a vector $\mathbf{F}$,

$$\begin{aligned}\mathbf{F} &= (F_\phi, F_{01}, F_{02}, ..., F_{(n-2)(n-1)}) \\ &\equiv (F_0, F_1, F_2, ..., F_{n(n-1)/2}).\end{aligned} \qquad (78)$$

We also define the vector $\mathbf{G}$,

$$\begin{aligned}\mathbf{G} &= (G_0^{(2)}, G_{01}^{(1)}, G_{02}^{(1)}, ..., G_{(n-2)(n-1)}^{(1)}) \\ &\equiv (G_0, G_1, G_2, ..., G_{n(n-1)/2}),\end{aligned} \qquad (79)$$

to write the brackets (75)–(76) in a vector form:

$$\{\mathbf{G}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx \mathbf{F}\delta(\mathbf{x} - \mathbf{y}). \qquad (80)$$

This vector equation can be "rotated" to have all the components of $\mathbf{F}$ but one equal to zero. In other words, the constraints $G_{ab}^{(1)}$ and $G_0^{(2)}$ can be rearranged by linearly combining them to have all the brackets (75)–(76) but one equal to zero. Therefore, the brackets (75)–(76) just mean that one combination of $G_{ab}^{(1)}$'s and $G_0^{(2)}$ will fail to be a first-class constraint. That combination together with $G_\pi^{(1)}$ will make up a (unique) pair of second-class constraints. As is known, the *pairs* of second-class constraints count as *individual* first-class constraints in the counting of d.o.f.

(35). So, although $f(T)$ gravity in the Jordan frame has an additional constraint $G_\pi^{(1)}$ compared with TEGR, the number to be subtracted in the counting of d.o.f. (35) will not change because one of the first-class constraints of TEGR has joined $G_\pi^{(1)}$ to make up a pair of second-class constraints. Since $f(T)$ gravity in the Jordan frame has an extra pair of canonical variables (ϕ, π), one concludes that $f(T)$ gravity contains an extra d.o.f. irrespective of the dimension of the spacetime. The extra d.o.f. is the other side of the coin of the reduction of the gauge freedom, since a combination of Lorentz constraints now takes part in a second-class constraint; thus, it stops generating a Lorentz gauge transformation. Therefore, the orientation of the tetrad in $f(T)$ gravity would be partially determined through the choice of the extra d.o.f. in the initial conditions. Which combination of Lorentz constraints no longer generates a gauge transformation will depend on the value of $\mathbf{F}$ for each solution; we just mention that F_ϕ will be dynamically zero [cf. Eq. (46)].

B. Lessons of the toy model for $f(T)$ gravity

Concerning the comparison between the toy model and $f(T)$ gravity, we see that the Poisson brackets (75) are analogous to its toy model counterpart $\{G^{(1)}, G_\pi^{(1)}\}$ defined in Eq. (51). The analogy implies that the functions F_{ab} somehow play a role analogous to $z\partial g/\partial z - \bar z\partial g/\partial\bar z$.[13] While $G^{(1)}$ corresponds to the rotational gauge symmetry of the toy model, $G_{ab}^{(1)}$ is related to the Lorentz gauge symmetry of TEGR. Both symmetries will be lost in the modified models, due to the nonvanishing of the brackets in Eqs. (51) and (75), respectively. However, in $f(T)$ there is still room for a subset of Lorentz transformations that keep being a symmetry of the theory; this subset is determined by the value of the vector $\mathbf{F}$ in each solution.

The analogy between $z\partial g/\partial z - \bar z\partial g/\partial\bar z$ and F_{ab} also appears at the Lagrangian level in the analysis of the pseudoinvariance of L and $L_{\text{TEGR}} = ET$. In fact, the change of L under an infinitesimal rotation of angle $\alpha(t)$ [i.e., $\delta z = i\alpha(t)z$],

$$\delta L = \delta\frac{dg}{dt} = \frac{d}{dt}\delta g = \frac{d}{dt}\left[i\alpha(t)\left(z\frac{\partial g}{\partial z} - \bar z\frac{\partial g}{\partial\bar z}\right)\right], \qquad (81)$$

is governed by $z\partial g/\partial z - \bar z\partial g/\partial\bar z$. On the other hand, the infinitesimal Lorentz transformation of L_{TEGR} can be obtained from the expression (12), rewritten as $L_{\text{TEGR}} = -ER + 2\partial_\mu(ET^\mu)$. In varying it, we must take into account that ER is locally invariant under Lorentz transformations

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[13]The toy model does not have an analogue for the bracket (76) because it is not invariant under reparametrizations. The reparametrization invariance can be considered by adding a *lapse* function $N(t)$ to the set of dynamical variables (see for instance Ref. [43]).

TABLE II.   Comparison between the modified rotationally pseudoinvariant toy model and $f(T)$ gravity.

	Modified toy model	$f(T)$ gravity
Boundary term	$\dot{z}\frac{dg}{dz} + \dot{\bar{z}}\frac{dg}{d\bar{z}}$	$\partial_\mu(ET^\mu)$
Poisson bracket	$\{G^{(1)}, G_\pi^{(1)}\} = -z\frac{dg}{dz} + \bar{z}\frac{dg}{d\bar{z}}$	$\{G_{ab}^{(1)}, G_\pi^{(1)}\} = F_{ab}$
Lost gauge symmetry	Rotation in the plane $(z, \bar{z})$	A linear combination of Lorentz transformations
Degrees of freedom	$|z| +$ scalar field	Two polarizations of $g_{\mu\nu} +$ scalar field

of the tetrad: it depends exclusively on the metric. Then, the variation of L_{TEGR} is equal to the variation of the boundary term $2\partial_\mu(ET^\mu)$, that is

$$\delta L_{\text{TEGR}} = 2\delta\partial_\mu(ET^\mu). \tag{82}$$

Since T^μ is invariant only under global Lorentz transformations of the tetrad field, Eq. (82) exhibits the pseudo-gauge-invariance of L_{TEGR}. Let us consider an infinitesimal local Lorentz transformation of the tetrad in the $a - b$ plane,

$$\delta_{ab}\mathbf{E}^g = -\alpha(t, \mathbf{x})\delta_{[a}^g\eta_{b]h}\mathbf{E}^h. \tag{83}$$

Then the change of $\partial_\mu(ET^\mu)$ is

$$\begin{aligned}
\delta_{ab}\partial_\mu(ET^\mu) &= \partial_\mu(E\delta_{ab}T^\mu) \\
&= -\partial_\mu[Eg^{\mu\nu}e_g^\rho(\delta_{[a}^g\eta_{b]h}E_\rho^h\partial_\nu\alpha - \delta_{[a}^g\eta_{b]h}E_\nu^h\partial_\rho\alpha)] \\
&= \partial_\mu(Eg^{\mu\nu}e_{[a}^\rho\eta_{b]h}E_\nu^h\partial_\rho\alpha) \\
&= \partial_\mu(Ee_{[a}^\rho e_{b]}^\mu\partial_\rho\alpha) = \partial_\rho\alpha\partial_\mu(Ee_{[a}^\rho e_{b]}^\mu). \tag{84}
\end{aligned}$$

In this calculation we have only kept terms involving derivatives of the parameter α, because we already know that L_{TEGR} is not sensitive to global Lorentz transformations [represented by $\alpha = \text{const}$ in Eq. (83)].[14] Using the standard formulas $\partial_\mu E = Ee_a^\nu\partial_\mu E_\nu^a$, $\partial_\mu e_b^\nu = -e_a^\nu e_b^\lambda\partial_\mu E_\lambda^a$, it is possible to show that

$$\partial_\mu(Ee_{[a}^\rho e_{b]}^\mu) = 3E\partial_\mu E_\lambda^c e_{[a}^\rho e_b^\mu e_{c]}^\lambda. \tag{85}$$

Comparing with Eq. (77) we see that $\partial_\mu(Ee_{[a}^0 e_{b]}^\mu) = -(3/4)F_{ab}$. Thus the variation (84) implies that

$$\delta_{ab}L_{\text{TEGR}} = -\frac{3}{2}\partial_0\alpha F_{ab} + 2\partial_i\alpha\partial_\mu(Ee_{[a}^i e_{b]}^\mu). \tag{86}$$

As seen, both F_{ab} and $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z}$ play a role in the pseudoinvariance of L_{TEGR} and L respectively.

The modified toy model has shown us that two types of solutions can exist when the original system possesses pseudoinvariance: the case-(i) solutions where

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[14]Instead, L in the toy model changes even if α is a constant, as seen in Eq. (81).

the Lagrangian is a constant to be chosen in the initial conditions (it is the extra d.o.f.), and the case-(iii) solutions where the extra d.o.f. does not manifest itself since it is subject to satisfying the condition $z\partial g/\partial z - \bar{z}\partial g/\partial\bar{z} = 0$. According to Eq. (81), the case-(iii) solutions are made of points of the configuration space where the Lagrangian is invariant rather than pseudoinvariant. Analogously, the case-(iii) solutions of $f(T)$ gravity do not exhibit the extra d.o.f. because it is subject to canceling out the F_{ab}'s. According to Eq. (86), the case-(iii) solutions of $f(T)$ gravity are made of configurations such that L_{TEGR} is invariant, rather than pseudoinvariant, under Lorentz transformations depending only on time [if $\alpha = \alpha(t)$, then $\delta_{ab}L_{\text{TEGR}} = -(3/2)\dot{\alpha}F_{ab}$].

The interest in case-(iii) solutions comes from the fact that they give new dynamics to the original gauge-invariant variables; in $f(T)$ gravity, they are apt to study modified gravity. In a case-(i) solution, instead, the dynamics for the components of the metric tensor is the same as in TEGR, except for the shift of the cosmological and Newton constants due to the fact that the determinant E is not encapsulated in the function f.

The previously remarked analogies between the modified toy model and $f(T)$ gravity are summarized in Table II.

C. Both types of solutions in flat FLRW cosmology

We will exemplify case-(i) and case-(iii) solutions in $f(T)$ gravity by revisiting cosmological solutions already existing in the literature, in the context of flat FLRW cosmology. The commonly used solution is the diagonal tetrad in Eq. (15) with $T = -6H^2$ [6,7]. It is easy to prove that all the coefficients F_{ab} are zero, since the tetrad (15) depends only on time, and F_{ab} just involve spatial derivatives. This is a case-(iii) solution; therefore, the extra d.o.f. does not manifest itself.

On the other hand, the tetrad (18) is a case-(i) solution. In fact, T is a constant T_o; besides some of the F_{ab}'s are different from zero, namely

$$F_{01} = -\frac{4}{3}ra(t)^2\sin\theta, \qquad F_{02} = -\frac{2}{3}ra(t)^2\cos\theta\cosh\lambda,$$

$$F_{12} = \frac{2}{3}ra(t)^2\cos\theta\sinh\lambda \tag{87}$$

(the rest of the antisymmetric components F_{ab} are zero on-shell). By replacing the tetrad (18) in the equations of

motion of $f(T)$ gravity, one obtains that the scale factor $a(t)$ fulfills the FLRW equations of general relativity with shifted gravitational and cosmological constants, as shown in Eq. (21). The extra d.o.f. $\phi = f'(T_o)$ is represented by T_o, which takes part in the tetrad field through the function $\lambda(t)$ in the same way that ϕ enters the phase of z in the case-(i) solutions of the toy model. Equation (87) suggests combining the constraints $G_{01}^{(1)}$, $G_{02}^{(1)}$ and $G_{12}^{(1)}$ as

$$\mathcal{G}_{01}^{(1)} = G_{01}^{(1)},$$
$$\mathcal{G}_{02}^{(1)} = \sinh \lambda G_{02}^{(1)} + \cosh \lambda G_{12}^{(1)},$$
$$\mathcal{G}_{12}^{(1)} = \cos \theta G_{01}^{(1)} - 2 \sin \theta (\cosh \lambda G_{02}^{(1)} + \sinh \lambda G_{12}^{(1)}), \quad (88)$$

to get on-shell

$$\{\mathcal{G}_{01}^{(1)}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx F_{01}\delta(\mathbf{x} - \mathbf{y}), \quad \{\mathcal{G}_{02}^{(1)}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx 0,$$
$$\{\mathcal{G}_{12}^{(1)}(\mathbf{x}), G_\pi^{(1)}(\mathbf{y})\} \approx 0. \quad (89)$$

Therefore, $\mathcal{G}_{01}^{(1)}$ and $G_\pi^{(1)}$ make up a pair of second-class constraints, and the rest of the constraints are first class. Of course, the second-class sector of the $\mathcal{G}_{ab}^{(1)}$'s is ambiguous, because the addition of a linear combination of first-class constraints to $\mathcal{G}_{01}^{(1)}$ will not change the result of the previous Poisson brackets.

We will take advantage of the simplicity of the case-(i) solution (18) to make some considerations about the relationship between the extra d.o.f. and the remnant gauge invariance. No local Lorentz transformation of the tetrad can modify the metric (2)–(4). But it could affect the $f(T)$ dynamics, since it will produce one of the following results:

(I) It affects the value of T; T is no longer a constant, so the transformed tetrad is not a case-(i) solution [it could be a case-(iii) solution or not a solution at all].

(II) It affects the value of T; T turns to be a different constant, so the transformed tetrad is another case-(i) solution because the extra d.o.f. has changed its value.

(III) The (constant) value of T is not affected; the local Lorentz transformation is a remnant gauge symmetry.

To exemplify these situations, let us show two local Lorentz transformations of the tetrad (18) that do not change the value of T (remnant symmetries):

(1) A rotation in the $(\mathbf{E}^2, \mathbf{E}^3)$ subspace [the local parameter $\alpha(\mathbf{x})$ is completely free],

$$\mathbf{E}^0 = \cosh \lambda \mathbf{dt} + \sinh \lambda a(t)\mathbf{dr}, \qquad \mathbf{E}^1 = \sinh \lambda \mathbf{dt} + \cosh \lambda a(t)\mathbf{dr},$$
$$\mathbf{E}^2 = a(t)r(\cos \alpha(\mathbf{x})\mathbf{d\theta} + \sin \alpha(\mathbf{x}) \sin \theta \mathbf{d\phi}), \qquad \mathbf{E}^3 = a(t)r(- \sin \alpha(\mathbf{x})\mathbf{d\theta} + \cos \alpha(\mathbf{x}) \sin \theta \mathbf{d\phi}). \quad (90)$$

(2) A boost along the φ direction [the local parameter $\beta(\mathbf{x})$ cannot depend on φ in order to keep the value $T = T_o$]:

$$\mathbf{E}^0 = \cosh \beta(\mathbf{x})(\cosh \lambda \mathbf{dt} + \sinh \lambda a(t)\mathbf{dr}) + \sinh \beta(\mathbf{x})a(t)r \sin \theta \mathbf{d\phi}, \qquad \mathbf{E}^1 = \sinh \lambda \mathbf{dt} + \cosh \lambda a(t)\mathbf{dr},$$
$$\mathbf{E}^2 = a(t)r\mathbf{d\theta}, \qquad \mathbf{E}^3 = \sinh \beta(\mathbf{x})(\cosh \lambda \mathbf{dt} + \sinh \lambda a(t)\mathbf{dr}) + \cosh \beta(\mathbf{x})a(t)r \sin \theta \mathbf{d\phi}. \quad (91)$$

The tetrads (18), (90) and (91) represent different gauges for the same solution of $f(T)$ gravity, since they share the value $T = T_o$ of the extra d.o.f. They can be distinguished by looking at non-gauge-invariant quantities, like the torsion vector T^μ and the axial vector $A_\mu \doteq E\epsilon_{\mu\nu\lambda\rho}T^{\nu\lambda\rho}$.[15]

On the other hand, a boost along the r direction—the transformation associated with $G_{01}^{(1)}$—is able to change the value of the constant T_o, thus passing to a different case-(i) solution. In fact, a boost along the r direction will leave the tetrad (91) unchanged, except for the replacement

$$\lambda(t, r) \to \lambda(t, r) + \gamma(\mathbf{x}), \quad (92)$$

where $\gamma(\mathbf{x})$ is the parameter of the boost. Therefore:

(I) If the parameter $\gamma(\mathbf{x})$ is arbitrary, then T will no longer be a constant. So the transformed tetrad will not be a case-(i) solution.

II, III) If the parameter $\gamma(\mathbf{x})$ has the form

$$\gamma(\mathbf{x}) = \Psi(ra(t)) - \frac{1}{4} ra(t)t\Delta T_o, \quad (93)$$

then the transformed tetrad will be a case-(i) solution with a different value of the extra d.o.f.: $T = T_o + \Delta T_o$. Thus, the function $a(t)$ will evolve with other effective cosmological and Newton constants.

V. MODIFYING A HIGHER-ORDER MECHANICAL SYSTEM WITH ROTATIONAL INVARIANCE

A. Rotationally invariant higher-order Lagrangian

In this section we will study another toy model and its modification, in order to show a qualitatively different

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[15]However the divergence of T^μ is a gauge-invariant quantity because it directly relates to T [see Eq. (82)].

mechanism for the generation of an extra degree of freedom. The idea is to mimic the Einstein-Hilbert Lagrangian which is composed of terms that are invariant under local Lorentz transformations in the tangent space (they depend just on the metric), but it exhibits a second-order boundary term to guarantee the invariance under local diffeomorphisms. So let us introduce a second-order Lagrangian displaying invariance under local rotations:

$$
\begin{aligned}
L &= 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 - U(z\bar{z}) + A(\ddot{z}\,\bar{z} + 2\dot{z}\,\dot{\bar{z}} + z\ddot{\bar{z}})\\
&= 2\left(\frac{d}{dt}\sqrt{z\bar{z}}\right)^2 - U(z\bar{z}) + \frac{d^2}{dt^2}(Az\bar{z}).
\end{aligned}
\tag{94}
$$

The last term is a total derivative which does not enter the Lagrange equations, so the dynamics is still governed by the equations (62). However the presence of second derivatives in the Lagrangian implies the use of Ostrogradsky's procedure to introduce the Hamilton equations; namely, we have to define momenta associated with both *canonical variables* z and $Z \equiv \dot{z}$:

$$
P_Z \equiv \frac{\partial L}{\partial \dot{Z}} = \frac{\partial L}{\partial \ddot{z}} = A\bar{z},
$$

$$
p_z \equiv \frac{\partial L}{\partial \dot{z}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{z}} = \frac{1}{z}\frac{d}{dt}(z\bar{z}) + A\dot{\bar{z}},
\tag{95}
$$

$$
P_{\bar{Z}} \equiv \frac{\partial L}{\partial \dot{\bar{Z}}} = \frac{\partial L}{\partial \ddot{\bar{z}}} = Az,
$$

$$
p_{\bar{z}} \equiv \frac{\partial L}{\partial \dot{\bar{z}}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\bar{z}}} = \frac{1}{\bar{z}}\frac{d}{dt}(z\bar{z}) + A\dot{z}.
\tag{96}
$$

Thus we get three primary constraints

$$
G^{(1)} \equiv z(p_z - A\bar{Z}) - \bar{z}(p_{\bar{z}} - AZ), \qquad G_Z^{(1)} \equiv P_Z - A\bar{z},
$$

$$
G_{\bar{Z}}^{(1)} \equiv P_{\bar{Z}} - Az,
\tag{97}
$$

that commute, since

$$
\{G^{(1)}, G_Z^{(1)}\} = 0, \quad \{G^{(1)}, G_{\bar{Z}}^{(1)}\} = 0, \quad \{G_Z^{(1)}, G_{\bar{Z}}^{(1)}\} = 0.
\tag{98}
$$

The canonical Hamiltonian is

$$
\begin{aligned}
H&(z, \bar{z}, Z, \bar{Z}, p_z, p_{\bar{z}}, P_Z, P_{\bar{Z}})\\
&= \dot{Z}P_Z + \dot{\bar{Z}}P_{\bar{Z}} + \dot{z}p_z + \dot{\bar{z}}p_{\bar{z}} - L\\
&= \frac{1}{8z\bar{z}}[z(p_z - A\bar{Z}) + \bar{z}(p_{\bar{z}} - AZ)]^2 + \phi U(z\bar{z}).
\end{aligned}
\tag{99}
$$

The primary Hamiltonian is

$$
H_p = H + uG^{(1)} + u_z G_Z^{(1)} + u_{\bar{z}} G_{\bar{Z}}^{(1)}.
\tag{100}
$$

As already expected, there is not a unique way of writing the canonical Hamiltonian, due to the presence of constraints. For instance, we can also write $H = Zp_z + \bar{Z}p_{\bar{z}} - (2z\bar{z})^{-1}(\bar{z}Z + z\bar{Z})^2 - 2AZ\bar{Z} + U(z\bar{z})$. However this apparently simpler form of H will lead to secondary constraints.[16]

The constraints remain zero when the system evolves,

$$
\dot{G}^{(1)} = \{G^{(1)}, \mathcal{H}_p\} = 0, \qquad \dot{G}_Z^{(1)} = \{G_Z^{(1)}, H_p\} = 0,
$$

$$
\dot{G}_{\bar{Z}}^{(1)} = \{G_{\bar{Z}}^{(1)}, H_p\} = 0.
\tag{101}
$$

So we have three first-class constraints in a phase space of dimension eight. The reduced phase space has dimension two, which means one degree of freedom.

B. Modified rotationally invariant higher-order Lagrangian

Let us deform the theory by replacing the invariant higher-order Lagrangian with a function of itself,

$$
\mathcal{L} = f(L),
\tag{102}
$$

which is dynamically equivalent to the Jordan-frame representation that includes an additional dynamical variable ϕ:

$$
\mathcal{L} = \phi L - V(\phi).
\tag{103}
$$

Again we apply Ostrogradsky's procedure. We will introduce not only the variable $Z \equiv \dot{z}$, but $\Phi \equiv \dot{\phi}$ as well. Thus the canonical momenta are

$$
\Pi \equiv \frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = 0, \qquad \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{\phi}} = 0, \tag{104}
$$

---

[16] The secondary constraints will be

$$
G^{(2)} \equiv \dot{G}^{(1)} = \frac{Z}{z}(zp_z - \bar{z}Z) - \frac{\bar{Z}}{\bar{z}}(\bar{z}p_{\bar{z}} - z\bar{Z}), G_Z^{(2)} \equiv \dot{G}_Z^{(1)}
$$

$$
= -p_z + A\bar{Z} + \frac{z\bar{Z} + \bar{z}Z}{z}, G_{\bar{Z}}^{(2)} \equiv \dot{G}_{\bar{Z}}^{(1)}
$$

$$
= -p_{\bar{z}} + AZ + \frac{z\bar{Z} + \bar{z}Z}{\bar{z}}.
$$

The constraints $G_Z^{(1)}$, $G_{\bar{Z}}^{(1)}$, $G_Z^{(2)}$, and $G_{\bar{Z}}^{(2)}$ are nothing but the definitions of P_Z, $P_{\bar{Z}}$, p_z, and $p_{\bar{z}}$. Besides $G_Z^{(2)}$, $G_{\bar{Z}}^{(2)}$ are not linearly independent of $G^{(1)}$; in fact, it is $G^{(1)} = \bar{z}G_{\bar{Z}}^{(2)} - zG_Z^{(2)}$. This nonindependence is an additional ingredient for the right counting of the degrees of freedom. The secondary constraints should prove to consistently evolve, which could lead to tertiary constraints.

$$P_Z \equiv \frac{\partial \mathcal{L}}{\partial \dot{Z}} = \frac{\partial \mathcal{L}}{\partial \ddot{z}} = \phi A \bar{z}, \qquad p_z \equiv \frac{\partial \mathcal{L}}{\partial \dot{z}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{z}} = \frac{\phi}{z}\frac{d}{dt}(z\bar{z}) + \phi A \dot{\bar{z}} - \dot{\phi} A \bar{z}, \tag{105}$$

$$P_{\bar{Z}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{Z}}} = \frac{\partial \mathcal{L}}{\partial \ddot{\bar{z}}} = \phi A z, \qquad p_{\bar{z}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \ddot{\bar{z}}} = \frac{\phi}{\bar{z}}\frac{d}{dt}(z\bar{z}) + \phi A \dot{z} - \dot{\phi} A z. \tag{106}$$

Thus, we obtain five primary constraints:

$$G_\Pi^{(1)} \equiv \Pi, \qquad G_\pi^{(1)} \equiv \pi, \tag{107}$$

$$G^{(1)} \equiv z(p_z - \phi A \bar{Z}) - \bar{z}(p_{\bar{z}} - \phi A Z), \tag{108}$$

$$G_Z^{(1)} \equiv P_Z - \phi A \bar{z}, \qquad G_{\bar{Z}}^{(1)} \equiv P_{\bar{Z}} - \phi A z. \tag{109}$$

The Poisson brackets are

$$\{G^{(1)}, G_Z^{(1)}\} = 0, \qquad \{G^{(1)}, G_{\bar{Z}}^{(1)}\} = 0, \qquad \{G^{(1)}, G_\Pi^{(1)}\} = 0, \qquad \{G^{(1)}, G_\pi^{(1)}\} = A(\bar{z}Z - z\bar{Z}), \tag{110}$$

$$\{G_Z^{(1)}, G_{\bar{Z}}^{(1)}\} = 0, \qquad \{G_Z^{(1)}, G_\Pi^{(1)}\} = 0, \qquad \{G_Z^{(1)}, G_\pi^{(1)}\} = -A\bar{z}, \tag{111}$$

$$\{G_{\bar{Z}}^{(1)}, G_\Pi^{(1)}\} = 0, \qquad \{G_{\bar{Z}}^{(1)}, G_\pi^{(1)}\} = -Az, \qquad \{G_\pi^{(1)}, G_\Pi^{(1)}\} = 0. \tag{112}$$

The canonical Hamiltonian is[17]

$$\mathcal{H}(z, \bar{z}, Z, \bar{Z}, \phi, \Phi, p_z, p_{\bar{z}}, P_Z, P_{\bar{Z}}, \pi, \Pi) = \dot{\Phi}\Pi + \dot{\phi}\pi + \dot{Z}P_Z + \dot{\bar{Z}}P_{\bar{Z}} + \dot{z}p_z + \dot{\bar{z}}p_{\bar{z}} - \mathcal{L}$$

$$= \frac{1}{8\phi z \bar{z}}[z(p_z - \phi A \bar{Z}) + \bar{z}(p_{\bar{z}} - \phi A Z) + 2\Phi A z \bar{z}]^2 - \Phi A(\bar{z}Z + z\bar{Z}) + \phi U(z\bar{z}) + V(\phi). \tag{113}$$

However $\Phi \equiv \dot{\phi}$ can be solved from the definitions of p_z, $p_{\bar{z}}$ as

$$\Phi \equiv \dot{\phi} = \frac{-z(p_z - A\phi\bar{Z}) - \bar{z}(p_{\bar{z}} - A\phi Z) + 2\phi(\bar{z}Z + z\bar{Z})}{2Az\bar{z}}. \tag{114}$$

So, on the primary constraint surface the canonical Hamiltonian can also be written as

$$\mathcal{H} = -\frac{\phi}{2z\bar{z}}(\bar{z}Z + z\bar{Z})^2 + \frac{z(p_z - A\phi\bar{Z}) + \bar{z}(p_{\bar{z}} - A\phi Z)}{2z\bar{z}}(\bar{z}Z + z\bar{Z}) + \phi U(z\bar{z}) + V(\phi).$$

The primary Hamiltonian is

$$\mathcal{H}_p = \mathcal{H} + uG^{(1)} + u_Z G_Z^{(1)} + u_{\bar{Z}} G_{\bar{Z}}^{(1)} + u_\pi G_\pi^{(1)} + u_\Pi G_\Pi^{(1)}. \tag{115}$$

The consistency equations are

$$\dot{G}^{(1)} = \{G^{(1)}, \mathcal{H}_p\} = A(z\bar{Z} - \bar{z}Z)(\Phi - u_\pi), \tag{116}$$

$$\dot{G}_Z^{(1)} = \{G_Z^{(1)}, H_p\} = A\bar{z}(\Phi - u_\pi), \tag{117}$$

$$\dot{G}_{\bar{Z}}^{(1)} = \{G_{\bar{Z}}^{(1)}, H_p\} = Az(\Phi - u_\pi), \tag{118}$$

---

[17]Notice that the definition $\Phi \equiv \dot{\phi}$ is necessary to write $\mathcal{H}$ in terms of canonical variables.

where Φ is given by Eq. (114), and

$$\dot{G}_\pi^{(1)} = \{G_\pi^{(1)}, \mathcal{H}_p\} = \frac{1+A}{2z\bar{z}}(z\bar{Z} + \bar{z}Z)^2 - U(z\bar{z}) - V'(\phi) - uA(\bar{z}Z - z\bar{Z}) + A(\bar{z}u_Z + zu_{\bar{z}}), \qquad (119)$$

$$\dot{G}_\Pi^{(1)} = \{G_\Pi^{(1)}, \mathcal{H}_p\} = 0. \qquad (120)$$

Thus, no secondary constraints will appear, since the consistency can be managed by properly choosing the Lagrange multipliers. From the algebra (110), (111), and (112) we recognize one first-class constraint $G_\Pi^{(1)}$ (so u_Π will be left as a free function of t). Among the other four Lagrange multipliers only two of them seem to have been determined: $u_\pi(t) = \Phi(t)$ (however the evolution of Φ is not determined by the Hamilton equations!), and some combination of u, u_Z, and $u_{\bar{z}}$ that makes the result (119) zero. Therefore, four Lagrange multipliers would be left free, which would imply that the evolution of four of the six variables z, $\bar{z}$, Z, $\bar{Z}$, ϕ, Φ are not determined by the Hamilton equations. The fact that some of the Lagrange multipliers u, u_Z, $u_{\bar{z}}$, u_π are not determined by the consistency equations means that the set $G^{(1)}$, $G_Z^{(1)}$, $G_{\bar{Z}}^{(1)}$, $G_\pi^{(1)}$ involves first-class constraints. In fact, the matrix

$$\{G_i^{(1)}, G_j^{(1)}\} = \begin{pmatrix} 0 & 0 & 0 & -A(z\bar{Z}-\bar{z}Z) \\ 0 & 0 & 0 & -A\bar{z} \\ 0 & 0 & 0 & -Az \\ A(z\bar{Z}-\bar{z}Z) & A\bar{z} & Az & 0 \end{pmatrix}$$

$$(121)$$

has rank 2; thus, by combining rows, we can make two of them zero. Concretely, the constraints can be combined to yield

$$\mathcal{G}_1^{(1)} = (z+\bar{z})G^{(1)} - (z\bar{Z}-\bar{z}Z)(G_Z^{(1)} + G_{\bar{Z}}^{(1)}),$$
$$\mathcal{G}_2^{(1)} = zG_Z^{(1)} - \bar{z}G_{\bar{Z}}^{(1)},$$
$$\mathcal{G}_3^{(1)} = zG_Z^{(1)} + \bar{z}G_{\bar{Z}}^{(1)},$$
$$\mathcal{G}_4^{(1)} = G_\pi^{(1)}. \qquad (122)$$

Thus, the algebra (110), (111), and (112) is replaced by

$$\{\mathcal{G}_i^{(1)}, \mathcal{G}_j^{(1)}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2Az\bar{z} \\ 0 & 0 & 2Az\bar{z} & 0 \end{pmatrix}. \qquad (123)$$

Therefore the first-class constraints are $\mathcal{G}_1^{(1)}$, $\mathcal{G}_2^{(1)}$, $G_\Pi^{(1)}$, and the second-class constraints are $\mathcal{G}_3^{(1)}$, $\mathcal{G}_4^{(1)} = G_\pi^{(1)}$.

Then $3+1$ degrees of freedom are removed from the canonical variables z, $\bar{z}$, Z, $\bar{Z}$, ϕ, Φ. Two genuine degrees of freedom are left. This toy model can be regarded as an analogue of $f(R)$ gravity. The extra d.o.f. is then analogous to the well-known propagating extra d.o.f. that results from the trace of the modified Einstein equations in $f(R)$ gravity.

VI. CONCLUSIONS

In order to better understand the nature of the extra degree of freedom in $f(T)$ gravity, we have developed in Sec. III a toy model endowed with local rotational pseudoinvariance, that mimics the pseudoinvariance of the TEGR Lagrangian under local Lorentz transformations of the tetrad field. The nonlinear modification of this system can then be taken as an analogue of $f(T)$ gravity. We have shown that the nonlinear modification of a pseudoinvariant system leads to two different scenarios. In general, one extra d.o.f. should be expected due to the loss of the local rotational pseudoinvariance in the modified system. In the so-called case-(i) solutions, the extra d.o.f manifests itself as a constant of motion affecting the phase of the dynamical variable z; it does not influence the gauge-invariant variable $|z|$, which evolves under the dictates of the (unmodified) original Lagrangian. The other scenario relates to the case-(iii) solutions, which make the modified dynamics work as if they come from an *invariant* Lagrangian [i.e., as if they were case-(ii) solutions]. These solutions do not exhibit an extra d.o.f., but they do exhibit a heavily modified dynamics for the gauge-invariant variables.

The counting of the number of d.o.f., both for the toy model (Sec. III) and $f(T)$ gravity (Sec. IV), relies on the Dirac-Bergmann formalism for constrained Hamiltonian systems, which has been designed to identify the constraints that generate gauge transformations, and to separate the spurious d.o.f. We have summarized the qualitative features of the toy model and TEGR in Table I; the same comparison between the modified toy model and $f(T)$ gravity is found in Table II. In both models, the distinctive feature is the deformation of the constraint algebra due to the loss of the pseudoinvariance. As a consequence, a subset of the Poisson brackets of the constraint algebra becomes different from zero; however, they could remain zero on some trajectories of the phase space, which is the key for the branching of the solutions into case (i) and case (iii). In the case-(i) solutions of $f(T)$ gravity, the scalar

torsion T is a genuine d.o.f. that behaves as a constant of motion. The dynamics of the original gauge-invariant d.o.f. (the components of the metric tensor) is dictated by the equations of TEGR (however the cosmological and Newton constants are shifted as a consequence of the role of E in the Lagrangian density). In the case-(iii) solutions the constraint algebra becomes (on-shell) trivial; the extra d.o.f. does not manifest itself but the metric gets a modified dynamics. Some remnant gauge symmetry can be left in both cases, since TEGR comes not with one but six local Lorentz pseudoinvariances (in $n = 4$ dimensions). We have exemplified the two different scenarios in the context of a FLRW flat cosmology. The present analysis strongly suggests the study of the branching of solutions to $f(T)$ gravity in cases other than the cosmological one. Some other examples of solutions with $T = $ const in modified teleparallel gravity have been documented in Refs. [32,44–48]. Naturally, the toy model cannot cover all the features of $f(T)$ gravity. The toy model is a mechanical pseudoinvariant system, whereas $f(T)$ gravity is a field theory. Because of this reason, there could still be room for $f(T)$ solutions exhibiting the extra d.o.f. but having T different from a constant. The point is that T should not evolve, as shown by the toy model, so we could consider solutions whose T only depends on the spatial coordinates. Such solutions could exhibit both the extra d.o.f. and an effect of modified gravity at the level of the metric tensor. In this regard, the study of exact wave solutions to $f(T)$ equations might be a fertile arena for future research.

Finally, in Sec. V we have contributed to deepening the comparison between $f(R)$ and $f(T)$ gravity by introducing a toy model that is intended to mimic $f(R)$ gravity. This model is *invariant* under local rotations; thus its nonlinear modification does not entail the loss of a local symmetry. However, the model comes with a second-order boundary term, which will be encapsulated in the function f of the modified dynamics. Thus, fourth-order Lagrange equations have to be expected for the modified dynamics, which will cause an extra d.o.f. This toy model is a good analogue of $f(R)$ gravity because the Einstein-Hilbert Lagrangian is made of terms that are separately invariant under local Lorentz transformations of the tetrad (they depend just on the metric). Besides, it includes an inoffensive second-order boundary term that is needed to achieve the invariance under local diffeomorphisms. As is well known, $f(R)$ gravity possesses a *propagating* extra d.o.f., whose dynamics is governed by the trace of the modified Einstein equations. This fact seems to constitute a remarkable difference when compared with $f(T)$ gravity.

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