# Generalized coherent states in QCD from asymptotic symmetries

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We investigate the relationship between asymptotic symmetries of QCD and vacuum transitions induced during scattering. Starting with the Fock vacuum in the far past, the infinity of conservation laws associated to (non-Abelian) asymptotic symmetries in QCD can be used to determine the vacuum in the far future. We show that the corresponding asymptotic Hilbert space in the future is generated by a class of dressed states in which each finite energy particle is dressed by a cloud of interacting soft gluons. We identify the precise structure of the dressing using conservation laws and show that the corresponding asymptotic states are equivalent to the generalized coherent states defined by Catani *et al.*

## I. INTRODUCTION

Since the seminal work of Strominger in [1], conservation laws associated to asymptotic symmetries have played an increasingly important role in uncovering the infrared structure of gauge theories and gravity. The associated constraints on the $S$ matrix of these theories are intimately connected to soft theorems. For the so-called leading asymptotic symmetries such as large gauge transformations in Yang-Mills theory and supertranslations in perturbative gravity, these constraints relate the residue of the Weinberg soft factor to the insertion of soft modes in the scattering states of the theory. This relationship between soft theorems and conservation laws has an interesting offshoot [2–6] in that the insertion of soft modes results in a shifted vacuum. Because of an infinity of soft modes parametrized by their location on the celestial sphere, one has an infinite degeneracy of vacua. It was shown in [2–5] that transitions between these vacua are constrained by conservation laws and starting with a vacuum in the far past, conservation laws imply that scattering states in the far future are such that the finite energy particles are dressed by a cloud of low frequency modes. In QED and gravity it has been shown that these "dressed states" are equivalent to the well-known Faddeev-Kulish states. Thus, starting with a number of technical assumptions primary among which is the infinity of conservation laws, one can obtain the dressed states of the theory in which the $S$ matrix is infrared finite.

We would like to ask precisely the same question in QCD. That is, starting with the infinite dimensional non-Abelian group of asymptotic symmetries, what does a given vacuum of perturbative QCD transit to in the far future? In a nutshell, we show that a close scrutiny of existing works on the relationship between dressed states and symmetries in perturbative gravity helps us to distill a minimal set of assumptions under which one obtains the asymptotic states in which finite energy partons are dressed by a cloud of (infinitely) correlated soft gluons. We argue that these states are equivalent to the well-known dressed states in QCD [7,8].

We emphasize that our analysis of determining the dressing only involves asymptotic symmetries and the so-called orthogonality condition. We do not use any multisoft gluon theorem (which is equivalent to Ward identities in undressed Fock states) to help us fix the dressing operator. This leads to a rather intricate analysis in which higher order (nested) Ward identities, orthogonality with respect to insertion of multiple gluons, and global color conservation come together to give the definition of a dressed state.

The paper is organized as follows. In Sec. II we recast the ideas in [2–5] to derive the Faddeev-Kulish states in gravity. In Sec. III we apply these ideas in the context of QCD so as to derive a set of dressed states in QCD. In Sec. IV we show that under a set of assumptions, one can recover the dressed states in QCD, which were obtained by Catani *et al.* in [8], from Ward identities associated to asymptotic symmetries in QCD.

## II. FROM ASYMPTOTIC SYMMETRIES TO DRESSED STATES IN GRAVITY

In this section we revisit the works of [2–5,9] with a motive of understanding the dressed states in gravity and their relationship with asymptotic symmetries. We begin by reviewing the relationship between Ward identities associated to Bondi-Metzner-Sachs (BMS) supertranslations

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and leading single soft graviton theorem [10,11] in Sec. II A. In Secs. II B and II C, a brief review of the work [4] in connecting the dressed states in gravity and BMS supertranslations is presented. Further in Secs. II D and II E, we show that under certain assumptions one can derive the Faddeev-Kulish states in gravity from BMS supertranslations. In Sec. II F an alternative asymptotic state to the Faddeev-Kulish state is presented.

## A. Review of BMS supertranslation Ward identity and leading single soft graviton theorem

We start with the definition of the supertranslation charge $Q[f]$ [10]. Supertranslation charges are associated to the class of vector fields that preserve the structure of an asymptotically flat metric. These charges are parametrized by an arbitrary function $f(z, \bar{z})$ on the sphere. $Q[f]$ can be written as the sum of a soft part ($Q_{\text{soft}}[f]$) and a hard part ($Q_{\text{hard}}[f]$). Upon asymptotic quantization of the charge, one can see that $Q_{\text{soft}}[f]$ is responsible for the creation or annihilation of a soft graviton mode. Additionally, one uses the Christodoulou-Klainerman (CK) condition[1] to relate the positive helicity soft graviton mode with the negative helicity soft graviton mode. Finally, the soft charge can be written as

$$Q_{\text{soft}}[f] = \lim_{E_p \to 0} \frac{E_p}{4\pi\kappa} \int d^2 w D_w^2 f(w, \bar{w})(a_-(E_p\hat{p}) + a_+^\dagger(E_p\hat{p})),$$

$$= \lim_{E_p \to 0} \frac{E_p}{4\pi\kappa} \int d^2 w D_{\bar{w}}^2 f(w, \bar{w})(a_+(E_p\hat{p}) + a_-^\dagger(E_p\hat{p})).$$

$$(2.1)$$

Here $\hat{p}$ denotes the direction of the soft graviton labeled by the coordinates $(w, \bar{w})$ and $\kappa^2 = 32\pi G$. $D_w/D_{\bar{w}}$ refers to the covariant derivative with respect to the two-sphere

The hard charge $Q_{\text{hard}}[f]$ receive contributions from two terms. One of them includes the contribution of the energy-momentum tensor of massless particles at null infinity, while the other includes the contribution of massive particles that will reach timelike infinity. The hard charge action on a massless and massive particle[2] can be written as

$$Q_{\text{hard}}[f]|\text{in}\rangle = \sum_{i=\text{in}} E_{k_i} f(\hat{k}_i)|\text{in}\rangle;$$

$$Q_{\text{hard}}[f]|\text{in}\rangle_{\text{massive}} = \sum_{i=\text{in}} m_i f_H(\vec{k}_i/m_i)|\text{in}\rangle_{\text{massive}},$$

$$\langle\text{out}|Q_{\text{hard}}[f] = \sum_{i=\text{out}} E_{k_i} f(\hat{k}_i)\langle\text{out}|;$$

$$_{\text{massive}}\langle\text{out}|Q_{\text{hard}}[f] = \sum_{i=\text{out}} m_i f_H(\vec{k}_i/m_i)\langle\text{out}|_{\text{massive}}. \quad (2.2)$$

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[1]The readers can refer to [10] for further details.
[2]The seminal work for outgoing/incoming particles being massive was done in [11].

Here, the sums $\sum_{\text{in}}$ and $\sum_{\text{out}}$ are over all the hard particles in the "in" and "out" states. Energy and mass of the hard particles are denoted by $E_{k_i}$ and $m_i$, respectively. The unit spatial vector and three-momentum of the $i$th particle are denoted by $\hat{k}_i$ and $\vec{k}_i$, respectively. For a massive particle, $f_H(\vec{k}_i/m_i)$ is defined by

$$f_H(\vec{k}_i/m_i) = \int d^2 w \mathcal{G}(\vec{k}_i/m_i; w, \bar{w}) f(w, \bar{w}). \quad (2.3)$$

where

$$\mathcal{G}(\vec{k}_i/m_i; w, \bar{w}) = -\frac{1}{2\pi} D_{\bar{w}}^2 \frac{((\epsilon^+(w, \bar{w})) \cdot (k_i/m_i))^2}{(p/E_p) \cdot (k_i/m_i)}. \quad (2.4)$$

Here $p \equiv E_p(1, \hat{p})$ denotes the four-momentum and $\epsilon^+(w, \bar{w})$ denotes the polarization of the soft graviton labeled by the coordinates $(w, \bar{w})$. The polarization vector $\epsilon^+(w, \bar{w})$ can be written as $\epsilon^+(w, \bar{w}) = 1/\sqrt{2}(\bar{w}, 1, -i, -\bar{w})$.

The Ward identity for supertranslation can be written as

$$\langle\text{out}|[Q[f], S]|\text{in}\rangle = 0 \Leftrightarrow \langle\text{out}|[Q_{\text{soft}}[f], S]|\text{in}\rangle$$

$$= -\langle\text{out}|[Q_{\text{hard}}[f], S]|\text{in}\rangle. \quad (2.5)$$

For simplicity let us consider the in and out states to contain only massive particles. Then, the Ward identity becomes

$$\lim_{E_p \to 0} \frac{E_p}{4\pi} \int d^2 w D_{\bar{w}}^2 f(w, \bar{w})(\langle\text{out}|a_+(E_p\hat{p})S|\text{in}\rangle$$

$$+ \langle\text{out}|S a_-^\dagger(E_p\hat{p})|\text{in}\rangle)$$

$$= -\kappa\left[\sum_{\text{out}} m_i f_H(\vec{k}_i/m_i) - \sum_{\text{in}} m_i f_H(\vec{k}_i/m_i)\right]\langle\text{out}|S|\text{in}\rangle. \quad (2.6)$$

Using crossing symmetry, the above equation can be written as

$$\lim_{E_p \to 0} \frac{E_p}{2\pi} \int d^2 w D_{\bar{w}}^2 f(w, \bar{w})(\langle\text{out}|a_+(E_p\hat{p})S|\text{in}\rangle)$$

$$= -\kappa\left[\sum_{\text{out}} m_i f_H(\vec{k}_i/m_i) - \sum_{\text{in}} m_i f_H(\vec{k}_i/m_i)\right]\langle\text{out}|S|\text{in}\rangle. \quad (2.7)$$

The Weinberg soft graviton theorem [12] is given by

$$\lim_{E_p \to 0} E_p\langle\text{out}|a_+(E_p\hat{p})S|\text{in}\rangle$$

$$= \frac{\kappa}{2}\left(\sum_{i=\text{out}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i=\text{in}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i}\right)$$

$$\times \langle\text{out}|S|\text{in}\rangle. \quad (2.8)$$

We adopt the notation,

$$S^{(0)}(\hat{p}; k_i) \equiv \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i}, \qquad (2.9)$$

with which the leading soft factor in the rhs of [12] can be written as

$$S^{(0)}(\hat{p}; \{k_i\}) \equiv \sum_{i=\text{out}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i=\text{in}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i}, \qquad (2.10)$$

$$\equiv \sum_{i=\text{out}} S^{(0)}(\hat{p}; k_i) - \sum_{i=\text{in}} S^{(0)}(\hat{p}; k_i). \qquad (2.11)$$

Now one can show the equivalence of the Ward identity with a soft theorem by choosing a particular $f(z, \bar{z})$, namely,

$$f(z, \bar{z}) = s(z, \bar{z}; w, \bar{w}) \equiv \frac{1 + w\bar{w}}{1 + z\bar{z}} \cdot \frac{\bar{w} - \bar{z}}{w - z}. \qquad (2.12)$$

In the rest of the paper, for simplicity of the calculations, we choose the specific form of $f(z, \bar{z})$ in (2.12) which led us to the soft theorem. After choosing this, we denote the soft charge and hard charge as $Q_{\text{soft}}(\hat{p})$ and $Q_{\text{hard}}(\hat{p})$, respectively. The soft charge therefore becomes

$$Q_{\text{soft}}(\hat{p}) = \lim_{E_p \to 0} \frac{1}{2} E_p(a_+(E_p\hat{p}) + a_-^\dagger(E_p\hat{p})). \qquad (2.13)$$

In this paper we are always concerned with the action of the soft operator at the level of scattering amplitudes. Therefore, using the notion of crossing symmetry we can always relate an outgoing positive helicity soft graviton with a negative helicity soft graviton. Hence, (2.13) can be further written as

$$Q_{\text{soft}}(\hat{p}) = \lim_{E_p \to 0} E_p a_+(E_p\hat{p}). \qquad (2.14)$$

The action of the hard charge can be written as

$$Q_{\text{hard}}(\hat{p})|k\rangle = -\frac{\kappa}{2} S^{(0)}(\hat{p}; k)|k\rangle, \qquad (2.15)$$

where $S^{(0)}(\hat{p}; k)$ is already defined in (2.9).

### B. Dressed states in gravity

Following [2], a number of recent works [3–5,13–18] have analyzed the intricate relationship between the spontaneous breaking of asymptotic symmetries, the

corresponding existence of soft modes as Goldstone modes, and the asymptotic Hilbert space which is comprised of the coherent states of such soft modes. A number of these works (notably [3–5,17]) have argued that under certain conditions, which we summarize below, the states in which asymptotic conservation laws are satisfied as Ward identities are precisely the well-known dressed states in which the $S$ matrix is infrared finite in the case of QED and gravity[9,19].

We revisit the earlier analysis [3,4] below in the context of perturbative gravity but with an eye towards QCD. More in detail, we attempt to recast some of the main ideas in [3,4] by asking the following question: if we assume that supertranslations are a symmetry of the perturbative $S$ matrix, what are the additional assumptions we need in order to derive the precise form of dressed states in perturbative gravity?

In the next section, we review and extract the key ideas contained in [3,4] which will help us determine the asymptotic Hilbert space from the existence of asymptotic symmetries in QCD.

### C. Faddeev-Kulish states in gravity from BMS supertranslations

In [4], the authors showed that conservation of the BMS charge leads to asymptotic states in which $S$-matrix elements are infrared finite.[3] The argument can be summarized as follows.

Let us consider all the degenerate vacua as eigenstates of the BMS supertranslation soft charge and consider a transition between scattering states built over such vacua. Let $|N\rangle$ denote the eigenstate of the soft operator $Q_{\text{soft}}(\hat{p})$ [defined as in (2.13)]. Let $|N_{\text{out}}, \text{out}\rangle / N_{\text{in}}, \text{in}\rangle$ denote the outgoing/incoming states which are the eigenstates of $Q_{\text{soft}}(\hat{p})$ with eigenvalues $N_{\text{out}}(\hat{p})/N_{\text{in}}(\hat{p})$; i.e.,

$$Q_{\text{soft}}(\hat{p})|N_{\text{out}}, \text{out}\rangle = N_{\text{out}}(\hat{p})|N_{\text{out}}, \text{out}\rangle,$$
$$Q_{\text{soft}}(\hat{p})|N_{\text{in}}, \text{in}\rangle = N_{\text{in}}(\hat{p})|N_{\text{in}}, \text{in}\rangle. \qquad (2.16)$$

The supertranslation Ward identity between such states can be written as

$$\langle N_{\text{out}}, \text{out}|[Q_{\text{soft}}(\hat{p}), S]|N_{\text{in}}, \text{in}\rangle$$
$$= -\langle N_{\text{out}}, \text{out}|[Q_{\text{hard}}(\hat{p}), S]|N_{\text{in}}, \text{in}\rangle. \qquad (2.17)$$

The above expression can be evaluated as

$$(N_{\text{out}}(\hat{p}) - N_{\text{in}}(\hat{p}))\langle N_{\text{out}}, \text{out}|S|N_{\text{in}}, \text{in}\rangle$$
$$= \Omega_{\text{soft}}(\hat{p})\langle N_{\text{out}}, \text{out}|S|N_{\text{in}}, \text{in}\rangle, \qquad (2.18)$$

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[3]The analogous analysis in QED was done in [2,5].

where $\Omega_{\mathrm{soft}}(\hat{p})$ is given by the soft factor (2.10). In evaluating the rhs of (2.17) one assumes that the hard charge has a trivial action on the degenerate vacua. Now the above equation suggests two possibilities:

$$N_{\mathrm{out}}(\hat{p}) - N_{\mathrm{in}}(\hat{p}) - \Omega_{\mathrm{soft}}(\hat{p}) = 0 \qquad (2.19)$$

or

$$\langle N_{\mathrm{out}}, \mathrm{out}|S|N_{\mathrm{in}}, \mathrm{in}\rangle = 0. \qquad (2.20)$$

Now, if we demand that the transition amplitude between such degenerate vacua is nontrivial then

$$N_{\mathrm{out}}(\hat{p}) - N_{\mathrm{in}}(\hat{p}) = \Omega_{\mathrm{soft}}(\hat{p}). \qquad (2.21)$$

The authors in [4] proposed an ansatz for the dressing operator for constructing such states which can be written as

$$e^{R_N} = \exp\left(\kappa \int d[k]\rho(k) \int^{\Lambda} d[q] N^{\mu\nu}(q;k)(a_{\mu\nu}^{\dagger}(q) - a_{\mu\nu}(q)\right). \qquad (2.22)$$

We will study this operator in detail in the next section. They further showed that the Faddeev-Kulish states belong to such a class of dressed states and these satisfy the constraint (2.21).

## D. A closer look at the derivation of dressed states from symmetries

As we recalled above, the derivation of dressed states consistent with the supertranslation conservation law relied on three key inputs.

  (i) The dressed state is an eigenstate of the soft charge.
  (ii) The hard charge has a trivial action on the dressing (as it had no gravitational contribution).
  (iii) $S$-matrix elements evaluated in the dressed states are nontrivial.

However, there are some caveats here that need to be emphasized from our perspective.

  (1) As we review below, due to the fact that vacuum is shifted by supertranslation soft charge, the dressed state for any choice of dressing cannot be an eigenstate of the soft charge. Because of this it may appear that the analysis presented in the previous Sec. II B is inconsistent. However, as we show below, this analysis can be made consistent if we demand that the dressed states satisfy a constraint called the orthogonality condition.[4] Analysis of the

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[4]The known construction of dressed states such as Faddeev-Kulish states do satisfy this condition as was shown in [4,5].

orthogonality condition will be central to us in the derivation of QCD asymptotic states.

  (2) If we work in linearized gravity where the hard charge has no contribution from the gravitational news tensor, it is indeed true that the hard charge commutes with the dressing. However, if the super-translation hard charge contains a contribution from the gravitational field, one needs to be careful with the commutators of the hard charge with the dressing. We will address this issue below by using the known action of the supertranslation charge on the Goldstone mode conjugate to the soft mode [10].

These caveats turn out to be even more severe in QCD and hence, in order to apply the analysis of [2–4], in that case, we determine the relationship between dressed states and asymptotic conservation laws in a slightly different manner, such that both the caveats mentioned above become explicit. Namely, our goal is to explore to what extent the infinity of supertranslation conservation laws can constrain the form of the dressed states. To address this question, we start with four assumptions.

  (1) Supertranslation symmetry is a symmetry of the quantum $S$ matrix.
  (2) Soft graviton modes satisfy what we call orthogonality relations with respect to the asymptotic states of the theory. The orthogonality condition simply means that if we consider a state $|\psi\rangle$ which is a tensor product of the (dressed) finite energy state and one or more soft graviton state, then this state $|\psi\rangle$ is orthogonal to all the (dressed) finite energy states.
  (3) The $S$-matrix elements are nontrivial.
  (4) The supertranslation hard charge has a trivial action on the dressing,

As we show below, by using all the assumptions mentioned above one can determine the dressed states in gravity. In Sec. II F we show that if one follows along the lines of [10] one obtains a different type of dressing. It turns out that in this analysis one need not use assumption (4).

To determine the dressed states which respect the above mentioned assumptions, we start with an ansatz for the dressing operator following [2,3,9]

$$e^{R_N} = \exp\left(\kappa \int d[k]\rho(k) \int^{\Lambda} d[q] N^{\mu\nu}(q;k)(a_{\mu\nu}^{\dagger}(q) - a_{\mu\nu}(q)\right), \qquad (2.23)$$

where $\rho(k) \equiv b^{\dagger}(k)b(k)$ is the number operator for the external massive particles [we are considering massive scalar particles for simplicity and $b^{\dagger}(k)$, $b(k)$ are the creation and annihilation operators for scalar particle respectively], $\kappa^2 = 32\pi G$, and $N^{\mu\nu}(q;k)$ is an arbitrary real function which has a pole in $E_q$. Here $d[k] \equiv \frac{d^3k}{(2\pi)^2 2E_k}$ is the Lorentz invariant measure. $\Lambda$ is an upper cutoff for the

integral over $q$ to ensure that the dressing comprises of only low energy gravitons. $a_{\mu\nu}^\dagger(q)$ and $a_{\mu\nu}(q)$ are the graviton creation and annihilation operators, respectively, which can be written in the polarization basis as

$$a_{\mu\nu}^\dagger(q) = \sum_{r=\pm} \epsilon_{\mu\nu}^r(q) a^{r\dagger}(q), \qquad a_{\mu\nu}(q) = \sum_{r=\pm} \epsilon_{\mu\nu}^{*r}(q) a^r(q).$$

(2.24)

These operators obey the commutation relation

$$[a^r(q), a^{s\dagger}(q')] = \delta^{rs}(2E_q)(2\pi)^3\delta^3(\vec{q} - \vec{q}').$$  (2.25)

A dressed state is constructed using the action of the asymptotic operator (2.23) on a "bare" state. Here we consider the dressing on a massive scalar field defined by

$$\varphi(x) = \int d[k][b(k)e^{ik.x} + b^\dagger(k)e^{-ik.x}].$$  (2.26)

The creation and annihilation operators of the scalar particle obey the commutation relation

$$[b(k), b^\dagger(k')] = (2\pi)^3(2E_k)\delta^3(\vec{k} - \vec{k}').$$  (2.27)

The dressed scalar state is then defined by

$$e^{R_N}|k\rangle = e^{R_N}b^\dagger(k)|0\rangle = [e^{R_N}, b^\dagger(k)]|0\rangle = e^{R_N(k)}b^\dagger(k)|0\rangle,$$

(2.28)

where

$$e^{R_N(k)} = \exp\left(\kappa \int^\Lambda d[q]N^{\mu\nu}(q;k)(a_{\mu\nu}^\dagger(q) - a_{\mu\nu}(q))\right).$$

(2.29)

The action of the asymptotic operator on a multiparticle state can also be found similarly,

$$e^{R_N}|k_1, k_2...k_n\rangle = e^{R_N(k_1)}e^{R_N(k_2)}...e^{R_N(k_n)}|k_1, k_2...k_n\rangle,$$

(2.30)

where $|k_1, k_2, ..., k_n\rangle$ denotes a multiparticle state generated by the action of creation operators $b^\dagger(k_1)$, $b^\dagger(k_2), ..., b^\dagger(k_n)$ on the Fock vacuum. From (2.30) it is clear that the dressing operator factorizes in the "hard" particle space.

Having defined the dressed state, we demonstrate caveat (1) which we discussed in the beginning of this section. Let us consider the action of the soft operator $Q_{\text{soft}}(\hat{p})$ (2.13) on a dressed state. The soft charge can be written as

$$Q_{\text{soft}}(\hat{p}) = \lim_{E_p\to0} \frac{E_p}{2}(a_+(E_p\hat{p}) + a_-^\dagger(E_p\hat{p})).$$  (2.31)

The action of $Q_{\text{soft}}(\hat{p})$ on the dressed state defined by (2.28) can be expressed as

$$Q_{\text{soft}}(\hat{p})e^{R_N}|k\rangle = [Q_{\text{soft}}(\hat{p}), e^{R_N(k)}]|k\rangle + e^{R_N(k)}Q_{\text{soft}}(\hat{p})|k\rangle.$$

(2.32)

Let us first compute the commutator term

$$[Q_{\text{soft}}(\hat{p}), e^{R_N(k)}]$$
$$= [Q_{\text{soft}}(\hat{p}), R_N(k)]e^{R_N(k)},$$
$$= \lim_{E_p\to0} \frac{E_p}{2}\left[a_+(E_p\hat{p}) + a_-^\dagger(E_p\hat{p}), \kappa\right.$$
$$\left.\times \int^\Lambda d[q]N^{\mu\nu}(q;k)(a_{\mu\nu}^\dagger(q) - a_{\mu\nu}(q))\right]e^{R_N(k)},$$
$$= \lim_{E_p\to0} \kappa E_p N^{\mu\nu}(p;k)\epsilon_{\mu\nu}^+(\hat{p})e^{R_N(k)},$$  (2.33)

where we used (2.25) and (2.24) to compute the last line from the second line. Therefore, (2.32) can be written as

$$Q_{\text{soft}}(\hat{p})e^{R_N}|k\rangle = \lim_{E_p\to0} \kappa E_p N^{\mu\nu}(p;k)\epsilon_{\mu\nu}^+(\hat{p})e^{R_N(k)}|k\rangle$$
$$+ e^{R_N(k)}Q_{\text{soft}}(\hat{p})|k\rangle.$$  (2.34)

The second term in the above expression can be written as

$$e^{R_N(k)}Q_{\text{soft}}(\hat{p})|k\rangle = e^{R_N(k)}b^\dagger(k)Q_{\text{soft}}(\hat{p})|0\rangle.$$  (2.35)

This corresponds to the action of the asymptotic operator on a one particle state built from a supertranslated vacuum.[5] If this term vanishes, then clearly the dressed state would be the eigenstate of the soft operator. But as the supertranslation charge shifts vacuum instead of annihilating it, this is not true.

In order to determine the dressed states in gravity let us start with our assumption (1) that supertranslation is a symmetry of the $S$ matrix. We then have the corresponding Ward identity

$$_d\langle\text{out}|[Q(\hat{p}), S]|\text{in}\rangle_d = 0,$$  (2.36)

where $Q(\hat{p}) = Q_{\text{soft}}(\hat{p}) + Q_{\text{hard}}(\hat{p})$ [defined in (2.13) and (2.15)]. $_d\langle\text{out}|, |\text{in}\rangle_d$ denotes the dressed out and in states, which are given as

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[5]One can also use the prescription (2.14) for $Q_{\text{soft}}(\hat{p})$ and claim that the extra term (2.35) vanishes. But this subtlety will again arise when one considers its action on an outgoing dressed state (bra).

$$_{\mathrm{d}}\langle \mathrm{out}| = \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}} = \langle \mathrm{out}| \exp\left(-\kappa \sum_{i=\mathrm{out}} \int^{\Lambda} d[q] N_i^{\mu\nu}(q;k_i)(a_{\mu\nu}^{\dagger}(q) - a_{\mu\nu}(q))\right),$$

$$|\mathrm{in}\rangle_{\mathrm{d}} = e^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle = \exp\left(\kappa \sum_{i=\mathrm{in}} \int^{\Lambda} d[q] N_i^{\mu\nu}(q;k_i)(a_{\mu\nu}^{\dagger}(q) - a_{\mu\nu}(q))\right)|\mathrm{in}\rangle. \tag{2.37}$$

And $k_i$ denotes the momentum of the external particles in the outgoing and incoming states. After expressing the charge as the sum of soft and hard parts, the Ward identity (2.36) becomes

$$\langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}[Q_{\mathrm{soft}}(\hat{p}), S]e^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle = -\langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}[Q_{\mathrm{hard}}(\hat{p}), S]e^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle. \tag{2.38}$$

Let us now consider the lhs of (2.38). This can be expanded as

$$\langle \mathrm{out}|Q_{\mathrm{soft}}(\hat{p})e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}Q_{\mathrm{soft}}(\hat{p})|\mathrm{in}\rangle$$

$$+ \langle \mathrm{out}|[e^{-R_N^{(\mathrm{out})}}, Q_{\mathrm{soft}}(\hat{p})]Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}S[Q_{\mathrm{soft}}(\hat{p}), e^{R_N^{(\mathrm{in})}}]|\mathrm{in}\rangle. \tag{2.39}$$

We can now use assumption (2), namely, the orthogonality condition, which implies that both of the first two terms of the above equation vanish. Therefore, we are left with

$$\langle \mathrm{out}|[e^{-R_N^{(\mathrm{out})}}, Q_{\mathrm{soft}}(\hat{p})]Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}S[Q_{\mathrm{soft}}(\hat{p}), e^{R_N^{(\mathrm{in})}}]|\mathrm{in}\rangle. \tag{2.40}$$

The commutators in the above expression can be evaluated using (2.33) as

$$[e^{-R_N^{(\mathrm{out})}}, Q_{\mathrm{soft}}(\hat{p})] = \lim_{E_p \to 0} \kappa E_p N_{\mathrm{out}}(p)e^{-R_N^{(\mathrm{out})}}, \qquad [Q_{\mathrm{soft}}(\hat{p}), e^{R_N^{(\mathrm{in})}}] = \lim_{E_p \to 0} \kappa E_p N_{\mathrm{in}}(p)e^{R_N^{(\mathrm{in})}}, \tag{2.41}$$

where

$$N_{\mathrm{out}}(p) = \sum_{i=\mathrm{out}} N_i^{\mu\nu}(p;k_i)\epsilon_{\mu\nu}^{+}(\hat{p}), \qquad N_{\mathrm{in}}(p) = \sum_{i=\mathrm{in}} N_i^{\mu\nu}(p;k_i)\epsilon_{\mu\nu}^{+}(\hat{p}). \tag{2.42}$$

Using these we can evaluate the lhs of (2.38) to

$$\langle \mathrm{out}|[e^{-R_N^{(\mathrm{out})}}, Q_{\mathrm{soft}}(\hat{p})]Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}S[Q_{\mathrm{soft}}(\hat{p}), e^{R_N^{(\mathrm{in})}}]|\mathrm{in}\rangle,$$

$$= \kappa \lim_{E_p \to 0} E_p(N_{\mathrm{out}}(p) - N_{\mathrm{in}}(p))\langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle. \tag{2.43}$$

The rhs of (2.38) can be expanded as

$$\langle \mathrm{out}|Q_{\mathrm{hard}}(\hat{p})e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}Q_{\mathrm{hard}}(\hat{p})|\mathrm{in}\rangle,$$

$$+ \langle \mathrm{out}|[e^{-R_N^{(\mathrm{out})}}, Q_{\mathrm{hard}}(\hat{p})]Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}S[Q_{\mathrm{hard}}(\hat{p}), e^{R_N^{(\mathrm{in})}}]|\mathrm{in}\rangle. \tag{2.44}$$

Using the action of the hard charges, the first two terms in the above expression gives

$$\langle \mathrm{out}|Q_{\mathrm{hard}}(\hat{p})e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle - \langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}Q_{\mathrm{hard}}(\hat{p})|\mathrm{in}\rangle = -\frac{\kappa}{2}S^{(0)}(\hat{p};\{k_i\})\langle \mathrm{out}|e^{-R_N^{(\mathrm{out})}}Se^{R_N^{(\mathrm{in})}}|\mathrm{in}\rangle, \tag{2.45}$$

where

$$S^{(0)}(\hat{p};\{k_i\}) = \sum_{i=\mathrm{out}} \frac{(\epsilon^{+}(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i} - \sum_{i=\mathrm{in}} \frac{(\epsilon^{+}(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i}. \tag{2.46}$$

Using expressions (2.45) and (2.43), the Ward identity (2.38) can finally be written as

$$\kappa \lim_{E_p \to 0} E_p (N_{\text{out}}(p) - N_{\text{in}}(p)) \langle \text{out}| e^{-R_N^{(\text{out})}} S e^{R_N^{(\text{in})}} |\text{in}\rangle,$$

$$= \frac{\kappa}{2} S^{(0)}(\hat{p}; \{k_i\}) \langle \text{out}| e^{-R_N^{(\text{out})}} S e^{R_N^{(\text{in})}} |\text{in}\rangle - \langle \text{out}|[e^{-R_N^{(\text{out})}}, Q_{\text{hard}}(\hat{p})] S e^{R_N^{(\text{in})}} |\text{in}\rangle + \langle \text{out}| e^{-R_N^{(\text{out})}} S [Q_{\text{hard}}(\hat{p}), e^{R_N^{(\text{in})}}] |\text{in}\rangle. \quad (2.47)$$

At this point let us recall caveat (2). If the hard charge has a contribution from the gravitational news tensor, then $Q_{\text{hard}}(\hat{p})$ does not commute with the dressing and one will not retrieve the usual conservation laws (2.19). In the Appendix B we show that the contribution from the commutator of the hard charge with the dressing is of $\mathcal{O}(\Lambda)$, where $\Lambda$ is the upper cutoff in the dressing operator. If $\Lambda$ is sufficiently small then one could ignore these terms. Hence, the last two terms in the rhs of (2.47) vanishes by assumption (4). Therefore,

$$\kappa \lim_{E_p \to 0} E_p (N_{\text{out}}(p) - N_{\text{in}}(p)) \langle \text{out}| e^{-R_N^{(\text{out})}} S e^{R_N^{(\text{in})}} |\text{in}\rangle, = \frac{\kappa}{2} S^{(0)}(\hat{p}; \{k_i\}) \langle \text{out}| e^{-R_N^{(\text{out})}} S e^{R_N^{(\text{in})}} |\text{in}\rangle. \quad (2.48)$$

We can now use assumption (3) to show that,

$$\lim_{E_p \to 0} E_p (N_{\text{out}}(p) - N_{\text{out}}(p)) = \frac{1}{2} S^{(0)}(\hat{p}; \{k_i\}) \quad (2.49)$$

or

$$\lim_{E_p \to 0} E_p \left( \sum_{i=\text{out}} N_i^{\mu\nu}(p; k_i) \epsilon_{\mu\nu}^+(\hat{p}) - \sum_{i=\text{in}} N_i^{\mu\nu}(p; k_i) \epsilon_{\mu\nu}^+(\hat{p}) \right) = \frac{1}{2} \left( \sum_{i=\text{out}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} - \sum_{i=\text{in}} \frac{(\epsilon^+(w, \bar{w}) \cdot k_i)^2}{(p/E_p) \cdot k_i} \right). \quad (2.50)$$

From the above expression we can associate naturally,

$$N_i^{\mu\nu}(p; k_i) = \frac{1}{2} \frac{k_i^\mu k_i^\nu}{p \cdot k_i}. \quad (2.51)$$

Thus, we have recovered the dressing factor for each external particle.

Substituting (2.51) in the dressing operator (2.29) we finally get the dressed state as

$$e^{R_N(k)} |k\rangle = \exp \left( \frac{\kappa}{2} \int^\Lambda d[q] \frac{k^\mu k^\nu}{q \cdot k} (a_{\mu\nu}^\dagger(q) - a_{\mu\nu}(q)) \right) |k\rangle. \quad (2.52)$$

This matches with the Faddeev-Kulish states in gravity [3].

### E. Orthogonality relations for multiple soft graviton insertions

Having derived the dressed states in gravity, we would now like to see whether these states decouple a finite number of soft graviton modes; i.e., we would like to see whether the orthogonality condition [assumption (2)] holds for more than one soft graviton mode. This analysis becomes important in QCD as it leads to a modification of the dressing factor in QCD. We will explain this detail in the QCD section. Since we already know that at the leading level, the multiple soft graviton theorems are not independent in the sense that each of the soft factors can be

determined using just the leading single soft factor; one does not expect any issue for the orthogonality condition involving multiple soft graviton modes. We will explicitly prove this in this section.

Without loss of generality, let us consider a case in which vacuum is shifted by two soft graviton modes; i.e., we would like to show

$$\langle \text{out}| Q_{\text{soft}}(\hat{p}_1) Q_{\text{soft}}(\hat{p}_2) e^{-R_N^{(\text{out})}} S e^{R_N^{(\text{in})}} |\text{in}\rangle = 0, \quad (2.53)$$

where $Q_{\text{soft}}(\hat{p}_1)$ and $Q_{\text{soft}}(\hat{p}_2)$ are soft graviton modes given by

$$Q_{\text{soft}}(\hat{p}_1) = \lim_{E_{p_1} \to 0} E_{p_1} a_+ (E_{p_1} \hat{p}_1), \quad (2.54)$$

$$Q_{\text{soft}}(\hat{p}_2) = \lim_{E_{p_2} \to 0} E_{p_2} a_+ (E_{p_2} \hat{p}_2), \quad (2.55)$$

and $e^{R_N}$ is the dressing operator already derived in the previous Sec. (2.52). The lhs of (2.53) can be evaluated as done in [3,5]. It will receive contributions from the three terms as shown in Fig. 1.

(1) Both the soft gravitons are connected to the external particles.

(2) One of the soft gravitons is connected to the external particle and the other is connected to the dressing operator.

(3) Both the soft gravitons are connected to the dressing operator.

The first contribution (1) can be evaluated by the leading double soft graviton theorem[6] in the undressed states as

$$(\langle \text{out}|Q_{\text{soft}}(\hat{p}_1)Q_{\text{soft}}(\hat{p}_2)e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle)_{\text{external}}$$

$$= \frac{\kappa^2}{4}S^{(0)}(\hat{p}_1;\{k_i\})S^{(0)}(\hat{p}_2;\{k_j\})\langle \text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle. \tag{2.56}$$

The second contribution (2) can be evaluated in the following way. The soft graviton connected to the external particle can be evaluated by single leading soft graviton theorem while the soft graviton connected to the dressing operator can be evaluated by the contraction of the soft operator with the dressing, i.e., $[Q_{\text{soft}}(\hat{p}), e^{R_N}]$. Therefore, this contribution can be written as

$$\frac{\kappa}{2}S^{(0)}(\hat{p}_1;k_i)(\langle \text{out}|[Q_{\text{soft}}(\hat{p}_2), e^{-R_N^{(\text{out})}}]Se^{R_N^{(\text{in})}}|\text{in}\rangle$$

$$+ \langle \text{out}|e^{-R_N^{(\text{out})}}S[Q_{\text{soft}}(\hat{p}_2), e^{R_N^{(\text{in})}}]|\text{in}\rangle)$$

$$+ \frac{\kappa}{2}S^{(0)}(\hat{p}_2;\{k_i\})(\langle \text{out}|[Q_{\text{soft}}(\hat{p}_1), e^{-R_N^{(\text{out})}}]Se^{R_N^{(\text{in})}}|\text{in}\rangle$$

$$+ \langle \text{out}|e^{-R_N^{(\text{out})}}S[Q_{\text{soft}}(\hat{p}_1), e^{R_N^{(\text{in})}}]|\text{in}\rangle). \tag{2.57}$$

Using the form of the dressing operator we derived in the previous section, we can finally evaluate the above expression to

$$(\langle \text{out}|Q_{\text{soft}}(\hat{p}_1)Q_{\text{soft}}(\hat{p}_2)e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle)_{\text{external+dressing}}$$

$$= -\frac{\kappa^2}{2}S^{(0)}(\hat{p}_1;\{k_i\})S^{(0)}(\hat{p}_2;\{k_j\})\langle \text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle. \tag{2.58}$$

The final contribution (3) can be evaluated by the contraction of the soft operators with the dressing. Hence, this term can be written as

$$\langle \text{out}|[Q_{\text{soft}}(\hat{p}_1), [Q_{\text{soft}}(\hat{p}_2), e^{-R_N^{(\text{out})}}]]Se^{R_N^{(\text{in})}}|\text{in}\rangle$$

$$+ \langle \text{out}|[Q_{\text{soft}}(\hat{p}_1), e^{-R_N^{(\text{out})}}]S[Q_{\text{soft}}(\hat{p}_2), e^{R_N^{(\text{in})}}]|\text{in}\rangle$$

$$+ \langle \text{out}|[Q_{\text{soft}}(\hat{p}_2), e^{-R_N^{(\text{out})}}]S[Q_{\text{soft}}(\hat{p}_1), e^{R_N^{(\text{in})}}]|\text{in}\rangle$$

$$+ \langle \text{out}|e^{-R_N^{(\text{out})}}S[Q_{\text{soft}}(\hat{p}_1), [Q_{\text{soft}}(\hat{p}_2), e^{R_N^{(\text{in})}}]]|\text{in}\rangle, \tag{2.59}$$

which evaluates to

$$(\langle \text{out}|Q_{\text{soft}}(\hat{p}_1)Q_{\text{soft}}(\hat{p}_2)e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle)_{\text{dressing}}$$

$$= \frac{\kappa^2}{4}S^{(0)}(\hat{p}_1;\{k_i\})S^{(0)}(\hat{p}_2;\{k_j\})\langle \text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle. \tag{2.60}$$

Adding up all the contributions (2.56), (2.58), and (2.60) we finally get

$$\langle \text{out}|Q_{\text{soft}}(\hat{p}_1)Q_{\text{soft}}(\hat{p}_2)e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle = 0. \tag{2.61}$$

Therefore, as expected, the two soft gravitons modes indeed decouple, if we are using the dressing operator (2.52). Without loss of generality, one can extend this analysis to multiple soft graviton modes and arrive at the same result, which suggests that the assumption (2) holds as expected.

As an aside, we can also use the orthogonality for one soft graviton mode and single soft graviton theorem to constrain the dressing ansatz we had started with. This is demonstrated in Appendix A.

In short, in perturbative gravity, it turns out that by the super translation Ward identity and the condition of orthogonality in the dressed states, the form of the dressed state is fixed. Additionally, one can use the orthogonality condition in the dressed states and the single soft theorem in the undressed states to constrain the dressing (Appendix A). Both of these constraints lead to the same conclusion.

## F. An alternative dressing

In all the previous analysis, for the determination of dressed states, we assumed (4) in which the soft gravitons are treated as a zero frequency limit of finite energy gravitons. This, together with the other assumptions, (1), (2) and (3), one naturally obtains the Faddeev-Kulish states. But, as pointed out by the authors in [10], for obtaining the correct BMS transformations, one needs to treat the soft modes as independent degrees of freedom. This is equivalent to extending the radiative phase space, so that one includes not only the free data $C_{zz}$ and $C_{\bar{z}\bar{z}}$,[7] but also the boundary modes (boundary of null infinity) defined by $C(z,\bar{z})$ and $N(z,\bar{z})$. These additional data are defined by

$$C_{zz}|_{\mathcal{I}^\pm_\mp} = D_{\bar{z}}^2 C, \tag{2.62}$$

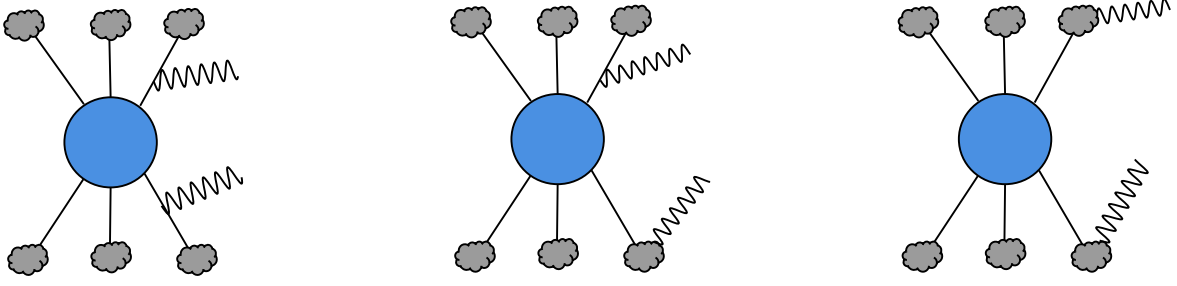$$\int_{-\infty}^{\infty} du N_{zz} = D_{\bar{z}}^2 N, \tag{2.63}$$

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FIG. 1. Diagrams illustrate the different ways to connect two soft graviton modes in the Feynman diagram. The first diagram shows both soft gravitons connected to the external legs. The second diagram shows one of the soft gravitons connected to the external leg while the other is connected to the dressing operator. The last diagram involves both soft gravitons connected to the dressing operator.

where $N_{zz} = \partial_u C_{zz}$ is the Bondi news tensor. $D_z$ refers to the covariant derivative with respect to the two-sphere. After taking this into account, the action of the BMS supertranslation charge $(Q[f])$ on these data can be written as

$$[Q[f], N_{zz}(u, z, \bar{z})] = f(z, \bar{z})\partial_u N_{zz}(u, z, \bar{z}),$$
$$[Q[f], C_{zz}(u, z, \bar{z}] = f(z, \bar{z})\partial_u C_{zz}(u, z, \bar{z}) - 2D_z^2 f(z, \bar{z}),$$
$$[Q[f], N(z, \bar{z})] = 0,$$
$$[Q[f], C(z, \bar{z})] = -2f(z, \bar{z}). \qquad (2.64)$$

In light of the above, we would additionally like to use the assumptions (1), (2), and (3) to determine a set of dressed states. Let us consider an ansatz for the dressed state of the following form:

$$e^{R_N} = \exp\left(\frac{\kappa}{2}\int d[k]\rho(k)\int d^2\hat{q}D_{\bar{z}}^2 N(\hat{q}; k)C(z, \bar{z})\right), \qquad (2.65)$$

where $\rho(k) = b^\dagger(k)b(k)$ is the number operator for the external particles.[8] Here $(z, \bar{z})$ are the coordinates for representing the direction $q$ in the integral over the sphere. $N(\hat{q}; k)$ is an arbitrary real function and $D_{\bar{z}}$ refers to the covariant derivative with respect to the two-sphere.

The action of the dressing operator on a bare state is similar to (2.28) and we can write the dressed state as

$$e^{R_N}|k\rangle = e^{R_N}b^\dagger(k)|0\rangle = [e^{R_N}, b^\dagger(k)]|0\rangle = e^{R_N(k)}b^\dagger(k)|0\rangle, \qquad (2.66)$$

where

$$e^{R_N(k)} = \exp\left(\frac{\kappa}{2}\int d^2\hat{q}D_{\bar{z}}^2 N(\hat{q}; k_i)C(z, \bar{z})\right). \qquad (2.67)$$

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[8]We consider massless scalar particles in this case for convenience.

In this section, for clarity let us start with the supertranslation charge $Q[f]$ for an arbitrary $f(z, \bar{z})$ and then later on in the calculations we will substitute for the particular $f(z, \bar{z})$ in (2.12). The supertranslation Ward identity between such states can be written as

$$\langle\text{out}|e^{-R_N^{(\text{out})}}[Q[f], S]e^{R_N^{(\text{in})}}|\text{in}\rangle = 0, \qquad (2.68)$$

where

$$e^{-R_N^{(\text{out})}} = \exp\left(-\frac{\kappa}{2}\sum_{i=\text{out}}\int d^2\hat{q}D_{\bar{z}}^2 N_i(\hat{q}; k_i)C(z, \bar{z})\right), \qquad (2.69)$$

$$e^{R_N^{(\text{in})}} = \exp\left(\frac{\kappa}{2}\sum_{i=\text{in}}\int d^2\hat{q}D_{\bar{z}}^2 N_i(\hat{q}; k_i)C(z, \bar{z})\right). \qquad (2.70)$$

The above equation can be expanded as

$$\langle\text{out}|Q[f]e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle - \langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}Q[f]|\text{in}\rangle$$
$$+ \langle\text{out}|[e^{-R_N^{(\text{out})}}, Q[f]]Se^{R_N^{(\text{in})}}|\text{in}\rangle$$
$$- \langle\text{out}|e^{-R_N^{(\text{out})}}S[Q[f], e^{R_N^{(\text{in})}}]|\text{in}\rangle = 0. \qquad (2.71)$$

If we use assumption (2) then the hard part of the charge only contributes to the first two terms in the above equation. The last two terms can be evaluated using (2.64). After a bit of calculation one arrives at

$$\kappa\left(\sum_{i=\text{out}}\int d^2\hat{q}N_i(\hat{q}; k_i)D_{\bar{z}}^2 f_i(\hat{q})\right.$$
$$\left.- \sum_{i=\text{in}}\int d^2\hat{q}N_i(\hat{q}; k_i)D_{\bar{z}}^2 f_i(\hat{q})\right)\langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle$$
$$- \kappa\left(\sum_{i=\text{out}}f_i(\hat{k}_i)E_i - \sum_{i=\text{in}}f_i(\hat{k}_i)E_i\right)$$
$$\times \langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle = 0. \qquad (2.72)$$

Now if one chooses $f(z, \bar{z})$ in (2.12), so that the integral over $\hat{q}$ gets localized in a particular direction, say $\hat{p}$, then the above equation simplifies to

$$\kappa\left(\sum_{i=\text{out}}N_i(\hat{p};k_i) - \sum_{i=\text{in}}N_i(\hat{p};k_i)\right)\langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle$$

$$-\frac{\kappa}{2}\left(\sum_{i=\text{out}}S^{(0)}(\hat{p};k_i) - \sum_{i=\text{in}}S^{(0)}(\hat{p};k_i)\right)$$

$$\times\langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle = 0. \tag{2.73}$$

We can now use the assumption (3) to get the constraint

$$\frac{\kappa}{2}\left(\sum_{i=\text{out}}S^{(0)}(\hat{p};k_i) - \sum_{i=\text{in}}S^{(0)}(\hat{p};k_i)\right)$$

$$= \kappa\left(\sum_{i=\text{out}}N_i(\hat{p};k_i) - \sum_{i=\text{in}}N_i(\hat{p};k_i)\right). \tag{2.74}$$

From the above equation it is natural to associate

$$N_i(\hat{p};k_i) = \frac{1}{2}S^{(0)}(\hat{p};k_i) = \frac{\epsilon_{\mu\nu}^+(\hat{p})k_i^\mu k_i^\nu}{2(p/E_p).k_i}. \tag{2.75}$$

Here $\epsilon_{\mu\nu}^+(\hat{p}) = \epsilon_\mu^+(\hat{p})\epsilon_\nu^+(\hat{p})$. Therefore the dressed state can be written as

$$e^{R_N}|k\rangle = e^{R_N(k)}|k\rangle$$

$$= \exp\left(\frac{\kappa}{2}\int d^2\hat{q}D_{\bar{z}}^2\left(\frac{\epsilon_{\mu\nu}^+(\hat{q})k^\mu k^\nu}{(q/E_q).k}\right)C(z,\bar{z})\right)|k\rangle. \tag{2.76}$$

We can again simplify the above expression due to the property

$$D_{\bar{z}}^2\left(\frac{\epsilon_{\mu\nu}^+(\hat{q})k^\mu k^\nu}{(q/E_q).k}\right) = (2\pi)E_k\delta^{(2)}(z-z_k), \tag{2.77}$$

where $(z_k,\bar{z}_k)$ refers to the direction of the external particle. Finally, the dressed particle can be written as

$$e^{R_N(k)}|k\rangle = \exp\left(\frac{\kappa}{2}E_kC(z_k,\bar{z}_k)\right)|k\rangle. \tag{2.78}$$

It is important to note that, unlike the Faddeev-Kulish states, these dressed states are made out of Goldstone modes conjugate to zero modes. Unlike the derivation for Faddeev-Kulish states from supertranslation Ward identity in which one has to neglect the $\mathcal{O}(\Lambda)$ terms where $\Lambda$ is the upper cutoff for the dressing, one need not make this assumption in this present derivation. The relationship between such states and the Faddeev-Kulish states has been analyzed in [2].

## III. FROM ASYMPTOTIC SYMMETRIES TO DRESSED STATES IN QCD

In this section we apply the same analysis as we have done to the case of gravity. We try to find a set of dressed states that are compatible with the Ward identities associated to large gauge transformations in Yang-Mills theory. In Sec. III A, the equivalence of Ward identities of large gauge transformations with the leading soft gluon theorem is reviewed. In Sec. III B we start with a simple ansatz (inspired from gravity) for a dressed state in QCD and constrain the dressing using certain assumptions which we will discuss later in the section. As a check we also verify the orthogonality condition for the dressing operator using the soft gluon theorem in Sec. III C.

### A. Review of single soft gluon theorem in undressed states

Here we briefly review the equivalence between leading single soft gluon theorem and asymptotic symmetries in Yang-Mills theory. The reader can refer to [1,20,21] for further details. The asymptotic charge defined at future null infinity can be written as

$$Q[\alpha] = \frac{1}{g^2}\int_{\mathcal{I}_-^+}\text{tr}(\alpha * \mathcal{F}), \tag{3.1}$$

where $\mathcal{I}_-^+$ is the past of future null infinity. $\mathcal{F}$ is the gauge field strength defined as $\mathcal{F} = \mathcal{F}_{\mu\nu}^aT^a$, where $\mathcal{F}_{\mu\nu}^a = \partial_\mu\mathcal{A}_\nu^a - \partial_\nu\mathcal{A}_\mu^a + gf^{abc}\mathcal{A}_\mu^b\mathcal{A}_\nu^c.\alpha \equiv \alpha^a(z,\bar{z})T^a$, where $\alpha^a(z,\bar{z})$ is an arbitrary function on the sphere and $T^a$ refers to the Lie algebra generator. $g$ is the coupling constant. Using the equation of motion ($D^\mu F_{\mu\nu} = g^2 j_\nu$, where $D^\mu$ is the gauge covariant derivative and $j_\nu$ the matter current) and the asymptotic fall off of the gauge fields $\mathcal{A}_\mu^a$'s near null infinity [1,20,21], we can write the asymptotic charge as sum of a soft part $Q_{\text{soft}}[\alpha]$ and a hard part $Q_{\text{hard}}[\alpha]$. After quantization, one can write the soft charge as [1]

$$Q_{\text{soft}}[\alpha] = -\lim_{E_p\to0}E_p\int d^2w\frac{\sqrt{2}\partial_{\bar{w}}\alpha^b(w,\bar{w})}{2(1+w\bar{w})}$$

$$\times(a_+^b(E_p\hat{p}) + a_-^{b\dagger}(E_p\hat{p})), \tag{3.2}$$

$$= -\lim_{E_p\to0}E_p\int d^2w\frac{\sqrt{2}\partial_w\alpha^b(w,\bar{w})}{2(1+w\bar{w})}$$

$$(a_-^b(E_p\hat{p}) + a_+^{b\dagger}(E_p\hat{p})). \tag{3.3}$$

Here, $\hat{p}$ refers to the direction of the soft gluon labeled by the coordinates $(w,\bar{w})$. Using crossing symmetry, one can rewrite the soft charge so as to include only one of the polarizations in each of the expressions. Then the above expression can be further written as

$$Q_{\text{soft}}[\alpha] = -\lim_{E_p \to 0} E_p \int d^2w \frac{\sqrt{2}\partial_{\bar{w}}\alpha^b(w,\bar{w})}{(1+w\bar{w})} a_+^b(E_p\hat{p}), \quad (3.4)$$

$$= -\lim_{E_p \to 0} E_p \int d^2w \frac{\sqrt{2}\partial_w\alpha^b(w,\bar{w})}{(1+w\bar{w})} a_-^b(E_p\hat{p}). \quad (3.5)$$

Similarly, one can write the action of the hard charge on an external state with momentum $k$ and color $c$ as[9]

$$Q_{\text{hard}}[\alpha]|(k,c)\rangle = g\alpha^b(\hat{k})T_{cd}^b|(k,d)\rangle, \quad (3.6)$$

where $\hat{k}$ refers to the direction of momentum of the external state.

The Ward identity for the large gauge transformation can be written as

$$\langle\text{out}|[Q[\alpha], S]|\text{in}\rangle = 0 \Leftrightarrow \langle\text{out}|[Q_{\text{soft}}[\alpha], S]|\text{in}\rangle$$
$$= -\langle\text{out}|[Q_{\text{hard}}[\alpha], S]|\text{in}\rangle. \quad (3.7)$$

Using the definition of soft and hard charge, the above Ward identity becomes

$$\lim_{E_p \to 0} E_p \int d^2w \frac{\sqrt{2}\partial_w\alpha^b(w,\bar{w})}{(1+w\bar{w})} \langle\text{out}|a_+^b(E_p\hat{p})S|\text{in}\rangle$$
$$= -g\left[\sum_{\text{out}}\alpha^b(\hat{k}_i)T_i^b - \sum_{\text{in}}\alpha^b(\hat{k}_i)T_i^b\right]\langle\text{out}|S|\text{in}\rangle. \quad (3.8)$$

The leading soft gluon theorem[10] [1] for a positive helicity soft gluon of color $a$ and in the direction $\hat{p}$ labeled by the coordinates $(w_p, \bar{w}_p)$ can be written as

$$\lim_{E_p \to 0} E_p\langle\text{out}|a_+^a(E_p\hat{p})S|\text{in}\rangle = gS^{(0)a}(\hat{p}; \{k_i\})\langle\text{out}|S|\text{in}\rangle, \quad (3.9)$$

where

$$S^{(0)a}(\hat{p}; \{k_i\}) = \sum_{i=\text{out}}S^{(0)a}(\hat{p}; k_i) - \sum_{i=\text{in}}S^{(0)a}(\hat{p}; k_i), \quad (3.10)$$

with

$$S^{(0)a}(\hat{p}; k_i) \equiv \frac{\epsilon^+(\hat{p}) \cdot k_i}{(p/E_p) \cdot k_i}T_i^a, \quad (3.11)$$

where $p \equiv E_p(1, \hat{p})$ denotes the four-momentum of the soft gluon and $\epsilon^+(\hat{p})$ refers to the polarization vector of the soft

gluon, which is given by $\epsilon^+(\hat{p}) = 1/\sqrt{2}(\bar{w}_p, 1, -i, -\bar{w}_p)$. $k_i$ denotes the four-momentum and $T_i^a$ denotes the Lie algebra generator in the representation of the $i$th hard particle.

If one chooses a particular $\alpha(w, \bar{w})$ which is

$$\alpha = \alpha^b(w, \bar{w})T^b = \frac{\delta^{ab}(1 + w_p\bar{w}_p)}{w - w_p}T^b \quad (3.12)$$

then the Ward identity (3.8) matches with the leading soft gluon theorem (3.9). For this particular choice of $\alpha$, we denote the soft and hard charges as $Q_{\text{soft}}^a(\hat{p})$ and $Q_{\text{hard}}^a(\hat{p})$, respectively, and are given by

$$Q_{\text{soft}}^a(\hat{p}) = \lim_{E_p \to 0} E_p a_+^a(E_p\hat{p}). \quad (3.13)$$

$$Q_{\text{hard}}^a(\hat{p})|(k,b)\rangle = -g(S^{(0)a}(\hat{p}; k))_{bc}|(k,c)\rangle. \quad (3.14)$$

We will be using $Q_{\text{soft}}^a(\hat{p})$ and $Q_{\text{hard}}^a(\hat{p})$ in the rest of our calculations.

It is important to note that in this work we are working with tree-level asymptotic charges only. Although we restrict our attention to the case where quarks are massless (as the understanding of QCD asymptotic symmetries is most developed in this context), the analysis of Sec. IV will not rely on this assumption.[11] We will be working with a gauge group $SU(N)$ in which the gauge generators satisfy

$$[T^a, T^b] = if^{abc}T^c \quad (3.15)$$

and are normalized as $\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The gluons transform in the adjoint representation $[(t^a)_{bc} = if^{abc}t^c]$ while quarks transform in the fundamental representation.

### B. Dressed states from Ward identity

In Sec. II C, it was shown that under certain assumptions one can recover a set of dressed states in perturbative gravity from asymptotic symmetries. In this section we will proceed along the same lines. We start with the following assumptions:

(1) (Non-Abelian) Large gauge transformation is a symmetry of the $S$ matrix. Because of the non-Abelian nature of the asymptotic symmetry, this assumption is more subtle than the corresponding assumption in gravity where supertranslations generate an Abelian group.

We quantify this assumption as a hierarchy of Ward identities

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[9]The above mentioned action of the hard charge is for tree level only. In this work, we are only considering the action of the hard charge at tree level and therefore the loop corrections to the charge are not considered.

[10]We are considering the tree-level $S$ matrix.

[11]As an aside, we note that assuming that quarks are massless is not an unreasonable assumption in the context of perturbative QCD where the $S$ matrix is well defined.

$$[Q^{a_1}(\hat{p}_1), [Q^{a_2}(\hat{p}_2), [\ldots[Q^{a_n}(\hat{p}_n), S]\ldots]] = 0 \quad \forall \; n,$$

(3.16)

where $Q^{a_1}(\hat{p}_1) = Q^{a_1}_{\text{soft}}(\hat{p}_1) + Q^{a_1}_{\text{hard}}(\hat{p}_1)$ are defined in (3.13) and (3.14).

(2) Soft gluon modes satisfy orthogonality relations with respect to asymptotic states of the theory. In this context, orthogonality condition means that a finite number of insertion of soft gluons generates a null state (i.e., the resulting state is orthogonal to all the asymptotic states of the theory without soft external gluons). This is equivalent to the equation,

$$\langle\text{out}|Q^{a_1}_{\text{soft}}(\hat{p}_1)Q^{a_2}_{\text{soft}}(\hat{p}_2)\ldots Q^{a_n}_{\text{soft}}(\hat{p}_n)U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle=0,$$

(3.17)

where $U^{\dagger\text{out}}_E$, $U^{\text{in}}_E$ are the dressing operator acting on the out and in states, respectively.

(3) The $S$-matrix elements are nontrivial.

As a warm up, let us first consider the simplest ansatz for the dressing operator which is motivated from the (infrared finite) asymptotic states of QED and gravity.

$$U_E = P_{\bar{E}}\exp\left(g\int d[k]\rho^{bc}(k)\int^E d[q](N^{a\mu}(q;k))_{bc}A^a_\mu(q)\right),$$

(3.18)

where

$$d[k] \equiv \frac{d^3k}{(2\pi)^3 2E_k}, \qquad d[q] \equiv \frac{d^3q}{(2\pi)^3 2E_q},$$

$$\rho^{bc}(k) \equiv b^{\dagger b}(k)b^c(k), \qquad A^a_\mu(q) \equiv a^{a\dagger}_\mu(q) - a^a_\mu(q).$$

(3.19)

Here $k$ refers to the external particle momenta while $q$ refers to the gluon momenta. $b^{\dagger b}(k)$ ($b^b(k)$) is the creation (annihilation) operator for the external particle with momentum $k$ and color $b$. $a^{a\dagger}_\mu(q)$, $a^a_\mu(q)$ are the creation and annihilation operators associated to the gluon field which can be written in the polarization basis as

$$a^{a\dagger}_\mu(q) = \sum_{r=\pm}\epsilon^r_\mu a^{a\dagger}_r(q) \quad \text{and} \quad a^a_\mu(q) = \sum_{r=\pm}\epsilon^{r*}_\mu a^a_r(q).$$

(3.20)

$a^a_r(q)$, $a^{b\dagger}_s(q')$ satisfy the normalization condition:

$$[a^a_r(q), a^{b\dagger}_s(q')] = (2E_q)(2\pi)^3\delta_{rs}\delta(\vec{q}-\vec{q}')\delta^{ab}.$$

(3.21)

$N^{a\mu}(k, q)$ is an arbitrary matrix valued function in the color space of external particles which has a pole in $E_q$. There is an upper cutoff $E$ for the dressing operator which ensures

that only low energy gluons are included in the dressing. Unlike gravity, due to the non-Abelian nature we choose a particular ordering of the operators $A^a_\mu(q)$ in (3.18). $A^a_\mu(q)$'s are ordered in such a way that the lowest energy operator will act first on the external particles. $\bar{P}_E$ denotes this energy ordering.

The action of the dressing operator (3.18) on an external single particle state with momentum $k$ and color index $b$ can be written as

$$U_E|(k, b)\rangle = (U_E(k))_{bc}|(k, c)\rangle,$$

(3.22)

where $(U_E(k))_{bc}$ is given by

$$(U_E(k))_{bc} = \left(\bar{P}_E \exp\left(g\int^E d[q](N^{a\mu}(q;k))A^a_\mu(q)\right)\right)_{bc}.$$

(3.23)

In a similar way, the action of the dressing operator on a multiparticle state can be found to be

$$U_E|(k_1, b_1), (k_2, b_2)\ldots(k_n, b_n)\rangle$$
$$= (U_E(k_1))_{b_1 c_1}(U_E(k_2))_{b_2 c_2}$$
$$\ldots(U_E(k_n))_{b_n c_n}|(k_1, c_1), (k_2, c_2)\ldots(k_n, c_n)\rangle. \quad (3.24)$$

Therefore, the dressing operator "factorizes" in the color space of external particles. In the rest of the calculations, we will suppress the color indices and will denote the dressing operator simply by $U_E(k_i)$.

We have defined the dressed state (3.23) to have a hierarchy in the softness of gluon momenta in such a way that the $n$th order term in the dressing operator takes the following form:

$$U_E(k)_{|g^n} = g^n\int^E d[q_1]\ldots\int^{E_{q_{n-1}}} d[q_n]N^{a_1\mu_1}(q_1;k)$$
$$\ldots N^{a_n\mu_n}(q_n;k)A^{a_1}_{\mu_1}(q_1)\ldots A^{a_n}_{\mu_n}(q_n). \quad (3.25)$$

The factor $(n!)$ in the denominator of the expansion of the exponential cancels with the $(n!)$ ways the energy ordering can be taken into account. Having defined the dressed state, let us start with assumption (1),

$$_\text{d}\langle\text{out}|[Q^a(\hat{p}), S]|\text{in}\rangle_\text{d} = 0,$$

(3.26)

where $Q^a(\hat{p}) = Q^a_{\text{soft}}(\hat{p}) + Q^a_{\text{hard}}(\hat{p})$ [which are already defined in (3.13) and (3.14)]. $_\text{d}\langle\text{out}|$ and $|\text{in}\rangle_\text{d}$ represents the dressed outgoing and incoming states which are given by

$$_\text{d}\langle\text{out}| = \langle\text{out}|U^{\dagger\text{out}}_E \quad \text{and} \quad |\text{in}\rangle_\text{d} = U^{\text{in}}_E|\text{in}\rangle.$$

Here,

$$U_E^{\dagger \text{out}} = \prod_{k_i=\text{out}} U_E^{\dagger \text{out}}(k_i) \quad \text{and} \quad U_E^{\text{in}} = \prod_{k_i=\text{in}} U_E^{\text{in}}(k_i).$$

After writing the charge as a sum of the soft and hard parts, we can write the Ward identity as

$$\langle \text{out}|U_E^{\dagger \text{out}}[Q_{\text{soft}}^a(\hat{p}), S]U_E^{\text{in}}|\text{in}\rangle$$
$$+ \langle \text{out}|U_E^{\dagger \text{out}}[Q_{\text{hard}}^a(\hat{p}), S]U_E^{\text{in}}|\text{in}\rangle = 0. \quad (3.27)$$

The first term can be expanded as

$$\langle \text{out}|Q_{\text{soft}}^a(\hat{p})U_E^{\dagger \text{out}}SU_E^{\text{in}}|\text{in}\rangle$$
$$+ \langle \text{out}|[U_E^{\dagger \text{out}}, Q_{\text{soft}}^a(\hat{p})]SU_E^{\text{in}}|\text{in}\rangle$$
$$- \langle \text{out}|U_E^{\dagger \text{out}}S[Q_{\text{soft}}^a(\hat{p}), U_E^{\text{in}}]|\text{in}\rangle. \quad (3.28)$$

Now by assumption (2) the first term in the above expression vanishes. Hence, (3.28) becomes

$$\langle \text{out}|[U_E^{\dagger \text{out}}, Q_{\text{soft}}^a(\hat{p})]SU_E^{\text{in}}|\text{in}\rangle$$
$$- \langle \text{out}|U_E^{\dagger \text{out}}S[Q_{\text{soft}}^a(\hat{p}), U_E^{\text{in}}]|\text{in}\rangle. \quad (3.29)$$

The commutator involved in the above expression has been evaluated in Appendix C, resulting in the following expression:

$$\langle \text{out}|U_E^{\dagger \text{out}}[Q_{\text{soft}}^a(\hat{p}), S]U_E^{\text{in}}|\text{in}\rangle$$
$$= g \lim_{E_p \to 0} E_p(\mathcal{N}_{\text{out}}^a(p; \{k_i\}) - \mathcal{N}_{\text{in}}^a(p; \{k_i\}))$$
$$\times \langle \text{out}|U_E^{\dagger \text{out}}SU_E^{\text{in}}|\text{in}\rangle, \quad (3.30)$$

where

$$\mathcal{N}_{\text{out}}^a(p; \{k_i\}) = \sum_{i=\text{out}} \mathcal{N}^{a\mu}(p; k_i)\epsilon_\mu^+(\hat{p}) \quad \text{and}$$
$$\mathcal{N}_{\text{in}}^a(p; \{k_i\}) = \sum_{i=\text{in}} \mathcal{N}^{a\mu}(p; k_i)\epsilon_\mu^+(\hat{p}). \quad (3.31)$$

Now let us consider the second term in Eq. (3.27), which can be expanded as

$$\langle \text{out}|Q_{\text{hard}}^a(\hat{p})U_E^{\dagger \text{out}}SU_E^{\text{in}}|\text{in}\rangle + \langle \text{out}|[U_E^{\dagger \text{out}}, Q_{\text{hard}}^a(\hat{p})]SU_E^{\text{in}}|\text{in}\rangle$$
$$- \langle \text{out}|U_E^{\dagger \text{out}}SU_E^{\text{in}}Q_{\text{hard}}^a(\hat{p})|\text{in}\rangle$$
$$- \langle \text{out}|U_E^{\dagger \text{out}}S[Q_{\text{hard}}^a(\hat{p}), U_E^{\text{in}}]|\text{in}\rangle. \quad (3.32)$$

The second and fourth terms involve the action of $Q_{\text{hard}}^a(\hat{p})$ on the dressing operator. This action is nontrivial as the hard charge acts on a gluon of arbitrary energy by rotating its color. However, as can be readily verified, the action of $Q_{\text{hard}}^a(\hat{p})$ on the dressing will produce terms which are 1

order higher in the coupling $g$ as compared to the rest of the terms. Thus, if we are interested in determining the dressing at leading order in $g$, these terms vanish. Without a perturbative expansion in $g$, the action of the hard charge on the dressing is nontrivial and we will come back to this issue in Sec. IV B.

The remaining terms can be evaluated by the action of the hard charge on the external state given by (3.14). Therefore, we get

$$\langle \text{out}|U_E^{\dagger \text{out}}[Q_{\text{hard}}^a(\hat{p}), S]U_E^{\text{in}}|\text{in}\rangle$$
$$= -gS^{(0)a}(\hat{p}; \{k_i\})\langle \text{out}|U_E^{\dagger \text{out}}SU_E^{\text{in}}|\text{in}\rangle. \quad (3.33)$$

Using (3.30) and (3.33), the Ward identity finally becomes

$$g[S^{(0)a}(\hat{p}; \{k_i\}) - \lim_{E_p \to 0} E_p(\mathcal{N}_{\text{out}}^a(p; \{k_i\})$$
$$- \mathcal{N}_{\text{in}}^a(p; \{k_i\}))]\langle \text{out}|U_E^{\dagger \text{out}}SU_E^{\text{in}}|\text{in}\rangle = 0. \quad (3.34)$$

Now we use the last assumption (3) and hence,

$$\sum_{i=\text{out}} S^{(0)a}(\hat{p}; k_i) - \sum_{i=\text{in}} S^{(0)a}(\hat{p}; k_i)$$
$$- \lim_{E_p \to 0} E_p(\mathcal{N}_{\text{out}}^a(p; \{k_i\}) - \mathcal{N}_{\text{in}}^a(p; \{k_i\})) = 0, \quad (3.35)$$

which can be also written as

$$\lim_{E_p \to 0} E_p\left(\sum_{i=\text{out}} \mathcal{N}^{a\mu}(p; k_i) - \sum_{i=\text{in}} \mathcal{N}^{a\mu}(p; k_i)\right)\epsilon_\mu^+(\hat{p})$$
$$= \sum_{i=\text{out}} \frac{k_i \cdot \epsilon^+(\hat{p})}{k_i \cdot (p/E_p)} T_i^a - \sum_{i=\text{in}} \frac{k_i \cdot \epsilon^+(\hat{p})}{k_i \cdot (p/E_p)} T_i^a.$$

As in gravity we can now associate,

$$\lim_{E_p \to 0} E_p \mathcal{N}^{a\mu}(p; k_i)\epsilon_\mu^+(\hat{p}) = \frac{k_i \cdot \epsilon^+(\hat{p})}{k_i \cdot (p/E_p)} T_i^a. \quad (3.36)$$

Hence

$$\mathcal{N}^{a\mu}(p; k_i) = \frac{k_i^\mu}{k_i \cdot p} T_i^a. \quad (3.37)$$

Substituting the above expression in the dressing ansatz we finally get the dressed state as

$$U_E|(k, b)\rangle = \bar{P}_E \exp\left(g \int^E d[q] \frac{k^\mu}{k \cdot q} T_k^a A_\mu^a(q)\right)|(k, b)\rangle. \quad (3.38)$$

## C. Orthogonality relations for single soft gluon insertion

We now ask if this dressed state (3.38) is such that a finite number of soft gluon modes satisfy the orthogonality relation (2). As we show below, this is not the case and we have to modify the ansatz.

Let us first consider the orthogonality condition involving one soft gluon mode; i.e.,

$$\langle \text{out}|Q_{\text{soft}}^a(\hat{p})U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle = 0. \quad (3.39)$$

Using the definition of soft charge, Eq. (3.39) can be written as

$$\lim_{E_p \to 0} E_p \langle \text{out}|a_+^a(E_p\hat{p})U_E^{\dagger}\text{out}SU_E^{\text{in}}|\text{in}\rangle = 0. \quad (3.40)$$

The above expression receive contributions from two terms as in gravity as shown in Fig. 2. One of them corresponds to the case when the soft gluon mode is connected to the external legs and the other corresponds to the case when it is connected to the dressing operator. The amplitude with the soft gluon mode connected to the external legs is given by the leading single soft gluon theorem and it evaluates to

$$(\langle \text{out}|Q_{\text{soft}}^a(\hat{p})U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle)_{\text{external}}$$
$$= gS^{(0)a}(\hat{p};\{k_i\})\langle \text{out}|U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle. \quad (3.41)$$

When the soft gluon mode is connected to the dressing operator, the corresponding amplitude can be evaluated by the contraction of the soft operator with the dressing operator,

$$(\langle \text{out}|Q_{\text{soft}}^a(\hat{p})U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle)_{\text{dressing}}$$
$$= \langle \text{out}|[Q_{\text{soft}}^a(\hat{p}), U_E^{\dagger\text{out}}]SU_E^{\text{in}}|\text{in}\rangle$$
$$+ \langle \text{out}|U_E^{\dagger\text{out}}S[Q_{\text{soft}}^a(\hat{p}), U_E^{\text{in}}]|\text{in}\rangle. \quad (3.42)$$

By using the commutators given in Appendix C, the above term evaluates to

$$(\langle \text{out}|Q_{\text{soft}}^a(\hat{p})U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle)_{\text{dressing}}$$
$$= -gS^{(0)a}(\hat{p};\{k_i\})\langle \text{out}|U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle. \quad (3.43)$$
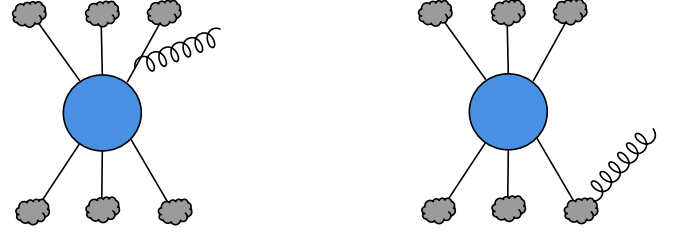
Summing (3.41) and (3.43) we finally get



FIG. 2. The figure demonstrates the different ways to connect a soft gluon mode in the Feynman diagram. The first diagram involves a soft gluon connected to the external leg while the second involves a soft gluon connected to the dressing operator.

$$g[S^{(0)a}(\hat{p};\{k_i\}) - S^{(0)a}(\hat{p};\{k_i\})]\langle \text{out}|U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle = 0. \quad (3.44)$$

We can see that the orthogonality relation involving one soft gluon mode is satisfied.

However, it is rather easy to see that these states do not satisfy the orthogonality condition when more than one soft gluon insertion is present. This is simply because such an orthogonality would imply that no consecutive soft emissions of gluons (that is, emission of one soft gluon from another soft gluon) can take place. In the next section we start with a rather general ansatz and show that the assumptions (1), (2), and (3) will lead to asymptotic states of the theory in which soft gluon correlations are taken into account.

## IV. GENERALIZED DRESSING OPERATOR

As explained in the previous section, the dressing ansatz we started with does not include gluon-gluon correlations and hence cannot satisfy the orthogonality condition involving multiple soft gluon insertions. In this section we modify the ansatz so as to take this into account.

The most natural modification of the ansatz of the previous section is given by

$$U_E|(k,b)\rangle = (U_E(k))_{bc}|(k,c)\rangle, \quad (4.1)$$

where

$$(U_E(k))_{bc} = \left( \bar{P}_E \exp\left( g \int^E d[q_1]\mathcal{N}_{(1)}^{a_1\mu_1}(q_1;k)A_{\mu_1}^{a_1}(q_1) \right.\right.$$
$$+ g^2 \int^E d[q_1] \int^{E_{q_1}} d[q_2]\mathcal{N}_{(2)}^{a_2\mu_2,a_1\mu_1}(q_1,q_2;k)A_{\mu_1}^{a_1}(q_1)A_{\mu_2}^{a_2}(q_2) \right)$$
$$+ g^3 \int^E d[q_1] \int^{E_{q_1}} d[q_2] \int^{E_{q_2}} d[q_3]\mathcal{N}_{(3)}^{a_3\mu_3,a_2\mu_2,a_1\mu_1}(q_1,q_2,q_3;k)A_{\mu_1}^{a_1}(q_1)A_{\mu_2}^{a_2}(q_2)A_{\mu_3}^{a_3}(q_3) \left.\left.\right) + \cdots \right)_{bc}, \quad (4.2)$$

where $\mathcal{N}^{a_1\mu_1}_{(1)}(q_1;k), \mathcal{N}^{a_2\mu_2,a_1\mu_1}_{(2)}(q_1,q_2;k), \mathcal{N}^{a_3\mu_3,a_2\mu_2,a_1\mu_1}_{(3)}(q_1, q_2,q_3;k)\ldots$ are all matrix valued functions in the color space of the external state. The rest of the notations follow from the initial ansatz (3.19).

The first term in the modified dressed state corresponds to the initial dressing we had started with. In order to include the gluon-gluon correlations we consider an infinite series of terms (series in coupling constant $g$) in the exponent.

As indicated earlier, we would like to determine each term in the exponent [at $\mathcal{O}(g^n)$] using the assumptions (1), (2), and (3). In fact, in order to determine the term in the dressing at order $g^n$ we only require the following.[12]

$$[Q^{a_1}(\hat{p}_1),[Q^{a_2}(\hat{p}_2),\ldots[Q^{a_n}(\hat{p}_n),S]\ldots]] = 0 \quad \text{at } \mathcal{O}(g^n),$$
$$\langle\text{out}|Q^{a_1}_{\text{soft}}(\hat{p}_1)\ldots Q^{a_n}_{\text{soft}}(\hat{p}_1)U^{\dagger}_E S U_E|\text{in}\rangle = 0 \quad \text{at } O(g^n).$$
$$(4.3)$$

Let us start with the Ward identity involving one asymptotic charge and demand that our states are such that the identity is satisfied at $\mathcal{O}(g)$:

$$_{\text{d}}\langle\text{out}|[Q^a(\hat{p}),S]|\text{in}\rangle_{\text{d}} = 0, \qquad (4.4)$$

where the out and in states are now dressed by the modified dressing operator. We repeat the same procedure we had done with the earlier ansatz by writing the charge as the sum of the soft and hard parts and separately evaluating these terms.

The soft part can be expanded as

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}),S]U^{\text{in}}_E|\text{in}\rangle$$
$$= \langle\text{out}|Q^a_{\text{soft}}(\hat{p})U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle$$
$$+ \langle\text{out}|[U^{\dagger\text{out}}_E, Q^a_{\text{soft}}(\hat{p})]S U^{\text{in}}_E|\text{in}\rangle$$
$$- \langle\text{out}|U^{\dagger\text{out}}_E S[Q^a_{\text{soft}}(\hat{p}), U^{\text{in}}_E]|\text{in}\rangle. \qquad (4.5)$$

The commutators involved in the term can be evaluated using the results in Appendix C [(C5) and (C6)] and the above expression evaluates to

$$\langle\text{out}|Q^a_{\text{soft}}(\hat{p})U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle + g\lim_{E_p\to 0}E_p(\mathcal{N}^a_{(1)\text{out}}(p;\{k_i\})$$
$$- \mathcal{N}^a_{(1)\text{in}}(p;\{k_i\}))\langle\text{out}|U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle + \mathcal{O}(g^2), \qquad (4.6)$$

where $\mathcal{N}^a_{(1)\text{out}}(p;\{k_i\}), \mathcal{N}^a_{(1)\text{in}}(p;\{k_i\})$ is defined as

$$\mathcal{N}^a_{(1)\text{out}}(p;\{k_i\}) = \sum_{i=\text{out}}\mathcal{N}^{a\mu}_{(1)}(p;k_i)\epsilon^+_\mu(\hat{p}) \quad \text{and}$$
$$\mathcal{N}^a_{(1)\text{in}}(p;\{k_i\}) = \sum_{i=\text{in}}\mathcal{N}^{a\mu}_{(1)}(p;k_i)\epsilon^+_\mu(\hat{p}). \qquad (4.7)$$

The hard part of Eq. (4.4) evaluates to

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{hard}}(\hat{p}),S]U^{\text{in}}_E|\text{in}\rangle$$
$$= -gS^{(0)a}(\hat{p};\{k_i\})\langle\text{out}|U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle. \qquad (4.8)$$

In deriving the above equation, we have used the fact that at $O(g)$ the action of $Q^a_{\text{hard}}(\hat{p})$ on the dressing operator is zero.

Therefore the Ward identity finally becomes

$$g[-S^{(0)a}(\hat{p};\{k_i\}) + \lim_{E_p\to 0}E_p(\mathcal{N}^a_{(1)\text{out}}(p;\{k_i\})$$
$$- \mathcal{N}^a_{(1)\text{in}}(p;\{k_i\}))]\langle\text{out}|U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle$$
$$+ \langle\text{out}|Q^a_{\text{soft}}(\hat{p})U^{\dagger\text{out}}_E S U^{\text{in}}_E|\text{in}\rangle + \mathcal{O}(g^2) = 0. \qquad (4.9)$$

Now if we use our assumption that the dressed states are such that the orthogonality condition for one soft gluon mode holds at $\mathcal{O}(g)$ and the assumption (3) then we finally get the conservation law

$$S^{(0)a}(\hat{p};\{k_i\}) - \lim_{E_p\to 0}E_p(\mathcal{N}^a_{(1)\text{out}}(p;\{k_i\})$$
$$- \mathcal{N}^a_{(1)\text{in}}(p;\{k_i\}) = 0, \qquad (4.10)$$

which is same as the initial conservation law (3.34) we got with the simplest ansatz we started with. Hence, from this conservation law we can write

$$\mathcal{N}^{a\mu}_{(1)}(p;k_i) = \frac{k^\mu_i}{k_i \cdot p}T^a_i. \qquad (4.11)$$

### A. Nested Ward identity

Having determined the first term in the dressing ansatz using the assumptions, we now use the same to extract the second order term in the ansatz. In order to proceed let us consider the Ward identity involving two asymptotic charges:

$$[Q^a(\hat{p}_1),[Q^b(\hat{p}_2),S]] = 0. \qquad (4.12)$$

We now use this Ward identity at $\mathcal{O}(g^2)$ to determine $\mathcal{N}^{a_2\mu_2,a_1\mu_1}_{(2)}(q_1,q_2;k)$:

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a(\hat{p}_1),[Q^b(\hat{p}_2),S]]U^{\text{in}}_E|\text{in}\rangle = 0. \qquad (4.13)$$

The charges can be written as the sum of the soft and hard parts and the above expression becomes

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1) + Q^a_{\text{hard}}(\hat{p}_1),[Q^b_{\text{soft}}(\hat{p}_2)$$
$$+ Q^b_{\text{hard}}(\hat{p}_2),S]]U^{\text{in}}_E|\text{in}\rangle = 0, \qquad (4.14)$$

where the soft and hard charges are defined in (3.13) and (3.14):

---

[12]We are assuming a particular ordering in energy associated to the soft charge $E_{p_1} > E_{p_2} > \ldots E_{p_n}$.

$$Q_{\text{soft}}^a(\hat{p}_1) = \lim_{E_{p_1} \to 0} E_{p_1} a_+^a(E_{p_1}\hat{p}_1), \qquad (4.15)$$

$$Q_{\text{hard}}^a(\hat{p}_1)|(k,b)\rangle = -g(S^{(0)a}(\hat{p}_1;k))_{bc}|(k,c)\rangle, \quad (4.16)$$

$$Q_{\text{soft}}^b(\hat{p}_2) = \lim_{E_{p_2} \to 0} E_{p_2} a_+^b(E_{p_2}\hat{p}_2), \qquad (4.17)$$

$$Q_{\text{hard}}^b(\hat{p}_2)|(k,c)\rangle = -g(S^{(0)b}(\hat{p}_2;k))_{cd}|(k,d)\rangle. \quad (4.18)$$

Using the action of charges, the expression (4.14) finally evaluates to [the details are given in Appendix D]

$$g^2\Big(\lim_{E_{p_2}\to 0} E_{p_2} \lim_{E_{p_1}\to 0} E_{p_1} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1,p_2;\{k_i\})\epsilon_\mu^+(\hat{p}_1)\epsilon_\nu^+(\hat{p}_2)$$
$$- S^{(0)b}(\hat{p}_2;p_1))_{ac}S^{(0)c}(\hat{p}_1;\{k_i\}))\langle\text{out}|U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle$$
$$+ \langle\text{out}|Q_{\text{soft}}^a(\hat{p}_1)Q_{\text{soft}}^b(\hat{p}_2)U_E^{\dagger\text{out}}SU_E^{\text{in}}|\text{in}\rangle + \mathcal{O}(g^3) = 0.$$
$$(4.19)$$

Now we can use the assumption (2) and demand that the orthogonality involving two soft gluon modes holds at $\mathcal{O}(g^2)$; then we get the following conservation law:

$$\lim_{E_{p_2}\to 0} E_{p_2} \lim_{E_{p_1}\to 0} E_{p_1} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1,p_2;\{k_i\})\epsilon_\mu^+(\hat{p}_1)\epsilon_\nu^+(\hat{p}_2)$$
$$= S^{(0)b}(\hat{p}_2;p_1))_{ac}S^{(0)c}(\hat{p}_1;\{k_i\}). \qquad (4.20)$$

It is important to note that the $S^{(0)b}(\hat{p}_2;p_1))_{ac}$ represents the soft factor associated to one soft gluon being emitted from another soft gluon. Therefore, the matrix involved in the soft factor will be in the adjoint representation. Using these we can finally write

$$\mathcal{N}_{(2)}^{a\mu,b\nu}(p_1,p_2;k_i) = \frac{p_2^\mu}{p_1.p_2}\frac{k_i^\nu}{k_i.p_1} if^{bac}T_{k_i}^c. \qquad (4.21)$$

This procedure can be done recursively to extract the $n$th order term in the exponent of the modified dressed state. For this, one needs to evaluate the nested Ward identity involving $n$ asymptotic charges along with the orthogonality condition for $n$ soft gluons. It turns out that the $\mathcal{N}_{(n)}^{a_n\mu_n,\dots,a_1\mu_1}(q_1,q_2,\dots,q_n;k)$-term will be picked by the commutators involving $n$ soft charges with the dressing operator, i.e., $[Q_{\text{soft}}^{a_1}(\hat{p}_1),[Q_{\text{soft}}^{a_2}(\hat{p}_2),[\dots[Q_{\text{soft}}^{a_n}(\hat{p}_n), U_E]\dots]$ and along with the commutators of the hard operator with the soft operator (D7) one is able to determine $\mathcal{N}_{(n)}^{a_n\mu_n,\dots,a_1\mu_1}(q_1,q_2,\dots,q_n;k)$. For example $\mathcal{N}_{(3)}^{a_3\mu_3,a_2\mu_2,a_1\mu_1} \times (q_1,q_2,q_3;k)$ can be found to be

$$\mathcal{N}_{(3)}^{a_3\mu_3,a_2\mu_2,a_1\mu_1}(q_1,q_2,q_3;k)$$
$$= \frac{k^{\mu_1}}{q_1\cdot k}\cdot\frac{q_1^{\mu_2}}{q_2\cdot q_1}\cdot\frac{q_1^{\mu_3}}{q_3\cdot q_1}(if^{a_1a_3a_4})(if^{a_4a_2a_5})T_k^{a_5}$$
$$+ \frac{k^{\mu_1}}{q_1\cdot k}\cdot\frac{q_1^{\mu_2}}{q_2\cdot q_1}\cdot\frac{q_2^{\mu_3}}{q_3\cdot q_2}(if^{a_2a_3a_4})(if^{a_1a_4a_5})T_k^{a_5}.$$
$$(4.22)$$

Note that from the Ward identities in Eq. (4.3), only the pole piece of each of the coefficients $\mathcal{N}_{(n)}^{a_n\mu_n,\dots,a_1\mu_1}(q_1, q_2,\dots,q_n;k)$ of the generalized dressing operator can be determined. Also, the orthogonality condition in Eq. (4.3) can be verified using the similar procedure in Sec. II E. One needs to use the consecutive multisoft gluon theorems in order to do so.

It is interesting to note that the ansatz we have started with has already been identified by Catani *et al.* in [8]. They showed that such states naturally emerge from the Faddeev-Kulish approach of asymptotic dynamics and the $S$ matrix between such states are shown to be infrared finite at the leading order. The properties of such states have been extensively studied in [8,22–27].

### B. Some thoughts on the color rotation of dressing operator

Throughout this paper, we faced a thorny issue of evaluating the action of $Q_{\text{hard}}^a(\hat{p})$ on the dressing operator. For the dressed states in gravity, the action of $Q_{\text{hard}}^a(\hat{p})$ on the dressing exponent did not modify the infrared structure of the dressing and in this sense its effect on the dressed states (as far as evaluating $S$-matrix elements are concerned) could be ignored. In QCD, the issue is far more subtle as (a) $Q_{\text{hard}}^a(\hat{p})$ on the dressing exponent rotates colors of the constituent soft gluons and (b) the "rotated dressing" obtained by the action of the hard charge has the same infrared singularity as the original dressing.

In our analysis, we bypassed this issue by considering Ward identities recursively in the coupling $g$ due to which the action of the hard charge on the dressing never appeared in our analysis. However, we will now like to speculate that precisely due to points (a) and (b) mentioned above, the hard charge action on the dressing is trivial as far as its contribution to $S$-matrix elements is concerned.

For simplicity, we consider the (finite) action of the hard charge as opposed to its infinitesimal action. Namely, let us introduce a (formal) group element $U_h(g_\epsilon) = \exp(iQ_h[\epsilon])$.[13] Then,

$$U_h(g_\epsilon)U_E U_h(g_\epsilon^{-1}) = U_E^{g_\epsilon}, \qquad (4.23)$$

---

[13]Recall that this group is infinite dimensional and hence our definition is rather formal, but suffices for the purposes of this section.

where in the new dressing each soft gluon is rotated in the color space by $g_e$. However, due to (a) and (b) and using the analysis of [19], we find it plausible that a dressed state obtained from $U_E^{g_e}$ will not be in the asymptotic Hilbert space obtained by using the original dressing operator $U_E$. Stated differently,

$$\langle \text{out}|U_E^\dagger S U_E^{g_e}|\text{in}\rangle = 0. \qquad (4.24)$$

This is because the exponential suppression due to virtual infrared soft factors is not canceled by the dressing as the incoming and outgoing states are dressed by inequivalent (color rotated) dressings. A detailed analysis of these speculations remain outside the scope of the paper.

## V. CONCLUSIONS

In this paper, we tried to build upon the work of [2–5] in the context of the perturbative QCD $S$ matrix. Our goal was to find out the extent to which (leading) asymptotic symmetries of QCD can be used to determine the asymptotic states of the theory. In order to do this, we revisited the analysis in gravity [4] and extracted out the minimal set of conditions under which supertranslation symmetries implied the existence of dressed states in gravity. We used these ideas to derive the structure of dressed states in QCD from asymptotic Ward identities. Under certain technical conditions, our analysis led us to the conclusion that the vacuum structure of QCD, which is consistent with asymptotic conservation laws, is equivalent to that derived by Catani *et al.* in [8]. We would once again like to emphasize that our analysis only relied on certain orthogonality conditions (defined in Sec. IV) and Ward identities. Nowhere did we make use of multiple soft gluon theorems.

In addition to the issue of understanding the action of the hard charge on the dressing operator, many important questions remain open. As we reviewed for perturbative gravity, if we do not start with an ansatz where the dressing operator involves contributions of low frequency modes (as opposed to soft modes which are precisely the zero frequency Goldstone modes) then the Ward identity leads us to an alternative dressing operator where the exponent only contains soft modes. This point was discussed in detail in QED [2,5] where it was shown how such an alternative dressing is related to the well-known Faddeev-Kulish dressing. It will be interesting to start with such an ansatz in QCD where dressing only contains soft (zero frequency) gluons and derives the dressing operator. The relationship of such dressings with the generalized coherent states of [8] may shed more light on the relationship between the infrared structure of QCD and asymptotic symmetries. In this work, we have restricted ourselves to charges at tree level. It would be also interesting to include loop-level corrections to the hard charge and its implications on the dressing operator. We would like to address these issues in a future work.

*Note added.*—While this manuscript was being prepared, a paper [28] appeared which also investigates the relationship between dressed states in QCD and asymptotic symmetries. In that paper, starting from the generalized coherent states of Catani *et al.*, the authors analyze the quantum correction to the asymptotic charges as a result of such a dressing. Our work is rather complementary in the sense that we ask if conservation laws associated to asymptotic symmetries can be used to derive the nature of dressed states when aided by certain well-motivated assumptions. In this sense we believe that the two papers complement each other.

## APPENDIX A: DRESSED STATES FROM SOFT THEOREM IN GRAVITY

In this Appendix we will use the orthogonality condition involving one soft graviton mode as a constraint for determining the dressing operator. In other words, we will the study the implications of this orthogonality condition on the ansatz $e^{R_N}$.

We start with the orthogonality condition

$$\langle \text{out}|Q_{\text{soft}}(\hat{p})e^{-R_N^{(\text{out})}}S e^{R_N^{(\text{in})}}|\text{in}\rangle = 0, \qquad (A1)$$

where $e^{R_N}$ is defined as (2.23).

Note that the lhs of this expression corresponds to two terms. One of the contributions comes from the soft graviton connected to the external legs and the other comes from the soft mode connected to the dressing. The soft graviton connected to the external legs can be computed through the leading single soft theorem in the undressed states and the other can be computed through contraction of the soft mode with the dressing operator $e^{R_N}$.

The external leg contribution can be computed through the leading single soft theorem in the undressed states (2.8) as

$$\left(\langle \text{out}|Q_{\text{soft}}(\hat{p})e^{-R_N^{(\text{out})}}S e^{R_N^{(\text{in})}}|\text{in}\rangle\right)_{\text{external}}$$
$$= \frac{\kappa}{2}S^{(0)}(\hat{p};\{k_i\})\langle \text{out}|e^{-R_N^{(\text{out})}}S e^{R_N^{(\text{in})}}|\text{in}\rangle, \qquad (A2)$$

where $S^{(0)}(\hat{p}; \{k_i\})$ is defined in (2.10). The contribution from the dressing can be evaluated as the contraction of the $Q_{\text{soft}}(\hat{p})$ with the dressing operator, i.e.,

$$
\begin{aligned}
&(\langle\text{out}|Q_{\text{soft}}(\hat{p})e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle)_{\text{dressing}} \\
&= \langle\text{out}|[Q_{\text{soft}}(\hat{p}), e^{-R_N^{(\text{out})}}]Se^{R_N^{(\text{in})}}|\text{in}\rangle \\
&\quad + \langle\text{out}|e^{-R_N^{(\text{out})}}S[Q_{\text{soft}}(\hat{p}), e^{R_N^{(\text{in})}}]|\text{in}\rangle, \\
&= -\kappa \lim_{E_p\to 0} E_p(N_{\text{out}}(p) - N_{\text{in}}(p))\langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle.
\end{aligned}
$$
(A3)

In going from the first line to the second we used the commutation relations (2.41). Here $N_{\text{out}}$ and $N_{\text{in}}$ is defined in (2.42) therefore the total contribution becomes the sum of (A2) and (A3). Hence,

$$
\begin{aligned}
&\langle\text{out}|Q_{\text{soft}}(\hat{p})e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle \\
&= \kappa\left(-\lim_{E_p\to 0}E_p(N_{\text{out}}(p) - N_{\text{in}}(p)) + \frac{1}{2}S^{(0)}(\hat{p}; \{k_i\})\right) \\
&\quad \times \langle\text{out}|e^{-R_N^{(\text{out})}}Se^{R_N^{(\text{in})}}|\text{in}\rangle = 0.
\end{aligned}
$$
(A4)

If we use the assumption (3) we get

$$
\lim_{E_p\to 0}E_p(N_{\text{out}}(p) - N_{\text{out}}(p)) = \frac{1}{2}S^{(0)}(\hat{p}; \{k_i\}),
$$
(A5)

which is the same constraint as (2.49).

To conclude, in perturbative gravity we can equivalently use the single orthogonality relation as well as single soft graviton theorem in the undressed states to constrain the dressing.

## APPENDIX B: ACTION OF HARD CHARGE ON DRESSED STATES

In this section we give an argument for our assumption (4). We are interested in evaluating the following terms:

$$
\begin{aligned}
&-\langle\text{out}|[e^{-R_N^{(\text{out})}}, Q_{\text{hard}}(\hat{p})]Se^{R_N^{(\text{in})}}|\text{in}\rangle \\
&\quad + \langle\text{out}|e^{-R_N^{(\text{out})}}S[Q_{\text{hard}}(\hat{p}), e^{R_N^{(\text{in})}}]|\text{in}\rangle.
\end{aligned}
$$
(B1)

The commutator in the first term of the above expression can be written as

$$
[e^{-R_N^{(\text{out})}}, Q_{\text{hard}}(\hat{p})] = -Ae^{-R_N^{(\text{out})}},
$$
(B2)

where

$$
A = \left([R_N^{(\text{out})}, Q_{\text{hard}}(\hat{p})] - \frac{[R_N^{(\text{out})}, [R_N^{(\text{out})}, Q_{\text{hard}}(\hat{p})]]}{2!} + \cdots\right).
$$
(B3)

Using the action of hard charge the first and second term in the above expression can be evaluated to

$$
\begin{aligned}
[R_N^{(\text{out})}, Q_{\text{hard}}(\hat{p})] = &-\int^{\Lambda}d[q]S^{(0)}(\hat{p}; q)N^{\mu\nu}(q; k_i)(a_{\mu\nu}^{\dagger}(q) \\
&+ a_{\mu\nu}(q)),
\end{aligned}
$$
(B4)

$$
\begin{aligned}
&[R_N^{(\text{out})}, [R_N^{(\text{out})}, Q_{\text{hard}}(\hat{p})]] \\
&= \int^{\Lambda}d[q]S^{(0)}(\hat{p}; q)N^{\mu\nu}(q; k_i)N^{\rho\sigma}(q; k_i)I_{\mu\nu\rho\sigma},
\end{aligned}
$$
(B5)

where

$$
I_{\mu\nu\rho\sigma} \equiv \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma},
$$
(B6)

and $S^{(0)}(\hat{p}; q)$ is already defined in (2.9). It is not difficult to see that both of the integrals in the above expressions are of the order $\mathcal{O}(\Lambda)$ where $\Lambda$ is the upper cutoff of the dressing operator. If the cutoff is assumed to be significantly small so that only low energy gravitons constitute the dressing then these integrals are vanishing. Hence,

$$
[e^{-R_N^{(\text{out})}}, Q_{\text{hard}}(\hat{p})] = \mathcal{O}(\Lambda).
$$
(B7)

A similar analysis also holds for $[Q_{\text{hard}}(\hat{p}), e^{R_N^{(\text{in})}}]$.

## APPENDIX C: ACTION OF SOFT CHARGE ON THE DRESSING OPERATOR

In this section we will derive the commutation relations of the soft gluon charge $Q_{\text{soft}}^a(\hat{p})$ with the dressing operator. Let us first consider the dressing operator:

$$
U_E(k) = \bar{P}_E\exp\left(g\int^{E}d[q]\mathcal{N}^{a\mu}(q;k)A_{\mu}^a(q)\right),
$$
(C1)

where $A_{\mu}^a(q) = a_{\mu}^{a\dagger}(q) - a_{\mu}^a(q)$ and $d[q] = \frac{d^3q}{(2\pi)^3(2E_q)}$.

As explained in Sec. III B, due to the non-Abelian nature the operators in (C1) are strictly ordered in energy. The gluon operator with the lowest energy will act first. The $n$th order expansion of the dressing operator can be written as

$$
\begin{aligned}
U_E(k)|_{g^n} = g^n\int^{E}d[q_1]\ldots\int^{E_{q_{n-1}}}d[q_n]\mathcal{N}^{a_1\mu_1}(q_1;k) \\
\ldots\mathcal{N}^{a_n\mu_n}(q_n;k)A_{\mu_1}^{a_1}(q_1)\ldots A_{\mu_n}^{a_n}(q_n).
\end{aligned}
$$
(C2)

Let us consider the commutator of the soft operator $Q_a^{\text{soft}}(\hat{p})$ [defined as in (3.13)] on the $n$th order term in the dressing operator; i.e., we consider

$$
\begin{aligned}
&[Q_{\text{soft}}^a(\hat{p}), g^n\int^{E}d[q_1]\ldots\int^{E_{q_{n-1}}}d[q_n]\mathcal{N}^{a_1\mu_1}(q_1;k) \\
&\ldots\mathcal{N}^{a_n\mu_n}(q_n;k)A_{\mu_1}^{a_1}(q_1)\ldots A_{\mu_n}^{a_n}(q_n)].
\end{aligned}
$$
(C3)

Since the integral is ordered in energy with the lowest energy gluon operator on the right, the soft operator will act only on the rightmost operator. Hence,

$$[Q^a_{\text{soft}}(\hat{p}), U_E(k)_{|g^n}] = gU_E(k)_{|g^{n-1}} \lim_{E_p \to 0} E_p \mathcal{N}^{a\mu}(p;k)\epsilon^+_\mu(\hat{p}). \tag{C4}$$

It is now easy to see that

$$[Q^a_{\text{soft}}(\hat{p}), U_E(k)] = gU_E(k) \lim_{E_p \to 0} E_p \mathcal{N}^{a\mu}(p;k)\epsilon^+_\mu(\hat{p}). \tag{C5}$$

Similarly one can also see that

$$[Q^a_{\text{soft}}(\hat{p}), U^\dagger_E(k)] = -g \lim_{E_p \to 0} E_p \mathcal{N}^{a\mu}(p;k)\epsilon^+_\mu(p)U^\dagger_E(k). \tag{C6}$$

If we now consider the modified dressing operator defined as

$$U_E(k) = \bar{P}_E \exp\left(g \int^E d[q_1]\mathcal{N}^{a_1\mu_1}_{(1)}(q_1;k)A^{a_1}_{\mu_1}(q_1)\right.$$
$$+ g^2 \int^E d[q_1] \int^{E_{q_1}} d[q_2]\mathcal{N}^{a_2\mu_2,a_1\mu_1}_{(2)}$$
$$\left. \times (q_1,q_2;k)A^{a_1}_{\mu_1}(q_1)A^{a_2}_{\mu_2}(q_2) + \dots\right). \tag{C7}$$

One could use the same analysis for the commutation relation of the soft operator with the modified dressing operator and verify that

$$[Q^a_{\text{soft}}(\hat{p}), U_E(k)] = gU_E(k) \lim_{E_p \to 0} E_p \mathcal{N}^{a\mu}(p;k)\epsilon^+_\mu(\hat{p})$$
$$+ \mathcal{O}(g^2) + \dots. \tag{C8}$$

Using this result we can also find the nested commutator.

$$[Q^a_{\text{soft}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2), U_E(k)]]$$
$$= g^2 U_E(k) \lim_{E_{p_2} \to 0} E_{p_2} \lim_{E_{p_1} \to 0} E_{p_1} \mathcal{N}^{a\mu}(p_1;k)\mathcal{N}^{b\nu}$$
$$\times (p_2;k)\epsilon^+_\mu(\hat{p}_1)\epsilon^+_\nu(\hat{p}_2)$$
$$+ g^2 U_E(k) \lim_{E_{p_2} \to 0} E_{p_2} \lim_{E_{p_1} \to 0} E_{p_1} \mathcal{N}^{a\mu,b\nu}_{(2)}$$
$$\times (p_1,p_2;k)\epsilon^+_\mu(\hat{p}_1)\epsilon^+_\nu(\hat{p}_2) + \mathcal{O}(g^3). \tag{C9}$$

## APPENDIX D: NESTED WARD IDENTITY WITH DRESSED STATES

In this section we evaluate (4.14) using the action of the charges. We start with the lhs of (4.14)

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1) + Q^a_{\text{hard}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2) + Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle, \tag{D1}$$

which can be written as

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2) + Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle + \langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{hard}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2) + Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle. \tag{D2}$$

Before proceeding with the computation, we note that the action of $Q^a_{\text{hard}}(\hat{p})$ on the dressing operator will produce terms of $O(g^2)$. This, together with the action of $Q^a_{\text{soft}}(\hat{p})$ on the external states or the dressing operator, will contribute at $O(g^3)$. Therefore, the terms involving the action of $Q^a_{\text{hard}}(\hat{p})$ on $U^{\dagger\text{out}}_E$ or $U^{\text{in}}_E$ in (D2) will not contribute at $O(g^2)$.

Let us consider the second term in the above equation first. Since the hard charge will only act on the external states, this becomes

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{hard}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2) + Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle = gS^{(0)a}(\hat{p}_1;\{k_i\})\langle\text{out}|U^{\dagger\text{out}}_E[Q^b(\hat{p}_2), S]U^{\text{in}}_E|\text{in}\rangle, = 0. \tag{D3}$$

In going from the first line to the second we used the Ward identity involving one asymptotic charge. Therefore in (D2) we are left with evaluating

$$\langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2) + Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle$$
$$= \langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1), [Q^b_{\text{hard}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle + \langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1), [Q^b_{\text{soft}}(\hat{p}_2), S]]U^{\text{in}}_E|\text{in}\rangle. \tag{D4}$$

The first term in the above expression can be expanded as

$$\langle\text{out}|Q^b_{\text{hard}}(\hat{p}_2)[U^{\dagger\text{out}}_E, Q^a_{\text{soft}}(\hat{p}_1)]SU^{\text{in}}_E|\text{in}\rangle - \langle\text{out}|[U^{\dagger\text{out}}_E, Q^a_{\text{soft}}(\hat{p}_1)]SU^{\text{in}}_E Q^b_{\text{hard}}(\hat{p}_2)|\text{in}\rangle$$
$$- \langle\text{out}|Q^b_{\text{hard}}(\hat{p}_2)U^{\dagger\text{out}}_E S[Q^a_{\text{soft}}(\hat{p}_1), U^{\text{in}}_E]|\text{in}\rangle + \langle\text{out}|U^{\dagger\text{out}}_E S[Q^a_{\text{soft}}(\hat{p}_1), U^{\text{in}}_E]Q^b_{\text{hard}}(\hat{p}_2)|\text{in}\rangle$$
$$+ \langle\text{out}|U^{\dagger\text{out}}_E[Q^a_{\text{soft}}(\hat{p}_1), Q^b_{\text{hard}}(\hat{p}_2)]SU^{\text{in}}_E|\text{in}\rangle + \langle\text{out}|U^{\dagger\text{out}}_E S[Q^b_{\text{hard}}(\hat{p}_2), Q^a_{\text{soft}}(\hat{p}_1)]U^{\text{in}}_E|\text{in}\rangle, \tag{D5}$$

which can be evaluated to

$$\langle \text{out}| U_E^{\dagger\text{out}} [Q_{\text{soft}}^a(\hat{p}_1), [Q_{\text{hard}}^b(\hat{p}_2), S]] U_E^{\text{in}} |\text{in}\rangle$$

$$= -(g^2 \lim_{E_{p_1}\to 0} E_{p_1} \mathcal{N}_{(1)}^{a\mu}(p_1; \{k_j\}) \epsilon_\mu^+(p_1) S^{(0)b}(\hat{p}_2; \{k_i\})$$

$$+ \lim_{E_{p_2}\to 0} E_{p_2} (S^{(0)b}(\hat{p}_2; p_1))_{ac} \mathcal{N}_{(1)}^{c\mu}(p_1; \{k_i\}) \epsilon_\mu^+(p_2)) \langle \text{out}| U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \mathcal{O}(g^3),$$

$$= -g^2 (S^{(0)a}(\hat{p}_1; \{k_j\}) S^{(0)b}(\hat{p}_2; \{k_i\})$$

$$+ (S^{(0)b}(\hat{p}_2; p_1))_{ac} S^{(0)c}(\hat{p}_1; \{k_i\})) \langle \text{out}| U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \mathcal{O}(g^3). \tag{D6}$$

For evaluating the last two terms in the expression (D5) we have used the commutator

$$[Q_{\text{soft}}^a(\hat{p}_1), Q_{\text{hard}}^b(\hat{p}_2)] = -g(S^{(0)b}(\hat{p}_2; p_1))_{ac} Q_{\text{soft}}^c(\hat{p}_1). \tag{D7}$$

(D7) can be easily derived using the definition of $Q_{\text{soft}}^a(\hat{p}_1)$ and $Q_{\text{hard}}^b(\hat{p}_2)$ in Eqs. (3.13) and (3.14)[14]

$$[Q_{\text{soft}}^a(\hat{p}_1), Q_{\text{hard}}^b(\hat{p}_2)] = [\lim_{E_p\to 0} E_{p_1} a_+^a(E_{p_1}\hat{p}_1), Q_{\text{hard}}^b(\hat{p}_2)]$$

$$= \lim_{E_p\to 0} E_{p_1} [a_+^a(E_{p_1}\hat{p}_1), Q_{\text{hard}}^b(\hat{p}_2)]$$

$$= -\lim_{E_{p_1}\to 0} E_{p_1} g(S^{(0)b}(\hat{p}_2, p_1))_{ac} a_+^c(E_{p_1}\hat{p}_1)$$

$$= -g(S^{(0)b}(\hat{p}_2, p_1))_{ac} Q_{\text{soft}}^c(\hat{p}_1). \tag{D8}$$

Similarly using the orthogonality condition involving one soft gluon mode, the second term in (D4) reduces to

$$\langle \text{out}| [[U_E^{\dagger\text{out}}, Q_{\text{soft}}^a(\hat{p}_1)], Q_{\text{soft}}^b(\hat{p}_2)] S U_E^{\text{in}} |\text{in}\rangle + \langle \text{out}| U_E^{\dagger\text{out}} S [Q_{\text{soft}}^b(\hat{p}_2), [Q_{\text{soft}}^a(\hat{p}_1), U_E^{\text{in}}]] |\text{in}\rangle$$

$$- \langle \text{out}| [U_E^{\dagger\text{out}}, Q_{\text{soft}}^a(\hat{p}_1)] S [Q_{\text{soft}}^b(\hat{p}_2), U_E^{\text{in}}] |\text{in}\rangle - \langle \text{out}| [U_E^{\dagger\text{out}}, Q_{\text{soft}}^b(\hat{p}_2)] S [Q_{\text{soft}}^a(\hat{p}_1), U_E^{\text{in}}] |\text{in}\rangle$$

$$+ \langle \text{out}| Q_{\text{soft}}^a(\hat{p}_1) Q_{\text{soft}}^b(\hat{p}_2) U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle. \tag{D9}$$

The commutators in the given expression have already been derived in Appendix C [(C8) and (C9)]. Hence, the above expression can be evaluated to be

$$g^2 \lim_{E_{p_2}\to 0} E_{p_2} \lim_{E_{p_1}\to 0} E_{p_1} (\mathcal{N}_{(1)}^{a\mu}(p_1; \{k_j\}) \mathcal{N}_{(1)}^{b\nu}(p_2; \{k_i\}) + \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; \{k_i\})) \epsilon_\mu^+(p_1) \epsilon_\nu^+(p_2)$$

$$\times \langle \text{out}| U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \langle \text{out}| Q_{\text{soft}}^a(\hat{p}_1) Q_{\text{soft}}^b(\hat{p}_2) U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle$$

$$= g^2 (S^{(0)a}(\hat{p}_1; \{k_j\}) S^{(0)b}(\hat{p}_2; \{k_i\}) + \lim_{E_{p_2}\to 0} E_{p_2} \lim_{E_{p_1}\to 0} E_{p_1} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; \{k_i\}) \epsilon_\mu^+(\hat{p}_1) \epsilon_\nu^+(\hat{p}_2))$$

$$\times \langle \text{out}| U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \langle \text{out}| Q_{\text{soft}}^a(\hat{p}_1) Q_{\text{soft}}^b(\hat{p}_2) U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \mathcal{O}(g^3), \tag{D10}$$

where

$$\mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; \{k_i\}) = \sum_{i=\text{out}} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; k_i) - \sum_{i=\text{in}} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; k_i). \tag{D11}$$

Therefore, summing the contributions (D6) and (D10) we finally get (D1) as

$$g^2 (\lim_{E_{p_2}\to 0} E_{p_2} \lim_{E_{p_1}\to 0} E_{p_1} \mathcal{N}_{(2)}^{a\mu,b\nu}(p_1, p_2; \{k_i\}) \epsilon_\mu^+(\hat{p}_1) \epsilon_\nu^+(\hat{p}_2) - S^{(0)b}(\hat{p}_2; p_1))_{ac} S^{(0)c}(\hat{p}_1; \{k_i\}))$$

$$\times \langle \text{out}| U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \langle \text{out}| Q_{\text{soft}}^a(\hat{p}_1) Q_{\text{soft}}^b(\hat{p}_2) U_E^{\dagger\text{out}} S U_E^{\text{in}} |\text{in}\rangle + \mathcal{O}(g^3). \tag{D12}$$

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[14]It is important to note that this is valid only at tree level. The hard charge receives loop corrections starting at $\mathcal{O}(g^2)$ and therefore (D7) receives loop corrections. But these will not affect our analysis since such contributions to (D1) will be at $\mathcal{O}(g^3)$ and higher.

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