$T\bar{T}$ deformations as TsT transformations

Alessandro Sfondrini[1,2,3,*] and Stijn J. van Tongeren[4,†]

[1] *Institut für theoretische Physik, ETH Zürich Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland*
[2] *Dipartimento di Fisica e Astronomia "Galileo Galilei," Universitá degli Studi di Padova via Marzolo 8, 35131 Padova, Italy*
[3] *Istituto Nazionale di Fisica Nucleare, Sezione di Padova via Marzolo 8, 35131 Padova, Italy*
[4] *Institut für Physik, Humboldt-Universität zu Berlin, IRIS Gebäude, Zum Grossen Windkanal 6, 12489 Berlin, Germany*

(Received 5 December 2019; accepted 25 February 2020; published 24 March 2020)

The relationship between $T\bar{T}$ deformations and the uniform light-cone gauge, first noted by Baggio and Sfondrini [Phys. Rev. D **98**, 021902 (2018)], provides a powerful generating technique for deformed models. We recall this construction, distinguishing between changes of the gauge frame, which do not affect the theory, and genuine deformations. We investigate the geometric interpretation of the latter and argue that they affect the global features of the geometry before gauge fixing. Exploiting a formal relation between uniform light-cone gauge and static gauge in a T-dual frame, we interpret such a change as a T-duality–shift–T-duality transformation involving the two light-cone coordinates. In the static-gauge picture, the $T\bar{T}$ Castillejo-Dalitz-Dyson factor then has a natural interpretation as a Drinfeld-Reshetikhin twist of the worldsheet S matrix. To illustrate these ideas, we find the geometries yielding a $T\bar{T}$ deformation of the worldsheet S matrix of pp-wave and Lin-Lunin-Maldacena backgrounds.

DOI: 10.1103/PhysRevD.101.066022

I. INTRODUCTION

The study of two-dimensional quantum field theories (QFTs) plays an important role in our understanding of condensed matter systems, string theory—where the string worldsheet is two dimensional—and QFT in general, providing useful toy models that may capture interesting physical features of higher-dimensional theories. Even among two-dimensional models, only some rather special theories can be understood in full detail, usually because they enjoy additional symmetries such as *conformal invariance* or *integrability*. Given such an exactly solvable theory, it is interesting to try and deform it while maintaining its solvability. A rather general class of such deformations can be constructed out of the conserved currents of a theory. A famous example is the *marginal* deformation of a conformal field theory (CFT) by a composite operator constructed out of one chiral and one antichiral current: a $J\bar{J}$ deformation. *Relevant* deformations of CFTs are also interesting, as they generate a renormalization group flow and can give rise to families of integrable theories.

More recently, *irrelevant* deformations have been considered, in particular the $T\bar{T}$ deformation. This deformation can be constructed for any two-dimensional Poincaré-invariant QFT—conformal, integrable, or not—and it is sourced by the determinant of the stress-energy tensor, $\det[T_{\alpha\beta}] = T_{00}T_{11} - T_{01}T_{10}$ [1]. Interestingly, this deformation acts in a simple way on the spectrum of the original theory: each energy level evolves according to an ordinary differential equation (ODE) [2,3]. In a similar way, the classical Hamiltonian and Lagrangian obey an ODE in the space of fields, which can often be solved in closed form [3,4]. Over the last three years, $T\bar{T}$ deformations of a number of integrable [3,5,6], as well as of more general [7–10] theories have been considered.[1]

A striking link has emerged between string theory and $T\bar{T}$ deformations, fueled by the initial observation that the $T\bar{T}$ deformation of a theory of free bosons is related to strings in flat space [3]; see also Refs. [30,31]. It was subsequently understood [24] that the link between strings and $T\bar{T}$ deformations is much more general and becomes particularly transparent in the *uniform light-cone gauge* of

*alessandro.sfondrini@unipd.it
†svantongeren@physik.hu-berlin.de

[1] Interesting applications to several classes of two-dimensional theories, such as supersymmetric theories [11–15], two-dimensional gravity [16–19] and AdS_3/CFT_2 holography [20–29], have also emerged.

Refs. [32–34]; see also Ref. [35] for a pedagogical review. In fact, this framework can be used as a powerful technique to generate $T\bar{T}$-deformed actions: finding the deformed Hamiltonian requires solving an *algebraic* equation rather than an ODE [11,36]. Moreover, this approach can be applied to general current-current deformations [37] of the type considered in Refs. [38–44].

The link between $T\bar{T}$ deformations and string theory in a uniform light-cone gauge is the focus of this paper. Given a two-dimensional model to be deformed, it can be uplifted to a reparametrization-invariant model by adding two extra fields, where fixing a particular light-cone gauge gives back the original model. The uniform light-cone gauge admits a family of gauge frames; however, a change of frame mimics the $T\bar{T}$ deformation. Changing the gauge parameter a affects the relation between volume, R, and energy $H_{\mathrm{w.s.}}$

$$R = R_0 + aH_{\mathrm{w.s.}}, \tag{1.1}$$

in a way that is typical for $T\bar{T}$ deformations [2,3]. In string-theory language $H_{\mathrm{w.s.}}$ is the *worldsheet Hamiltonian* which also depends on a, precisely so that the a dependence cancels in physical quantities like the spectrum. It is then important to distinguish between mere changes of gauge frame, and genuine deformations.

For genuine deformations the change of the Hamiltonian density *is not compensated* by a redefinition of the worldsheet length, and hence the spectrum changes as for a $T\bar{T}$ deformation. We consider this case and study the effect of the deformation *on the uplifted geometry*. We will argue that this deformation does not affect the geometry *locally*, but does so *globally*. Exploiting a formal relation between uniform light-cone gauge and static gauge [45], we can make the geometric interpretation of the deformation more transparent, and recast it as a T-duality–shift–T-duality (TsT) transformation [46] involving the two longitudinal coordinates. Indeed, in a string sigma model, such TsT transformations can equivalently be understood as a twist of the boundary conditions of the involved coordinates [47–49], rather than a genuine modification of the local geometry. For integrable models, such a twist of the boundary conditions results in a twist of the Bethe-Yang equations [50]. Equivalently, from the point of view of the deformed geometry, a TsT transformation in general leads to a Drinfeld-Reshetikhin twist [51,52] of the worldsheet S matrix [53,54]. Taking this view, we can interpret the Castillejo-Dalitz-Dyson (CDD) factor [55] arising from $T\bar{T}$ deformation [2,3] as such a Drinfeld-Reshetikhin twist based on the Cartan charges corresponding to the two longitudinal directions. This reinforces the identification between $T\bar{T}$ deformations and gauge fixing. In fact, the $T\bar{T}$ CDD factor can be taken as a *definition* of such a deformation [3].

We can apply these ideas to construct integrable deformations of superstring backgrounds. The resulting geometries are such that once a light-cone gauge is fixed, the associated worldsheet S matrix differs from the undeformed one precisely by the $T\bar{T}$ CDD factor. In the case of $\mathrm{AdS}_5 \times \mathrm{S}^5$, we can for instance construct a string background which yields a $T\bar{T}$ deformation of Beisert's S matrix [56] in the "string frame" of Ref. [57], preserving integrability by virtue of being a $T\bar{T}$ deformation. We can also consider nonintegrable geometries, although the resulting spectral problem will be less tractable. As an illustration we consider Lin-Lunin-Maldacena (LLM) backgrounds, where the deformation has a particularly clean interpretation.[2]

This paper is structured as follows. In Sec. II we review the uniform light-cone gauge and its relation with $T\bar{T}$ deformations. In Sec. III we discuss the geometrical interpretation of such deformations, the relation to TsT transformations, and the interpretation of the CDD factor as a Drinfeld-Reshetikhin twist. In Secs. IV and V we illustrate our arguments on pp-wave and LLM backgrounds respectively. We present some concluding remarks in Sec. VI. Our results can be straightforwardly generalized to the case of current-current deformations involving a $\mathfrak{u}(1)$ current J, such as $J\bar{T}$ or $T\bar{J}$ deformations; we briefly discuss this in the Appendix.

II. $T\bar{T}$ DEFORMATIONS AND UNIFORM LIGHT-CONE GAUGE

The relationship between $T\bar{T}$ deformations and uniform light-cone gauge[3] was first noted in Ref. [24] and subsequently exploited to construct $T\bar{T}$-deformed Lagrangians; see Ref. [11] and in particular Refs. [36,37]. We will briefly review this construction as it is central to our subsequent discussion.

A. Uniform light-cone gauge

Consider a two-dimensional nonlinear sigma model with action

$$S = -\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma (\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)), \tag{2.1}$$

where $G_{\mu\nu}(X)$ denotes the metric, $B_{\mu\nu}(X)$ is the B field, and X collectively denotes the fields of the model. $\gamma^{\alpha\beta}$ denotes the two-dimensional worldsheet metric, which we take to have unit determinant and signature $(-,+)$, matching the overall sign of the action.

Classically this model is reparametrization invariant. We are currently interested in the classical theory and will not

[2]The (non)integrability of LLM geometries was discussed in Ref. [58].

[3]The uniform light-cone gauge was introduced in Refs. [32–34] and was reviewed in detail in Ref. [35].

assume that the metric and B field describe a string background. We do assume that the metric has at least two shift isometries: one for a time-like coordinate t, $t \to t + \delta t$, and one for a space-like coordinate ϕ, $\phi \to \phi + \delta\phi$. By Noether's theorem these yield two conserved charges

$$E = -\int_0^R \mathrm{d}\sigma\, p_t, \quad \text{and} \quad J = \int_0^R \mathrm{d}\sigma\, p_\phi, \tag{2.2}$$

for shifts in t and ϕ respectively. Here we introduced the momenta p_μ, canonically conjugated to X^μ

$$p_\mu = \frac{\delta S}{\delta \partial_\tau X^\mu} = -\gamma^{0\beta}\partial_\beta X^\nu G_{\mu\nu}(X) - \acute{X}^\nu B_{\mu\nu}(X), \tag{2.3}$$

and we use primes for space derivatives, $\acute{X}^\nu \equiv \partial_\sigma X^\mu$.

In the first-order formalism the action takes the form

$$S = \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma \left(p_\mu \dot{X}^\mu + \frac{\gamma^{01}}{\gamma^{00}} \mathcal{C}_1 + \frac{1}{2\gamma^{00}} \mathcal{C}_2 \right), \tag{2.4}$$

where the worldsheet metric acts as a Lagrange multiplier giving the Virasoro constraints:

$$\begin{aligned} 0 &= \mathcal{C}_1 = p_\mu \acute{X}^\mu, \\ 0 &= \mathcal{C}_2 = p_\mu p_\nu G^{\mu\nu} + \acute{X}^\mu \acute{X}^\nu G_{\mu\nu} + 2G^{\mu\nu} B_{\nu\rho} p_\mu \acute{X}^\rho \\ &\quad + G^{\mu\nu} B_{\mu\rho} B_{\nu\lambda} \acute{X}^\rho \acute{X}^\lambda. \end{aligned} \tag{2.5}$$

Choice of the light-cone coordinates.—We now use the isometric coordinates t and ϕ to construct light-cone coordinates $X^\pm$, to be used in the gauge fixing. These coordinates are typically introduced as

$$X^\pm = \frac{1}{2}(\phi \pm t), \tag{2.6}$$

but it is convenient to generalize this choice by introducing two parameters a and b

$$\begin{aligned} &X^+ = a\phi + (1-a)t, \qquad X^- = (1-b)\phi - bt, \\ &\Delta_{ab} \equiv 1 - a - b + 2ab \neq 0, \end{aligned} \tag{2.7}$$

so that we have

$$p_+ = \frac{b}{\Delta_{ab}} p_\phi + \frac{1-b}{\Delta_{ab}} p_t, \qquad p_- = \frac{1-a}{\Delta_{ab}} p_\phi - \frac{a}{\Delta_{ab}} p_t. \tag{2.8}$$

Let us note that if $b = 1$, then $p_+ \sim p_\phi$ with no dependence on p_t. We will see below that this case is pathological, so we shall always assume $b \neq 1$.

Uniform light-cone gauge fixing.—The uniform light-cone gauge is fixed by imposing

$$X^+ = \tau, \qquad p_- = \frac{1}{1-b}, \tag{2.9}$$

identifying the worldsheet time τ with the target space direction X^+, and making the momentum density for p_- constant. The choice of this constant is a matter of future convenience; it is compatible with our requirement that $b \neq 1$. We can then eliminate the two remaining longitudinal degrees of freedom X^- and p_+ through the Virasoro constraints (2.5), obtaining

$$\begin{aligned} 0 = \mathcal{C}_1 = p_+ \acute{X}^+ + p_- \acute{X}^- + p_i \acute{X}^i \Rightarrow \\ \acute{X}^- = -(1-b) p_i \acute{X}^i, \end{aligned} \tag{2.10}$$

while $\mathcal{C}_2 = 0$ gives a quadratic equation for p_+.[4] The above fixes $\acute{X}^-$ but not X^- itself, as is to be expected for an isometric coordinate; the action depends only on dX^-. Of course X^- satisfies appropriate boundary conditions, which we take to be periodic.[5] This gives the level matching constraint

$$0 = \int_0^R \mathrm{d}\sigma \acute{X}^- = \int_0^R \mathrm{d}\sigma(-p_i \acute{X}^i) = P_{\mathrm{w.s.}}, \tag{2.11}$$

where we identified the final integral with the total momentum on the worldsheet $P_{\mathrm{w.s.}}$ since $-p_i \acute{X}^i$ is the charge density for the symmetry $\sigma \to \sigma + \delta\sigma$.

In the end, the action (2.4) depends only on transverse degrees of freedom, becoming

$$\begin{aligned} S &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma\, p_\mu \dot{X}^\mu \\ &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma (p_i \dot{X}^i - (-p_+)), \end{aligned} \tag{2.12}$$

where we dropped a total derivative $\dot{X}^-$. This identifies $-p_+$ as the worldsheet Hamiltonian,

$$H_{\mathrm{w.s.}} = -\int_0^R \mathrm{d}\sigma\, p_+(X^i, \acute{X}^i, p_i), \tag{2.13}$$

which is expected because $H_{\mathrm{w.s.}}$ is canonically conjugated to τ and hence to X^+. As for p_-, we find that in this gauge

$$P_- = \int_0^R \mathrm{d}\sigma\, p_- = \frac{R}{1-b}. \tag{2.14}$$

To conclude, the worldsheet Hamiltonian $H_{\mathrm{w.s.}}$ and volume R are related to the target space energy E and (angular) momentum J as

[4]This may degenerate into a linear equation should $G^{++} = 0$ for some particular choice of a and b.

[5]It is possible to consider more general boundary conditions, for instance involving winding along ϕ if its range is compact; see e.g., Refs. [35,36].

$$H_{\text{w.s.}}=\frac{(1-b)E-bJ}{\Delta_{ab}},\qquad R=\frac{1-b}{\Delta_{ab}}((1-a)J+aE)=J+aH_{\text{w.s.}}. \tag{2.15}$$

Clearly $b=1$ is a singular choice, as we would be matching the worldsheet Hamiltonian with the potentially quantized (angular) momentum J. Finally, unless $a=0$, the volume R in which the theory will be quantized will be *state dependent*: it depends on the energy of each given state. This is a first indication of a relation with $T\bar{T}$ deformations.

Choices of the parameters a and b.—Let us briefly comment on some features of this perhaps somewhat unconventional gauge choice. The parameter b allows us to change the relation between $H_{\text{w.s.}}$ and E. Strictly speaking, the uniform light-cone gauge corresponds to the choice where $H_{\text{w.s.}}$ is the light-cone Hamiltonian, $H_{\text{w.s.}}=E-J$,[6] achieved at $b=1/2$:

$$b=\frac{1}{2}\colon\ H_{\text{w.s.}}=E-J,\qquad R=J+aH_{\text{w.s.}}. \tag{2.16}$$

Another simple choice is $b=0$, basically identifying the worldsheet Hamiltonian with E:

$$b=0\colon\ H_{\text{w.s.}}=\frac{E}{1-a},\qquad R=J+aH_{\text{w.s.}}. \tag{2.17}$$

In either case, the choice $a=0$ looks simple as it fixes the volume of the theory in terms of the charge J, and hence does not depend on the state, or more precisely, different choices of J yield different superselection sectors that may be studied separately.

B. Changing the gauge frame

We now come to the relation between the light-cone gauge—in particular, the parameter a introduced in Eq. (2.7)—and $T\bar{T}$ deformations. This was first discussed in Ref. [24] and in greater detail in Ref. [36], building on existing literature on the uniform light-cone gauge [32–35].

Changes of gauge frame and the Hamiltonian.—Varying the parameters a and b introduced in Eq. (2.7) *cannot* have any physical consequence. It is simple to understand this for a variation of b, with a fixed. Such a change of course modifies the spectrum of $H_{\text{w.s.}}$, but will not affect the spectrum of E, defined through Eq. (2.15); it is quite simply a linear redefinition of the operator whose spectrum we are computing. When varying a things are more subtle (keeping b fixed for simplicity). Now R varies, and moreover the Hamiltonian density $-p_+(X^i,\acute{X}^i,p_i)$ depends explicitly on a. Hence formally we must have

$$0=\frac{\mathrm{d}}{\mathrm{d}a}H_{\text{w.s.}}=-\frac{\mathrm{d}}{\mathrm{d}a}\int_0^{J+aH_{\text{w.s.}}}\mathrm{d}\sigma\,p_+(X^i,\acute{X}^i,p_i;a). \tag{2.18}$$

This property is well known in the context light-cone gauge-fixed strings [35], and has also been verified perturbatively for a number of models; see e.g., Refs. [27,59,60].

Changes of gauge frame and the S matrix.— It is instructive to consider the condition (2.18) for models described by a factorized S-matrix and Bethe ansatz. In terms of particles corresponding to the fields X^i, with worldsheet momentum p and energy $\omega_i(p)$,[7] the interactions of $H_{\text{w.s.}}$ translate to a nontrivial S matrix. If this S matrix is factorizable we need only the 2-to-2 scattering matrix $S_{i_1i_2}^{i'_2i'_1}(p_1,p_2;a)$, which depends on a, like $H_{\text{w.s.}}$. The energy of a state with momenta $p_1,\dots p_M$ can be computed for asymptotic states, where all particles are approximately free and

$$P_{\text{w.s.}}=\sum_{k=1}^{M}p_k,\qquad H_{\text{w.s.}}=\sum_{k=1}^{M}\omega_{i_k}(p_k). \tag{2.19}$$

In finite volume R the momenta are quantized, as prescribed by the Bethe-Yang equations, which for diagonal scattering take the form[8]

$$e^{ip_jR(a)}\prod_{k\neq j}^{M}S_{i_ji_k}^{i_ki_j}(p_j,p_k;a)=1. \tag{2.20}$$

Already in Ref. [57] it was argued that the a dependence of the S matrix takes the form

$$S_{i_ji_k}^{i_ki_j}(p_j,p_k;a)=e^{ia\Phi(p_j,p_k)}S_{i_ji_k}^{i_ki_j}(p_j,p_k), \tag{2.21}$$

with

$$\Phi(p_j,p_k)=p_k\omega_{i_j}(p_j)-p_j\omega_{i_k}(p_k). \tag{2.22}$$

[6]For several string backgrounds this choice preserves some manifest supersymmetry, protecting the corresponding vacuum from quantum corrections and simplifying quantization of the theory.

[7]The worldsheet momentum p should not be confused with the conjugate momenta p_μ. The index i denotes the flavor of the particle.

[8]Nondiagonal S matrices can be incorporated by the nested Bethe ansatz, and the following arguments can be repeated to arrive at the same conclusion. The exact spectrum also incorporates finite-size effects (exponentially suppressed in R) [61,62]. These can be accounted for by the thermodynamic Bethe ansatz [63,64], with again the same conclusion.

This is a CDD factor [55], meaning that it solves the homogeneous crossing equation, regardless of the specific form of $\omega_i(p)$. Using that $P_{\mathrm{w.s.}} = 0$, we get

$$e^{ip_k(J+aH_{\mathrm{w.s.}})}e^{-iap_kH_{\mathrm{w.s.}}}\prod_{k\neq j}^{M} S^{i_k i_j}_{i_j i_k}(p_j,p_k) = 1, \qquad (2.23)$$

which indeed is a independent.

C. $T\bar{T}$ deformations vs gauge-frame choices

The relation between the uniform light-cone gauge and $T\bar{T}$ deformations [24,36] is now clear. First, the dependence of the volume R on the energy $H_{\mathrm{w.s.}}$ is precisely such as to reproduce the Burgers equation [2,3]. Second, the phase factor $\Phi(p_j,p_j)$ is precisely the $T\bar{T}$ "CDD factor" of Refs. [3,30,31]. Indeed for a relativistic theory with $p = m\sinh\theta$ and $\omega(p) = m\cosh\theta$ we have $\Phi(p_j,p_k) = m_j m_k \sinh(\theta_k - \theta_j)$. What is important to note is that the change of gauge frames described above *does not* generate a new theory; indeed we have stressed that a change of a does not affect the spectrum of $H_{\mathrm{w.s.}}$ [see Eq. (2.18)]. What would generate a deformation of the $T\bar{T}$ type is *to deform the Hamiltonian density* $-p_+(X^i,\acute{X}^i,p_i;a)$ by tuning a, *without redefining the volume* R accordingly. In this sense the a dependence of the light-cone gauge frame can be used to generate $T\bar{T}$-deformed Hamiltonian and Lagrangian densities [11,36]. In a similar way, a variation of the frame parameter b also induces a deformation if we vary the Hamiltonian density $-p_+(X^i,\acute{X}^i,p_i;b)$ without changing the relation between $H_{\mathrm{w.s.}}$, E and J of Eq. (2.15).

Our next goal will be to understand such deformations, in particular those related to a, in geometric terms. Let us introduce an *ad hoc* notation to denote deformations (as opposed to changes of the gauge frame),

$$a \to \bar{a} = a - \delta a, \qquad b \to \bar{b} = b - \delta b, \qquad (2.24)$$

meaning that δa and δb are *deformation parameters*, which generate genuinely new theories. In particular, the parameter δa is proportional to the $T\bar{T}$ deformation parameter.

III. DEFORMED BACKGROUNDS FROM $T\bar{T}$

We just reviewed how the $T\bar{T}$ deformation of a bosonic theory can be described by coupling it to two additional isometric coordinates t and ϕ and endowing it with parametrization invariance. Then the $T\bar{T}$-deformed Hamiltonian (or Lagrangian) density may be obtained from gauge fixing this parent theory and varying the gauge-frame parameter a while keeping the worldsheet size R fixed.[9] It is natural to ask what the geometrical interpretation of the deformed parent theory is. For instance, let us take a string background, fix uniform light-cone gauge, and then vary the parameters a, b in $-p_+(X^i,\acute{X}^i,p_i;a,b)$ but not in Eq. (2.15). What geometry would lead to such a gauge-fixed theory?

A. $T\bar{T}$ deformations as a coordinate shift

Let us begin by considering the $T\bar{T}$ deformation in terms of reparametrizing the light-cone coordinates. The effect of changing a and b in our light-cone parametrization amounts to

$$X^+ \to X^+ + \delta a\,\frac{X^- + (2\bar{b}-1)X^+}{\Delta_{\bar{a}\bar{b}}},$$
$$X^- \to X^- - \delta b\,\frac{X^+ - (2\bar{a}-1)X^-}{\Delta_{\bar{a}\bar{b}}}, \qquad (3.1)$$

where the $X^\pm$ on the right-hand side are our new light-cone coordinates. It may seem that such a redefinition is trivial. Indeed this linear map is certainly a local diffeomorphism. Hence *locally* the new metric that we obtain from such a shift will be equivalent to the original one. This does not mean that the geometry will be the same *globally*, unless we also modify the boundary conditions of the field $X^\pm$ according to the shift (3.1), and unless we redefine the interpretation of the charges $P_\pm$. Purely the coordinates result in a different spectrum for the gauge-fixed theory. It is instructive to work this out in some detail for some examples, such as pp waves and flat space, or $\mathrm{AdS}_5\times\mathrm{S}^5$ and LLM geometries. We will do so in Secs. IV and V. Before doing so we will discuss a more general and transparent way to understand the geometric effect of the shift (3.1), by exploiting a formal relation between the uniform light-cone gauge and the static gauge [45].

B. From uniform light-cone gauge to static gauge

In the Hamiltonian or first-order formalism one fixes a light-cone gauge by fixing $X^+ = \tau$ and $p_- = (1-b)^{-1}$, as in Eq. (2.9). Alternatively, as shown in Ref. [45], we can obtain the same result, by T dualizing the action in X^-, integrating out the worldsheet metric, and fixing $X^+ = \tau$ and the T-dual coordinate $\tilde{X}^- = \sigma/(1-b)$, i.e., fixing a static gauge. Let us briefly review why this is the case.

To perform T duality in the X^- direction we gauge the shift symmetry for X^-, replacing

$$\partial_\alpha X^- \to \partial_\alpha X^- + A_\alpha \qquad (3.2)$$

in the Lagrangian, and adding the term $\tilde{X}^-\epsilon^{\alpha\beta}\partial_\alpha A_\beta$,

[9] More general actions and deformations may be studied in the same way, and we refer the reader to Refs. [36,37] for a detailed discussion of these points.

$$L(\partial_\alpha X^+, \partial_\alpha X^-, X^i) \to L(\partial_\alpha X^+, \partial_\alpha X^- + A_\alpha, X^i) + \tilde{X}^- \epsilon^{\alpha\beta}\partial_\alpha A_\beta, \tag{3.3}$$

where the Lagrange multiplier field $\tilde{X}^-$ ensures that A_α is flat and hence pure gauge. Integrating out $\tilde{X}^-$ gives back the original Lagrangian, while integrating out A_α gives the Lagrangian of the T-dual model. Upon integrating out A_α we in particular need to take into account the equation of motion for A_τ

$$\partial_\sigma \tilde{X}^- = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = p_-, \tag{3.4}$$

where p_- is the momentum conjugate to the *original* light-cone coordinate X^-. We see that the gauge condition $p_- = 1/(1-b)$ translates to

$$\tilde{X}^- = \frac{\sigma}{1-b}, \tag{3.5}$$

in the T-dual picture. The range of σ in the T-dual picture is fixed by the requirement that $\tilde{X}^-$ winds an integer number of times. This matches with the intuition that T duality interchanges winding and momentum modes, so that a vacuum with nonzero momentum P_- along X^- has nonzero winding along $\tilde{X}^-$. On the other hand, since we considered no winding along X^- in the original theory, we will have no momentum along $\tilde{P}_-$.[10] To understand the physical meaning of $\tilde{P}_-$ we recall that $\tilde{p}_-$ is canonically conjugated to $\tilde{X}^- \sim \sigma$. Indeed using the Virasoro constraint $\mathcal{C}_1$ we have

$$0 = \mathcal{C}_1 = 2\tilde{p}_- + p_i \acute{X}^i \Rightarrow \tilde{P}_- = \frac{1}{2} P_{\text{w.s.}}, \tag{3.6}$$

so that a state with zero winding in the original theory is level matched in the T-dual description. In summary, fixing a uniform light-cone gauge is equivalent to T dualizing in X^- and fixing a static gauge instead. This procedure has been applied in setups of increasing generality in Refs. [59,65,66].

C. $T\bar{T}$ in the T-dual picture

Now let us compare light-cone gauge fixing with two different choices of "gauge" parameter from the T-dual perspective, having in mind to keep R fixed. Starting with a parent theory $\mathcal{T}(a,b)$ with gauge parameters a and b, we can T dualize in X^- to obtain a dual model, $\tilde{\mathcal{T}}(a,b)$, whose static gauge version is equivalent to the light-cone gauge version of the original. In the parent theory we can vary our choice of gauge parameters, where $a \to \bar{a} = a - \delta a$ and $b \to \bar{b} = b - \delta b$, corresponds to the coordinate redefinition (3.1). In this resulting theory, we can fix a light-cone gauge with respect to our new light-cone gauge coordinates, and again view this from a T-dual perspective. All in all this gives us two theories that in the static gauge are related by a change of the gauge parameters a and b:

$$\begin{array}{ccc} \mathcal{T}(a,b) & \xleftrightarrow{\text{redefinition (3.1)}} & \mathcal{T}(\bar{a},\bar{b}) \\ \updownarrow \text{T duality} & & \text{T duality} \updownarrow \\ \tilde{\mathcal{T}}(a,b) & \longleftrightarrow & \tilde{\mathcal{T}}(\bar{a},\bar{b}) \end{array} \tag{3.7}$$

where all arrows can be traversed in the opposite direction as well of course. Clearly, $\tilde{\mathcal{T}}(a,b)$ and $\tilde{\mathcal{T}}(\bar{a},\bar{b})$ are related by a T duality in $\tilde{X}^-$, followed by the coordinate redefinition (3.1), followed by another T duality in X^-. If we specialize this to the case corresponding to a $T\bar{T}$ transformation *only*, i.e., $\delta b = 0$ and $b = \bar{b} = 1/2$, the transformation (3.1) is simply a shift,

$$X^+ \to Y^+ = X^+ + 2\delta a X^-, \qquad X^- \to Y^- = X^-. \tag{3.8}$$

Hence the diagram above yields precisely a TsT sequence:

$$\begin{array}{ccc} \mathcal{T}(a) & & \mathcal{T}(\bar{a}) \\ (X^+, X^-) & \xleftrightarrow{\text{shift (3.8)}} & (Y^+, Y^-) \\ \updownarrow \text{T duality} & & \text{T duality} \updownarrow \\ \tilde{\mathcal{T}}(a) & & \tilde{\mathcal{T}}(\bar{a}) \\ (X^+, \tilde{X}^-) & \xleftrightarrow{\text{TsT}} & (Y^+, \tilde{Y}^-) \end{array} \tag{3.9}$$

As we remarked, changing the light-cone gauge parameters while keeping the string length fixed—a $T\bar{T}$ deformation—results in a change of the original background that is rather subtle, as it affects the *global* aspects of the geometry. However, things change considerably by T dualizing and viewing the $T\bar{T}$ deformation as a TsT transformation. In the TsT picture, the deformation is a true deformation of the metric, and cannot be removed by a diffeomorphism (at least in general). This gives us a family of backgrounds, which in static gauge manifestly give us a Lagrangian density equal to the $T\bar{T}$ deformation of the original light-cone gauge-fixed string. If we treat the parameter in this family of backgrounds as a gauge parameter, i.e., we also vary the string length $[P_- = P_-(a)]$, we do nothing. In the dual picture, we would have to adjust the periodicity conditions of $\tilde{X}^-$, because here R is related to the range of $\tilde{X}^-$, and momentum becomes winding:

[10] Winding (dual momentum) can be incorporated in the gauge fixing; see e.g., Ref. [37]. Here we focus on the simplest setting, which suffices to obtain the relation between backgrounds.

$$\boxed{\begin{array}{c}\mathcal{T}(a)\\ X^+=\tau,\quad p_-=2,\\ 2R=\int_0^R \mathrm{d}\sigma\, p_-\end{array}} \quad\Longleftrightarrow\quad \boxed{\begin{array}{c}\widetilde{\mathcal{T}}(a)\\ X^+=\tau,\quad \widetilde{X}^-=2\sigma,\\ 2R=\int_0^R \mathrm{d}\sigma\,\partial_\sigma\widetilde{X}_-\end{array}} \tag{3.10}$$

This is in agreement with the fact that a TsT transformation can be undone by a twist of the boundary conditions of the coordinates involved [47], and in line with our expectation that only *global* features of the geometry are affected. Here, the nontrivial metric deformation is exactly what we want to keep. In other words, doing a $T\bar{T}$ deformation instead of a gauge transformation from the T-dual perspective amounts to redefining the metric without keeping track of any twist of the boundary conditions. Hence the TsT approach makes more manifest the *geometrical* effect of a $T\bar{T}$ deformation.

D. *TsT* and boundary conditions

As we mentioned, it is well established that a TsT transformation of a sigma model is classically equivalent to twisting the boundary conditions of the sigma model before the TsT transformation [47–49]. These twisted boundary conditions affect the fields associated with the TsT transformation, in our case X^+ and X^-. Concretely a TsT transformation of the type (3.9) corresponds to the boundary conditions

$$\begin{aligned} Y^+(R)-Y^+(0)&=X^+(R)-X^+(0)+2\delta a\tilde{P}_-,\\ \tilde{Y}^-(R)-\tilde{Y}^-(0)&=\tilde{X}^-(R)-\tilde{X}^-(0)-2\delta aP_+. \end{aligned} \tag{3.11}$$

Such a twist of the boundary conditions can usually be equivalently viewed as a Drinfeld-Reshetikhin twist [51,52] of the S matrix, of the form

$$\mathbb{S}\to e^{i\gamma\epsilon^{kl}\hat{Q}_k\otimes\hat{Q}_l}\mathbb{S}, \tag{3.12}$$

for some $\gamma\in\mathbb{R}$ and depending on the Cartans $\hat{Q}_j$ relative to the twisted coordinates. This picture, and the effect of this twist, is quite clear when such Cartans act linearly on the particles of the theory; in the simplest case, they correspond to the particle flavors, and $\hat{Q}_j$ is proportional to the number operator for a given particle flavor. In our case the situation is not as transparent, because the charges corresponding to P_+ and $\tilde{P}_-$ are not number operators in the Fock space. In general, the charges corresponding to the longitudinal isometries may not be linearly realized on the Fock space. However for our particular gauge choice, both P_+ and $\tilde{P}_-$ act diagonally on a single-particle state. To evaluate the value of P_+ and $\tilde{P}_-$ on a one-particle state of momentum p_j we have to recall the static-gauge fixing, which for $b=1/2$ takes the form $X^+=\tau$, $\tilde{X}^-=2\sigma$. Then as we have seen in Eqs. (2.12) and (3.6) we have that $H_{\text{w.s.}}=-P_+$ and $P_{\text{w.s.}}=2P_-$, so that

$$P_+(p_j)=-\omega_j(p_j),\qquad \tilde{P}_-(p_j)=\frac{1}{2}p_j. \tag{3.13}$$

Based on this, we expect the S matrix to undergo a Drinfeld-Reshetikhin twist of the form (3.12). Considering for simplicity an S matrix of the form (2.21) such a twist would yield

$$\begin{aligned} S^{i_ki_j}_{i_ji_k}(p_j,p_k)&\to S^{i_ki_j}_{i_ji_k}(p_j,p_k;\delta a)\\ &=e^{2i\delta a[\tilde{P}_+(p_j)P_-(p_k)-P_-(p_j)\tilde{P}_+(p_k)]}S^{i_ki_j}_{i_ji_k}(p_j,p_k)\\ &=e^{i\delta a[p_j\omega_{i_k}(p_k)-p_k\omega_{i_j}(p_j)]}S^{i_ki_j}_{i_ji_k}(p_j,p_k). \end{aligned} \tag{3.14}$$

We see that this precisely matches the CDD factor (2.22). Below we will illustrate these ideas with some examples.

IV. FIRST EXAMPLE: *pp*-WAVE GEOMETRIES

Let us consider a pp-wave metric

$$ds^2=4\mathrm{d}X^+\mathrm{d}X^- - V(X^i)\mathrm{d}X^+\mathrm{d}X^+ + \mathrm{d}X^i\mathrm{d}X^i. \tag{4.1}$$

We will consider the case where the theory has a quadratic action and is hence solvable, which is the case when

$$V=\text{const},\quad \text{or}\quad V(X^i)=\sum_i(\mu_iX^i)^2. \tag{4.2}$$

In practice we could complete this to a supersymmetric model, as well as possibly include a nontrivial B field with $H=dB=C_{ij}\mathrm{d}X^+\wedge\mathrm{d}X^i\wedge\mathrm{d}X^j$,[11] but we will refrain from doing so to avoid cluttering our analysis. In fact, our analysis will be perhaps most interesting in the simplest case $V(X^i)=\text{const}$, i.e., for a flat spacetime.

Shift of the light-cone coordinates. We can consider changing the gauge parameters $a\to\bar{a}=a-\delta a$ and $b\to\bar{b}=b-\delta b$ introduced above. This changes the form of the light-cone components of the metric. It is insightful to consider two simple cases. Let us first consider changing $b\to b-\delta b$. In terms of the new light-cone coordinates, the original metric now gives light-cone components

$$\begin{aligned} G_{+-}&=4+\delta b\frac{4(1-2a)}{\Delta_{ab}},\\ G_{++}&=-V+\delta b\frac{4}{\Delta_{ab}},\qquad G_{--}=0. \end{aligned} \tag{4.3}$$

[11] Such a B field plays an important role in particular in $AdS_3/CFT2$ [67–69] where it allows for a particularly simple exact S matrix [24,25,27,70].

We can see that, up to rescaling X^+, we have a simple change of the potential $V(X^i)$. The most interesting case, and the one related to $T\bar{T}$ deformations, is changing $a \to a - \delta a$, which we do for simplicity at $b = 1/2$. This gives

$$G_{+-} = 2 - 2\delta a V, \quad G_{++} = -V, \quad G_{--} = 4\delta a(2 - \delta a V),$$
$$G^{+-} = \frac{1 - \delta a V}{2}, \quad G^{++} = -\delta a(2 - \delta a V), \quad G^{--} = \frac{V}{4}, \tag{4.4}$$

where we suppressed the X^i dependence in V.

A. Hamiltonian and spectrum of the deformed theories

Let us now fix light-cone gauge with $X^+ = \tau$ and $p_- = 2$ (for $b = 1/2$). The Hamiltonian can be easily found from the Virasoro constraints [35]

$$\begin{aligned} -p_+ = {} & [(1 + 2\delta a(p_i p_i + \acute{X}^i \acute{X}^i) \\ & + \delta a^2(16(\acute{X}^-)^2 - (p_i p_i + \acute{X}^i \acute{X}^i)V) \\ & - 16\delta a^3(\acute{X}^-)^2 V + 4\delta a^4(\acute{X}^-)^2 V^2)^{1/2} - (1 + \delta a V)] \\ & \times [\delta a(2 - \delta a V)]^{-1}, \end{aligned} \tag{4.5}$$

where $\acute{X}^- = -p_i \acute{X}^i/2$. This is not a particularly transparent equation. However, expanding in the deformation parameter we recover

$$\begin{aligned} -p_+ = {} & \frac{1}{2} p_i p_i + \frac{1}{2} \acute{X}^i \acute{X}^i + \frac{1}{2} V(X^i) \\ & - \frac{\delta a}{4} [(p_i p_i + \acute{X}^i \acute{X}^i + 4\acute{X}^-) \\ & \times (p_i p_i + \acute{X}^i \acute{X}^i - 4\acute{X}^-) - V(X^i)^2] + O(\delta a^2), \end{aligned} \tag{4.6}$$

which is the free pp-wave Hamiltonian at $\delta a = 0$, corrected by quartic interaction terms at leading order in δa.

B. Spectrum of the deformed theory

The spectrum of the deformed theory can be found in principle from the Hamiltonian (4.5). However, it is simplest to derive this from the form of the deformed S matrix. The undeformed theory at $\delta a = 0$ is free. The dispersion relation is

$$\omega_i(p) = \sqrt{c^2 p^2 + \mu_i^2}, \tag{4.7}$$

where c depends on the string tension, and the S matrix is the identity. Hence the spectrum, for $b = 1/2$ and $a = 0$, is fixed by the quantization condition

$$1 = e^{i p_j R} = e^{i p_j J} \Rightarrow p_j = \frac{2\pi n_j}{J}, \qquad j = 1, \ldots M, \tag{4.8}$$

subject to the level-matching constraint $\sum_j n_j = 0$ so that

$$H_{\text{w.s.}} = E - J = \sum_{j=1}^{M} \omega_{i_j}\left(\frac{2\pi}{J} n_j\right). \tag{4.9}$$

If we consider the deformed theory we have that the quantization condition is modified by

$$1 = e^{i p_j (R + \delta a H_{\text{w.s.}})} \Rightarrow p_j = \frac{2\pi n_j}{J + \delta a H_{\text{w.s.}}}, \tag{4.10}$$

so that for the energy we have

$$H_{\text{w.s.}} = E - J = \sum_{j=1}^{M} \omega_{i_j}\left(\frac{2\pi n_j}{J + \delta a H_{\text{w.s.}}}\right). \tag{4.11}$$

The case of flat space.—The above equation cannot be solved in closed form unless $\mu_i = 0$, which is the flat-space case. In that case we have $\omega(p) = c|p|$, so that we can introduce left- and right-movers with

$$N = \sum_{i: n_i > 0} n_i, \qquad \tilde{N} = -\sum_{i: n_i < 0} n_i. \tag{4.12}$$

Hence we get the familiar equation

$$\begin{aligned} H_{\text{w.s.}} &= \frac{4\pi c}{J - \delta a H_{\text{w.s.}}}, \\ H_{\text{w.s.}} &= E - J = \frac{\sqrt{J^2 + 16\pi c \delta a N} - J}{2\delta a}, \end{aligned} \tag{4.13}$$

where we used that $N = \tilde{N}$. We recover the fact that going from $a = 0$ to $a = 1/2$, with $\delta a = 1/2$, sends us from the free pp-wave geometry $\mathrm{d}s^2 = 4\mathrm{d}X^+\mathrm{d}X^- + \mathrm{d}X^i\mathrm{d}X^i$ to the flat-space one, where indeed

$$E = \sqrt{J^2 + 8\pi c N}. \tag{4.14}$$

C. Geometric interpretation of the shift

We have seen that a transformation with $\delta a = 1/2$ sends us from a metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}X^+\mathrm{d}X^+ + 2\mathrm{d}X^+\mathrm{d}X^- + \mathrm{d}X^i\mathrm{d}X^i \tag{4.15}$$

to one of the form

$$\mathrm{d}s^2 = -\mathrm{d}Y^+\mathrm{d}Y^+ + \mathrm{d}Y^-\mathrm{d}Y^- + \mathrm{d}X^i\mathrm{d}X^i. \tag{4.16}$$

Both these metrics define flat spaces, yet the string spectra are substantially different. This is because the two resulting manifolds, despite being *locally* isomorphic, are *globally*

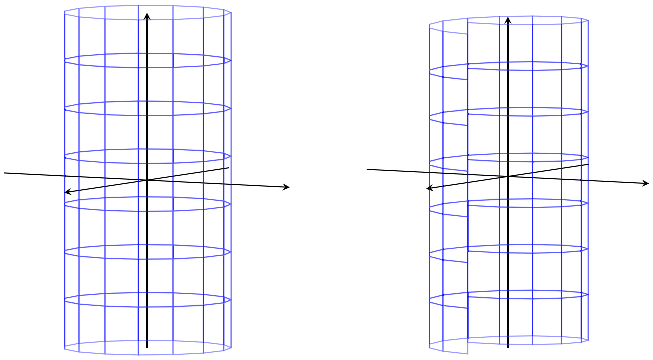

FIG. 1. The embedding of (Y^+, Y^-) in $\mathbb{R}^{1,2}$ before and after the shift. This submanifold corresponds to the target space geometry; in the static gauge $Y^+ \sim \tau$ and $Y^- \sim \sigma$ the string worldsheet has the same topology. Left: Before the shift Eq. (4.17) has periodic boundary conditions. Right: After the shift Eq. (4.18) has twisted boundary conditions proportional to δa.

different unless we define nontrivial boundary conditions for the metric (4.15). In Eq. (4.16) Y^+ is the time coordinate, with range $\mathbb{R}$, while Y^- is a space coordinate with some e.g., range $2\pi R_Y$. The whole cylinder can be embedded in $\mathbb{R}^{1,2} \ni (t, z_1, z_2)$ as

$$(t, z_1, z_2) = \left(Y^+, \cos\frac{Y^-}{R_Y}, \sin\frac{Y^-}{R_Y}\right). \tag{4.17}$$

Under a true diffeomorphism we would have a different embedding

$$(t, z_1, z_2) = \left(Y^+ - 2\delta a Y^-, \cos\frac{Y^-}{R_Y}, \sin\frac{Y^-}{R_Y}\right). \tag{4.18}$$

We can conclude that the linear transformation $Y^+ = X^+ - 2\delta a X^-$ which relates Eq. (4.16) to Eq. (4.15) is not a diffeomorphism unless we correctly keep track of the boundary conditions of the fields; see Fig. 1. The difference will become even more transparent in static gauge, as we shall see in the next section.

D. *TsT*-deformed geometry

If the take the view that a deformation $a \to \bar{a} = a - \delta a$ should be seen from the static gauge, then the background undergoes a TsT transformation. Starting from the geometry (4.1), we would like to T dualize in X^-. This however is problematic since X^- is null. Fortunately this problem disappears for any other member of our family of deformed backgrounds. Put differently, we want to consider the TsT transformation of a T dual of a background, but since two of the T dualities cancel out, we are really just considering an "sT" transformation, and after the shift we no longer have issues with null coordinates. Indeed, if we shift our coordinates as in Eq. (3.8) we obtain

$$ds^2 = 4(1 - \delta a V)\mathrm{d}Y^+\mathrm{d}Y^- - V\mathrm{d}Y^+\mathrm{d}Y^+ + 4\delta a(2 - \delta a V)\mathrm{d}Y^-\mathrm{d}Y^- + \mathrm{d}X^i\mathrm{d}X^i. \tag{4.19}$$

As long as δa is nonzero, Y^- is not null. T dualizing in Y^- now gives

$$ds^2 = \frac{-4\mathrm{d}Y^+\mathrm{d}Y^+ + \mathrm{d}\tilde{Y}^-\mathrm{d}\tilde{Y}^-}{4\delta a(2 - \delta a V)} + \mathrm{d}X^i\mathrm{d}X^i,$$
$$B = -\frac{1}{\delta a}\frac{1 - \delta a V}{2 - \delta a V}\mathrm{d}Y^+ \wedge \mathrm{d}\tilde{Y}^-. \tag{4.20}$$

This is our TsT-transformed background.[12] The problem in the geometry at $\delta a = 0$ reflects our inability to T dualize in a null direction. Taking this geometry and fixing a static gauge, by definition gives the gauge-fixed Hamiltonian density of Eq. (4.5), which is nevertheless finite (and free) at $\delta a = 0$. In the flat-space case, where $V = \text{const}$, we get the flat Minkowski metric with an overall scale in front of Y^+, $\tilde{Y}^-$ and a constant B field. Once again this affects the spectrum when we impose the static gauge conditions.

V. SECOND EXAMPLE: LIN-LUNIN-MALDACENA GEOMETRIES

One of the reasons to consider $T\bar{T}$ deformations is to construct new integrable models starting from known ones. In the context of string sigma models, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ type IIB superstring [71,72] is a prime example to consider deforming. At the same time, our methods are not restricted to integrable models. As a second illustrative example, let us therefore consider a more general, not generically integrable, class of string backgrounds containing $\mathrm{AdS}_5 \times \mathrm{S}^5$, where the $T\bar{T}$ deformation can be neatly accounted for: LLM geometries [73].

A. Some essential facts about LLM geometries

The geometries constructed in Ref. [73] manifestly preserve a $\mathfrak{so}(4) \oplus \mathfrak{so}(4) \oplus \mathfrak{u}(1)$ bosonic algebra. Furthermore, they are required to preserve half of the maximal amount of supercharges, i.e., 16 real supercharges. These assumptions result in an ansatz for the whole supergeometry [73], where the line element is

$$\mathrm{d}s^2 = -y(e^G + e^{-G})(\mathrm{d}t + V_i\mathrm{d}x^i)^2 + \frac{\mathrm{d}y^2 + \mathrm{d}x^i\mathrm{d}x^i}{y(e^G + e^{-G})} + ye^G\mathrm{d}\Omega_3{}^2 + ye^{-G}\mathrm{d}\Omega_3'^2, \tag{5.1}$$

[12] Put differently, if we TsT transform this, the first T duality takes us back to Eq. (4.19), the shift then amounts to changing the value of δa, and the second T duality brings us back to the above background (4.20) with a different value of δa. In other words, for generic δa Eq. (4.20) gives the TsT transformation of the T-dual geometry of the plane wave. It just happens to degenerate at $\delta a = 0$, the point of would-be null T duality.

where the potential $V_1(y, x_1, x_2)$, $V_2(y, x_1, x_2)$ as well as the function $G(y, x_1, x_2)$ are fixed in terms of a single function $z(y, x_1, x_2)$:

$$z = \frac{1}{2}\frac{e^{2G} - 1}{e^{2G} + 1}, \qquad y\partial_y V_i = \epsilon_{ij}\partial_j z,$$
$$y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z. \tag{5.2}$$

Moreover, the y dependence in $z(y, x_i)$ is fixed by a Laplace-like equation and that on the plane $y = 0$ the function is piecewise constant, $z(0, x_i) = \pm\frac{1}{2}$. Using this, it is possible to consider a vast class of geometries, including pp-wave ones.

Geometries with additional rotation symmetry.—For our purposes it is convenient to restrict ourselves to geometries that possess one further $\mathfrak{u}(1)$ isometry, corresponding to rotations in the (x_1, x_2) plane. Calling (r, φ) the radial and angular coordinates in that plane, the metric (5.1) simplifies and

$$\mathrm{d}s^2 = -y(e^G + e^{-G})(\mathrm{d}t + V_\varphi \mathrm{d}\varphi)^2 + \frac{\mathrm{d}y^2 + \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2}{y(e^G + e^{-G})}$$
$$+ ye^G\mathrm{d}\Omega_3{}^2 + ye^{-G}\mathrm{d}\Omega_3'^2, \tag{5.3}$$

and now G and $V_\varphi = -r\sin\varphi V_1 + r\cos\varphi V_2$ depend only on (y, r). Furthermore, on the $y = 0$ plane $z(0, r)$ is given by rings where values of $z = \pm\frac{1}{2}$ alternate. The general solution for $z(y, r)$ is then [73]

$$z(y, r) = \frac{(-1)^M}{2} + \sum_{i=0}^{M}(-1)^{i+1}\zeta(y, r; r_i), \tag{5.4}$$

with

$$\zeta(y, r; r_i) = \frac{1}{2}\left(\frac{r^2 - r_i^2 + y^2}{\sqrt{(r^2 + r_i^2 + y^2)^2 - 4r_i^2 r^2}} - 1\right). \tag{5.5}$$

Indeed $\zeta(0, r; r_i) = (\mathrm{sgn}[r^2 - r_i^2] - 1)/2$, so that $z(0, r)$ asymptotes to $(-1)^M$ at large r and is always $-1/2$ at $r = 0$.[13] We can also solve the equation (5.2) for V_φ to find

$$V_\varphi(y, r) = \psi_\varphi(r) + \sum_{i=1}^{M}(-1)^{i+1}v(y, r; r_i), \tag{5.6}$$

with

$$v(y, r; r_i) = -\frac{1}{2}\left(\frac{r^2 + y^2 + r_i^2}{\sqrt{(r^2 + y^2 + r_i^2)^2 - 4r_i^2 r^2}} - 1\right). \tag{5.7}$$

[13]This is a slightly different normalization with respect to Ref. [73], as we will be interested in changing the large-r behavior later on.

This solution differs from the one in Ref. [73] by the function $\psi_\varphi(r)$ which, looking back at Eq. (5.2), must be y independent and should yield an irrotational vector field (ψ_1, ψ_2) in the (x_1, x_2) plane. If we require V_φ to be well defined at $r = 0$ and $r = \infty$, it must be that $\psi_\varphi(r) = 0$.

Undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$—. Among the many LLM geometries, we can recover undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ by simply setting $M = 0$, $\psi(r) = 0$, and performing the change of variables [73]

$$y = r_0 \sin\theta \sinh\rho, \quad r = r_0 \cos\theta \cosh\rho \quad \varphi = \phi - t. \tag{5.8}$$

This gives the line element of $\mathrm{AdS}_5 \times \mathrm{S}^5$ in global coordinates

$$\mathrm{d}s^2 = r_0[-\cosh^2\rho\, \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\, \mathrm{d}\Omega_3{}^2 + \cos^2\theta\, \mathrm{d}\phi^2$$
$$+ \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\Omega_3'^2]. \tag{5.9}$$

B. Deforming the LLM geometries

It is natural to ask whether the deformation discussed above can be applied to an LLM geometry to obtain a geometry of the same type. We may address this question in the direct geometry or in the T-dual one. Here it is most illustrative to work in terms of the direct geometry, where we consider the shift (3.8).[14] The shift deformation makes sense in the case where we have an $\mathfrak{u}(1)^{\oplus 2}$ symmetry on top of the $\mathfrak{so}(4)^{\oplus 2}$, because (a combination of) the two $\mathfrak{u}(1)$ directions will play the role of the shift symmetries $X^\pm$ appearing in the light-cone gauge fixing. Moreover, by construction, the shift deformation preserves the full $\mathfrak{so}(4)^{\oplus 2} \oplus \mathfrak{u}(1)^{\oplus 2}$ symmetry. For $\mathrm{AdS}_5 \times \mathrm{S}^5$, it clearly will also preserve the $\mathfrak{su}(2|2)^{\oplus 2}$ (centrally extended) symmetry which is manifest after gauge fixing [35]. It is actually relatively straightforward to reverse engineer what the shift of Sec. III is in the LLM language. Since the shift does not affect the angular part of the line element, it is reasonable to look for a transformation affecting V_φ only. Consider the redefinition

$$V_\varphi(y, x_1, x_2) \mapsto V_\varphi(y, x_1, x_2) + \alpha. \tag{5.10}$$

In Cartesian components this amounts to $V_i \mapsto V_i + \alpha\psi_i$ with $\psi_i = \epsilon_{ij}\partial_j \log r$. This is clearly irrotational wherever it is defined, and yields a new solution of the LLM constraints. To compare with the shift transformation discussed in Eq. (3.8) it is convenient to introduce light-cone coordinates. As evidenced by Eq. (5.8), ϕ is already a light-cone coordinate, and in our notation of Eq. (2.6), $\varphi = 2X^-$ while $t = X^+ - X^-$. Hence the line element (5.3) becomes

[14]We illustrate the dual TsT deformation of $\mathrm{AdS}_n \times \mathrm{S}^n$ in the conclusions.

$$\mathrm{d}s^2 = -y(e^G + e^{-G})(\mathrm{d}X^+ + (2V_\varphi - 1)\mathrm{d}X^-)^2 + \frac{\mathrm{d}y^2 + \mathrm{d}r^2 + 4r^2(\mathrm{d}X^-)^2}{y(e^G + e^{-G})} + \ldots, \quad (5.11)$$

where the ellipsis denotes the angular part of the line element, which is unchanged. We can see that the modification

$$V_\varphi \mapsto V_\varphi + \alpha \quad \text{is equivalent to} \quad X^+ \mapsto X^+ + 2\delta a X^- \quad \text{for } \alpha = \delta a, \quad (5.12)$$

while leaving X^- unchanged. This is precisely the deformation of Eq. (3.8). This is completely general, holding for any LLM geometry with an additional $\mathfrak{u}(1)$ symmetry.

VI. CONCLUSIONS AND OUTLOOK

The uniform light-cone gauge formalism for string theory [32–34] allows one to readily construct $T\bar{T}$ deformations of various models [11,24,36,37]. This starts by uplifting the original model to a reparametrization-invariant model in two higher dimensions, and then gauge fixing appropriately. In this paper we asked what happens to this uplifted geometry under a $T\bar{T}$ deformation, i.e., what the $T\bar{T}$ deformation of a gauge-fixed (string) sigma model means at the level of the target space geometry. Operatively, we tune the would-be gauge parameter in the worldsheet Lagrangian *only*, and not in the identification of conserved charges or volume R of the model. The effect of this deformation is subtle from the point of view of the original geometry for our light-cone gauge picture, but becomes more transparent when taking a T-dual point of view [45], where we exchange light-cone gauge for *static* gauge fixing. In the T-dual frame, a $T\bar{T}$ deformation affects the local geometry directly, taking the form of a TsT transformation.[15] This TsT picture then also gives a natural interpretation to the $T\bar{T}$ CDD factor as a Drinfeld-Reshetikhin twist of the S matrix; this is particularly transparent thanks to the static-gauge identification of target-space charges with worldsheet momentum and energy. Computationally, for the purpose of generating deformed Lagrangians, this static-gauge approach is equivalent to the uniform light-cone gauge treatment of Refs. [11,24,36,37]; conceptually however, we feel that it further clarifies why $T\bar{T}$ deformations are so intimately related to gauge-fixed sigma models, and may help further uncover some of the features of this important class of deformations. Let us remark that our discussion of $T\bar{T}$ deformations can be quite straightforwardly extended to $T\bar{J}$ and $J\bar{T}$ deformations, as well as to more general deformations along the lines of Ref. [36]. We briefly comment on this in the Appendix.

It would be interesting to extend our approach to include fermions and to consider supergeometries. First steps have been taken while investigating the relation between $T\bar{T}$ and supersymmetry, as well as in Ref. [36] for more general theories. However, a complete analysis of such a setup, including the role of κ symmetry, has not yet been performed. It would also be interesting to extend this analysis to the nonrelativistic deformations of Refs. [38–44], which can indeed be understood in the framework of light-cone gauge [37], and further explore its relation with null dipole-deformed CFT [74,75], which can indeed be understood in AdS/CFT by means of TsT transformations involving light-cone directions.

Another especially interesting case is that of integrable *string* sigma models. Here, the $T\bar{T}$ CDD factor can be readily taken into account in their Bethe ansatz. As we saw, in the special case of flat space, the $T\bar{T}$ deformation can trivialize the S matrix. In general, however, the S matrix will remain nontrivial, and be nontrivially modified. This is certainly the case for all integrable string backgrounds involving Ramond-Ramond fluxes, where the form of the light-cone symmetry algebra fixes the S matrix to be nondiagonal.[16] Still, it would be interesting to study the corresponding deformations of (the T duals of) such integrable backgrounds, as at least we have good control over the spectral problem. In this paper we have considered two classes of backgrounds: pp-wave geometries, which are integrable, and LLM geometries, which are not generally integrable, with the important exception of $\mathrm{AdS}_5 \times \mathrm{S}^5$. In both cases we derived explicit expressions for the deformed backgrounds. In particular, for $\mathrm{AdS}_5 \times \mathrm{S}^5$, we have described a "shifted" geometry which would yield a $T\bar{T}$ deformation of Beisert's S matrix. It is presently not clear what interpretation this would have in the gauge-theory dual.

One could also study deformed AdS backgrounds in the T-dual frame, by means of a TsT transformation rather than a shift. As an illustration, for $\mathrm{AdS}_2 \times \mathrm{S}^2$ in global coordinates

$$ds^2 = -(1+\rho^2)dt^2 + \frac{d\rho^2}{1+\rho^2} + (1-r^2)d\phi^2 + \frac{dr^2}{1-r^2}, \quad (6.1)$$

with isometry coordinates t and ϕ as input for the light-cone coordinates, the dual deformed geometry takes the form

[15] In this paper we only discussed NSNS backgrounds explicitly, but RR fields can of course be added and TsT transformed.

[16] The relationship between light-cone symmetry algebra and the integrable S matrix was originally worked out for $\mathrm{AdS}_5 \times \mathrm{S}^5$ in Refs. [57,76].

$$ds^2 = \frac{-(1-r^2)(1+\rho^2)dY^+dY^+ + \frac{1}{4}d\tilde{Y}^-d\tilde{Y}^-}{1-r^2+2\delta a(1-r^2)-\delta a^2(r^2+\rho^2)} + \frac{d\rho^2}{1+\rho^2} + \frac{dr^2}{1-r^2},$$
$$B = -\frac{1-r^2-\delta a(r^2+\rho^2)}{1-r^2+2\delta a(1-r^2)-\delta a^2(r^2+\rho^2)} dY^+ \wedge d\tilde{Y}^-, \tag{6.2}$$

where we deform away from $a = 0$.[17] As the $T\bar{T}$ deformation preserves integrability, it would be interesting to combine it with other integrable deformations of strings, such as Yang-Baxter deformations [77–79]. These, as a nice contrast, contain TsT transformations of the *direct* (as opposed to T-dual) geometry [80]; see also Refs. [81–83].

Integrable AdS_3 backgrounds have some particularly interesting features. They can be supported by a mixture of Ramond-Ramond (RR) and Neveu-Schwarz–Neveu-Schwarz (NSNS) fluxes (see Ref. [84] for a review of integrability in this setup), and the kinematics depends both on the RR strength h and the NSNS strength k. When no RR fluxes are present, $h = 0$ and $k \in \mathbb{N}$ is the level of the $\mathfrak{sl}(2) \oplus \mathfrak{su}(2)$ supersymmetric Wess-Zumino-Witten (WZW) model, giving a chiral model even after gauge fixing. In this case the perturbative worldsheet S matrix is *proportional to the identity*, and takes a universal form dependent on the chirality, but not the masses, of the scattered particles [27,70]. This allows to solve for the spectrum in closed form [24,25,27], similarly to flat space as discussed in Sec. IV B. However, unlike flat space, the scattering *cannot* be completely trivialized by a $T\bar{T}$ deformation.[18] Interestingly, for this theory it also possible to consider a $T\bar{T}$ deformation of the *dual* conformal field theory. It was proposed [21,22] that these too can be studied *on the worldsheet*, namely that a $T\bar{T}$ deformation on the boundary should correspond to a $J\bar{J}$ deformation on the worldsheet (which can be then analyzed by worldsheet-CFT tools). Such $J\bar{J}$ deformations *can also be understood as TsT transformations* [85]. This scenario can be generalized to nonrelativistic $J\bar{T}$ deformations, and in that case too deformations of the dual CFT_2 can be interpreted as TsT transformations [86,87].[19] This points to the fact that in pure-NSNS AdS_3/CFT_2, a rich interplay arises between deformations on the worldsheet and in the two-dimensional dual, which is yet to be explored. We hope to revisit some of these questions in the near future.

ACKNOWLEDGMENTS

We would like to thank Sergey Frolov, Ben Hoare, Leonardo Rastelli and Fiona K. Seibold for insightful discussions and Sergey Frolov for comments on a draft of this manuscript. A. S. would like to thank the participants of the workshop *A Fresh Look at AdS3/CFT2* at Villa Garbald for stimulating discussions that paved the way to this research, as well as the organizers and participants of the CERN Theory Institute *Exact computations in* AdS/CFT for the stimulating environment where the last phases of this work were completed, and in particular Nikolay Bobev for discussions on $AdS_5 \times S^5$. He is also grateful to Christian Ferko, Hongliang Jiang, Sav Sethi and Gabriele Tartaglino-Mazzucchelli for ongoing collaboration on related topics. A. S.'s work is funded by ETH Career Seed Grant No. SEED-23 19-1, as well as by the NCCR *SwissMAP*, funded by the Swiss National Science Foundation. The work of S. T. is supported by the German Research Foundation (DFG) via the Emmy Noether program "Exact Results in Extended Holography". S. T. is supported by L. T.

APPENDIX: $T\bar{J}$, $J\bar{T}$, AND TsT

In the main text we discuss the geometric interpretation of $T\bar{T}$ deformations as TsT transformations in the T-dual frame. It is natural to ask whether a similar interpretation exists for deformations of $T\bar{J}$ and $J\bar{T}$ types. This is indeed the case, even though only a limited number of such deformations have a simple geometric interpretation in a given T-dual frame.

In order to be able to consider generalized deformations we need to assume that our background has a further $\mathfrak{u}(1)$ isometry commuting with two light-cone isometries. Let us fix coordinates such that this extra isometry acts as a shift in X^1. This direction can now be mixed into TsT transformations. In general, given m commuting isometries we can consider $m(m-1)/2$ independent TsT transformations, giving us three with isometries in $X^+, \tilde{X}^-$ and X^1.

For concreteness let us consider a TsT transformation in $(\tilde{X}^-, X^1)$. Since we are doing a TsT transformation starting from the static-gauge frame, from the point of view of the light-cone-gauge description we are doing an sT transformation, shifting $X^1 \to X^1 + \alpha X^-$, and T dualizing $X^- \to \tilde{X}^-$. This shift in the original geometry is precisely what corresponds to the canonical transformation giving a JT_μ deformation with $\mu = \sigma$, the spatial direction on the worldsheet. Indeed, as discussed in Ref. [37], cf. point 3 in Sec. II B, this canonical transformation is

$$X^1 \to X^1 - a_{1-}X^-, \qquad X^- \to X^-, \qquad p_1 \to p_1,$$
$$p_- \to p_- + a_{1-}p_1. \tag{A1}$$

[17] Unlike the pp-wave example of the last section, here we generically never encounter a null direction in the T duality. Of course we can see the problem reappear by taking $r, \rho \to 0$ and taking $\delta a = -1/2$.

[18] This is because in this case $p_1\omega(p_2) - p_2\omega(p_1) \neq \pm 2p_1p_2$, nor does it vanish for same-chirality scattering; this is crucial to reproduce the spectrally flowed sectors of the WZW description (see Refs. [25,27]).

[19] See the Appendix for a discussion of $J\bar{T}$ deformations on the worldsheet of the gauge-fixed theory.

For $\alpha = -a_{1-}$ this agrees exactly with our shift; the shift in momenta follows directly from the shift of coordinates. To complete the picture we just perform one more T duality in $\tilde{X}^-$, which takes us back to the static-gauge picture.

In the main text we see that a TsT in $(\tilde{X}^-, X^+)$ gives the $T\bar{T}$ deformation, and we just discussed that one in $(\tilde{X}^-, X^1)$ gives a JT_σ deformation. The last option is a TsT in (X^1, X^+), which is similarly easily seen to correspond to the JT_τ deformation as given in Ref. [37]. Of course it is possible to take (linear) combinations of the JT_μ as well as $T\bar{T}$ deformations. In general J can be any (not necessarily chiral) conserved $\mathfrak{u}(1)$ current.

Various further deformations can be realized via canonical transformations in the light-cone gauge-fixing picture of Ref. [37], and many of them can be obviously cast as TsT transformations. However, these would not all be TsT transformations in our natural T-dual frame for the $T\bar{T}$ deformation. For example, the $\tilde{J}T_\mu$, $\mu = \tau$, deformation of Ref. [37] can be naturally viewed as a TsT transformation in $(\tilde{X}^1, X^+)$, i.e., it can be viewed as a TsT transformation in a geometry where we have first T dualized in X^1.

---

[1] A. B. Zamolodchikov, Expectation value of composite field T anti-T in two-dimensional quantum field theory, arXiv:hep-th/0401146.

[2] F. A. Smirnov and A. B. Zamolodchikov, On space of integrable quantum field theories, Nucl. Phys. **B915**, 363 (2017).

[3] A. Cavaglià, S. Negro, I. M. Szécsényi, and R. Tateo, $T\bar{T}$-deformed 2D Quantum Field Theories, J. High Energy Phys. 10 (2016) 112.

[4] G. Bonelli, N. Doroud, and M. Zhu, $T\bar{T}$-deformations in closed form, J. High Energy Phys. 06 (2018) 149.

[5] R. Conti, S. Negro, and R. Tateo, Conserved currents and $T\bar{T}_s$ irrelevant deformations of 2D integrable field theories, arXiv:1904.09141.

[6] R. Conti, L. Iannella, S. Negro, and R. Tateo, Generalised Born-Infeld models, Lax operators and the $T\bar{T}$ perturbation, J. High Energy Phys. 11 (2018) 007.

[7] B. Chen, L. Chen, and P.-X. Hao, Entanglement entropy in $T\bar{T}$-deformed CFT, Phys. Rev. D **98**, 086025 (2018).

[8] O. Aharony, S. Datta, A. Giveon, Y. Jiang, and D. Kutasov, Modular invariance and uniqueness of $T\bar{T}$ deformed CFT, J. High Energy Phys. 01 (2019) 086.

[9] J. Cardy, $T\bar{T}$ deformations of non-Lorentz invariant field theories, arXiv:1809.07849.

[10] T. Araujo, E. Colgáin, Y. Sakatani, M. M. Sheikh-Jabbari, and H. Yavartanoo, Holographic integration of $T\bar{T}$ & $J\bar{T}$ via $O(d, d)$, J. High Energy Phys. 03 (2019) 168.

[11] M. Baggio, A. Sfondrini, G. Tartaglino-Mazzucchelli, and H. Walsh, On $T\bar{T}$ deformations and supersymmetry, J. High Energy Phys. 06 (2019) 063.

[12] C.-K. Chang, C. Ferko, and S. Sethi, Supersymmetry and $T\bar{T}$ deformations, J. High Energy Phys. 04 (2019) 131.

[13] H. Jiang, A. Sfondrini, and G. Tartaglino-Mazzucchelli, $T\bar{T}$ deformations with $\mathcal{N} = (0, 2)$ supersymmetry, arXiv:1904.04760.

[14] C.-K. Chang, C. Ferko, S. Sethi, A. Sfondrini, and G. Tartaglino-Mazzucchelli, $T\bar{T}$ flows and (2,2) supersymmetry, arXiv:1906.00467.

[15] N. Cribiori, F. Farakos, and R. von Unge, The 2D Volkov-Akulov model as a $T\bar{T}$ deformation, arXiv:1907.08150.

[16] S. Dubovsky, V. Gorbenko, and M. Mirbabayi, Asymptotic fragility, near AdS_2 holography and $T\bar{T}$, J. High Energy Phys. 09 (2017) 136.

[17] S. Dubovsky, V. Gorbenko, and G. Hernández-Chifflet, $T\bar{T}$ partition function from topological gravity, J. High Energy Phys. 09 (2018) 158.

[18] R. Conti, S. Negro, and R. Tateo, The $T\bar{T}$ perturbation and its geometric interpretation, J. High Energy Phys. 02 (2019) 085.

[19] T. Ishii, S. Okumura, J.-I. Sakamoto, and K. Yoshida, Gravitational perturbations as $T\bar{T}$-deformations in 2D dilaton gravity systems, arXiv:1906.03865.

[20] L. McGough, M. Mezei, and H. Verlinde, Moving the CFT into the bulk with $T\bar{T}$, J. High Energy Phys. 04 (2018) 010.

[21] A. Giveon, N. Itzhaki, and D. Kutasov, $T\bar{T}$ and LST, J. High Energy Phys. 07 (2017) 122.

[22] A. Giveon, N. Itzhaki, and D. Kutasov, A solvable irrelevant deformation of AdS_3/CFT_2, J. High Energy Phys. 12 (2017) 155.

[23] M. Asrat, A. Giveon, N. Itzhaki, and D. Kutasov, Holography beyond AdS, Nucl. Phys. **B932**, 241 (2018).

[24] M. Baggio and A. Sfondrini, Strings on NS-NS backgrounds as integrable deformations, Phys. Rev. D **98**, 021902 (2018).

[25] A. Dei and A. Sfondrini, Integrable spin chain for stringy Wess-Zumino-Witten models, J. High Energy Phys. 07 (2018) 109.

[26] V. Gorbenko, E. Silverstein, and G. Torroba, dS/dS and $T\bar{T}$, J. High Energy Phys. 03 (2019) 085.

[27] A. Dei and A. Sfondrini, Integrable S matrix, mirror TBA and spectrum for the stringy $AdS_3 \times S^3 \times S^3 \times S^1$ WZW model, J. High Energy Phys. 02 (2019) 072.

[28] A. Giveon, Comments on $T\bar{T}$, $J\bar{T}$ and String Theory, arXiv:1903.06883.

[29] G. Giribet, $T\bar{T}$-deformations, AdS/CFT and correlation functions, J. High Energy Phys. 02 (2018) 114.

[30] S. Dubovsky, R. Flauger, and V. Gorbenko, Solving the simplest theory of quantum gravity, J. High Energy Phys. 09 (2012) 133.

[31] M. Caselle, D. Fioravanti, F. Gliozzi, and R. Tateo, Quantisation of the effective string with TBA, J. High Energy Phys. 07 (2013) 071.

[32] G. Arutyunov and S. Frolov, Integrable Hamiltonian for classical strings on $AdS_5 \times S^5$, J. High Energy Phys. 02 (2005) 059.

[33] G. Arutyunov and S. Frolov, Uniform light-cone gauge for strings in $AdS_5 \times S^5$: Solving $\mathfrak{su}(1|1)$ sector, J. High Energy Phys. 01 (2006) 055.

[34] G. Arutyunov, S. Frolov, and M. Zamaklar, Finite-size effects from giant magnons, Nucl. Phys. **B778**, 1 (2007).

[35] G. Arutyunov and S. Frolov, Foundations of the $AdS_5 \times S^5$ superstring. part I, J. Phys. A **42**, 254003 (2009).

[36] S. Frolov, TTbar deformation and the light-cone gauge, arXiv:1905.07946.

[37] S. Frolov, $T\bar{T}$, $\tilde{J}J$, JT and $\tilde{J}T$ deformations, arXiv: 1907.12117.

[38] M. Guica, An integrable Lorentz-breaking deformation of two-dimensional CFTs, SciPost Phys. **5**, 048 (2018).

[39] A. Bzowski and M. Guica, The holographic interpretation of $J\bar{T}$-deformed CFTs, J. High Energy Phys. 01 (2019) 198.

[40] Y. Nakayama, Very Special $T\bar{J}$ deformed CFT, arXiv: 1811.02173.

[41] S. Chakraborty, A. Giveon, and D. Kutasov, $J\bar{T}$ deformed CFT_2 and string theory, J. High Energy Phys. 10 (2018) 057.

[42] B. Le Floch and M. Mezei, Solving a family of $T\bar{T}$-like theories, arXiv:1903.07606.

[43] M. Guica, On correlation functions in $J\bar{T}$-deformed CFTs, arXiv:1902.01434.

[44] S. Chakraborty, A. Giveon, and D. Kutasov, $T\bar{T}$, $J\bar{T}$, $T\bar{J}$ and string theory, arXiv:1905.00051.

[45] M. Kruczenski and A. A. Tseytlin, Semiclassical relativistic strings in S^5 and long coherent operators in $\mathcal{N} = 4$ SYM theory, J. High Energy Phys. 09 (2004) 038.

[46] O. Lunin and J. M. Maldacena, Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals, J. High Energy Phys. 05 (2005) 033.

[47] S. Frolov, Lax pair for strings in Lunin-Maldacena background, J. High Energy Phys. 05 (2005) 069.

[48] L. F. Alday, G. Arutyunov, and S. Frolov, Green-Schwarz strings in TsT-transformed backgrounds, J. High Energy Phys. 06 (2006) 018.

[49] S. J. Van Tongeren, On Yang–Baxter models, twist operators, and boundary conditions, J. Phys. A **51**, 305401 (2018).

[50] N. Beisert and R. Roiban, Beauty and the twist: The Bethe ansatz for twisted $N = 4$ SYM, J. High Energy Phys. 08 (2005) 039.

[51] V. Drinfeld, Quasi Hopf algebras, Alg. Anal. **1N6**, 114 (1989).

[52] N. Reshetikhin, Multiparameter quantum groups and twisted quasitriangular Hopf algebras, Lett. Math. Phys. **20**, 331 (1990).

[53] C. Ahn, Z. Bajnok, D. Bombardelli, and R. I. Nepomechie, Twisted Bethe equations from a twisted S-matrix, J. High Energy Phys. 02 (2011) 027.

[54] C. Ahn, M. Kim, and B.-H. Lee, Worldsheet S-matrix of beta-deformed SYM, Phys. Lett. B **719**, 458 (2013).

[55] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Low's scattering equation for the charged and neutral scalar theories, Phys. Rev. **101**, 453 (1956).

[56] N. Beisert, The $\mathfrak{su}(2|2)$ dynamic S-matrix, Adv. Theor. Math. Phys. **12**, 948 (2008).

[57] G. Arutyunov, S. Frolov, and M. Zamaklar, The Zamolodchikov-Faddeev algebra for $AdS_5 \times S^5$ superstring, J. High Energy Phys. 04 (2007) 002.

[58] Y. Chervonyi and O. Lunin, (Non)-integrability of geodesics in D-brane backgrounds, J. High Energy Phys. 02 (2014) 061.

[59] T. Klose, T. McLoughlin, R. Roiban, and K. Zarembo, Worldsheet scattering in $AdS_5 \times S^5$, J. High Energy Phys. 03 (2007) 094.

[60] P. Sundin and L. Wulff, Worldsheet scattering in AdS_3/CFT_2, J. High Energy Phys. 07 (2013) 007.

[61] M. Lüscher, Volume dependence of the energy spectrum in massive quantum field theories. 1. Stable particle states, Commun. Math. Phys. **104**, 177 (1986).

[62] M. Lüscher, Volume dependence of the energy spectrum in massive quantum field theories. 2. Scattering states, Commun. Math. Phys. **105**, 153 (1986).

[63] C.-N. Yang and C. P. Yang, Thermodynamics of one-dimensional system of bosons with repulsive delta function interaction, J. Math. Phys. (N.Y.) **10**, 1115 (1969).

[64] A. B. Zamolodchikov, Thermodynamic Bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models, Nucl. Phys. **B342**, 695 (1990).

[65] K. Zarembo, Worldsheet spectrum in AdS_4/CFT_3 correspondence, J. High Energy Phys. 04 (2009) 135.

[66] G. Arutyunov and S. J. van Tongeren, Double Wick rotating Green-Schwarz strings, J. High Energy Phys. 05 (2015) 027.

[67] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills, J. High Energy Phys. 04 (2002) 013.

[68] J. G. Russo and A. A. Tseytlin, On solvable models of type IIB superstring in NS-NS and R-R plane wave backgrounds, J. High Energy Phys. 04 (2002) 021.

[69] A. Dei, M. R. Gaberdiel, and A. Sfondrini, The plane-wave limit of $AdS_3 \times S^3 \times S^3 \times S^1$, J. High Energy Phys. 08 (2018) 097.

[70] B. Hoare and A. A. Tseytlin, On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: Tree-level S-matrix, Nucl. Phys. **B873**, 682 (2013).

[71] R. R. Metsaev and A. A. Tseytlin, Type IIB superstring action in $AdS_5 \times S^5$ background, Nucl. Phys. **B533**, 109 (1998).

[72] I. Bena, J. Polchinski, and R. Roiban, Hidden symmetries of the $AdS_5 \times S^5$ superstring, Phys. Rev. D **69**, 046002 (2004).

[73] H. Lin, O. Lunin, and J. M. Maldacena, Bubbling AdS space and 1/2 BPS geometries, J. High Energy Phys. 10 (2004) 025.

[74] M. Alishahiha and O. J. Ganor, Twisted backgrounds, PP waves and nonlocal field theories, J. High Energy Phys. 03 (2003) 006.

[75] M. Guica, F. Levkovich-Maslyuk, and K. Zarembo, Integrability in dipole-deformed $\mathcal{N} = 4$ super Yang-Mills, J. Phys. A **50**, 394001 (2017).

[76] G. Arutyunov, S. Frolov, J. Plefka, and M. Zamaklar, The off-shell symmetry algebra of the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, J. Phys. A **40**, 3583 (2007).

[77] C. Klimcik, On integrability of the Yang-Baxter sigma-model, J. Math. Phys. (N.Y.) **50**, 043508 (2009).

[78] F. Delduc, M. Magro, and B. Vicedo, An Integrable Deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring Action, Phys. Rev. Lett. **112**, 051601 (2014).

[79] I. Kawaguchi, T. Matsumoto, and K. Yoshida, Jordanian deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, J. High Energy Phys. 04 (2014) 153.

[80] D. Osten and S. J. van Tongeren, Abelian Yang–Baxter deformations and TsT transformations, Nucl. Phys. **B915**, 184 (2017).

[81] T. Matsumoto and K. Yoshida, Lunin-Maldacena backgrounds from the classical Yang-Baxter equation—towards the gravity/CYBE correspondence, J. High Energy Phys. 06 (2014) 135.

[82] S. J. van Tongeren, On classical Yang-Baxter based deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring, J. High Energy Phys. 06 (2015) 048.

[83] R. Borsato and L. Wulff, Target space supergeometry of η and λ-deformed strings, J. High Energy Phys. 10 (2016) 045.

[84] A. Sfondrini, Towards integrability for $\mathrm{AdS}_3/\mathrm{CFT}_2$, J. Phys. A **48**, 023001 (2015).

[85] R. Borsato and L. Wulff, Marginal deformations of WZW models and the classical Yang–Baxter equation, J. Phys. A **52**, 225401 (2019).

[86] L. Apolo and W. Song, Strings on warped AdS_3 via $\mathrm{T}\bar{\mathrm{J}}$ deformations, J. High Energy Phys. 10 (2018) 165.

[87] L. Apolo and W. Song, Heating up holography for single-trace $J\bar{T}$ deformations, arXiv:1907.03745.