Scheme-independent series for anomalous dimensions of higher-spin operators at an infrared fixed point in a gauge theory

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We consider an asymptotically free vectorial gauge theory, with gauge group G and N_f fermions in a representation R of G, having an infrared fixed point of the renormalization group. We calculate scheme-independent series expansions for the anomalous dimensions of higher-spin bilinear fermion operators at this infrared fixed point up to $O(\Delta_f^3)$, where Δ_f is an N_f-dependent expansion variable. Our general results are evaluated for several special cases, including the case $G = \mathrm{SU}(N_c)$ with R equal to the fundamental and adjoint representations.

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I. INTRODUCTION

An asymptotically free gauge theory with sufficiently many massless fermions evolves from the deep ultraviolet (UV) to an infrared fixed point (IRFP) of the renormalization group at a zero of the beta function. The theory at this IRFP exhibits scale invariance due to the vanishing of the beta function. The properties of the theory at this IRFP are of fundamental field-theoretic interest. Among the basic properties are the anomalous dimensions $\gamma_{\rm IR}^{(\mathcal{O})}$ of various gauge-invariant operators $\mathcal{O}$.

In this paper we consider an asymptotically free vectorial gauge theory of this type, with a general gauge group G and N_f copies ("flavors") of massless Dirac fermions ψ_i, $i = 1, \ldots, N_f$, transforming according to a representation R of G [1]. We present scheme-independent series expansions of the anomalous dimensions of gauge-invariant higher-spin operators that are bilinear in the fermion fields, up to $O(\Delta_f^3)$ inclusive, at the infrared fixed point, where Δ_f is an N_f-dependent expansion variable defined below, in Eq. (1.8). The operators that we consider are of the form (suppressing flavor indices) $\bar\psi\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$ and $\bar\psi\sigma_{\lambda\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$, where D_μ is the covariant derivative for the gauge theory, and it is understood here and below that the operators are symmetrized over the Lorentz indices μ_i, $1 \le i \le j$ and have Lorentz traces subtracted, and $\sigma_{\lambda\mu_1}$ is the commutator of two Dirac matrices [defined in Eq. (2.3)]. We consider the cases $1 \le j \le 3$.

The operators $\bar\psi\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$ were considered early on in the analysis of approximate Bjorken scaling in deep inelastic lepton scattering and the associated development of the theory of quantum chromodynamics (QCD). We briefly review this background [2–13]. In Euclidean quantum field theory, the short-distance operator product expansion (OPE) expresses the product of two operators $A(x)$ and $B(y)$ as a sum of local operators $\mathcal{O}_i$ multiplied by coefficient functions $c_{\mathcal{O}_i}$,

$$A(x)B(y) = \sum_i c_{\mathcal{O}_i}(x-y)\mathcal{O}_i((x+y)/2), \quad (1.1)$$

in the limit where $x - y \to 0$. Let us denote the Maxwellian (i.e., free-field) dimension of an operator $\mathcal{O}$ in mass units as $d_{\mathcal{O}}$. Then the (free-field) dimension of the coefficient function is $d_{c_{\mathcal{O}_i}} = d_A + d_B - d_{\mathcal{O}_i}$, so

$$c_{\mathcal{O}_i}(x-y) \sim |x-y|^{d_{\mathcal{O}_i}-d_A-d_B}, \quad (1.2)$$

where $|x-y|$ refers to the Euclidean distance. Hence, in the short-distance OPE, the operators with the lowest dimensions dominate, since they are multiplied by the smallest powers of $|x-y|$. However, deep inelastic scattering and the associated Bjorken limit probe the light cone limit, $(x-y)^2 \to 0$ with $x - y \neq 0$ in Minkowski space, where $x^2 = x_\mu x^\mu$. With the arguments of two illustrative Lorentz-scalar operators denoted in a symmetric manner as $\pm x/2$, the light-cone OPE for $A(x/2)B(-x/2)$ is

$$A(x/2)B(-x/2) = \sum_{i,n} \bar{c}_{i,n}(x^2) x^{\mu_1} \cdots x^{\mu_n} \mathcal{O}_{i,n;\mu_1,\ldots,\mu_n}(0) \tag{1.3}$$

in the limit $x^2 \to 0$, where the coefficient functions have been written in a form that explicitly indicates the factor $x^{\mu_1} \cdots x^{\mu_n}$ and the operator $\mathcal{O}_{i,n;\mu_1,\ldots,\mu_n}$ has spin $j = n$. Here (suppressing the Lorentz indices on $\mathcal{O}_{i,n;\mu_1,\ldots,\mu_n}$) the dependence of $\bar{c}_{i,n}$ on x^2 is

$$\bar{c}_{i,n}(x^2) \sim (x^2)^{(d_{\mathcal{O}_{i,n}} - n - d_A - d_B)/2} \tag{1.4}$$

(with logarithmic corrections in QCD due to anomalous dimensions). Consequently, the operators that have the strongest singularity in their coefficient function $\bar{c}_{i,n}(x^2)$ as $x^2 \to 0$ and hence make the dominant contribution to the right-hand side of the light-cone OPE, Eq. (1.3) are those with minimal "twist" τ [7], where τ is the dimension minus the spin j of the operator, i.e.,

$$\tau_{\mathcal{O}_{i,n}} = d_{\mathcal{O}_{i,n}} - j_{\mathcal{O}_{i,n}}, \tag{1.5}$$

with $j_{\mathcal{O}_{i,n}} = n$ here. Thus, among bilinear fermion operators, in addition to $\bar{\psi}\gamma_{\mu_1}\psi$ with dimension 3, spin 1, and hence $\tau = 2$, there are the operators $\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j}\psi$, with dimension $3 + (j-1)$ and spin j, which also have $\tau = 2$. These are the minimum-twist bilinear fermion operators that contribute to the light-cone OPE (1.3) [14]. In a similar manner, twist-2 operators make the dominant contribution to the right-hand side of the light-cone OPE for the product of two electromagnetic or weak currents. The other operators that we consider, namely $\bar{\psi}\sigma_{\lambda\mu_1} D_{\mu_2} \ldots D_{\mu_j}\psi$, have been relevant for the study of transversity distributions in QCD [15].

Our approach here is complementary to these previous analyses of higher-spin operators, which have focused on applications to QCD. In contrast, we study the anomalous dimensions of these operators at an infrared fixed point in a (deconfined) chirally symmetric non-Abelian Coulomb phase (NACP), where the theory is scale invariant and is inferred to be conformally invariant [16], hence the commonly used term "conformal window." The goal of our calculations is to gain information about the properties of the conformal field theory that is defined at this IRFP.

Let us recall some further relevant background for our work. The evolution of the running gauge coupling $g = g(\mu)$, as a function of the momentum scale, μ, is described by the renormalization-group (RG) beta function $\beta = d\alpha/d\ln\mu$, where $\alpha(\mu) = g(\mu)^2/(4\pi)$. From the one-loop term in the beta function [10,11], it follows that the property of asymptotic freedom restricts N_f to be less than an upper (u) bound, N_u, where [17]

$$N_u = \frac{11C_A}{4T_f}. \tag{1.6}$$

Here, C_A is the quadratic Casimir invariant for the group G and T_f is the trace invariant for the representation R [18]. If N_f is slightly less than N_u, then this theory has an infrared zero in the (perturbatively calculated) beta function, i.e., an IR fixed point of the renormalization group, at a value that we shall denote $\alpha_{\rm IR}$ [19,20]. In the two-loop beta function (with $N_f < N_u$ as required by asymptotic freedom), this IR zero is present if N_f is larger than a lower (ℓ) value N_ℓ, where [19]

$$N_\ell = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}. \tag{1.7}$$

As the scale μ decreases from large values in the UV to small values in the IR, $\alpha(\mu)$ approaches $\alpha_{\rm IR}$ from below as $\mu \to 0$. Here we consider the properties of the theory at this IRFP in the perturbative beta function. (For a discussion of an IR zero in a nonperturbatively defined beta function and its application to QCD, see [21].)

Since the anomalous dimensions of gauge-invariant operators evaluated at the IRFP are physical, they must be independent of the scheme used for regularization and renormalization. In the conventional approach, one first expresses these anomalous dimensions as series expansions in powers of α or equivalently $a = g^2/(16\pi^2) = \alpha/(4\pi)$, calculated to n-loop order; second, one computes the IR zero of the beta function, denoted $\alpha_{{\rm IR},n}$, to the same n-loop order; and third, one sets $\alpha = \alpha_{{\rm IR},n}$ in the series expansion for the given anomalous dimension to obtain its value at the IR zero of the beta function to this n-loop order. For the operator $\bar{\psi}\psi$ this conventional approach to calculate anomalous dimensions at an IR fixed point was carried out to the four-loop level in [22–24] and to the five-loop level in [25]. However, these conventional series expansions in powers of α, calculated to a finite order, are scheme-dependent beyond the leading terms. This is a well-known property of higher-order QCD calculations used to fit actual experimental data, which, in turn, has motivated many studies to reduce scheme dependence [26]. These studies dealt with the UV fixed point (UVFP) at $\alpha = 0$, as is appropriate for QCD. Studies of scheme dependence of quantities calculated in a conventional manner at an IR fixed point at $\alpha_{\rm IR}$ were carried out in [27–31]. In particular, it was shown that many scheme transformations that are admissible in the vicinity of the UVFP at $\alpha = 0$ in an asymptotically free theory are not admissible away from the origin because of various pathological properties. For example, the scheme transformation $ra = \tanh(ra')$ (depending on a parameter r) is an admissible transformation in the neighborhood of $\alpha = \alpha' = 0$. However, the inverse of this transformation is $a' = (2r)^{-1}\ln[(1+ra)/(1-ra)]$, which is singular at an IRFP with $a_{\rm IR} \geq 1/r$, i.e.,

$\alpha_{IR} \geq 4\pi/r$, so that the transformation is not admissible at this IRFP. References [27] derived and studied an explicit scheme transformation that removes terms of loop order 3 and higher from the beta function in the local vicinity of $\alpha = 0$, as is relevant to the UVFP in QCD [32], but also showed that such a scheme transformation cannot, in general, be used at an IRFP away from the origin owing to various pathologies, one of which was illustrated above.

It is thus desirable to use a theoretical framework in which the series expansions for physical quantities, such as anomalous dimensions of gauge-invariant operators at the IRFP, are scheme-independent at any finite order in an expansion variable. Because $\alpha_{IR} \to 0$ as N_f approaches N_u from below (where N_f is formally generalized here from a non-negative integer to a non-negative real number [17]), one can reexpress the expansions for physical quantities at the IRFP as power series in the manifestly scheme-independent quantity [20,33]

$$\Delta_f = N_u - N_f. \tag{1.8}$$

In previous work we have calculated scheme-independent expansions for anomalous dimensions of several types of gauge-invariant operators at an IRFP in an asymptotically free gauge theory [34–40]. We have compared the resultant values for anomalous dimensions with lattice measurements where available [35–37,41,42].

In the present paper we extend these calculations to the case of the higher-spin operators $\bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$ and $\bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$ for $1 \leq j \leq 3$. In addition to general formulas, we present results for several different special cases, including the case where $G = \mathrm{SU}(N_c)$ and the fermions are in the fundamental (F) and adjoint (Adj) representations. We also give results for the limit $N_c \to \infty$ and $N_f \to \infty$ with the ratio N_f/N_c fixed and finite. Our calculations show that these scheme-independent expansions of the anomalous dimensions of the operators are reasonably accurate throughout much of the non-Abelian Coulomb phase. Our results give further insight into the properties of a theory at an IRFP and should be useful to compare with lattice measurements of the anomalous dimensions of these higher-spin operators when such measurements will be performed [43].

This paper is organized as follows. Some relevant background and methods are discussed in Sec. II. General structural forms for the anomalous dimensions of higher-spin bilinear fermion operators are given in Sec. III. In Sec. IV we present our scheme-independent calculations of the anomalous dimensions of these higher-spin Wilson operators for a general gauge group G and fermion representation R. In Sec. V we give results for the case where $G = \mathrm{SU}(N_c)$ and R is the fundamental representation, and in Sec. VI we present the special case of these results for the limit $N_c \to \infty$ and $N_f \to \infty$ with N_f/N_c fixed and finite. Anomalous dimension calculations for the case where $G = \mathrm{SU}(N_c)$ and R is the adjoint representation are presented in Sec. VII. Our conclusions are given in Sec. VIII and some auxiliary results are included in Appendix.

II. CALCULATIONAL METHODS

Let us consider a (gauge-invariant) operator $\mathcal{O}$. Because of the interactions, the full scaling dimension of this operator, denoted $D_{\mathcal{O}}$, differs from its free-field value, $D_{\mathcal{O},\mathrm{free}} \equiv d_{\mathcal{O}}$:

$$D_{\mathcal{O}} = D_{\mathcal{O},\mathrm{free}} - \gamma_{\mathcal{O}}, \tag{2.1}$$

where $\gamma_{\mathcal{O}}$ is the anomalous dimension of the operator [44]. Since $\gamma_{\mathcal{O}}$ arises from the gauge interaction, it can be expressed as the power series

$$\gamma^{(\mathcal{O})} = \sum_{\ell=1}^{\infty} c_{\gamma,\ell}^{(\mathcal{O})} a^{\ell}, \tag{2.2}$$

where $c_{\gamma,\ell}^{\mathcal{O}}$ is the ℓ-loop coefficient.

As stated in the introduction, we shall consider the gauge-invariant operators $\mathcal{O}_{\mu_1\ldots\mu_j} = \bar{\psi}\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$ and $\mathcal{O}_{\lambda\mu_1\ldots\mu_j} = \bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}\ldots D_{\mu_j}\psi$, where

$$\sigma_{\lambda\mu_1} = \frac{i}{2}[\gamma_\lambda, \gamma_{\mu_1}]. \tag{2.3}$$

We focus on the operators with $1 \leq j \leq 3$. We introduce the following compact notation for these operators:

$$\mathcal{O}^{(\gamma D)}_{\mu_1\mu_2} \equiv \bar{\psi}\gamma_{\mu_1}D_{\mu_2}\psi, \tag{2.4}$$

$$\mathcal{O}^{(\gamma DD)}_{\mu_1\mu_2\mu_3} \equiv \bar{\psi}\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}\psi, \tag{2.5}$$

$$\mathcal{O}^{(\gamma DDD)}_{\mu_1\mu_2\mu_3\mu_4} \equiv \bar{\psi}\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}\psi, \tag{2.6}$$

$$\mathcal{O}^{(\sigma D)}_{\lambda\mu_1\mu_2} \equiv \bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}\psi, \tag{2.7}$$

$$\mathcal{O}^{(\sigma DD)}_{\lambda\mu_1\mu_2\mu_3} \equiv \bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_3}\psi, \tag{2.8}$$

and

$$\mathcal{O}^{(\sigma DDD)}_{\lambda\mu_1\mu_2\mu_3} \equiv \bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}\psi. \tag{2.9}$$

For brevity of notation, we suppress the flavor indices on the fields ψ.

For a given operator $\mathcal{O}$, we write the scheme-independent expansion of its anomalous dimension $\gamma^{(\mathcal{O})}$ evaluated at the IRFP, denoted $\gamma_{IR}^{(\mathcal{O})}$, as

$$\gamma_{\rm IR}^{(\mathcal{O})} = \sum_{n=1}^{\infty} \kappa_n^{(\mathcal{O})} \Delta_f{}^n. \qquad (2.10)$$

The truncation of right-hand side of Eq. (2.10) to maximal power p is denoted

$$\gamma_{{\rm IR},\Delta_f^p}^{(\mathcal{O})} = \sum_{n=1}^{p} \kappa_n^{(\mathcal{O})} \Delta_f{}^n. \qquad (2.11)$$

We use a further shorthand notation for the anomalous dimensions in which the superscript in $\gamma_{\rm IR}^{(\mathcal{O})}$ is replaced by a symbol for the quantity standing between $\bar\psi$ and ψ in the operator $\mathcal{O}$. These shorthand symbols are as follows: $\gamma_{\rm IR}^{(\gamma D)}$ for the anomalous dimension of the operator $\mathcal{O}_{\mu_1\mu_2}^{(\gamma D)} = \bar\psi\gamma_{\mu_1} D_{\mu_2}\psi$ at the IRFP, and so forth for the other operators. In comparing with our previous calculations in [34–39], we also use the notation $\gamma_{\rm IR}^{(1)}$ and $\gamma_{\rm IR}^{(\sigma)}$ for the anomalous dimensions of $\bar\psi\psi$ and $\bar\psi\sigma_{\lambda\mu_1}\psi$ at the IRFP. (The anomalous dimension $\gamma_{\rm IR}^{(\sigma)}$ was denoted $\gamma_{T,{\rm IR}}$ in [36], where the subscript T referred to the Dirac tensor $\sigma_{\mu\nu}$.)

As discussed in [34,36], the calculation of the coefficient $\kappa_n^{(\mathcal{O})}$ in Eq. (2.10) requires, as inputs, the beta function coefficients at loop order $1 \le \ell \le n+1$ and the anomalous dimension coefficients $c_{\gamma,\ell}^{(\mathcal{O})}$ at loop order $1 \le \ell \le n$. The method of calculation requires that the IR fixed point must be exact, which is the case in the non-Abelian Coulomb phase. As in our earlier work [34–39], we thus restrict our consideration to the non-Abelian Coulomb phase (conformal window) [45]. For a given gauge group G and fermion representation R, the conformal window extends from an upper end at $N_f = N_u$ to a lower end at a value that is commonly denoted $N_{f,cr}$. In contrast to the exactly known value of N_u [given in Eq. (1.6)], the value of $N_{f,cr}$ is not precisely known and has been investigated extensively for several groups G and fermion representations R [41,42,45]. For values of N_f in the non-Abelian Coulomb phase such that Δ_f is not too large, one may expect the expansion (2.10) of $\gamma_{\rm IR}^{(\mathcal{O})}$ in a series in powers of Δ_f to yield reasonably accurate perturbative calculations of the anomalous dimension. In our earlier works, using our explicit calculations, we have shown that this is, in fact, the case.

We recall some relevant properties of the theory regarding global flavor symmetries. Because the N_f fermions are massless, the Lagrangian is invariant under the classical global flavor (fl) symmetry $G_{fl,cl} = {\rm U}(N_f)_L \otimes {\rm U}(N_f)_R$, or equivalently,

$$G_{fl,cl} = {\rm SU}(N_f)_L \otimes {\rm SU}(N_f)_R \otimes {\rm U}(1)_V \otimes {\rm U}(1)_A \qquad (2.12)$$

(where V and A denote vector and axial-vector). The ${\rm U}(1)_V$ represents fermion number, which is conserved by the bilinear operators that we consider. The ${\rm U}(1)_A$ symmetry is broken by instantons, so the actual nonanomalous global flavor symmetry is

$$G_{fl} = {\rm SU}(N_f)_L \otimes {\rm SU}(N_f)_R \otimes {\rm U}(1)_V. \qquad (2.13)$$

This G_{fl} symmetry is respected in the non-Abelian Coulomb phase, since there is no spontaneous chiral symmetry breaking in this phase [41,42]. For our operators, the flavor matrix between $\bar\psi$ and ψ is either the identity or the operator T_a, a generator of ${\rm SU}(N_f)$, which can be viewed as acting either to the right on ψ or to the left on $\bar\psi$. These yield the same anomalous dimensions [46]. As a consequence of the unbroken global flavor symmetry, our operators transform as representations of the global flavor group G_{fl}. The invariance under the full G_{fl} in the non-Abelian Coulomb phase is different from the situation in the QCD-like phase at smaller N_f, where the chiral part of G_{fl} is spontaneously broken by the QCD bilinear quark condensate to the vectorial subgroup ${\rm SU}(N_f)_V$ and operators are classified according to whether they are singlet or nonsinglet (adjoint) under this vectorial ${\rm SU}(N_f)$ symmetry. In particular, in the consideration of flavor-singlet operators, in QCD, one must take into account mixing with gluonic operators. Here, in contrast, there is no mixing between any of our bilinear fermion operators and gluonic operators, since the latter are singlets under G_{fl}.

The operators $\mathcal{O}$ with an even number of Dirac γ matrices, symbolically denoted Γ_e, link left with right chiral components of ψ, while the operators $\mathcal{O}$ with an odd number of Dirac γ matrices, Γ_o, link left with left and right with right components:

$$\bar\psi\Gamma_e\psi = \bar\psi_L\Gamma_e\psi_R + \bar\psi_R\Gamma_e\psi_L, \qquad (2.14)$$

$$\bar\psi\Gamma_o\psi = \bar\psi_L\Gamma_o\psi_L + \bar\psi_R\Gamma_o\psi_R, \qquad (2.15)$$

where $\bar\psi = \psi^\dagger\gamma_0$. In the non-Abelian Coulomb phase where the flavor symmetry is (2.13), one may regard the T_b in the term $\bar\psi_L T_b \psi_R$ acting to the right as an element of ${\rm SU}(N_f)_R$ and acting to the left as an element of ${\rm SU}(N_f)_L$.

Given that the theory at the IR fixed point is conformally invariant [16], there is an important lower bound on the full dimension of an operator $\mathcal{O}$ and hence, with our definition (2.1), an upper bound on the anomalous dimension $\gamma^{(\mathcal{O})}$ that follows from the conformal invariance. To state this, we first recall that a (finite-dimensional) representation of the Lorentz group is specified by the set (j_1, j_2), where j_1 and j_2 take on non-negative integral or half-integral values [47]. A Lorentz scalar operator thus transforms as (0, 0), a Lorentz vector as $(1/2, 1/2)$, an antisymmetric tensor like the field-strength tensor $F_{\mu\nu}^a$ as $(1,0) \oplus (0,1)$, etc. Then the requirement of unitarity in a conformally invariant theory implies the lower bound [48]

$$D_{\mathcal{O}} \geq j_1 + j_2 + 1, \tag{2.16}$$

i.e., the upper bound

$$\gamma_{\mathcal{O}} \leq D_{\mathcal{O},\mathrm{free}} - (j_1 + j_2 + 1). \tag{2.17}$$

We have studied the constraints from the upper bound (2.17) in our previous calculations of anomalous dimensions in [22,25,36–39]. Anticipating the results given below, since our calculations yield negative values for the anomalous dimensions of higher-spin Wilson operators, they obviously satisfy these conformality upper bounds.

III. SOME GENERAL STRUCTURAL PROPERTIES OF $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$

From our previous calculations [34–39] for the anomalous dimensions of $\bar{\psi}\psi$ and $\bar{\psi}\sigma_{\mu\nu}\psi$, in conjunction with our new results on the anomalous dimensions $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$ of higher-spin twist-2 bilinear fermion operators $\mathcal{O}$, we find some general structural properties of the coefficients $\kappa_n^{(\mathcal{O})}$ in the scheme-independent series expansions of the anomalous dimensions $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$. These involve various group invariants, including the quadratic Casimir invariants $C_A \equiv C_2(G)$, $C_f \equiv C_2(R)$, the trace invariant $T(R)$, and the quartic trace invariants $d_R^{abcd} d_{R'}^{abcd}/d_A$, where d_A denotes the dimension of the adjoint representation [18,49]. For compact notation, it is convenient to define a factor that occurs in the denominators of these $\kappa_n^{(\mathcal{O})}$ coefficients, namely.

$$D = 7C_A + 11C_f \tag{3.1}$$

(not to be confused with covariant derivative). We exhibit this general form here, using $a_{j,k}^{(\mathcal{O})}$ for various (constant) numerical coefficients:

$$\kappa_1^{(\mathcal{O})} = c_1^{(\mathcal{O})} \frac{C_f T_f}{C_A D}, \tag{3.2}$$

$$\kappa_2^{(\mathcal{O})} = \frac{C_f T_f^2 (a_{2,1}^{(\mathcal{O})} C_A^2 + a_{2,2}^{(\mathcal{O})} C_A C_f + a_{2,3}^{(\mathcal{O})} C_f^2)}{C_A^2 D^3}, \tag{3.3}$$

and

$$\kappa_3^{(\mathcal{O})} = \frac{C_f T_f}{C_A^4 D^5} \Bigg[a_{3,1}^{(\mathcal{O})} C_A^5 T_f^2 + a_{3,2}^{(\mathcal{O})} C_A^4 C_f T_f^2 + a_{3,3}^{(\mathcal{O})} C_A^3 C_f^2 T_f^2 + a_{3,4}^{(\mathcal{O})} C_A^2 C_f^3 T_f^2 + a_{3,5}^{(\mathcal{O})} C_A C_f^4 T_f$$
$$+ a_{3,6}^{(\mathcal{O})} C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A} + a_{3,7}^{(\mathcal{O})} C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A} + a_{3,8}^{(\mathcal{O})} C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A} + a_{3,9}^{(\mathcal{O})} C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}$$
$$+ a_{3,10}^{(\mathcal{O})} C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A} + a_{3,11}^{(\mathcal{O})} C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A} \Bigg]. \tag{3.4}$$

IV. ANOMALOUS DIMENSIONS $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$ OF HIGHER-SPIN OPERATORS

A. General

In this section we present the results of our calculations of the coefficients in the scheme-independent series expansions up to $O(\Delta_f^3)$ for the various higher-spin operators considered here. As was noted above, the calculation of the $O(\Delta_f^n)$ coefficient, $\kappa_n^{(\mathcal{O})}$, for the anomalous dimension of an operator $\mathcal{O}$ at the IRFP requires, as inputs, the beta function coefficients at loop order $1 \leq \ell \leq n+1$ and the anomalous dimension coefficients $c_\ell^{(\mathcal{O})}$ at loop order $1 \leq \ell \leq n$. Hence, we use the beta function coefficients from one-loop up to the four-loop level [10,19,50,51], together with the anomalous dimension coefficients calculated in the conventional series expansion in powers of a up to the three-loop level [11,46,52–57]. The higher-order terms in the beta function and anomalous dimensions that we use have been calculated in the $\overline{\mathrm{MS}}$ scheme [58], but our results are independent of this since they are scheme-independent. (The beta function has actually been calculated up to five-loop order [59,60], but these results will not be needed here.)

B. $\gamma_{\mathrm{IR}}^{(\gamma D)}$

For the anomalous dimension $\gamma_{\mathrm{IR}}^{(\gamma D)}$ of the operator $\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \psi$ at the IRFP, we calculate

$$\kappa_1^{(\gamma D)} = -\frac{2^6 C_f T_f}{3^2 C_A D}, \tag{4.1}$$

$$\kappa_2^{(\gamma D)} = \frac{2^5 C_f T_f^2 (693 C_A^2 - 3104 C_A C_f - 1540 C_f^2)}{3^5 C_A^2 D^3}, \tag{4.2}$$

and

$$\kappa_3^{(\gamma D)} = -\frac{2^5 C_f T_f}{3^8 C_A^4 D^5}\Big[C_A^5 T_f^2(-202419+1016064\zeta_3) + C_A^4 C_f T_f^2(2764440+145152\zeta_3) + C_A^3 C_f^2 T_f^2(8940028-5038848\zeta_3) + C_A^2 C_f^3 T_f^2(-7341488-1140480\zeta_3) + C_A C_f^4 T_f^2(3841024+5018112\zeta_3) + C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-161280+4257792\zeta_3) + C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-253440+6690816\zeta_3) + C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(2838528-27675648\zeta_3) + C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(4460544-43490304\zeta_3) + C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-10733184+23417856\zeta_3) + C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-16866432+36799488\zeta_3)\Big]. \tag{4.3}$$

In these expressions and the following ones, we have indicated the factorizations of the numbers in the denominators, since they are rather simple. In general, the numbers in the numerators do not have such simple factorizations.

With these coefficients, the anomalous dimension $\gamma_{\rm IR}^{(\gamma D)}$ calculated to order $O(\Delta_f^p)$, denoted $\gamma_{{\rm IR},F,\Delta_f^p}^{(\gamma D)}$, is given by Eq. (2.11) with $\mathcal{O} = \bar\psi\gamma_{\mu_1} D_{\mu_2}\psi$. Our results here yield $\gamma_{{\rm IR},F,\Delta_f^p}^{(\gamma D)}$ with $p = 1, 2, 3$. Analogous statements apply to the anomalous dimensions of the other operators for which we have performed calculations, and we proceed to present the coefficients for these next.

C. $\gamma_{\rm IR}^{(\gamma DD)}$

For the anomalous dimension $\gamma_{\rm IR}^{(\gamma DD)}$ of the operator $\bar\psi\gamma_{\mu_1} D_{\mu_2} D_{\mu_3}\psi$ at the IRFP, we calculate

$$\kappa_1^{(\gamma DD)} = -\frac{100 C_f T_f}{3^2 C_A D}, \tag{4.4}$$

$$\kappa_2^{(\gamma DD)} = \frac{10 C_f T_f^2(5103 C_A^2 - 14017 C_A C_f - 9383 C_f^2)}{3^5 C_A^2 D^3}, \tag{4.5}$$

and

$$\kappa_3^{(\gamma DD)} = -\frac{10 C_f T_f}{3^8 C_A^4 D^5}\Big[C_A^5 T_f^2(1538649+2794176\zeta_3) + C_A^4 C_f T_f^2(14860881+399168\zeta_3) + C_A^3 C_f^2 T_f^2(40821518-13856832\zeta_3) + C_A^2 C_f^3 T_f^2(-35403412-3136320\zeta_3) + C_A C_f^4 T_f^2(19308575+13799808\zeta_3) + C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-806400+21288960\zeta_3) + C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-1267200+33454080\zeta_3) + C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(14192640-138378240\zeta_3) + C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(22302720-217451520\zeta_3) + C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-53665920+117089280\zeta_3) + C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-84332160+183997440\zeta_3)\Big]. \tag{4.6}$$

D. $\gamma_{\rm IR}^{(\gamma DDD)}$

Proceeding to the anomalous dimension $\gamma_{\rm IR}^{(\gamma DDD)}$ of the operator $\bar\psi\gamma_{\mu_1} D_{\mu_2} D_{\mu_3} D_{\mu_4}\psi$ at the IRFP, we find

$$\kappa_1^{(\gamma DDD)} = -\frac{628 C_f T_f}{3^2 \cdot 5 C_A D}, \tag{4.7}$$

$$\kappa_2^{(\gamma DDD)} = \frac{2 C_f T_f^2(4550175 C_A^2 - 10373329 C_A C_f - 7719767 C_f^2)}{3^5 \cdot 5^3 C_A^2 D^3}, \tag{4.8}$$

and

$$\kappa_3^{(\gamma DDD)} = \frac{2C_fT_f}{3^8\cdot 5^5 C_A^4 D^5}\bigg[C_A^5T_f^2(-67181774625-45691128000\zeta_3)+C_A^4C_fT_f^2(-318706112025-6527304000\zeta_3)$$
$$+C_A^3C_f^2T_f^2(-720947009518+226590696000\zeta_3)+C_A^2C_f^3T_f^2(709569531572+51285960000\zeta_3)$$
$$+C_AC_f^4T_f^2(-433168554247-225658224000\zeta_3)$$
$$+C_AT_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(15825600000-417795840000\zeta_3)+C_fT_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(24868800000-6565363200000\zeta_3)$$
$$+C_A^2T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(-278530560000+2715672960000\zeta_3)$$
$$+C_AC_fT_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(-437690880000+4267486080000\zeta_3)$$
$$+C_A^3\frac{d_R^{abcd}d_R^{abcd}}{d_A}(1053193680000-2297877120000\zeta_3)$$
$$+C_A^2C_f\frac{d_R^{abcd}d_R^{abcd}}{d_A}(1655018640000-3610949760000\zeta_3)\bigg]. \tag{4.9}$$

E. $\gamma_{\mathrm{IR}}^{(\sigma D)}$

For the anomalous dimension $\gamma_{\mathrm{IR}}^{(\sigma D)}$ of the operator $\bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}\psi$ at the IRFP, we calculate

$$\kappa_1^{(\sigma D)} = -\frac{8C_fT_f}{C_AD}, \tag{4.10}$$

$$\kappa_2^{(\sigma D)} = \frac{4C_fT_f^2(77C_A^2-348C_AC_f-176C_f^2)}{3C_A^2D^3}, \tag{4.11}$$

$$\kappa_3^{(\sigma D)} = \frac{4C_fT_f}{3^4C_A^4D^5}\bigg[13083C_A^5T_f^2-240492C_A^4C_fT_f^2-819408C_A^3C_f^2T_f^2+738144C_A^2C_f^3T_f^2-662112C_AC_f^4T_f^2$$
$$+C_AT_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(17920-473088\zeta_3)+C_fT_f^2\frac{d_A^{abcd}d_A^{abcd}}{d_A}(28160-743424\zeta_3)$$
$$+C_A^2T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(-315392+3075072\zeta_3)+C_AC_fT_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(-495616+4832256\zeta_3)$$
$$+C_A^3\frac{d_R^{abcd}d_R^{abcd}}{d_A}(1192576-2601984\zeta_3)+C_A^2C_f\frac{d_R^{abcd}d_R^{abcd}}{d_A}(1874048-4088832\zeta_3)\bigg]. \tag{4.12}$$

F. $\gamma_{\mathrm{IR}}^{(\sigma DD)}$

For the anomalous dimension $\gamma_{\mathrm{IR}}^{(\sigma DD)}$ we calculate

$$\kappa_1^{(\sigma DD)} = -\frac{104C_fT_f}{3^2C_AD}, \tag{4.13}$$

$$\kappa_2^{(\sigma DD)} = \frac{4C_fT_f^2(12537C_A^2-36292C_AC_f-22352C_f^2)}{3^5C_A^2D^3}, \tag{4.14}$$

and

$$\kappa_3^{(\sigma DD)} = -\frac{2^2 C_f T_f}{3^8 C_A^4 D^5}\Big[C_A^5 T_f^2(2935737 + 4064256\zeta_3) + C_A^4 C_f T_f^2(39906468 + 580608\zeta_3)$$
$$+ C_A^3 C_f^2 T_f^2(107242456 - 20155392\zeta_3) + C_A^2 C_f^3 T_f^2(-102128048 - 4561920\zeta_3)$$
$$+ C_A C_f^4 T_f^2(43045024 + 20072448\zeta_3)$$
$$+ 3C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-698880 + 18450432\zeta_3) + 3C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-1098240 + 28993536\zeta_3)$$
$$+ 3C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(12300288 - 119927808\zeta_3) + 3C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(19329024 - 188457984\zeta_3)$$
$$+ 3C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-46510464 + 101477376\zeta_3) + 3C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-73087872 + 159464448\zeta_3)\Big]. \quad (4.15)$$

G. $\gamma_{\mathrm{IR}}^{(\sigma DDD)}$

Finally, for the anomalous dimension $\gamma_{\mathrm{IR}}^{\sigma DDD)}$ we obtain

$$\kappa_1^{(\sigma DDD)} = -\frac{2^7 C_f T_f}{3^2 C_A D}, \quad (4.16)$$

$$\kappa_2^{(\sigma DDD)} = \frac{2^3 C_f T_f^2(9219C_A^2 - 21185C_A C_f - 15664C_f^2)}{3^5 C_A^2 D^3}, \quad (4.17)$$

and

$$\kappa_3^{(\sigma DDD)} = -\frac{2^3 C_f T_f}{3^8 C_A^4 D^5}\Big[C_A^5 T_f^2(5213502 + 2667168\zeta_3) + C_A^4 C_f T_f^2(25185069 + 381024\zeta_3)$$
$$+ C_A^3 C_f^2 T_f^2(58268711 - 13226976\zeta_3) + C_A^2 C_f^3 T_f^2(-56962840 - 2993760\zeta_3)$$
$$+ C_A C_f^4 T_f^2(36476660 + 13172544\zeta_3)$$
$$+ C_A T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-1290240 + 34062336\zeta_3) + C_f T_f^2 \frac{d_A^{abcd} d_A^{abcd}}{d_A}(-2027520 + 53526528\zeta_3)$$
$$+ C_A^2 T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(22708224 - 221405184\zeta_3) + C_A C_f T_f \frac{d_R^{abcd} d_A^{abcd}}{d_A}(35684352 - 347922432\zeta_3)$$
$$+ C_A^3 \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-85865472 + 187342848\zeta_3) + C_A^2 C_f \frac{d_R^{abcd} d_R^{abcd}}{d_A}(-134931456 + 294395904\zeta_3)\Big]. \quad (4.18)$$

V. EVALUATION OF $\kappa_n^{(\mathcal{O})}$ FOR $G = \mathrm{SU}(N_c)$ AND $R = F$

In this section we evaluate our general results for these anomalous dimensions $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$ in the important special case where the gauge group is $G = \mathrm{SU}(N_c)$ and the N_f fermions are in the fundamental representation of this group, $R = F$.

A. $\gamma_{\mathrm{IR},\mathrm{SU}(N_c),F}^{(\gamma D)}$

Substituting $G = \mathrm{SU}(N_c)$ and $R = F$ in our general results (4.1)–(4.3), we obtain

$$\kappa_{1,\mathrm{SU}(N_c),F}^{(\gamma D)} = -\frac{2^5(N_c^2 - 1)}{3^2 N_c(25N_c^2 - 11)}, \quad (5.1)$$

$$\kappa_{2,\mathrm{SU}(N_c),F}^{(\gamma D)} = -\frac{2^5(N_c^2 - 1)(1244N_c^4 - 2322N_c^2 + 385)}{3^5 N_c^2(25N_c^2 - 11)^3}, \quad (5.2)$$

and

$$\kappa^{(\gamma D)}_{3,\mathrm{SU}(N_c),F} = -\frac{2^6(N_c^2-1)}{3^8 N_c^3(25N_c^2-11)^5}[2137786N_c^8 + (1831104 - 9784800\zeta_3)N_c^6 + (-15928259 + 36575712\zeta_3)N_c^4 + (6282342 - 14911776\zeta_3)N_c^2 + 240064 + 313632\zeta_3]. \tag{5.3}$$

Then, for this case $G = \mathrm{SU}(3)$, $R = F$, the anomalous dimension $\gamma^{(\gamma D)}_{\mathrm{IR}}$ calculated to order $O(\Delta_f^p)$, denoted $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^p}$, is given by Eq. (2.11) with $\mathcal{O} = \bar{\psi}\gamma_{\mu_1} D_{\mu_2}\psi$.

B. $\gamma^{(\gamma DD)}_{\mathrm{IR},\mathrm{SU}(N_c),F}$

Substituting $G = \mathrm{SU}(N_c)$ and $R = F$ in our general results (4.4)–(4.6), we obtain

$$\kappa^{(\gamma DD)}_{1,\mathrm{SU}(N_c),F} = -\frac{50(N_c^2-1)}{3^2 N_c(25N_c^2-11)}, \tag{5.4}$$

$$\kappa^{(\gamma DD)}_{2,\mathrm{SU}(N_c),F} = -\frac{5(N_c^2-1)(17005N_c^4 - 46800N_c^2 + 9383)}{2\cdot 3^5 N_c^2(25N_c^2-11)^3}, \tag{5.5}$$

and

$$\kappa^{(\gamma DD)}_{3,\mathrm{SU}(N_c),F} = -\frac{5(N_c^2-1)}{2^2\cdot 3^8 N_c^3(25N_c^2-11)^5}[207341255N_c^8 + (160969860 - 841104000\zeta_3)N_c^6 + (-1281330310 + 2919058560\zeta_3)N_c^4 + (499565484 - 1152911232\zeta_3)N_c^2 + 19308575 + 13799808\zeta_3]. \tag{5.6}$$

C. $\gamma^{(\gamma DDD)}_{\mathrm{IR},\mathrm{SU}(N_c),F}$

In a similar manner, from our general formulas (4.7)–(4.9), we find

$$\kappa^{(\gamma DDD)}_{1,\mathrm{SU}(N_c),F} = -\frac{314(N_c^2-1)}{3^2\cdot 5N_c(25N_c^2-11)}, \tag{5.7}$$

$$\kappa^{(\gamma DDD)}_{2,\mathrm{SU}(N_c),F} = -\frac{(N_c^2-1)(10265725N_c^4 - 36186192N_c^2 + 7719767)}{2\cdot 3^5\cdot 5^3 N_c^2(25N_c^2-11)^3}, \tag{5.8}$$

and

$$\kappa^{(\gamma DDD)}_{3,\mathrm{SU}(N_c),F} = -\frac{(N_c^2-1)}{2^2\cdot 3^8\cdot 5^5 N_c^3(25N_c^2-11)^5}[4581316819375N_c^8 + (3455659520100 - 16739946000000\zeta_3)N_c^6 + (-25230047265878 + 57258530640000\zeta_3)N_c^4 + (9616576686156 - 22465759536000\zeta_3)N_c^2 + 433168554247 + 225658224000\zeta_3)]. \tag{5.9}$$

D. $\gamma^{(\sigma D)}_{\mathrm{IR},\mathrm{SU}(N_c),F}$

From our general results (4.10)–(4.12), we obtain

$$\kappa^{(\sigma D)}_{1,\mathrm{SU}(N_c),F} = -\frac{4(N_c^2-1)}{N_c(25N_c^2-11)}, \tag{5.10}$$

$$\kappa^{(\sigma D)}_{2,\mathrm{SU}(N_c),F} = -\frac{4(N_c^2-1)(141N_c^4 - 262N_c^2 + 44)}{3N_c^2(25N_c^2-11)^3}, \tag{5.11}$$

and

$$\kappa^{(\sigma D)}_{3,\mathrm{SU}(N_c),F} = -\frac{2^3(N_c^2-1)}{3^3N_c^3(25N_c^2-11)^5}[64843N_c^8 + (78610 - 422400\zeta_3)N_c^6 + (-565316 + 1347456\zeta_3)N_c^4 + (209836 - 511104\zeta_3)N_c^2 + 13794]. \tag{5.12}$$

E. $\gamma^{(\sigma DD)}_{\mathrm{IR},\mathrm{SU}(N_c),F}$

From our general results (4.13)–(4.15), we obtain

$$\kappa^{(\sigma DD)}_{1,\mathrm{SU}(N_c),F} = -\frac{52(N_c^2-1)}{3^2N_c(25N_c^2-11)}, \tag{5.13}$$

$$\kappa^{(\sigma DD)}_{2,\mathrm{SU}(N_c),F} = -\frac{4(N_c^2-1)(11197N_c^4 - 29322N_c^2 + 5588)}{3^5N_c^2(25N_c^2-11)^3}, \tag{5.14}$$

and

$$\kappa^{(\sigma DD)}_{3,\mathrm{SU}(N_c),F} = -\frac{2^3(N_c^2-1)}{3^8N_c^3(25N_c^2-11)^5}[31831693N_c^8 + (30539268 - 141782400\zeta_3)N_c^6 + (-214403216 + 473734656\zeta_3)N_c^4 + (84228606 - 183845376\zeta_3)N_c^2 + 2690314 + 1254528\zeta_3]. \tag{5.15}$$

F. $\gamma^{(\sigma DDD)}_{\mathrm{IR},\mathrm{SU}(N_c),F}$

For this case we have

$$\kappa^{(\sigma DDD)}_{1,\mathrm{SU}(N_c),F} = -\frac{2^6(N_c^2-1)}{3^2N_c(25N_c^2-11)}, \tag{5.16}$$

$$\kappa^{(\sigma DDD)}_{2,\mathrm{SU}(N_c),F} = -\frac{2^2(N_c^2-1)(10579N_c^4 - 36849N_c^2 + 7832)}{3^5N_c^2(25N_c^2-11)^3}, \tag{5.17}$$

and

$$\kappa^{(\sigma DDD)}_{3,\mathrm{SU}(N_c),F} = -\frac{2^2(N_c^2-1)}{3^8N_c^3(25N_c^2-11)^5}[(90949802N_c^8 + (70557192 - 347943600\zeta_3)N_c^6 + (-511679503 + 1166243184\zeta_3)N_c^4 + (194401944 - 453269520\zeta_3)N_c^2 + 9119165 + 3293136\zeta_3]. \tag{5.18}$$

Below, where the meaning is clear, we will often omit the SU(3) in the subscript.

We remark on the signs of these coefficients. It is evident from Eqs. (4.1), (4.4), (4.7), (4.10), (4.13), and (4.16) that $\kappa_1^{(\gamma D)}$, $\kappa_1^{(\gamma DD)}$, $\kappa_1^{(\gamma DDD)}$, $\kappa_1^{(\sigma D)}$, $\kappa_1^{(\sigma DD)}$, and $\kappa_1^{(\sigma DDD)}$ are all negative for any G and R. We find that the $O(\Delta_f^2)$ and $O(\Delta_f^3)$ coefficients, $\kappa_2^{(\mathcal{O})}$ and $\kappa_3^{(\mathcal{O})}$, for these operators are also negative for the theory with $G = \mathrm{SU}(N_c)$ and fermions in the fundamental representation, $R = F$, in the full range $N_c \geq 2$ of relevance here. In Table I we list the signs of these coefficients $\kappa_n^{(\mathcal{O})}$ for the operators in this theory. For comparison, we also include the signs of $\kappa_n^{(1)}$ for $\bar{\psi}\psi$ and $\kappa_n^{(\sigma)}$ for $\bar{\psi}\sigma_{\mu\nu}\psi$ that we obtained in our earlier calculations (which hold for all N_c).

It is interesting to note that for all of the higher-spin operators $\mathcal{O}$ that we consider, the anomalous dimensions $\gamma_{\mathrm{IR}}^{(\mathcal{O})}$ that we calculate are negative (with our sign convention in (2.1) [44]). They thus have the same sign as the

TABLE I. Signs of scheme-independent expansion coefficients $\kappa_n^{(\mathcal{O})}$ for gauge group $G = \mathrm{SU}(N_c)$ with $N_c \geq 2$ and fermion representation $R = F$ (fundamental).

$\mathcal{O}$	$\kappa^{(\mathcal{O})}_{1,\mathrm{SU}(N_c),F}$	$\kappa^{(\mathcal{O})}_{2,\mathrm{SU}(N_c),F}$	$\kappa^{(\mathcal{O})}_{3,\mathrm{SU}(N_c),F}$
$\bar\psi\psi$	+	+	+
$\bar\psi\sigma_{\lambda\mu_1}\psi$	−	−	+
$\bar\psi\gamma_{\mu_1}D_{\mu_2}\psi$	−	−	−
$\bar\psi\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}\psi$	−	−	−
$\bar\psi\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}\psi$	−	−	−
$\bar\psi\sigma_{\lambda\mu_1}D_{\mu_2}\psi$	−	−	−
$\bar\psi\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_3}\psi$	−	−	−
$\bar\psi\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}\psi$	−	−	−

TABLE II. Values of the anomalous dimension $\gamma^{(\gamma D)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \leq p \leq 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.377	−0.446	−0.481
9	−0.332	−0.386	−0.411
10	−0.288	−0.328	−0.344
11	−0.244	−0.273	−0.282
12	−0.199	−0.219	−0.224
13	−0.155	−0.167	−0.169
14	−0.111	−0.117	−0.118
15	−0.0665	−0.0686	−0.0688
16	−0.02215	−0.0224	−0.0224

sign of the anomalous dimension of the operator $\bar\psi\sigma_{\mu\nu}\psi$ and are opposite in sign relative to the anomalous dimensions that we calculated for $\bar\psi\psi$ in our previous work [22,34–39].

In Tables II–VIII we list values of the anomalous dimensions $\gamma_{\mathrm{IR}}^{(\gamma D)}$, $\gamma_{\mathrm{IR}}^{(\gamma DD)}$, $\gamma_{\mathrm{IR}}^{(\gamma DDD)}$, $\gamma_{\mathrm{IR}}^{(\sigma)}$, $\gamma_{\mathrm{IR}}^{(\sigma D)}$, $\gamma_{\mathrm{IR}}^{(\sigma DD)}$, and $\gamma_{\mathrm{IR}}^{(\sigma DDD)}$ for the theory with $G = \mathrm{SU}(3)$ and fermions in the fundamental representation, $R = F$, calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^p}$, etc., with $p = 1, 2, 3$, as functions of N_f for a relevant range of N_f values extending downward from the upper end of the conformal regime at $N_f = N_u$ (i.e., $\Delta_f = 0$) within this conformal window [61]. The numbers in Table V are evaluations of our analytic results given in [36] and are included for comparison.

TABLE III. Values of the anomalous dimension $\gamma^{(\gamma DD)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \leq p \leq 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.588	−0.654	−0.724
9	−0.519	−0.570	−0.618
10	−0.450	−0.488	−0.520
11	−0.381	−0.408	−0.427
12	−0.3115	−0.330	−0.340
13	−0.242	−0.253	−0.258
14	−0.173	−0.179	−0.180
15	−0.104	−0.106	−0.106
16	−0.0346	−0.0348	−0.0349

TABLE IV. Values of the anomalous dimension $\gamma^{(\gamma DDD)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \leq p \leq 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.739	−0.794	−0.900
9	−0.652	−0.695	−0.7675
10	−0.565	−0.598	−0.645
11	−0.478	−0.501	−0.530
12	−0.391	−0.407	−0.422
13	−0.304	−0.314	−0.321
14	−0.217	−0.222	−0.225
15	−0.130	−0.132	−0.133
16	−0.0435	−0.0437	−0.0437

TABLE V. Values of the anomalous dimension $\gamma^{(\sigma)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\sigma)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \leq p \leq 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\sigma)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\sigma)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\sigma)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.141	−0.223	−0.207
9	−0.125	−0.188	−0.1775
10	−0.108	−0.156	−0.149
11	−0.0914	−0.125	−0.121
12	−0.0748	−0.0976	−0.0953
13	−0.05815	−0.07195	−0.0709
14	−0.0415	−0.0486	−0.0482
15	−0.0249	−0.0275	−0.0274
16	−0.00831	−0.00859	−0.00859

TABLE VI. Values of the anomalous dimension $\gamma^{(\sigma D)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\sigma D)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \leq p \leq 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\sigma D)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\sigma D)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\sigma D)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.424	−0.503	−0.527
9	−0.374	−0.436	−0.452
10	−0.324	−0.3705	−0.381
11	−0.274	−0.307	−0.314
12	−0.224	−0.247	−0.250
13	−0.174	−0.188	−0.190
14	−0.125	−0.131	−0.132
15	−0.0748	−0.0772	−0.0774
16	−0.0249	−0.0252	−0.0252

TABLE VII. Values of the anomalous dimension $\gamma^{(\sigma DD)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\sigma DD)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \le p \le 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\sigma DD)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\sigma DD)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\sigma DD)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.612	−0.682	−0.748
9	−0.540	−0.594	−0.640
10	−0.468	−0.509	−0.539
11	−0.396	−0.425	−0.443
12	−0.324	−0.344	−0.353
13	−0.252	−0.264	−0.268
14	−0.180	−0.186	−0.188
15	−0.108	−0.110	−0.111
16	−0.0360	−0.03624	−0.03625

TABLE VIII. Values of the anomalous dimension $\gamma^{(\sigma DDD)}_{\mathrm{IR},F}$ calculated to $O(\Delta_f^p)$, denoted $\gamma^{(\sigma DDD)}_{\mathrm{IR},F,\Delta_f^p}$, with $1 \le p \le 3$, for $G = \mathrm{SU}(3)$, as a function of N_f.

N_f	$\gamma^{(\sigma DDD)}_{\mathrm{IR},F,\Delta_f}$	$\gamma^{(\sigma DDD)}_{\mathrm{IR},F,\Delta_f^2}$	$\gamma^{(\sigma DDD)}_{\mathrm{IR},F,\Delta_f^3}$
8	−0.753	−0.811	−0.913
9	−0.665	−0.709	−0.779
10	−0.576	−0.610	−0.655
11	−0.487	−0.511	−0.539
12	−0.399	−0.415	−0.430
13	−0.310	−0.320	−0.327
14	−0.222	−0.2265	−0.229
15	−0.133	−0.135	−0.135
16	−0.0443	−0.0445	−0.0445

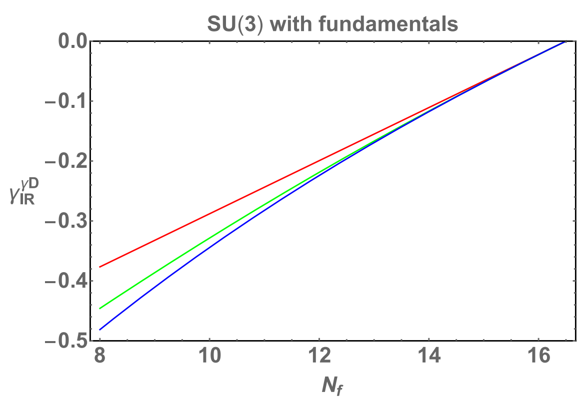

FIG. 1. Plot of the anomalous dimension $\gamma^{(\gamma D)}_{\mathrm{IR},F}$ of the operator $\bar\psi\gamma_{\mu_1}D_{\mu_2}\psi$ at the IRFP for the theory with $G = \mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p = 1, 2, 3$. Denoting the anomalous dimension calculated to order $O(\Delta_f^p)$ as $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^p}$, the curves, from top to bottom, refer to $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f}$ (red), $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^2}$ (green), and $\gamma^{(\gamma D)}_{\mathrm{IR},F,\Delta_f^3}$ (blue).

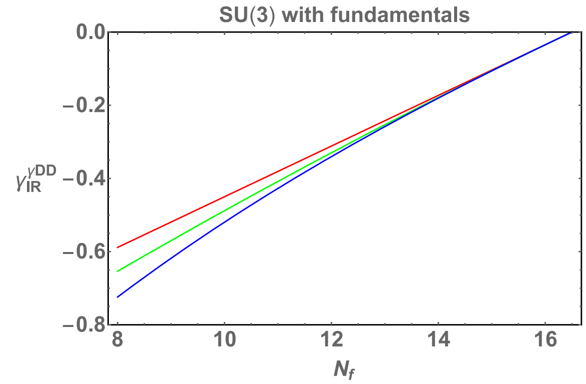

FIG. 2. Plot of the anomalous dimension $\gamma^{(\gamma DD)}_{\mathrm{IR},F}$ of the operator $\bar\psi\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}\psi$ at the IRFP for $G = \mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p = 1, 2, 3$. Denoting the anomalous dimension calculated to order $O(\Delta_f^p)$ as $\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^p}$, the curves, from top to bottom, refer to $\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f}$ (red), $\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^2}$ (green), and $\gamma^{(\gamma DD)}_{\mathrm{IR},F,\Delta_f^3}$ (blue).

In Figs. 1–7 we show plots of these anomalous dimensions for the SU(3) theory with $R = F$. The plot of the anomalous dimension for $\bar\psi\sigma_{\lambda\mu_1}\psi$ is based on the analytic results of our earlier paper [36] but was not given there and is new here. As can be seen from these tables and figures, the higher-order terms in the Δ_f expansion are sufficiently small that it is expected to be reliable throughout much of the non-Abelian Coulomb phase (i.e., conformal window). As is obvious, since our calculations are finite series expansions in powers of Δ_f, they are most accurate in the upper part of the NACP, where this expansion parameter Δ_f is small. This is similar to what we found in our earlier scheme-independent calculations of anomalous dimensions

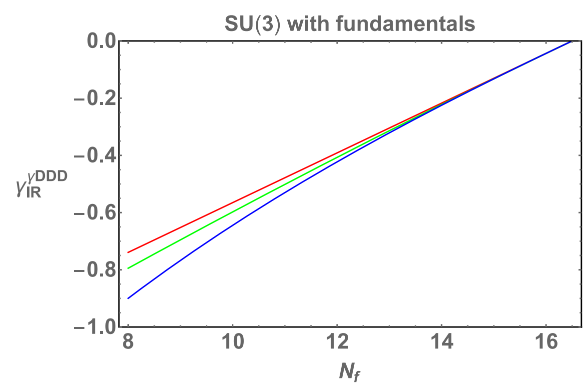

FIG. 3. Plot of the anomalous dimension $\gamma^{(\gamma DDD)}_{\mathrm{IR},F}$ of the operator $\bar\psi\gamma_{\mu_1}D_{\mu_2}D_{\mu_3}D_{\mu_4}\psi$ at the IRFP for $G = \mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p = 1, 2, 3$. Denoting the calculation to order $O(\Delta_f^p)$ as $\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^p}$, from top to bottom, the colors refer to $\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f}$ (red), $\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^2}$ (green), and $\gamma^{(\gamma DDD)}_{\mathrm{IR},F,\Delta_f^3}$ (blue).

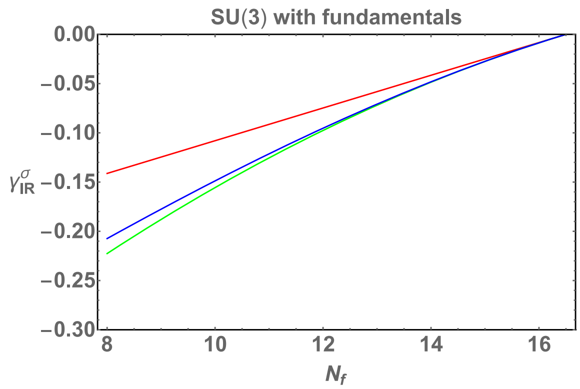

FIG. 4. Plot of the anomalous dimension $\gamma_{\mathrm{IR},F}^{(\sigma)}$ of the operator $\bar{\psi}\sigma_{\lambda\mu_1}\psi$ at the IRFP for $G=\mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p=1,2,3$. Denoting the calculation to order $O(\Delta_f^p)$ as $\gamma_{\mathrm{IR},F,\Delta_f^p}^{(\sigma)}$, the colors refer to $\gamma_{\mathrm{IR},F,\Delta_f}^{(\sigma)}$ (red), $\gamma_{\mathrm{IR},F,\Delta_f^2}^{(\sigma)}$ (green), and $\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\sigma)}$ (blue).

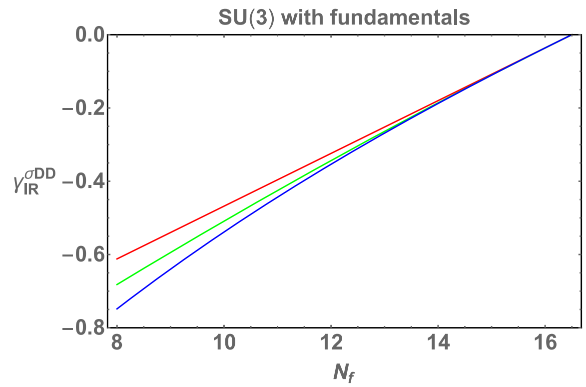

FIG. 6. Plot of the anomalous dimension $\gamma_{\mathrm{IR},F}^{(\sigma DD)}$ of the operator $\bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_3}\psi$ at the IRFP for $G=\mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p=1,2,3$. Denoting the calculation to order $O(\Delta_f^p)$ as $\gamma_{\mathrm{IR},F,\Delta_f^p}^{(\sigma DD)}$, from top to bottom, the colors refer to $\gamma_{\mathrm{IR},F,\Delta_f}^{(\sigma DD)}$ (red), $\gamma_{\mathrm{IR},F,\Delta_f^2}^{(\sigma DD)}$ (green), and $\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\sigma DD)}$ (blue).

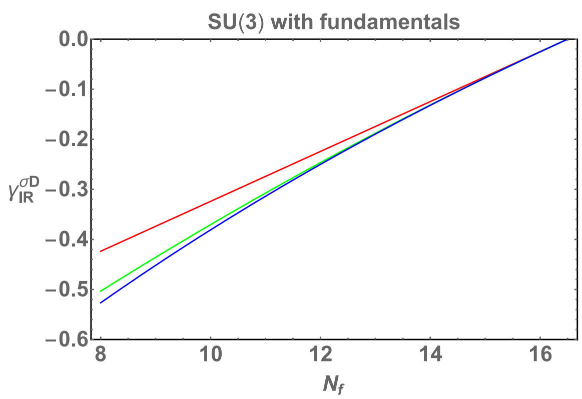

FIG. 5. Plot of the anomalous dimension $\gamma_{\mathrm{IR},F}^{(\sigma D)}$ of the operator $\bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}\psi$ at the IRFP for $G=\mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p=1,2,3$. Denoting the calculation to order $O(\Delta_f^p)$ as $\gamma_{\mathrm{IR},F,\Delta_f^p}^{(\sigma D)}$, from top to bottom, the colors refer to $\gamma_{\mathrm{IR},F,\Delta_f}^{(\sigma D)}$ (red), $\gamma_{\mathrm{IR},F,\Delta_f^2}^{(\sigma D)}$ (green), and $\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\sigma D)}$ (blue).

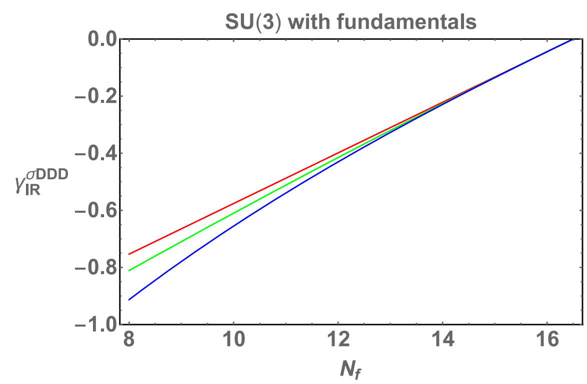

FIG. 7. Plot of the anomalous dimension $\gamma_{\mathrm{IR},F}^{(\sigma DDD)}$ of the operator $\bar{\psi}\sigma_{\lambda\mu_1}D_{\mu_2}D_{\mu_2}D_{\mu_3}\psi$ at the IRFP for $G=\mathrm{SU}(3)$, and N_f fermions in the fundamental representation, calculated to order $O(\Delta_f^p)$, where $p=1,2,3$. Denoting the calculation to order $O(\Delta_f^p)$ as $\gamma_{\mathrm{IR},F,\Delta_f^p}^{(\sigma DDD)}$, from top to bottom, the colors refer to $\gamma_{\mathrm{IR},F,\Delta_f}^{(\sigma DDD)}$ (red), $\gamma_{\mathrm{IR},F,\Delta_f^2}^{(\sigma DDD)}$ (green), and $\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\sigma DDD)}$ (blue).

[34–39]. In the figures, this is evident from the fact that the curves for the anomalous dimensions calculated to $O(\Delta_f^3)$ are reasonably close to the corresponding curves for these anomalous dimensions calculated to order $O(\Delta_f^2)$. As is evident from the values of $\gamma_{\mathrm{IR},F,\Delta_f^p}^{(\mathcal{O})}$ that we have listed for the various operators $\mathcal{O}$ in Tables II–VIII, the fractional differences $R_{3,2}^{(\mathcal{O})} \equiv (\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\mathcal{O})} - \gamma_{\mathrm{IR},F,\Delta_f^2}^{(\mathcal{O})})/\gamma_{\mathrm{IR},F,\Delta_f^3}^{(\mathcal{O})}$ are $\sim O(10^{-3})$ for N_f values near the upper end of the conformal window and increase as N_f decreases; at $N_f=12$, the $R_{3,2}^{(\mathcal{O})}$ are a few per cent, and at $N_f=8$, $R_{3,2}^{(\mathcal{O})} \sim 0.1$. For a given N_f, $R_{3,2}^{(\mathcal{O})}$ increases slightly with the spin of the operator $\mathcal{O}$; for example, for $N_f=12$, $R_{3,2}^{(\gamma D)}=0.024$, $R_{3,2}^{(\gamma DD)}=0.032$, and $R_{3,2}^{(\gamma DDD)}=0.037$, while $R_{3,2}^{(\sigma D)}=0.014$, $R_{3,2}^{(\sigma DD)}=0.028$, and $R_{3,2}^{(\sigma DDD)}=0.035$.

VI. LNN LIMIT FOR $\gamma_{\mathrm{IR},\mathrm{SU}(N_c),F}^{(\mathcal{O})}$

In a theory with gauge group $\mathrm{SU}(N_c)$ and fermions in the fundamental representation, $R=F$, it is of interest to consider the limit

$$N_c \to \infty, \qquad N_F \to \infty \quad \text{with} \quad r \equiv \frac{N_F}{N_c} \text{ fixed and finite}$$
$$\text{and} \quad \xi(\mu) \equiv \alpha(\mu) N_c \text{ is a finite function of } \mu. \tag{6.1}$$

This limit is denoted as $\lim_{\rm LNN}$ [where "LNN" connotes "large N_c and N_F" with the constraints in Eq. (6.1) imposed]. It is also often called the 't Hooft-Veneziano limit. It has the simplifying feature that rather than depending on N_c and N_f, the properties of the theory only depend on their ratio, r. The scheme-independent expansion parameter in this LNN limit is

$$\Delta_r \equiv \lim_{\rm LNN} \frac{\Delta_f}{N_c} = \frac{11}{2} - r. \tag{6.2}$$

$$r_u = \lim_{\rm LNN} \frac{N_u}{N_c}, \tag{6.3}$$

and

$$r_\ell = \lim_{\rm LNN} \frac{N_\ell}{N_c}, \tag{6.4}$$

with values

$$r_u = \frac{11}{2} = 5.5 \tag{6.5}$$

and

$$r_\ell = \frac{34}{13} = 2.6154. \tag{6.6}$$

With $I_{\rm IRZ}\colon N_\ell < N_f < N_u$, it follows that the corresponding interval in the ratio r is

$$I_{{\rm IRZ},r}\colon \ \frac{34}{13} < r < \frac{11}{2}, \text{i.e.,} \ 2.6154 < r < 5.5. \tag{6.7}$$

Here we evaluate these scheme-independent anomalous dimension coefficients in a theory with $G = {\rm SU}(N_c)$ and $R = F$, in the LNN limit. The rescaled coefficients that are finite in the LNN limit are

$$\hat{\kappa}_n^{(\mathcal{O})} = \lim_{N_c \to \infty} N_c^n \kappa_n^{(\mathcal{O})}. \tag{6.8}$$

The anomalous dimension $\gamma_{\rm IR}^{(\mathcal{O})}$ is also finite in this limit and is given by

$$\lim_{\rm LNN} \gamma^{(\mathcal{O})}_{{\rm IR},{\rm SU}(N_c),F} = \sum_{n=1}^{\infty} \kappa_n^{(\mathcal{O})} \Delta_f^n = \sum_{n=1}^{\infty} \hat{\kappa}_n^{(\mathcal{O})} \Delta_r^n. \tag{6.9}$$

As r decreases from its upper limit, r_u, to r_ℓ, the expansion variable Δ_r increases from 0 to

$$(\Delta_r)_{\rm max} = \frac{75}{26} = 2.8846 \quad \text{for } r \in I_{{\rm IRZ},r}. \tag{6.10}$$

In this LNN limit, the values of $\hat{\kappa}_n^{(\mathcal{O})}$ with $1 \le n \le 3$ for the operators $\mathcal{O}$ considered here are listed in Table IX. For comparison, we also include the corresponding values of $\hat{\kappa}_n^{(\mathcal{O})}$ for the operators $\bar\psi\psi$ and $\bar\psi\sigma_{\mu\nu}\psi$ that we had calculated in [36].

TABLE IX. Values of the $\hat{\kappa}_n^{(\mathcal{O})}$ coefficients for $G = {\rm SU}(N_c)$ and $R = F$ in the LNN limit. The operators are indicated by their shorthand symbols, so 1 refers to $\bar\psi\psi$; σ refers to $\bar\psi\sigma_{\lambda\mu_1}\psi$; γD to $\bar\psi\gamma_{\mu_1}D_{\mu_2}\psi$, etc. The notation ae-n means $a \times 10^{-n}$.

$\mathcal{O}$	$\hat{\kappa}_1^{(\mathcal{O})}$	$\hat{\kappa}_2^{(\mathcal{O})}$	$\hat{\kappa}_3^{(\mathcal{O})}$
1	0.160000	0.0376320	0.832074e-2
σ	−0.0533333	−0.969956e-2	2.33189e-4
γD	−0.142222	−1.04844e-2	−2.135375e-3
γDD	−0.222222	−1.11967e-2	−0.404507e-2
γDDD	−0.279111	−1.08149e-2	−0.572019e-2
σD	−0.160000	−0.0120320	−1.967385e-3
σDD	−0.231111	−1.17960e-2	−0.397447e-2
σDDD	−0.284444	−1.114495e-2	−0.567795e-2

VII. EVALUATION OF ANOMALOUS DIMENSIONS $\gamma_{\rm IR}^{(\mathcal{O})}$ FOR $G = {\rm SU}(N_c)$ AND $R = Adj$

For the case where $G = {\rm SU}(N_c)$ and the fermions are in the adjoint representation, $R = Adj$, our general results for the scheme-independent expansion coefficients for the anomalous dimensions of the operators under consideration are as follows::

$$\kappa^{(\gamma D)}_{1,{\rm SU}(N_c),Adj} = -\frac{2^5}{3^4} = -0.395062, \tag{7.1}$$

$$\kappa^{(\gamma D)}_{2,{\rm SU}(N_c),Adj} = -\frac{1756}{3^9} = -0.0892140, \tag{7.2}$$

$$\kappa^{(\gamma D)}_{3,{\rm SU}(N_c),Adj} = -\frac{88129}{3^{14}} + \frac{4736}{3^{10}N_c^2} = -0.0184256 + \frac{0.0802046}{N_c^2}, \tag{7.3}$$

$$\kappa^{(\gamma DD)}_{1,{\rm SU}(N_c),Adj} = -\frac{50}{3^4} = -0.617284, \tag{7.4}$$

$$\kappa^{(\gamma DD)}_{2,{\rm SU}(N_c),Adj} = -\frac{10165}{2^2 \cdot 3^9} = -0.129109, \tag{7.5}$$

$$\kappa^{(\gamma DD)}_{3,{\rm SU}(N_c),Adj} = -\frac{2272255}{2^4 \cdot 3^{14}} + \frac{7400}{3^{10}N_c^2} = -0.0296920 + \frac{0.125320}{N_c^2}, \tag{7.6}$$

$$\kappa^{(\gamma DDD)}_{1,{\rm SU}(N_c),Adj} = -\frac{314}{3^4 \cdot 5} = -0.775309, \tag{7.7}$$

$$\kappa^{(\gamma DDD)}_{2,{\rm SU}(N_c),Adj} = -\frac{1504769}{2^2 \cdot 3^9 \cdot 5^3} = -0.152900, \tag{7.8}$$

and

$$\kappa^{(\gamma DDD)}_{3,\mathrm{SU}(N_c),Adj} = -\frac{9206650603}{2^4 \cdot 3^{14} \cdot 5^5} + \frac{46472}{3^{10} \cdot 5N_c^2} = -0.0384976 + \frac{0.1574015}{N_c^2}, \tag{7.9}$$

$$\kappa^{(\sigma D)}_{1,\mathrm{SU}(N_c),Adj} = -\frac{2^2}{3^2} = -0.444444, \tag{7.10}$$

$$\kappa^{(\sigma D)}_{2,\mathrm{SU}(N_c),Adj} = -\frac{149}{2 \cdot 3^6} = -0.102195, \tag{7.11}$$

and

$$\kappa^{(\sigma D)}_{3,\mathrm{SU}(N_c),Adj} = -\frac{10801}{2^3 \cdot 3^{10}} + \frac{592}{3^8 N_c^2} = -0.0228645 + \frac{0.0902301}{N_c^2}, \tag{7.12}$$

$$\kappa^{(\sigma DD)}_{1,\mathrm{SU}(N_c),Adj} = -\frac{52}{3^4} = -0.641975, \tag{7.13}$$

$$\kappa^{(\sigma DD)}_{2,\mathrm{SU}(N_c),Adj} = -\frac{5123}{2 \cdot 3^9} = -0.130138, \tag{7.14}$$

$$\kappa^{(\sigma DD)}_{3,\mathrm{SU}(N_c),Adj} = -\frac{984949}{2^3 \cdot 3^{14}} + \frac{7696}{3^{10} N_c^2} = -0.0257410 + \frac{0.130332}{N_c^2}, \tag{7.15}$$

$$\kappa^{(\sigma DDD)}_{1,\mathrm{SU}(N_c),Adj} = -\frac{2^6}{3^4} = -0.790123, \tag{7.16}$$

$$\kappa^{(\sigma DDD)}_{2,\mathrm{SU}(N_c),Adj} = -\frac{3070}{3^9} = -0.155972, \tag{7.17}$$

and

$$\kappa^{(\sigma DDD)}_{3,\mathrm{SU}(N_c),Adj} = -\frac{378247}{2 \cdot 3^{14}} + \frac{9472}{3^{10} N_c^2} = -0.0395410 + \frac{0.160409}{N_c^2}. \tag{7.18}$$

For all of these operators $\mathcal{O}$, the coefficients $\kappa^{(\mathcal{O})}_{n,\mathrm{SU}(N_c),Adj}$ are negative for $n = 1$ and $n = 2$ and for all N_c. The coefficient $\kappa^{(\sigma D)}_{3,\mathrm{SU}(N_c),Adj}$ is negative for all N_c, while the coefficients $\kappa^{(\mathcal{O})}_{3,\mathrm{SU}(N_c),Adj}$ for the other operators are positive for $N_c = 2$, i.e., $G = \mathrm{SU}(2)$, and are negative for $N_c \geq 3$.

VIII. CONCLUSIONS

In conclusion, in this paper we have calculated scheme-independent expansions up to $O(\Delta_f^3)$ inclusive for the anomalous dimensions of the higher-spin, twist-2 bilinear fermion operators $\bar{\psi}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j}\psi$ and $\bar{\psi}\sigma_{\lambda\mu_1} D_{\mu_2} \ldots D_{\mu_j}\psi$ with j up to 3, evaluated at an IR fixed point in the non-Abelian Coulomb phase of an asymptotically free gauge theory with gauge group G and N_f fermions transforming according to a representation R of G. Our general results are evaluated for several special cases, including the case $G = \mathrm{SU}(N_c)$ with R equal to the fundamental and adjoint representations. We have presented our results in convenient tabular and graphical formats. For fermions in the fundamental representation, we also analyze the limit $N_c \to \infty$ and $N_f \to \infty$ with N_f/N_c fixed and finite. A comparison with our previous scheme-independent calculations of the corresponding anomalous dimensions of $\bar{\psi}\psi$ and $\bar{\psi}\sigma_{\mu\nu}\psi$ has also been given. Our new results further elucidate the properties of conformal field theories. With the requisite inputs, one could extend these scheme-independent calculations to higher-spin operators and to higher order in powers of Δ_f. It is hoped that lattice measurements of these anomalous dimensions of higher-spin operators in the conformal window will be performed in the future, and it will be of interest to compare our calculations with lattice results when they will become available.

ACKNOWLEDGMENTS

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APPENDIX: PREVIOUS RESULTS ON $\gamma^{(1)}$ AND $\gamma^{(\sigma)}$ FOR $G = \mathrm{SU}(3)$ AND $R = F$

In this appendix, for comparison with our new results, we list our previous results from [36] (see also [37]) for the scheme-independent series expansions of the anomalous dimensions $\gamma^{(\mathcal{O})}_{\mathrm{IR}}$ for $\mathcal{O} = \bar{\psi}\psi$ and $\mathcal{O} = \bar{\psi}\sigma_{\mu\nu}\psi$. Following the same shorthand notation as in the text, we denote the coefficients at order $O(\Delta_f^n)$ in the scheme-independent series expansions (2.10) for these anomalous dimensions as $\kappa_n^{(1)}$ and $\kappa_n^{(\sigma)}$. We calculated

$$\kappa_1^{(1)} = \frac{8T_f C_f}{C_A D}, \tag{A1}$$

$$\kappa_2^{(1)} = \frac{4T_f^2 C_f (5C_A + 88C_f)(7C_A + 4C_f)}{3C_A^2 D^3}, \tag{A2}$$

and

$$\kappa_3^{(1)} = \frac{4T_fC_f}{3^4C_A^4D^5}\Bigg[-55419T_f^2C_A^5 + 432012T_f^2C_A^4C_f + 5632T_f^2C_f\frac{d_A^{abcd}d_A^{abcd}}{d_A}(-5+132\zeta_3) + 16C_A^3\left(122043T_f^2C_f^2 + 6776\frac{d_R^{abcd}d_R^{abcd}}{d_A}(-11+24\zeta_3)\right) + 704C_A^2\left(1521T_f^2C_f^3 + 112T_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(4-39\zeta_3) + 242C_f\frac{d_R^{abcd}d_R^{abcd}}{d_A}(-11+24\zeta_3)\right) + 32T_fC_A\left(53361T_fC_f^4 - 3872C_f\frac{d_R^{abcd}d_A^{abcd}}{d_A}(-4+39\zeta_3) + 112T_f\frac{d_A^{abcd}d_A^{abcd}}{d_A}(-5+132\zeta_3)\right)\Bigg] \tag{A3}$$

where the denominator factor D was defined in Eq. (3.1)]. In [37,39] we presented results for the next-higher order coefficient, $\kappa_4^{(1)}$, but these are not needed here.

For the $\kappa_n^{(\sigma)}$ we found

$$\kappa_1^{(\sigma)} = -\frac{8C_fT_f}{3C_AD} \tag{A4}$$

$$\kappa_2^{(\sigma)} = -\frac{4C_fT_f^2(259C_A^2 + 428C_AC_f - 528C_f^2)}{9C_A^2D^3} \tag{A5}$$

and

$$\kappa_3^{(\sigma)} = \frac{4C_fT_f}{3^5C_A^4D^5}\Bigg[3C_AT_f^2\{C_A^4(-11319+188160\zeta_3) + C_A^3C_f(-337204+64512\zeta_3) + C_A^2C_f^2(83616-890112\zeta_3) + C_AC_f^3(1385472-354816\zeta_3) + C_f^4(-212960+743424\zeta_3)\} - 512T_f^2D(-5+132\zeta_3)\frac{d_A^{abcd}d_A^{abcd}}{d_A} - 15488C_A^2D(-11+24\zeta_3)\frac{d_R^{abcd}d_R^{abcd}}{d_A} + 11264C_AT_fD(-4+39\zeta_3)\frac{d_R^{abcd}d_A^{abcd}}{d_A}\Bigg]. \tag{A6}$$

For $G = \mathrm{SU}(N_c)$ and $R = F$, in the LNN limit, these yield the rescaled coefficients

$$\hat{\kappa}_1^{(1)} = \frac{4}{5^2} = 0.1600, \tag{A7}$$

$$\hat{\kappa}_2^{(1)} = \frac{588}{5^6} = 0.037632, \tag{A8}$$

$$\hat{\kappa}_3^{(1)} = \frac{2193944}{3^3\cdot 5^{10}} = 0.83207\times 10^{-2}, \tag{A9}$$

$$\hat{\kappa}_1^{(\sigma)} = -\frac{4}{3\cdot 5^2} = -0.053333, \tag{A10}$$

$$\hat{\kappa}_2^{(\sigma)} = -\frac{1364}{3^2\cdot 5^6} = -(0.969956\times 10^{-2}), \tag{A11}$$

and

$$\hat{\kappa}_3^{(\sigma)} = \frac{184456}{3^4\cdot 5^{10}} = 2.3319\times 10^{-4}. \tag{A12}$$

[1] This assumption of massless fermions does not entail any loss of generality, since a fermion with nonzero mass m would be integrated out of the low-energy effective field theory that describes the physics at Euclidean momentum scales $\mu < m$ and hence would not affect the infrared limit $\mu \to 0$ that we consider here.

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[18] We recall the definitions of relevant group invariants. Let T_a denote the generators of a Lie algebra of G, and let $d(R)$ denote the dimension of a representation R of G. Then the quadratic Casimir invariant $C_2(R)$ is defined by $\sum_a T_R^a T_R^a = C_2(R)I$, where I is the $d_R \times d_R$ identity matrix. We utilize the common notation $C_2(Adj) \equiv C_2(G) \equiv C_A$. The trace invariant $T(R)$ is defined by $\mathrm{Tr}_R(T_R^a T_R^b) = T(R)\delta^{ab}$. We use standard normalizations for $\mathrm{SU}(N_c)$ so that, e.g., for the fundamental representation F, $T(F) = 1/2$ and $C_2(F) = (N_c^2 - 1)/(2N_c)$.

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[45] In contrast, in the confined QCD-type phase at lower N_f, although the UV beta function may exhibit a formal IR zero, as the theory flows from the UV to the IR, the running coupling becomes large enough so that spontaneous chiral symmetry breaking occurs, giving the fermions dynamical masses, so that they are integrated out of the low-energy effective field theory that is applicable deeper in the IR, and hence the resultant beta function is that of a pure gauge theory, which does not have a (perturbative) IR zero. If N_f is only slightly less than $N_{f,cr}$, these theories thus exhibit quasiconformal behavior. If the boundary in N_f between the NACP and the QCD-type regime involves a continuous, although nonanalytic, change in the values of anomalous dimensions, then our previous scheme-independent results in [34–39] and our results here may give some approximate guide to the values of the corresponding anomalous dimensions in quasiconformal theories with N_f slightly less than $N_{f,cr}$.

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